Reinforcement Problems for Elliptic Equations and Variational Inequalities (*) (**).

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Summary. – *Consider the Dirichlet problem for an elliptic equation in a domain Ω, with coefficients having discontinuity on a surface Γ. Suppose Γ divides Ω into $\Omega_1 \cup \Omega_2$ (Ω_2 the inner core), the thickness of Ω_1 is of order of magnitude ε, and the modulus of ellipticity in Ω_1 is of order magnitude λ_1. The asymptotic behavior of the solution is studied as $\varepsilon \to 0$, $\lambda_1 \to 0$, provided* lim (ε/λ_1) *exists. Other questions of this type are studied both for elliptic equations and for elliptic variational inequalities.*

Introduction.

In this paper we consider elliptic equations, and also elliptic variational inequalities, with piecewise continuous coefficients. The coefficients have jumps along certain hypersurfaces. Thus we can write the domain Ω, where the solution is considered, as a disjoint union of regions Ω_i such that the coefficients of the operator are smooth in each $\bar{\Omega}_i$, but have a jump across the boundary $\partial\Omega_i$.

More specifically we assume that the coefficients of the operator, in one specific region Ω_{i_0}, have an order of magnitude λ_{i_0}, and consider the situation when $\lambda_{i_0} \to 0$ whereas the « thickness » of Ω_{i_0} may shrink to zero at the same time. The region Ω_{i_0} may be considered as a « reinforcement » of $\Omega \setminus \Omega_{i_0}$. The smallness of λ_{i_0} has the interpretation, at least in problems of elasticity, that the reinforcement is made up of an « extremely hard » material.

Such problems of reinforcement for 2-dimensional elliptic equations and variational inequalities have been studied by Caffarelli and Friedman in the dam problem [1] and in elastoplastic problems [2]. In these papers the limiting solutions (as $\lambda_{i_0} \to 0$) have been identified and uniform estimates on the convergence have been obtained.

In this paper we derive L^2 estimates for the rate of convergence. Unlike [1], [2], the present approach applies for any number of dimensions.

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In Part I we consider a boundary reinforcement for elliptic equations. That means that Ω_{i_0} is a layer whose boundary contains the boundary of Ω. For simplicity we take the coefficients in $\Omega\setminus\Omega_{i_0}$ to be continuous. The limiting problem depends on the thickness ε of Ω_{i_0} or, more precisely, on $\lim(\varepsilon/\lambda_0)$. There are actually three types of possible limit problems, and they are dealt with in Sections 2-4; some auxiliary estimates are derived in Section 1.

In Part II we deal with interior reinforcement, that is, $\bar{\Omega}_{i_0}$ lies in Ω. Here again there are several possible limiting solutions, depending on $\lim(\varepsilon/\lambda_0)$.

In Part III we establish results analogous to those of Parts I, II for variational inequalities.

SANCHEZ-PALENCIA [8] has studied the case of interior reinforcement in the special case where the reinforcing material occupies a lense-shaped region Ω_{i_0} around a smooth surface S; Ω_{i_0} shrinks to S as $\lambda_0\to 0$. His limit problems are similar to those obtained in Part. II. He works with H^1 a priori estimates (resulting from the « variational » approach) whereas our approach is based on deriving H^2 a priori estimates; the latter approach can be used to obtain better estimates on the rate of convergence to the limit problem of the corresponding solutions.

In case Ω_{i_0} is a *fixed* domain interior to Ω and $\Omega\setminus\bar{\Omega}_{i_0}$ is connected, the « variational » approach of SANCHEZ-PALENCIA [8] was extended by LIONS [4] to yield an asymptotic series (in λ_0, as $\lambda_0\to 0$) for the solution.

We finally mention that the methods of this paper apply also to interior-boundary reinforcement problems; such problems are studied in [1] where another method is used for deriving the necessary a priori estimates. Our methods (like those of [8]) apply also to parabolic equations.

PART I. BOUNDARY REINFORCEMENT FOR ELLIPTIC EQUATIONS

1. – A priori estimates.

Let Γ, S be $C^{1,1}$ connected hypersurfaces in R^n, such that Γ lies in the interior of S. Let Ω_2 be the bounded domain with boundary Γ, and denote by Ω the bounded domain with boundary S. The boundary of the domain $\Omega_1=\Omega\setminus\bar{\Omega}_2$ is the $\Gamma\cup S$.

Let $(a_{ij}^k(x))$ $(k=1,2)$ be a positive definite matrix with elements in $C^{1+\bar{\varepsilon}}(\bar{\Omega})$ for some $\bar{\varepsilon}>0$, and define the conormal derivative $\partial u/\partial\nu^k$ on Γ by

$$\frac{\partial u}{\partial\nu^k}=\sum_{i,j=1}^{n}a_{ij}^k(x)\cos(x_i,\nu)\frac{\partial u}{\partial x_j}$$

where ν is the normal to Γ pointing toward Ω_1. The vector $\nu^k=\Big(\sum_i a_{ij}^k(x)\cos(x_i,\nu)\Big)_{j=1}^n$ is called the conormal vector.

Let f^k $(k = 1, 2)$ be given functions in $C^{1+\bar{\varepsilon}}(\bar{\Omega})$ and let λ_k be positive numbers. Consider the problem: Find $u = (u^1, u^2)$ satisfying:

$$\lambda_k \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial u^k}{\partial x_j}\right) = f^k \quad \text{in } \Omega_k \ (k = 1, 2), \tag{1.1}$$

$$u^1 = u^2 \quad \text{on } \Gamma, \tag{1.2}$$

$$\lambda_1 \frac{\partial u^1}{\partial \nu^1} = \lambda_2 \frac{\partial u^2}{\partial \nu^2} \quad \text{on } \Gamma, \tag{1.3}$$

$$u^1 = 0 \quad \text{on } S. \tag{1.4}$$

According to [3], this problem has a unique classical solution, that is, u^k is in $C^1(\bar{\Omega}_k) \cap C^2(\Omega_k)$ and (1.1)-(1.4) are satisfied in the usual sense.

A weaker formulation of (1.1)-(1.4) is given by the variational principle: Find $u = (u^1, u^2)$ in $H_0^1(\Omega)$ which minimizes

$$\sum_{k=1}^{2} \int_{\Omega_k} \left(\sum_{i,j=1}^{n} \lambda_k a_{ij}^k \frac{\partial u^k}{\partial x_i}\frac{\partial u^k}{\partial x_j} + 2f^k u^k\right) dx\,, \tag{1.5}$$

or, equivalently, find $u = (u^1, u^2)$ in $H_0^1(\Omega)$ such that

$$\sum_{k=1}^{2} \int_{\Omega_k} \left(\sum_{i,j=1}^{n} \lambda_k a_{ij}^k \frac{\partial u^k}{\partial x_i}\frac{\partial v}{\partial x_j} + f^k v\right) dx = 0 \quad \text{for any } v \in H_0^1(\Omega). \tag{1.6}$$

A unique solution for this problem exists under weaker conditions on a_{ij}^k, f^k and Γ, S than those stated above (see [3] [7]).

In this part I we shall consider the asymptotic behavior of the solution as $\lambda_1 \to 0$. We shall first deal with the case where the thickness of Ω_1 shrinks to zero as $\lambda_1 \to 0$.

Let $h(x)$ be a positive function defined on Γ and let $\mu_x (x \in \Gamma)$ denote the ray initiating at x in the conormal direction $\nu^1(x)$. Denote by x_ε $(\varepsilon > 0)$ the point on μ_x such that $\overline{x_\varepsilon x} = \varepsilon h(x)/\|\nu^1(x)\|$. If ε is sufficiently small then the set of points x_ε, when x varies over Γ, form a manifold, designated by Γ_ε. We shall now assume that

$$S = \Gamma_\varepsilon\,, \tag{1.7}$$

and prove the following lemma.

LEMMA 1.1. – *There exists a positive constant C such that, for all ε, λ_1 sufficiently small,*

$$\int_{\Omega_1} |u^1|^2\, dx + \int_{\Omega_2} |u^2|^2\, dx \le C\frac{\varepsilon^2}{\lambda_1^2} + C, \tag{1.8}$$

$$\lambda_1 \int_{\Omega_1} |Du^1|^2\,dx + \int_{\Omega_2} |Du^2|^2 \leqslant C\frac{\varepsilon}{\lambda_1} + C\,, \tag{1.9}$$

$$\lambda_1 \int_{\Omega_1} |D^2u^1|^2\,dx + \int_{\Omega_2} |D^2u^2|^2\,dx \leqslant C\frac{\varepsilon}{\lambda_1} + C\,. \tag{1.10}$$

Here $D^\alpha u$ is the vector with components $\partial^{|\alpha|}u/\partial x_1^{\alpha_1}\cdots\partial x_k^{\alpha_n}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n = j$.

PROOF. – In what follows C will denote a generic positive constant independent of ε, λ_1. Let x^0 be any point of Γ and let V be a small neighborhood of x^0. Recall that in V we can represent Γ and S in the form

$$\Gamma \cap V: x_i = g_i(s)\,, \qquad s = (s_1, \ldots, s_{n-1}) \qquad (1 \leqslant i \leqslant n)$$

$$S \cap V: x_i = g_i(s) + \varepsilon\nu_i^1 h(s)/\|\nu^1\|^2 \qquad (1 \leqslant i \leqslant n)$$

where $\nu^1 = (\nu_1^1, \ldots, \nu_n^1)$. Assume, for definiteness, that

$$\frac{\partial(x_1, \ldots, x_{n-1})}{\partial(s_1, \ldots, s_{n-1})} \neq 0 \qquad \text{on } \Gamma \cap V.$$

Then we can represent $\Gamma \cap V$ (with perhaps a smaller neighborhood V) in the form

$$x_n = g(x_1, \ldots, x_{n-1})\,,$$

and $S \cap V$ in the form

$$x_n = g(x_1, \ldots, x_{n-1}) - \varepsilon k(x_1, \ldots, x_{n-1}, \varepsilon)$$

where $|D^\alpha g| \leqslant C$, $|D^\alpha k| \leqslant c$ if $|\alpha| \leqslant 2$.

We now introduce a change of coordinates

$$y_i = x_i \quad (1 \leqslant i \leqslant n-1), \qquad y_n = \frac{x_n - g(x_1, \ldots, x_{n-1})}{k(x_1, \ldots, x_{n-1}, \varepsilon)}\,. \tag{1.11}$$

Then, with $\tilde{u}^k(y) = u^k(x)$, we have

$$\sum a_{ij}^k \frac{\partial u^k}{\partial x_i}\frac{\partial u^k}{\partial x_j} = \sum \tilde{a}_{ij}^k \frac{\partial \tilde{u}^k}{\partial y_i}\frac{\partial \tilde{u}^k}{\partial y_j}\,.$$

Denote by $\tilde{V}$ the image of V under (1.11), and denote by $\tilde{\Gamma} \cap \tilde{V}$, $\tilde{S} \cap \tilde{V}$ the images of $\Gamma \cap V$, $S \cap V$ respectively under (1.11). Then

$$\tilde{\Gamma} \cap \tilde{V} \quad \text{is contained in } y_n = 0,$$

$$\tilde{S} \cap \tilde{V} \quad \text{is contained in } y_n = -\varepsilon.$$

We may assume that $\tilde{V}$ is the n-dimensional cube $|y_i| < \delta_0$, $1 \leq i \leq n$. Finally, denote by $\tilde{\Omega}_k \cap \tilde{V}$ the image of $\Omega_k \cap V$ under (1.11).

Since $u \in H_0^1(\Omega)$,

$$\tilde{u}^1 = \int_{-\varepsilon}^{y_n} \tilde{u}^1_{y_n}\, dy_n\,.$$

Therefore, with $y^* = (y_1, \ldots, y_{n-1})$,

$$\int_{\tilde{\Gamma} \cap \tilde{V}} |\tilde{u}^1|^2\, dy^* \leq C\varepsilon \int_{\tilde{\Omega}_1 \cap \tilde{V}} |D\tilde{u}^1|^2\, dy\,.$$

Also

$$\int_{\tilde{\Omega}_1 \cap \tilde{V}} |\tilde{u}^1|^2\, dy \leq C\varepsilon \int_{\tilde{\Omega}_1 \cap \tilde{V}} |D\tilde{u}^1|^2\, dy\,.$$

Going back to the x-coordinates we get a similar inequality in $\Omega_1 \cap V$. Taking a finite open covering of $\bar{\Omega}_1$, we conclude that

$$\int_{\Gamma} |u^1|^2\, dS \leq C\varepsilon \int_{\Omega_1} |Du^1|^2\, dx\,, \tag{1.12}$$

$$\int_{\Omega_1} |u^1|^2\, dx \leq C\varepsilon^2 \int_{\Omega_1} |Du^1|^2\, dx\,. \tag{1.13}$$

Using the standard inequality

$$\int_{\Omega_2} |u^2|^2\, dx \leq C \int_{\Gamma} |u^2|^2\, dS + C \int_{\Omega_2} |Du^2|^2\, dx\,,$$

the relation $u^2 = u^1$ on Γ, and (1.12), we get

$$\int_{\Omega_2} |u^2|^2\, dx \leq C\varepsilon \int_{\Omega_1} |Du^1|^2\, dx + C \int_{\Omega_2} |Du^2|^2\, dx\,.$$

Combining this with (1.13), we obtain

$$\int_{\Omega_1} |u^1|^2\, dx + \int_{\Omega_2} |u^2|^2\, dx \leq C\varepsilon \int_{\Omega_1} |Du^1|^2\, dx + C \int_{\Omega_2} |Du^2|^2\, dx\,. \tag{1.14}$$

Taking $v = (u^1, u^2)$ in (1.6), we obtain

$$\lambda_1 \int_{\Omega_1} |Du^1|^2\, dx + \int_{\Omega_2} |Du^2|^2\, dx \leq C \int_{\Omega_1} |u^1|\, dx + C \int_{\Omega_2} |u^2|\, dx\,. \tag{1.15}$$

Hence

$$\int_{\Omega_2} |Du^2|^2 \leq C + \frac{1}{C_1} \int_{\Omega_1} |u^1|^2 + \frac{1}{C_1} \int_{\Omega_2} |u^2|^2$$

for any large C_1. Substituting into (1.14), we get

$$\int_{\Omega_1}|u^1|^2+\int_{\Omega_2}|u^2|^2\leqslant C\varepsilon\int_{\Omega_1}|Du^1|^2+C\leqslant\frac{C\varepsilon}{\lambda_1}\left(\int_{\Omega_1}|u^1|+\int_{\Omega_1}|u^2|\right)+C \tag{1.16}$$

where (1.15) was used. Since

$$\frac{C\varepsilon}{\lambda_1}\int_{\Omega_k}|u^k|\leqslant\int_{\Omega_k}\frac{|u^k|^2}{C_1}+\frac{C\varepsilon^2}{\lambda_1}$$

for any large C_1, (1.16) gives

$$\int_{\Omega_1}|u^1|^2+\int_{\Omega_2}|u^2|^2\leqslant C+C\frac{\varepsilon^2}{\lambda_1^2},$$

that is, the assertion (1.8) holds.

Using (1.8) we obtain

$$\int_{\Omega_1}|u^1|+\int_{\Omega_2}|u^2|\leqslant C\left(\int_{\Omega_1}|u^1|^2+\int_{\Omega_2}|u^2|\right)^{\frac{1}{2}}\leqslant C\frac{\varepsilon}{\lambda_1}+C\,.$$

Substituting this into the right hand side of (1.15), the assertion (1.9) follows.

To prove (1.10), let $\tilde{\zeta}(y)$ be a $C_0^\infty(\tilde{V})$ function such that $\tilde{\zeta}=1$ in

$$\tilde{V}_0=\left\{y\colon\ |y_i|<\frac{\delta_0}{2}\text{for } 1\leqslant i\leqslant n\right\}.$$

Suppose first that

$$\begin{cases}\Gamma \text{ is three times continuously differentiable;}\\ \tilde{u}^k \text{ is three times continuously differentiable in } \tilde{\Omega}_k\cap\tilde{V}.\end{cases} \tag{1.17}$$

Let

$$\tilde{v}^k=\tilde{\varsigma}\sum_{l=1}^{n-1}\frac{\partial^2\tilde{u}^k}{\partial y_l^2}\,,\ v^k(x)=\tilde{v}^k(y)\,.$$

In view of (1.17), (v^1, v^2) belongs to $H_0^1(\Omega)$ and, therefore, (1.6) holds for $v=(v^1,v^2)$. Going into the y-coordinates, we get

$$\sum_{k=1}^{2}\lambda_k\left\|\int_{\tilde{\Omega}\cap\tilde{V}}\sum_{i,j=1}^{n}\tilde{a}_{ij}^k\frac{\partial\tilde{u}^k}{\partial y_i}\frac{\partial}{\partial y_j}\left(\tilde{\varsigma}\sum_{l=1}^{n-1}\frac{\partial^2\tilde{u}^k}{\partial y_l^2}\right)dy\right|\leqslant C\sum_{k=1}^{2}\int_{\tilde{\Omega}_k\cap\tilde{V}}\sum_{l=1}^{n-1}\left|\frac{\partial\tilde{u}^k}{\partial y_l}\right|dy \tag{1.18}$$

$\left(\text{since } \left|\int \tilde{\varsigma}\frac{\partial^2 \tilde{u}^k}{\partial y_l^2} f^k\right| = \left|\int \frac{\partial \tilde{u}^k}{\partial y_l}\frac{\partial}{\partial y_l}(\tilde{\varsigma} f^k)\right| \leq C\int \left|\frac{\partial \tilde{u}^k}{\partial y_l}\right|\right)$. Integrating by parts we find that the left-hand side is equal to

$$\sum_{k=1}^{2} \lambda_k \int_{\tilde{\Omega}_k \cap \tilde{V}} \tilde{a}_{ij}^k \frac{\partial \tilde{u}^k}{\partial y_i}\frac{\partial \varsigma}{\partial y_j}\sum_{l=1}^{n-1}\frac{\partial^2 \tilde{u}^k}{\partial y_l^2}\,dy$$

$$-\sum_{k=1}^{2} \lambda_k \int_{\tilde{\Omega}_k \cap \tilde{V}} \sum_{l=1}^{n-1}\sum_{i,j=1}^{n}\frac{\partial(\tilde{a}_{ij}^k\tilde{\varsigma})}{\partial y_l}\frac{\partial \tilde{u}^k}{\partial y_i}\frac{\partial^2 \tilde{u}^k}{\partial y_j \partial y_l}\,dy\,.$$

$$-\sum_{k=1}^{2} \lambda_k \int_{\tilde{\Omega}_k \cap \tilde{V}} \tilde{\varsigma}\sum_{l=1}^{n-1}\sum_{i,j=1}^{n}\tilde{a}_{ij}^k\frac{\partial^2 \tilde{u}}{\partial y_i \partial y_l}\frac{\partial^2 \tilde{u}^k}{\partial y_j \partial y_l}\,dy\,.$$

Hence, (1.18) gives

$$\sum_{k=1}^{2} \lambda_k \int_{\tilde{\Omega}_k \cap \tilde{V}_0} \sum_{l=1}^{n-1}\sum_{i=1}^{n}\left|\frac{\partial^2 \tilde{u}^k}{\partial y_i \partial y_l}\right|^2 dy \leq C\sum_{k=1}^{2}\int_{\tilde{\Omega}_k \cap \tilde{V}} [|\lambda_k D\tilde{u}^k||\hat{D}^2 u^k| + |D\tilde{u}^k|]\,dy \tag{1.19}$$

where $\hat{D}^2$ is the vector with components $\partial^2/\partial y_i \partial y_l$, $1 \leq i \leq n$, $1 \leq l \leq n-1$.

By Schwarz's inequality,

$$C\int_{\tilde{\Omega}_k \supset \tilde{V}_0} \lambda_k |D\tilde{u}^k||\hat{D}^2\tilde{u}^k| \leq \frac{\lambda_k}{M}\int_{\tilde{\Omega}_k \cap \tilde{V}} |\hat{D}^2\tilde{u}^k|^2 + C\lambda_k M\int_{\tilde{\Omega}_k \cap \tilde{V}} |D\tilde{u}^k|^2$$

where M is a positive constant to be determined later on (independently of ε, λ_1). Substituting this into the right hand side of (1.19) and using (1.9), we get

$$\sum_{k=1}^{2}\int_{\tilde{\Omega}_k \cap \tilde{V}_0} \lambda_k |\hat{D}^2\tilde{u}^k|^2 \leq \frac{1}{M}\sum_{k=1}^{2}\int_{\tilde{\Psi}_k \cap \tilde{V}} \lambda_k |\hat{D}^2\tilde{u}^k|^2 + CM\left(\frac{\varepsilon}{\lambda_1} + C\right) + C\left(\frac{\varepsilon}{\lambda_1} + C\right)^{\frac{1}{2}}. \tag{1.20}$$

From the differential equation

$$-\lambda_k \sum \frac{\partial}{\partial y_j}\left(\tilde{a}_{ij}^k \frac{\partial \tilde{u}^k}{\partial y_i}\right) = \tilde{f}^k \qquad (\tilde{f}^k(y) = f^k(x))$$

we can estimate $\lambda_k(\partial^2\tilde{u}^k/\partial y_n^2)$ in terms of the other terms. Using this estimate and (1.20), we get, after using (1.9),

$$\sum_{k=1}^{2}\int_{\tilde{\Omega}_k \cap \tilde{V}_0} \lambda_k |D^2\tilde{u}^k|^2 \leq \frac{1}{M}\sum_{k=1}^{2}\int_{\tilde{\Omega}_k \cap \tilde{V}} \lambda_k |\hat{D}^2\tilde{u}^k|^2 + C + C\frac{\varepsilon}{\lambda_1}\,. \tag{1.21}$$

We now go back to the x-coordinates and obtain an estimate on

$$\sum \lambda_k \int\limits_{\Omega_k \cap V_0} |D^2 u^k|^2 \tag{1.22}$$

where V_0 is the inverse image of $\tilde{V}_0$ under the mapping (1.11).

Let W_δ be a δ-neighborhood of Γ. If δ is sufficiently small, we can cover W_δ with a finite number of neighborhoods of the form V_0. Collecting all the estimates on the terms (1.22) we obtain

$$\sum_{k=1}^{2} \int\limits_{\Omega_1 \cap W_\delta} \lambda_k |D^2 u^k|^2 \leqslant \frac{C}{M} \sum_{k=1}^{2} \int\limits_{\Omega_k \cap W_\delta} \lambda_k |D^2 u^k|^2 + C + C\frac{\varepsilon}{\lambda_1} + C\left(\frac{\varepsilon}{\lambda_1}\right)^{\frac{1}{2}}$$

where C is a constant independent of M. Choosing $M \geqslant 2C$, the assertion (1.10) follows.

In the above proof we have made the smoothness assumptions of (1.17). In the general case, we approximate Γ, S by C^∞ surfaces Γ_m, S_m. Then (see [6]) the condition (1.17) is satisfied for the corresponding solution u^k_m. As easily checked from the above proof of (1.10), the constant C occurring in (1.10), for $u^k = u^k_m$, depends only on bounds on the first two derivatives of the local representation of Γ_m, S_m; hence it can be taken to be independent of m. Taking $m \to \infty$, the proof of (1.10) follows.

REMARK. – Lemma 1.1 and, in fact, all the results of this paper, extend to more general elliptic operators obtained by adding $\lambda_k[\sum b^k_j(x)\partial u/\partial x_j + c^k(x)u]$ to the left hand side of (1.1); $c^k \geqslant 0$.

2. – The case $\varepsilon/\lambda_1 \to \alpha$, $\alpha < \infty$.

THEOREM 2.1. – *If $\lambda_1 \to 0$, $\varepsilon/\lambda_1 \to \alpha$ where $0 \leqslant \alpha < \infty$, then*

(2.1) $u^2 \to w$ *uniformly in compact subsets of Ω_2, where w is the solution of*

$$\lambda_2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^k_{ij}\frac{\partial w}{\partial x_j}\right) = f^2 \quad \text{in } \Omega_2, \tag{2.2}$$

$$\alpha\lambda_2 h(x)\frac{\partial w}{\partial \nu^2} + w = 0 \quad \text{on } \Gamma. \tag{2.3}$$

PROOF. – Let $x^0 \in \Gamma$ and let l be the ray in the conormal direction ν^k initiating at x^0; it intersects S at a point x^1. By (1.2), (1.3),

$$u^2(x^0) + \frac{\lambda_2}{\lambda_1}\varepsilon h(x^0)\frac{\partial u^2(x^0)}{\partial \nu^2} = u^1(x^0) + \varepsilon\frac{h(x^0)}{\|\nu^1\|}\frac{\partial u^1(x^0)}{\partial \mu} \tag{2.4}$$

where μ is the unit vector in the direction ν^1.

Clearly

$$u^1(x^0) = u^1(x^0) - u^1(x^1) = -\int_0^{\varepsilon k(x^0)} \frac{\partial u^1}{\partial \mu}\, ds \tag{2.5}$$

where $k(x^0) = h(x^0)/\|\nu^1\|$ and s measures the length on l from x^0 to x^1. Also, for any x in $l \cap \Omega_1$,

$$\frac{\partial u^1(x^0)}{\partial \mu} - \frac{\partial u^1(x)}{\partial \mu} = \int \frac{\partial^2 u^1}{\partial \mu^2}\, ds\,.$$

Substituting this into (2.5), and substituting the resulting expression for $u^1(x^0)$ into (2.4), we obtain

$$u^2(x^0) + \frac{\lambda_2}{\lambda_1}\varepsilon h(x^0)\frac{\partial u^2(x^0)}{\partial \nu^2} = -\int_0^{\varepsilon k(x^0)} ds \int_0^s \frac{\partial^2 u^1}{\partial \mu^2}\, d\sigma\,. \tag{2.6}$$

Using (1.10) we get

$$\int_\Gamma \left| u^2(x) + \frac{\lambda_2}{\lambda_1}\varepsilon h(x)\frac{\partial u^2(x)}{\partial \nu^2}\right|^2 ds \leqslant C\frac{\varepsilon^3}{\lambda_1}\left(\frac{\varepsilon}{\lambda_1} + C\right) \leqslant C\varepsilon^2 \tag{2.7}$$

since $\varepsilon \leqslant C\lambda_1$.

By (1.9), (1.10), we also have

$$\int_\Gamma \left|\frac{\partial u^2(x)}{\partial \nu^2}\right|^2 ds \leqslant C\int_{\Omega_2} |Du^2(x)|^2\, dx + C\int_{\Omega_2} |D^2 u^2(x)|^2\, dx \leqslant C\,.$$

Hence

$$\int_\Gamma \left| u^2(x) + \alpha\lambda_2 h(x)\frac{\partial u^2(x)}{\partial \nu^2}\right|^2 ds \to 0 \tag{2.8}$$

if $(\varepsilon/\lambda_1) \to \alpha$. We can now complete the proof of the theorem, in case $\alpha > 0$, by representing u^2 in terms of Robin's function, and in case $\alpha = 0$ by representing u^2 in terms of Green's function.

3. – The case $\varepsilon/\lambda_1 \to \infty$, $\varepsilon \to 0$.

THEOREM 3.1. – *If $\varepsilon/\lambda_1 \to \infty$, $\varepsilon \to 0$ then there exist constants $A_{\lambda_1\varepsilon}$ such that*

(3.1) $u^2 - A_{\lambda_1\varepsilon} \to w$ *uniformly in compact subsets of Ω_2, where w is the solution of* (2.2) *subject to the Neumann boundary condition*

$$\lambda_2 h(x)\frac{\partial w(x)}{\partial \nu^2} = \gamma \quad on\ \Gamma, \qquad \gamma = \left(\int_{\Omega_2} f^2\, dx\right)\Big/\left(\int_\Gamma \frac{ds}{h(x)}\right). \tag{3.2}$$

PROOF. – The first inequality in (2.7) is still valid. Multiplying it by $\lambda_1^2/\varepsilon^2$ we get

$$\int_\Gamma \left| \frac{\lambda_1}{\varepsilon} u^2(x) + \lambda_2 h(x) \frac{\partial u^2(x)}{\partial \nu^2} \right|^2 ds \leqslant C\varepsilon^2 . \tag{3.3}$$

By (1.12), (1.9),

$$\int_\Gamma \frac{\lambda_1^2}{\varepsilon^2} |u^2(x)|^2 dS \leqslant C \frac{\lambda_1^2}{\varepsilon^2} \varepsilon \int_{\Omega_1} |Du^1(x)|^2 dx \leqslant C \frac{\lambda_1}{\varepsilon} \left(C \frac{\varepsilon}{\lambda_1} + C \right) \leqslant C .$$

Hence (3.3) gives

$$\int_\Gamma \left| \frac{\partial u^2}{\partial \nu^2} \right|^2 dS \leqslant C . \tag{3.4}$$

We now represent u^2 in terms of Neumann's function and use (3.4). We conclude that there is a constant $c = c(\lambda_1, \varepsilon)$ such that every sequence $(\lambda_1', \varepsilon')$ has a subsequence $(\lambda_1'', \varepsilon'')$ such that

(3.5) $u^2(x) - c(\lambda_1'', \varepsilon'')$ is uniformly convergent in compact subsets of Ω_2,

if $(\lambda_1'', \varepsilon\,) \to 0$.

Now let Γ_* be a surface parallel to Γ and lying in Ω_2 such that, for any $x \in \Gamma$ there is a unique point x_* of Γ_* lying on the inner normal $-\nu$ to Γ at x, and $\overline{x_* x} = \delta$ is small and independent of x.

Since

$$u^2(x) - u^2(x_*) = \int \frac{\partial}{\partial \nu} u^2 \, ds ,$$

we have

$$\int_\Gamma |u^2(x) - u^2(x_*)|^2 dS_* \leqslant C\delta\varepsilon \int_{\Omega_*} |Du^2|^2$$

where Ω_* the region bounded by Γ_*, Γ. Multiplying both sides by $(\lambda_1/\varepsilon)^2$ and using (1.9), we get

$$\int_\Gamma \left| \frac{\lambda_1}{\varepsilon} u^2(x) - \frac{\lambda_1}{\varepsilon} u^2(x_*) \right|^2 dS_* \leqslant C\delta\lambda_1 . \tag{3.6}$$

By (3.5), for fixed point x_0 in Ω_2,

$$|u^2(x_*) - u^2(x_0)| \leqslant C .$$

Combining this with (3.6) we find that

$$\int_\Gamma \left| \frac{\lambda_1}{\varepsilon} u^2(x) - C_{\lambda_1 \varepsilon} \right|^2 dS_x \leqslant C \frac{\lambda_1}{\varepsilon} \tag{3.7}$$

where $C_{\lambda_1\varepsilon} = \lambda_1 u^2(x_0)/\varepsilon$ is a constant. Using this in (3.3), we obtain

$$\int_\Gamma \left|\lambda_2 h(x)\frac{\partial u^2(x)}{\partial \nu^2} + C_{\lambda_1\varepsilon}\right|^2 dS_x \leqslant C\left(\frac{\lambda_1}{\varepsilon} + \varepsilon^2\right). \tag{3.8}$$

Since

$$\int_{\Omega_2} f^2\,dx = \lambda_2 \int_{\Omega_2} \sum \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial u^2}{\partial x_j}\right) dx = \int_\Gamma \lambda_2 \frac{\partial u^2}{\partial \nu^2}\,dS$$

and, by (3.8),

$$\int_\Gamma \left|\lambda_2 \frac{\partial u^2}{\partial \nu^2} + \frac{1}{h(x)} C_{\lambda_1\varepsilon}\right|^2 dS \to 0 \quad \text{if } \varepsilon \to 0, \frac{\varepsilon}{\lambda_1} \to \infty,$$

we deduce that

$$\lim C_{\lambda_1\varepsilon} = \left(\int_{\Omega_2} f^2\,dx\right)\Big/\left(\int_\Gamma \frac{1}{h}\,dS\right). \tag{3.9}$$

Representing u^2 in terms of Neumann's function and using (3.8), (3.9), the assertion of the theorem follows.

REMARK. – If

$$\int_{\Omega_2} f^2\,dx \neq 0$$

then $A_{\lambda_1\varepsilon} \to \infty$. Indeed, otherwise we get from (3.3)

$$\int \left|\frac{\partial u^2}{\partial \nu}\right|^1 dS \to 0$$

for a sequence $(\varepsilon', \lambda_1') \to 0$. Consequently w will have to satisfy (3.2) with $\gamma = 0$, which is impossible.

4. – The case of thick reinforcement.

We shall now study the case where $\lambda_1 \to 0$ but ε does not shrink to zero. In fact we shall take S to be fixed.

The pair $(v^1, v^2) = (\lambda_1 u^1, \lambda_2 u^2)$ satisfies $\mu_k \sum \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial v^k}{\partial x_j}\right) = f^k$ in Ω_k,

$$\begin{aligned} v^1 &= v^2 && \text{on } \Gamma,\\ \mu_1 \frac{\partial v^1}{\partial \nu^1} &= \mu_2 \frac{\partial v^2}{\partial \nu^2} && \text{on } \Gamma,\\ v^1 &= 0 && \text{on } S \end{aligned}$$

where $\mu_1 = 1$, $\mu_2 = \lambda_2/\lambda_1$. As $\lambda_1 \to 0$, μ_1 tends to ∞.

Let $z(x)$ be the solution of the variational problem

$$\int_{\Omega_1} \sum_{i,j=1}^{n} a^1_{ij} \frac{\partial z}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx = \int_{\Omega} f^1 v\, dx \tag{4.1}$$

for any

$$v \in H^1_0(\Omega)\,, \qquad v = \text{const in } \Omega_2 \tag{4.2}$$
$$z \in H^1_0(\Omega)\,, \qquad z = \text{const in } \Omega_2\,.$$

We can determine z as follows: Let ζ be the solution of

$$\begin{aligned} &\sum \frac{\partial}{\partial x_i}\left(a^1_{ij} \frac{\partial \zeta}{\partial x_j}\right) = 0 \qquad && \text{in } \Omega_1\,, \\ &\zeta = 0 \quad \text{on } S\,, && \zeta = 1 \quad \text{on } \Gamma, \end{aligned} \tag{4.3}$$

and let η be the solution of

$$\begin{aligned} &\sum \frac{\partial}{\partial x_i}\left(a^1_{ij} \frac{\partial \eta}{\partial x_j}\right) = f^1 \quad \text{in } \Omega_1\,, \\ &\eta = 0 \quad \text{on } S \cup \Gamma. \end{aligned} \tag{4.4}$$

Then (see [2]) the solution of (4.1), (4.2) is $C_0\zeta + \eta$ where C_0 is a constant determined by

$$C_0 \int_{\Gamma} \frac{\partial \zeta}{\partial \nu^2} + \int_{\Gamma} \frac{\partial \eta}{\partial \nu^2} = \int_{\Omega_2} f^2\,. \tag{4.5}$$

The following result is proved in [2]:

$$\int_{\Omega_1} |\lambda_1 u^1 - z|^2\, dx \to 0 \qquad \text{if } \lambda_1 \to 0. \tag{4.6}$$

For δ positive and sufficiently small, denote by $\tilde{\Gamma}_\delta$ the set of points in Ω_2 obtained from Γ by moving each point x of Γ a distance δ along the direction $-\nu^2(x)$. Denote by x_δ the point on $\tilde{\Gamma}_\delta$ such that $\boldsymbol{x\,x_\delta}$ is in the direction $-\nu^2(x)$. Write

$$\lambda_2 \frac{\partial u^2(x)}{\partial \nu^2} = \lambda_1 \frac{\partial u^1(x)}{\partial \nu^1} = \frac{\partial v^1(x_\delta)}{\partial \nu^1} + \left[\frac{\partial v^1(x)}{\partial \nu^1} - \frac{\partial v^1(x_\delta)}{\partial \nu^1}\right]. \tag{4.7}$$

In view of (4.6),

$$\frac{\partial}{\partial \nu^1} v^1(x_\delta) - \frac{\partial}{\partial \nu^1} z(x_\delta) \to 0 \text{ uniformly on } \tilde{\Gamma}_\delta, \text{ as } \lambda_1 \to 0. \tag{4.8}$$

Also,

$$\int_{\Gamma} \left| \frac{\partial}{\partial \nu^1} v^1(x) - \frac{\partial}{\partial \nu^1} v^1(x_\delta) \right|^2 dS_x \leqslant C\delta \int_{G_\delta} |D^2(\lambda_1 u^1)|^2 dx \tag{4.9}$$

where G_δ is the region bounded by Γ_δ, Γ.

The proof of Lemma 1.1 extends to the present case with minor changes, provided we replace ε by 1 in all the estimates. Thus (1.8)-(1.10) hold and, in particular, it follows that the right-hand side of (4.9) is bounded by $C\delta$. Using this and (4.8), we deduce from (4.7) that

$$\overline{\lim_{\lambda_1 \to 0}} \int_{\Gamma} \left| \lambda_2 \frac{\partial u^2(x)}{\partial \nu^2} - \frac{\partial}{\partial \nu_1} z(x_\delta) \right|^2 dS \leqslant C\delta .$$

The function z is continuously differentiable in $\bar{\Omega}_2$. Hence from the last relation we can deduce that

$$\overline{\lim_{\lambda_1 \to 0}} \int_{\Gamma} \left| \lambda_2 \frac{\partial u^2(x)}{\partial \nu^2} - \frac{\partial}{\partial \nu_1} z(x) \right|^2 dS \leqslant \eta(\delta)$$

where $\eta(\delta) \to 0$ if $\delta \to 0$. It follows that the left-hand side vanishes. Representing u^2 in terms of the Neumann function we then obtain the following result.

THEOREM 4.1. – *If $\lambda_1 \to 0$ and S is fixed, then $\lambda_1 u^1 \to z$ in $L^2(\Omega_1)$ where z is the solution of* (4.1), (4.2). *Further, there exist constants A_{λ_1} such that*

$$u^2(x) - A_{\lambda_1} \to w(x) \text{ uniformly in compact subsets of } \Omega_2 , \tag{4.10}$$

where w is the solution of (2.2) *suject to the Neumann boundary condition*

$$\lambda_2 \frac{\partial w}{\partial \nu^2} = \frac{\partial z}{\partial \nu^1} \quad \text{on } \Gamma. \tag{4.11}$$

PART II. INTERIOR REINFORCEMENT FOR ELLIPTIC EQUATIONS

5. – A priori estimates.

Let Γ_2, Γ_1, Γ_0 be connected $C^{1,1}$ hypersurfaces such that Γ_{i+1} lies in the interior of Γ_i. Denote by Ω_3 the domain with boundary Γ_2, by Ω_2 the domain with boundary $\Gamma_2 \cup \Gamma_1$, and by Ω_1 the domain with boundary $\Gamma_1 \cup \Gamma_0$.

Set $\Omega = \Omega_1 \cup \bar{\Omega}_2 \cup \bar{\Omega}_3$.

Let (a_{ij}^k) $(k = 1, 2, 3)$ be positive definite and $C^{1+\bar{\varepsilon}}$ matrices defined in $\bar{\Omega}$, and let f^k be functions in $C^{1+\bar{\varepsilon}}(\Omega)$. For $\lambda_k > 0$, consider the problem:

$$\lambda_k \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial u^k}{\partial x_j}\right) = f^k \quad \text{in } \Omega_k \ (k = 1, 2, 3), \tag{5.1}$$

$$u^{k+1} = u^k \quad \text{on } \Gamma_k \ (k = 1, 2), \tag{5.2}$$

$$\lambda_{k+1}\frac{\partial u^{k+1}}{\partial \nu^{k+1}} = \lambda_k \frac{\partial u^k}{\partial \nu^k} \quad \text{on } \Gamma_k \ (k = 1, 2), \tag{5.3}$$

$$u^1 = 0 \quad \text{on } \Gamma_0 . \tag{5.4}$$

This problem has a unique classical solution. It can be given also in the variational form (1.6) where k ranges over 1, 2, 3.

We shall consider Ω_2 as a layer of reinforcement, and study the asymptotic behavior as $\lambda_2 \to 0$.

Denote by $(\Gamma_2)_\varepsilon$ the manifold consisting of the points

$$x_\varepsilon = x + \varepsilon h(x)\nu^2(x)/\|\nu^2(x)\|^2$$

where x varies over Γ_2; $h(x)$ is a given positive function. We shall assume that

$$\Gamma_1 = (\Gamma_2)_\varepsilon \tag{5.5}$$

and prove the following lemma.

LEMMA 5.1. – *There exists a positive constant C such that, for all ε, λ_1 sufficiently small,*

$$\sum_{k=1}^{3} \int_{\Omega_k} |u^k|^2 \, dx \leqslant C\frac{\varepsilon^2}{\lambda_2^2} + C , \tag{5.6}$$

$$\sum_{k=1}^{3} \lambda_k \int_{\Omega_k} |Du^k|^2 \, dx \leqslant C\frac{\varepsilon}{\lambda_2} + C , \tag{5.7}$$

$$\sum_{k=1}^{3} \lambda_k \int_{\Omega_k} |D^2 u^k|^2 \, dx \leqslant C\frac{\varepsilon}{\lambda_2} + C . \tag{5.8}$$

PROOF. – By standard inequalities,

$$\int_{\Omega_1} |u^1|^2 \leqslant C \int_{\Omega_1} |Du^1|^2 , \tag{5.9}$$

$$\int_{\Gamma_1} |u^1|^2 \leqslant C \int_{\Omega_1} |Du^1|^2 . \tag{5.10}$$

As in the proof of Lemma 1.1 we have

$$\int_{\Gamma_2}|u^2|^2 \leqslant C\int_{\Gamma_1}|u^2|^2 + C\varepsilon\int_{\Omega_2}|Du^2|^2 ,$$
$$\int_{\Omega_2}|u^2|^2 \leqslant C\varepsilon\int_{\Gamma_1}|u^2|^2 + C\varepsilon^2\int_{\Omega_2}|Du^2|^2 .$$

Using (5.10) we get

$$\int_{\Gamma_2}|u^2|^2 \leqslant C\int_{\Omega_1}|Du^1|^2 + C\varepsilon\int_{\Omega_2}|Du^2|^2 ,$$

$$\int_{\Omega_2}|u^2|^2 \leqslant C\varepsilon\int_{\Omega_1}|Du^1|^2 + C\varepsilon\int_{\Omega_2}|D^2u^2|^2 . \tag{5.12}$$

Finally,

$$\int_{\Omega_3}|u^3|^2 \leqslant C\int_{\Gamma_2}|u^2|^2 + C\int_{\Omega_3}|Du^3|^2 ,$$

so that, upon using (5.11),

$$\int_{\Omega_3}|u^3|^2 \leqslant C\int_{\Omega_1}|Du^1|^2 + C\varepsilon\int_{\Omega_2}|Du^2|^2 + C\int_{\Omega_3}|Du^3|^2 . \tag{5.13}$$

From the variational principle (1.6) (with $k = 1, 2, 3$ and $v = u^k$ in Ω_k), we obtain

$$\sum_{k=1}^{3}\lambda_k\int_{\Omega_k}|Du^k|^2 \leqslant C\sum_{k=1}^{3}\int_{\Omega_k}|u^k| . \tag{5.14}$$

Now, for any $M > 1$,

$$\int_{\Omega_1}|u^1| \leqslant CM + \frac{1}{M}\int_{\Omega_1}|u^1|^2 \leqslant CM + \frac{C}{M}\int_{\Omega_1}|Du^1|^2$$

where (5.9) was used; the constants C in the sequel will not depend on the choice of M (which will be made below).

Also,

$$\int_{\Omega_2}|u^2| \leqslant CM + \frac{C}{M}\int_{\Omega_2}|u^2|^2 \leqslant CM + \frac{C}{M}\left(\varepsilon\int_{\Omega_1}|Du^1|^2 + \varepsilon^2\int_{\Omega_2}|Du^2|^2\right)$$

by (5.12). Finally,

$$\int_{\Omega_3}|u^3| \leqslant CM + \frac{C}{M}\left(\int_{\Omega_1}|Du^1|^2 + \varepsilon\int_{\Omega_2}|Du^2|^2 + \int_{\Omega_3}|Du^3|^2\right)$$

by (5.13). Thus,

$$\sum_{k=1}^{3}\int_{\Omega_k}|u^k|\leqslant C+CMN\max\left(\frac{\varepsilon}{\lambda_2},1\right)+\frac{C}{M}\int_{\Omega_1}|Du^1|^2+\frac{C\varepsilon}{M}\int_{\Omega_2}|Du^2|^2+\frac{C}{N}\int_{\Omega_3}|Du^3|^2 .$$

Substituting this into (5.14) and choosing $M=N+N\varepsilon/\lambda_2$, N sufficiently large, we obtain the inequality (5.7).

The assertion (5.6) follows from (5.7) and (5.9), (5.12), (5.13).

Finally, the proof of (5.8) is similar to the proof of (1.10); it is based on locally transforming Γ_1, Γ_2 into planar regions lying on $\tilde{x}_n=0$, $\tilde{x}_n=-\varepsilon$ respectively, substituting test functions $\tilde{\zeta}\sum_{l=1}^{n-1}\partial^2\tilde{u}^k/\partial\tilde{x}_l^2$ and making use of (5.7).

6. – Asymptotic estimates as $\varepsilon/\lambda_2\to\alpha$, $\alpha<\infty$.

Let $x^1\in\Gamma_1$, $x^2\in\Gamma_2$ be such that x^1 lies on the conormal $\nu^2(x^2)$ initiating at x^2. Denote by μ the unit vector in the direction $\nu^2(x^2)$.

From (5.3) we have

$$\lambda_1\frac{\partial u^1(x^1)}{\partial\nu^1(x^1)}-\lambda_3\frac{\partial u^3(x^2)}{\partial\nu^3(x^2)}=\lambda_2\left(\frac{\partial u^2(x^1)}{\partial\nu^2(x^1)}-\frac{\partial u^2(x^2)}{\partial\nu^2(x^2)}\right). \tag{6.1}$$

Clearly,

$$\lambda_2^2\int_{\Gamma_2}\left|\frac{\partial u^2(x^1)}{\partial\nu^2(x^2)}-\frac{\partial u^2(x^2)}{\partial\nu^2(x^2)}\right|^2 dS_{x^2}\leqslant C\lambda_2^2\int_{\Gamma_2}\varepsilon\int\left|\frac{\partial^2u^2(x_s)}{\partial\mu^2}\right|^2 ds\,dS_{x^2}\leqslant C\lambda_2\varepsilon\left(\frac{\varepsilon}{\lambda_2}+1\right)$$

where (5.8) has been used.

Also, since

$$|\nu^2(x^1)-\nu^2(x^2)|\leqslant C\varepsilon ,$$

we have

$$\lambda_2^2\int_{\Gamma_2}\left|\frac{\partial u^2(x^2)}{\partial\nu^2(x^1)}-\frac{\partial u^2(x^2)}{\partial\nu(x^2)}\right|^2 dS_{x^2}\leqslant C\lambda_2^2\varepsilon\int_{\Gamma_2}|Du^2|^2\leqslant C\lambda_2^2\varepsilon\int_{\Gamma_2}|Du^3|^2$$
$$\leqslant C\lambda_2^2\varepsilon\int_{\Omega_3}\left(|Du^3|^2+|D^2u^3|^2\right)\leqslant C\lambda_2^2\varepsilon\left(L+\frac{\varepsilon}{\lambda_2}\right)$$

where (5.7), (5.8) have been used. It thus follows from (6.1) that

$$\int_{\Gamma_2}\left|\lambda_1\frac{\partial u^1(x^1)}{\partial\nu^1(x^1)}-\lambda_3\frac{\partial u^3(x^3)}{\partial\nu^3(x^2)}\right| dS_{x^2}\leqslant C(\lambda_2\varepsilon+\lambda_2\varepsilon^2). \tag{6.2}$$

From (5.2) we have

$$u^1(x^1) - u^3(x^2) = u^2(x^1) - u^2(x^2) = \int \frac{\partial}{\partial \mu} u^2 = \varepsilon h(x^2) \frac{\partial u^2(x^2)}{\partial \nu^2(x^2)} + \iint \frac{\partial^2}{\partial \mu^2} u^2 , \tag{6.3}$$

so that

$$\int_{\Gamma_2} \left| u^1(x^1) - u^3(x^2) - \frac{\lambda_3 \varepsilon}{\lambda_2} h(x^2) \frac{\partial u^3(x^2)}{\partial \nu^3(x^2)} \right|^2 \leqslant C\varepsilon^3 \int_{\Omega_2} |D^2 u^2|^2 \leqslant C \frac{\varepsilon^3}{\lambda^2}\left(1 + \frac{\varepsilon}{\lambda_2}\right), \tag{6.4}$$

where (5.8) has been used. Thus, if $\varepsilon \leqslant C\lambda_2$,

$$\int_{\Gamma_2} \left| u^1(x^1) - u^3(x^2) - \frac{\varepsilon h(x^2)\, \lambda_3}{\lambda_2} \frac{\partial u^3(x^2)}{\partial \nu^2(x^2)} \right| \leqslant C\varepsilon^2 . \tag{6.5}$$

From (6.2), (6.4) we see that if $\varepsilon/\lambda_2 \to \alpha$, $u^k \to w^k$ for $k = 1, 3$ then, formally, (w^1, w^3) satisfy the relations

$$\lambda_1 \frac{\partial w^1}{\partial \nu^1} = \lambda_3 \frac{\partial w^3}{\partial \nu^3} \quad \text{on } \Gamma_2, \tag{6.6}$$

$$w^1 - w^2 = \alpha \lambda_3 h(x) \frac{\partial w^3}{\partial \nu^3} \quad \text{on } \Gamma_2 , \tag{6.7}$$

and, of course, also

$$\lambda_k \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x) + \frac{\partial w^k}{\partial x_j}\right) = f^k \quad \text{in } \tilde{\Omega}_k \ (k = 1, 3) \tag{6.8}$$

where $\tilde{\Omega}_3 = \Omega_3$, $\tilde{\Omega}_1 = \Omega_2 \cup \Gamma_1 \cup \Omega_1$, and

$$w^1 = 0 \quad \text{on } \Gamma_0 . \tag{6.9}$$

Thus we are led to the following theorems.

THEOREM 6.1. – *If $\lambda_2 \to 0$, $\varepsilon/\lambda_2 \to \alpha$ where $0 \leqslant \alpha < \infty$ then $u^k \to w^k$ $(k = 1, 3)$ uniformly in compact subsets of Ω_k, where w^1, w^2 form the solution of* (6.6)-(6.9).

PROOF. – We introduce a diffeomorphism Q of $\Omega_1 \cup \Omega_3$ onto $\tilde{\Omega}_1 \cup \tilde{\Omega}_3$ such that the derivatives of Q are bounded by $O(\varepsilon)$ (cf. [1; Sec. 8]). Set

$$\tilde{u}^k = u^k \circ Q \quad (k = 1, 3).$$

Then $(\tilde{u}^1, \tilde{u}^3)$ satisfy a system similar to (w^1, w^2) with coefficients $\tilde{a}_{ij}^k$, $\tilde{f}^k$ such that

$$|\tilde{a}_{ij}^k - a_{ij}^k| \leqslant C\varepsilon, \qquad |\tilde{f}^k - f^k| \leqslant C\varepsilon \tag{6.10}$$

and with additive terms η_1, η_2 in the corresponding right hand sides of (6.6), (6.7) where

$$\int_{\Gamma_1} |\eta_k|^2 \leqslant C\varepsilon + C\left(\frac{\varepsilon}{\lambda_2} - \alpha\right)^2; \tag{6.11}$$

here Lemma 5.1 is used in obtaining (6.11).

From (6.10) we get

$$\left|\int_{\tilde{\Omega}_k} (\tilde{a}_{ij}^k - a_{ij}^k) \frac{\partial \tilde{u}^k}{\partial x_i} \frac{\partial \tilde{u}^k}{\partial x_j}\right| \leqslant C\varepsilon^2 \int_{\tilde{\Omega}_k} |D\tilde{u}^k|^2 \leqslant C\varepsilon$$

where (5.7) has been used.

Now let

$$z^k = \tilde{u}^k - w^k \qquad (k = 1, 2).$$

Then, on Γ_2,

$$\lambda_1 \frac{\partial z^1}{\partial \nu^1} - \lambda_3 \frac{\partial z^3}{\partial \nu^3} = \eta_1, \qquad z^1 - z^3 = \alpha h \lambda_3 \frac{\partial z^2}{\partial \nu^2} + \eta_2. \tag{6.12}$$

Multiplying the elliptic equation for z^k by z^k $(k = 1, 3)$, integrating over $\tilde{\Omega}_k$ and adding, we get

$$\int_{\tilde{\Omega}_1} |Dz^1|^2 + \int_{\tilde{\Omega}_3} |Dz^3|^2 + \int_{\Gamma_2} \left(\lambda_1 \frac{\partial z^1}{\partial \nu^1} z^1 - \lambda_3 \frac{\partial z^3}{\partial \nu^3} z^3\right) \leqslant C\varepsilon + C\left(\frac{\varepsilon}{\lambda_2} - \alpha\right)^2. \tag{6.13}$$

By (6.12), the integral over Γ_2 is equal to

$$\int_{\Gamma_2} \left(\lambda_1 \frac{\partial z^1}{\partial \nu^1} - \lambda_3 \frac{\partial z^3}{\partial \nu^3}\right) z^1 + \int_{\Gamma_2} \lambda_3 \frac{\partial z^3}{\partial \nu^3} \alpha h \frac{\partial z^3}{\partial \nu^3} + I \tag{6.14}$$

where

$$|I| \leqslant C \int_{\Gamma_2} \left|\frac{\partial z^1}{\partial \nu^1}\right| |\eta_2| + C \int_{\Gamma_2} \left|\frac{\partial z^3}{\partial \nu^3}\right| |\eta_2| \leqslant C\left(\sqrt{\varepsilon} + \left|\frac{\varepsilon}{\lambda_2} - \alpha\right|\right); \tag{6.15}$$

here we have used Schwarz's inequality, (6.11) and Lemma 5.1.

By (6.11), (6.12), the first integral in (6.14) is bounded by $C(\sqrt{\varepsilon} + |\alpha - \varepsilon/\lambda_2|)$. The second integral in (6.14) is positive. Using this and (6.15), we obtain from (6.13) the inequality

$$\int_{\tilde{\Omega}_1} |Dz^1|^2 + \int_{\tilde{\Omega}_3} |Dz^3|^2 \leqslant C(\sqrt{\varepsilon} + |\alpha - \varepsilon/\lambda_2|).$$

The assertion of Theorem 6.1 now readily follows.

7. – Asymptotic estimates in case $\varepsilon/\lambda_2 \to \infty$.

Consider now the case where $\varepsilon/\lambda_1 \to \infty$, $\varepsilon \to 0$. We multiply (6.4) by $(\lambda_2/\varepsilon)^2$ and obtain

$$\int_\Gamma \left| \lambda_3 h(x) \frac{\partial u^3(x^2)}{\partial \nu^3(x^2)} - \frac{\lambda_2}{\varepsilon} \left(u^1(x^1) - u^3(x^2)\right) \right|^2 \leqslant C\varepsilon^2 . \tag{7.1}$$

Since

$$|u^1(x^1) - u^3(x^2)|^2 = \left| \int \frac{\partial}{\partial \mu} u^2 \right|^2 \leqslant \varepsilon \int \left| \frac{\partial u^2}{\partial \mu} \right|^2 ,$$

we also have

$$\int_{\Gamma_1} \left(\frac{\lambda_2}{\varepsilon}\right)^2 (u^1(x^1) - u^3(x^2))^2 \leqslant C \left(\frac{\lambda_2}{\varepsilon}\right)^2 \varepsilon \int_{\Omega_2} |Du^2|^2 \leqslant C$$

where (5.7) has been used. Thus (7.1) implies

$$\int_{\Gamma_2} \left| \frac{\partial u^3}{\partial \nu^3} \right|^2 \leqslant C . \tag{7.2}$$

Let

$$\Omega_{3,\delta} = \{x \in \Omega_3,\ \text{dist.}\,(x, \Gamma_2) > \delta\} .$$

Representing u^3 in terms of the Neumann function and using (7.2), it follows that

$$|u^3(x) - u^3(x^0)| \leqslant C \qquad (x \in \Omega_{3,\delta})$$

where x^0 is any point in Ω_3. Further, any sequence $(\varepsilon', \lambda_2')$ has a subsequence $(\varepsilon'', \lambda_2'')$ such that

(7.3) $\qquad u^3 - u^3(x^0)$ is uniformly convergent in $\Omega_{3,\delta}$ for any $\delta > 0$,

provided $(\varepsilon'', \lambda_2'') \to 0$.

Since, by (5.7),

$$\int_{\Gamma_1} \left| \frac{\lambda_2}{\varepsilon} u^1(x^1) \right|^2 \leqslant C \frac{\lambda_2^2}{\varepsilon^2} \int_{\Omega_1} |Du^1|^2 \leqslant C \frac{\lambda_2}{\varepsilon} ,$$

(7.1) gives

$$\int_{\Gamma_1} \left| \lambda_3 h(x) \frac{\partial u^3}{\partial \nu^3} + \frac{\lambda_2}{\varepsilon} u^3 \right|^2 \leqslant C \frac{\lambda_2}{\varepsilon} + C\varepsilon^2 . \tag{7.4}$$

For any $x \in \Gamma_2$, denote by x^δ the nearest point to x on $\partial\Omega_{3,\delta}$. Then

$$\int_{\Gamma_2} \left| \frac{\lambda_2}{\varepsilon} u^3(x) - \frac{\lambda_2}{\varepsilon} u^3(x^\delta) \right|^2 \leqslant C\delta \frac{\lambda_2^2}{\varepsilon^2} \int_{\Gamma_2} \int \left| \frac{\partial u^3}{\partial \nu} \right|^2 \leqslant C\delta \frac{\lambda_2^2}{\varepsilon^2} \frac{\varepsilon}{\lambda_2} = C \frac{\lambda_2}{\varepsilon}.$$

Using this in (7.4) and using also (7.3), we find as in Section 3, that

$$u^3 - C_{\lambda_2\varepsilon} \to w^3 \quad \text{uniformly in compact subsets of } \Omega_3, \tag{7.5}$$

where $C_{\lambda_2\varepsilon}$ are some constants and

$$\lambda_3 h(x) \frac{\partial w^3}{\partial \nu} = \gamma \quad \text{on } \Gamma_2, \qquad \gamma = \Big(\int_{\Omega_3} f^3\Big) \Big/ \Big(\int_{\Gamma_2} \frac{1}{h}\Big). \tag{7.6}$$

As for u^1, using (6.2) one can show that, for some constants $\hat{C}_{\lambda_2\varepsilon}$,

$$u^1 \to w^1 \quad \text{uniformly in compact subsets of } \Omega \diagdown \bar{\Omega}_3, \tag{7.7}$$

where

$$\lambda_1 \frac{\partial w^1}{\partial \nu^1} = \lambda_3 \frac{\partial w}{\partial \nu^3} = \frac{\gamma}{h(x)} \qquad \text{on } \Gamma_2; \tag{7.8}$$

also, of course, w^1 and w^3 satisfy (6.8) and w^1 satisfies (6.9). The proof of (7.7) uses a diffeomorphism of Ω_1 onto $\tilde{\Omega}_1 = \Omega_1 \cup \Gamma_1 \cup \Omega_2$ and the corresponding argument of Section 4; the details are omitted.

We sum up:

Theorem 7.1. – *If* $\varepsilon/\lambda_1 \to \infty$, $\varepsilon \to 0$ *then* (7.5), (7.7) *hold, where* w^3 *is the solution of* (6.8) *with* $k = 3$ *satisfying the Neumann boundary condition* (7.6), *and* w^1 *is the solution of* (6.8) *with* $k = 1$ *satisfying the boundary conditions* (7.8), (6.9).

Thick reinforcement. We consider briefly the case of thick reinforcement, that is, Γ_1, Γ_2 are now fixed and the parameter λ_2 tends to zero. As in Section 4 we begin with the fact established in [2] that

$$\lambda_2 u^2 \to z \quad \text{uniformly in compact subsets of } \Omega_2, \tag{7.9}$$

where z is the minimum of the functional

$$\int_{\Omega_2} \Big(\sum_{i,j=1}^{n} a_{ij}^2 \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} + 2f^2\zeta \Big) dx\,, \qquad \zeta \in H_0^1(\bar{\Omega}_3 \cup \Omega_2)\,, \qquad \zeta = \text{const in } \Omega_3:$$

We can then show (as in Section 4) that there exist constants A_{λ_2} such that

(7.10) $u^3 - A_{\lambda_2} \to w^3$, $u^1 \to w^1$ uniformly in compact subsets of Ω_3 and Ω_1, respectively,

where w^k $(k = 1, 3)$ satisfies (5.1) and

$$\lambda_3 \frac{\partial w^3}{\partial \nu^3} = \frac{\partial z}{\partial \nu^2} \quad \text{on } \Gamma_2 , \tag{7.11}$$

$$\lambda_1 \frac{\partial w^1}{\partial \nu^1} = \frac{\partial z}{\partial \nu^2} \quad \text{on } \Gamma_1 . \tag{7.12}$$

$$w^1 = 0 \quad \text{on } \Gamma_0 . \tag{7.13}$$

PART III. REINFORCEMENT FOR VARIATIONAL INEQUALITIES

8. - A priori estimates.

In this section we extend Lemma 1.1 to the elliptic variational inequality

$$\sum_{k=1}^{2} \int_{\Omega_k} \left(\lambda_k \sum_{i,j=1}^{n} a_{ij}^k \frac{\partial u^k}{\partial x_j} \frac{\partial (v - u^k)}{\partial x_j} + f^k(v - u^k) \right) dx \geqslant 0 \tag{8.1}$$

$$\text{for every } v \in H_0^1(\Omega),\ v \geqslant \varphi,\ u^k \geqslant \varphi,\ u = (u^1, u^2) \in H_0^1(\Omega);$$

the function φ is in $C^2(\bar{\Omega})$ and $\varphi \leqslant 0$ on $\partial\Omega$.

Let $\beta_\delta(t)$ be C^∞ functions in $t \in \boldsymbol{R}$, for any $\delta > 0$, such that

$$\begin{aligned} &\beta_\delta(t) = 0 && \text{if } t \geqslant 0, \\ &\beta_\delta(t) \to -\infty && \text{if } t < 0,\ \delta \to 0, \\ &\beta_\delta'(t) \geqslant 0 . \end{aligned}$$

Consider the elliptic problem (the «penalized problem»)

$$\sum_{k=1}^{2} \int_{\Omega_2} \left(\lambda_k \sum_{i,j=1}^{n} a_{ij}^k \frac{\partial u^k}{\partial x_i} \frac{\partial v}{\partial x_j} + f^k v + \beta_\delta(u^k - \varphi) v \right) dx = 0 \tag{8.2}$$

$$\text{for any } v \in H_0^1(\Omega),\ u = (u^1, u^2) \in H_0^1(\Omega).$$

Denote by $u_\delta = (u_\delta^1, u_\delta^2)$ the unique solution of (8.2). Then, as is well known, as $\delta \to 0$

$$(u_\delta^1, u_\delta^2) \to (u^1, u^2) \quad \text{in } L^2(\Omega),$$

where (u^1, u^2) is the solution of (8.1).

LEMMA 8.1. – *There exists a constant C independent of δ such that*

$$\int_{\Omega_k} |\beta_\delta(u_\delta^k - \varphi)|^2 \leqslant C \tag{8.3}$$

and such that the estimates of Lemma 1.1 *hold for* (u_δ^1, u_δ^2); *consequently, the estimates of Lemma* 1.1 *hold also for the solution* (u^1, u^2) *of* (8.1).

PROOF. – The proof of (1.8), (1.9) extends to the present case provided we substitute $v = u_\delta - \varphi$ in (8.2) to get an auxiliary inequality analogous to (1.15). Substituting $v = \beta_\delta(u_\delta - \varphi)$ in (8.2) and using (1.9), we obtain the inequality (8.3). We now proceed to derive (1.10), taking for simplicity $\varphi \equiv 0$. In the inequality analogous to (1.19) we now have an additional term on the left hand side, namely,

$$-\sum_{k=1}^{2} \int_{\tilde{\Omega}_k \supset \tilde{V}_0} \beta_\delta(u^k)\tilde{\varsigma} \sum_{l=1}^{n-1} \frac{\partial^2 \tilde{u}^k}{\partial y_l^2}\, dy = \sum_{k=1}^{2} \Bigg[\int_{\tilde{\Omega}_k \cap \tilde{V}_0} \beta_\delta(u^k) \frac{\partial \tilde{\varsigma}}{\partial y_l} \sum_{l=1}^{n-1} \frac{\partial \tilde{u}^k}{\partial y_l}\, dy + \int_{\tilde{\Omega}_k \cap \tilde{V}_0} \tilde{\varsigma}\, \beta_\delta'(u_k) \sum_{l=1}^{n-1} \left(\frac{\partial \tilde{u}^k}{\partial y_l}\right)^2 dy \Bigg]$$

$$\geqslant -C \sum_{k=1}^{2} \Bigg[\int_{\tilde{\Omega}_k \cap \tilde{V}_0} (\beta_\delta(u^k))^2 \Bigg]^{\frac12} \Bigg[\int_{\tilde{\Omega}_k \cap \tilde{V}_0} |Du^k|^2 \Bigg]^{\frac12} \geqslant -C\left(\frac{\varepsilon}{\lambda_1} + C\right)^{\frac12}$$

by (1.9) and (8.3). Thus, (1.20) follows.

Using the equation (8.2) and (1.20), (8.3), the estimate (1.21) also follows, and the proof of (1.10) can now be completed as before.

9. – Boundary reinforcement.

THEOREM 9.1. – *If* $\lambda_1 \to 0$, $\varepsilon/\lambda_1 \to \alpha$ *where* $0 \leqslant \alpha < \infty$, *then*

(9.1) $u^2 \to w$ *uniformly in compact subsets of* Ω_2, *where* w *is the unique solution of the variational inequality*

$$\int_{\Omega_2} \lambda_2 \sum_{i,j=1}^{n} a_{ij}^2 \frac{\partial w}{\partial x_i} \frac{\partial (v-w)}{\partial x_j}\, dx + \int_{\Omega_2} f^2 (v-w)\, dx + \int_{\partial\Omega} \frac{w}{\alpha\lambda_2 h} (v-w)\, dS \geqslant 0 \tag{9.2}$$

for every $v \in H^1(\Omega_2)$, $v \geqslant \varphi$, $w \in H^1(\Omega_2)$, $w \geqslant \varphi$.

PROOF. – Multiply the equation for u_δ^2 by $v - u_\delta^2$ and integrate over Ω_2. Using (2.8) for u_δ^2 (which is established as in Section 2, using Lemma 8.1) and the relations

$$\beta_\varepsilon(v-\varphi) = 0 \quad \text{ if } v \geqslant \varphi,$$

$$(z_1 - z_2)(\beta(z_1) - \beta(z_2)) \geqslant 0 \quad \text{ for } z_1 = v - \varphi,\ z_2 = u_\delta^2 - \varphi,$$

we obtain, after letting $\delta \to 0$,

$$\int_{\Omega_2} \lambda_2 \sum a_{ij}^2 \frac{\partial u^2}{\partial x_i} \frac{\partial (v - u^2)}{\partial x_j} + \int_{\Gamma} \frac{u^2}{\alpha \lambda_2 h} (v - u^2) + \int_{\Omega_2} f^2 (v - u^2) + \int_{\Gamma} \gamma_{\varepsilon\lambda_1} (v - u^2) \geqslant 0$$

where

$$\int_{\Gamma} |\gamma_{\varepsilon\lambda_1}|^2 \, dS \to 0 \qquad \text{if } \varepsilon \to 0,\ \lambda_1 \to 0,\ \frac{\varepsilon}{\lambda_1} \to \alpha .$$

By [6] there exists a unique solution of w of (9.2) and (by stability) $u^2 \to w$ in compact subsets of Ω_2. This completes the proof.

We shall now consider the case where $\varepsilon/\lambda_2 \to \infty$. For simplicity we take $\varphi \equiv 0$. We shall assume that

$$\int_{\Omega_2} f^2 \, dx < 0 . \tag{9.3}$$

LEMMA 9.2. – *If $\varphi \equiv 0$ and (9.3) holds, then there exists a unique solution w of the variational inequality*

$$\int_{\Omega_2} \lambda_2 \sum_{i,j=1}^{n} a_{ij}^2 \frac{\partial w}{\partial x_i} \frac{\partial (v - w)}{\partial x_j} \, dx + \int_{\lambda_2} f^2 (v - w) \, dx \geqslant 0 \tag{9.4}$$

for every $v \in H^1(\Omega_2)$, $v \geqslant 0$, $w \in H^1(\Omega_2)$, $w \geqslant 0$.

This result is due to LIONS and STAMPACCHIA [5].

THEOREM 9.3. – *If $\varphi \equiv 0$ and (9.3) holds and if*

$$\varepsilon \to 0 , \qquad \lambda_1 \to 0 , \ \frac{\varepsilon}{\lambda_1} \to \infty , \qquad \frac{\varepsilon^3}{\lambda_1} \to 0$$

then

$$\int_{\Omega_2} |D(u^2 - w)|^2 \, dx \to 0 \tag{9.5}$$

where w is the unique solution of (9.4).

PROOF. – Take for simplicity $a_{ij}^2 = \delta_{ij}$, $\lambda_2 = 1$. Then

$$- \Delta u^2 + \beta(u^2) \ni f^2 , \qquad - \Delta w + \beta(w) \ni f^2$$

where $\beta(t)$ is the graph $\beta(t) = \emptyset$ if $t < 0$, $\beta(0) = (-\infty, 0]$, and $\beta(t) = \{0\}$ if $t > 0$. Thus

$$- \Delta(u^2 - w) + \beta(u^2) - \beta(w) \ni 0 .$$

Multiplying by $u^2 - w$ and integrating over Ω_2, we get

$$-\int_{\Gamma} \frac{\partial(u^2 - w)}{\partial\nu}(u^2 - w) + \int_{\Omega_2} |\nabla(u^2 - w))|^2 \leqslant 0 .$$

Since $u^2 \geqslant 0$, $\partial w/\partial\nu \geqslant 0$ on Γ, we have

$$\frac{\partial w}{\partial\nu} u_2 \geqslant \text{on } \Gamma .$$

Also $w(\partial w/\partial\nu) = 0$ on Γ. Hence

$$(9.6)\qquad -\int_{\Gamma} \frac{\partial u^2}{\partial\nu} u^2 + \int_{\Gamma} \frac{\partial u^2}{\partial\nu} w + \int_{\Omega_2} |\nabla(u^2 - w)|^2 \leqslant 0 .$$

Suppressing the third term on the left hand side and using (2.6), we get

$$(9.7)\qquad \frac{\varepsilon}{\lambda_1}\int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right|^2 \leqslant C\int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right|\left|\int_0^{\varepsilon k}\left(\int_0^s \frac{\partial^2 u^1}{\partial\mu^2}\, d\sigma\right) ds\right| dS + C\int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right| |w|\, dS \equiv I + J .$$

By Cauchy's and Schwarz's inequalities,

$$I \leqslant \frac{\varepsilon}{4\lambda_1}\int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right|^2 + C\frac{\lambda_1}{\varepsilon}\varepsilon^3\int_{\Gamma}\int_0^{\varepsilon k}\left|\frac{\partial^2 u^1}{\partial\mu^2}\right|^2 ds\, dS$$

and, by Lemma 8.1, the last term is bounded by $C\varepsilon^3/\lambda_1$. Also,

$$J \leqslant \frac{\varepsilon}{4\lambda_1}\int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right|^2 + C\frac{\lambda_1}{\varepsilon} .$$

Thus we conclude from (9.7) that

$$(9.8)\qquad \int_{\Gamma}\left|\frac{\partial u^2}{\partial\nu}\right|^2 dS \leqslant C\varepsilon^2 + C\frac{\lambda_1^2}{\varepsilon^2} .$$

From (2.6),

$$-\int_{\Gamma}\frac{\partial u^2}{\partial\nu} u^2 \geqslant \int_{\Gamma}\frac{\partial u^2}{\partial\nu}\int_0^{sk}\left(\int_0^{\varepsilon}\frac{\partial^2 u^1}{\partial\nu^2}\, d\sigma\right) ds\, dS .$$

Using the estimate (1.10) of Lemma 8.1 and (9.8), we find that

$$-\int\frac{\partial u^2}{\partial\nu} u^2 \geqslant -C\left(\varepsilon^2 + \frac{\lambda_1^2}{\varepsilon^2}\right)^{\frac{1}{2}}\left(\frac{\varepsilon^4}{\lambda_1^2}\right)^{\frac{1}{2}} = -C\left(\varepsilon + \frac{\varepsilon^3}{\lambda_1}\right) \to 0 .$$

Since also

$$\left|\int_\Gamma \frac{\partial u^2}{\partial \nu} w\right| \leqslant C\left(\int_\Gamma \left|\frac{\partial u^2}{\partial \nu}\right|^2\right)^{\frac{1}{2}} \leqslant C\left(\varepsilon + \frac{\lambda_1}{\varepsilon}\right) \to 0 ,$$

the assertion (9.5) follows from (9.6).

REMARK 1. – If we also assume that $\varepsilon^2/\lambda_1 \leqslant C$ then from (2.6) and (9.8) we deduce that

$$\int_\Gamma |u^2| \leqslant C ;$$

hence also

$$\int_{\Omega_2} |u^2|^2 \leqslant C . \tag{9.9}$$

For any subsequence of u^2's such that $u^2 \to \zeta$ weakly in $L^2(\Omega_2)$ we have $\nabla u^2 \to \nabla\zeta = \Delta w$; thus ζ is determined up to a constant. This and (9.9) imply that there exist bounded constants $A_{\lambda_1 \varepsilon}$ such that

$$u^2 - A_{\lambda_1 \varepsilon} \to w \quad \text{ in } H^1(\Omega_2).$$

REMARK 2. – If

$$\int_{\Omega_2} f^2 \, dx < 0 \tag{9.10}$$

then the variational inequality (9.4) does not have a solution, since the functional

$$\mathfrak{A}(v) = \frac{\lambda_2}{2}\int_{\Omega_2} \sum a_{ij}^2 \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} dx + \int_{\Omega_2} f^2 v$$

does not have a minimum in the set: $v \in H^1(\Omega_2)$, $v \geqslant 0$. Indeed, for a sequence of constants $C_m \uparrow \infty$, $\mathfrak{A}(C_m) \to -\infty$. Notice also that in this case,

$$\int_\Gamma |u^2| \to \infty$$

if $\varepsilon \to 0$, $(\varepsilon/\lambda_1) \to \infty$, $(\varepsilon^2/\lambda_1) \to 0$. Indeed, otherwise, by multiplying the differential inequality for u^2 by $v - u^2$ and integrating we find that a subsequence of u^2 is convergent to a solution of (9.9). However, such a solution cannot exist when (9.10) holds.

10. – Interior reinforcement.

In this section we consider the variational inequality analogous to (5.1)-(5.4) and generalize the results of section 6. The variational inequality can be written in the form

$$\tag{10.1} \sum_{k=1}^{3}\int_{\Omega_k}\Bigl(\sum_{i,j=1}^{n} a_{ij}^k \frac{\partial u^k}{\partial x_i}\frac{\partial(v-u^k)}{\partial x_j} + f^k(v-u^k)\Bigr)dx \geqslant 0$$

$$\text{for every } v\in H_0^1(\Omega),\ v\geqslant\varphi,\ u=(u^1,u^2,u^3)\in H_0^1(\Omega).$$

Introduce the form

$$\tag{10.2} \mathcal{B}(z) = \frac{\lambda_1}{2}\int_{\tilde\Omega_1}\sum_{i,j=1}^{n} a_{ij}^1 \frac{\partial z^1}{\partial x_i}\frac{\partial z^2}{\partial x_j}dx + \frac{\lambda_3}{2}\int_{\tilde\Omega_2}\sum_{i,j=1}^{n} a_{ij}^3\frac{\partial z^3}{\partial x_i}\frac{\partial z^3}{\partial x_j}dx + \int_{\Gamma_2}\frac{1}{\alpha h(x)}(z^1-z^3)^2 dS + \int_{\tilde\Omega_1} f^1 z^1 dx + \int_{\tilde\Omega_3} f^3 z^3 dx$$

over the set K consisting of all functions $z=(z^1,z^2)$ such that $z^1\in H^1(\tilde\Omega_1)$, $z^2\in H^1(\tilde\Omega_2)$, $z^1\geqslant 0$, $z^2\geqslant 0$, and the trace of z^2 on $\partial\Omega$ is equal to zero. Notice that z^1 and z^2 are not required to agree along Γ_2

Consider the variational problem:

$$\min_{z\in K}\mathcal{B}(z)\,.$$

Since the quadratic part of $\mathcal{B}(z)$ is coercive, there exists a unique solution (w^1, w^3). Obviously, (w^1, w^3) is the unique solution of the variational inequality

$$\tag{10.3} \lambda_1\int_{\tilde\Omega_1}\sum a_{ji}^1\frac{\partial w^1}{\partial x_i}\frac{\partial(v^1-w^1)}{\partial x_j}dx + \lambda_3\int_{\tilde\Omega_2}\sum a_{ij}^3\frac{\partial w^3}{\partial x_i}\frac{\partial(v^3-w^3)}{\partial x_j}dx + \int_{\Gamma_2}\frac{1}{\alpha h(x)}(w^1-w^3)\bigl((v^1-v^3)-(w^1-w^3)\bigr)dS + \int_{\tilde\Omega_1} f^1(v^1-w^1)dx + \int_{\tilde\Omega_3} f^3(v^3-w^3)dx \geqslant 0$$

for any $(v^1,v^3)\in K$, $(w^1,w^3)\in K$.

It is easily seen that on the subset of Γ_2 where $w^1>0$, $w^3>0$, the relations (6.6), (6.7) are satisfied.

THEOREM 10.1. *If* $\varphi\equiv 0$ *and* $\varepsilon\to 0$, $\lambda_2\to 0$, $(\varepsilon/\lambda_2)\to\alpha$ *where* $0<\alpha<\infty$, *then*

$$u^k\to w^k \textit{ uniformly in compact subsets of } \Omega_k \qquad (k=1,3)$$

where (w^1,w^3) *is the solution of* (10.3).

We shall briefly describe the proof. First we note that Lemma 5.1 extends to the present case; the proof is the same, except for some modifications as in the proof of Lemma 8.1. We can now conclude that the estimate (6.2), (6.4) are valid.

Next we write the penalized problem for u_δ^1, u_δ^2, u_δ^3, and multiply the first equation by $v^1 - u_\delta^1$ and the third equation by $v^3 - u_\delta^3$, and integrate over Ω_1 and Ω_3 respectively. Now we perform a diffeomorphism of $\Omega_1 \cup \Omega_3$ onto $\tilde{\Omega}_1 \cup \tilde{\Omega}_3$ as at the beginning of the proof of Theorem 6.1. It follows, after letting $\delta \to 0$, that u^1, u^3 satisfy a variational inequality similar to (10.3) with

$$a_{ij}^k,\ f^k \text{ replaced by } \tilde{a}_{ij}^k,\ \tilde{f}^k$$

and with additional terms

$$\int_{\Gamma_2} \eta_k (v - u^k) \qquad (k = 1, 3)$$

where

$$\int_{\Gamma_2} |\eta_k|^2 \to 0 \qquad (\text{cf. } (6.11));$$

also, (6.10) holds.

Using a stability result for variational inequalities, the assertion of the theorem readily follows.

REMARK 1. – If $\alpha = 0$ then (u^1, u^3) converges to the solution of the variational inequality

$$\lambda_1 \int_{\tilde{\Omega}_1} \sum a_{ij}^1 \frac{\partial w^1}{\partial x_i} \frac{\partial (v^1 - w^1)}{\partial x_j} + \lambda_3 \int_{\tilde{\Omega}_3} \sum a_{ij}^3 \frac{\partial w^3}{\partial x_i} \frac{\partial (v^3 - w^3)}{\partial x_j}$$
$$+ \int_{\tilde{\Omega}_1} f^1 (v^1 - w^1) + \int_{\tilde{\Omega}_3} f^3 (v^3 - w^3) \geqslant 0 \qquad \text{for any } (v^1, v^3) \in \tilde{K},\ (w^1, w^3) \in \tilde{K}$$

where $\tilde{K}$ consists of all $v \in H_0^1(\Omega)$, $v \geqslant 0$. The proof is similar to the proof of Theorem 10.1.

REMARK 2. – For the case $\alpha = \infty$ one can probably derive asymptotic results provided the f^k or the u^k are appropriately scaled (compare [1]).

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