The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants (*).

ALFRED GRAY (College Park, Maryland, U.S.A.)
LUIS M. HERVELLA (Santiago de Compostela, Spain)

Summary. – *It is shown that in a natural way there are precisely sixteen classes of almost Hermitian manifolds.*

1. – Introduction.

In order to generalize Kähler geometry various authors have studied certain types of almost Hermitian manifolds, e.g., Hermitian manifolds, almost Kähler manifolds. These types of manifolds bear sufficient resemblance to Kähler manifolds so that it is possible to generalize a portion of Kähler geometry to each type. In [20] KOTŌ established inclusion relations between various classes. In [9], [10] these inclusion relations were shown to be strict by the method of constructing explicit examples.

The main point of the present paper is to fit all of these classes into a general system, which in a reasonable sense is complete. This will be accomplished by means of a detailed study of a representation of the unitary group $U(n)$ on a certain space W. Geometrically, W can be interpreted as the space of tensors which satisfy the same identities as the covariant derivative of the Kähler form of an almost Hermitian manifold.

Our scheme provides a general framework in which to study almost Hermitian maniolfds. On the one hand the classes of nearly Kähler manifolds, almost Kähler manifolds, etc., fit nicely into our pattern. In addition, locally conformal Kähler and almost Kähler manifolds occur. There are sixteen classes in all. Furthermore our scheme is important in the study of invariants of almost Hermitian manifolds.

The sixteen classes come about in the following way. The representation of $U(n)$ on W has four irreducible components, $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. It is possible to form sixteen different invariant subspaces from these four. Each invariant subspace corresponds to a different class of almost Hermitian manifolds. For example, W_1 corresponds to the class of nearly Kähler manifolds, W_2 to the class of almost Kähler manifolds, and $W_3 \oplus W_4$ to the class of Hermitian manifolds.

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In section 2 we define precisely the space W, and using Weyl's theorem on invariants we prove the irreducibility of the components in the decomposition of W. We show in section 3 how each invariant subspace of W corresponds to a class of almost Hermitian manifolds.

We construct in section 4 a certain tensor field μ which measures the failure of an almost Hermitian manifold to be locally conformally equivalent to a Kähler manifold; the tensor field is analogous to the Weyl conformal tensor field of Riemannian geometry. Using μ we determine which of the classes are preserved under conformal changes of metric. The results of section 4 are used in section 5 to find all possible inclusion relations between the sixteen types, and to demonstrate that all of the inclusions are strict.

A particularly interesting class that arises in our scheme is the class corresponding to W_4. This is the class $\mathcal{W}_4$ of almost Hermitian manifolds M satisfying the identity

$$\nabla_X(F)(Y, Z) = \frac{-1}{2(n-1)} \{\langle X, Y\rangle \delta F(Z) - \langle X, Z\rangle \delta F(Y) - \langle X, JY\rangle \delta F(JZ) + \langle X, JZ\rangle \delta F(JY)\}, \tag{1.1}$$

where $2n$ is the real dimension of M. Three facts about the class $\mathcal{W}_4$ are noteworthy: (1) Any manifold in $\mathcal{W}_4$ automatically has an integrable almost complex structure. (2) Any manifold locally conformally equivalent to a Kähler manifold is in $\mathcal{W}_4$. (3) Let the *Lee form* θ of an almost Hermitian manifold M be defined by $\theta = \delta F \cdot J$. Suppose $M \in \mathcal{W}_4$; then M is locally or globally conformally Kählerian according to whether θ is closed or exact.

In section 6 we give many examples of almost Hermitian manifolds in each of the sixteen classes. A large number of the examples are compact homogeneous spaces.

In section 7 we discuss the invariants of $U(n)$ from a more general point of view, and we determine all invariants of the representation of $U(n)$ on the space of tensors involving two derivatives of the components of the metric tensor and almost complex structure. These are the six invariants τ, τ^*, $\|\nabla F\|^2$, $\|dF\|^2$, $\|\delta F\|^2$, and $\|S\|^2$. Furthermore, we determine the linear relations between these invariants, and in this way we compute the space of invariants of order 2 for each of the 16 classes. Our work generalizes that of GILKEY [8] who did the computation for Hermitian manifolds. See also [4].

Finally in section 8 we show that there are naturally four classes of almost symplectic manifolds.

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2. – The space of covariant derivatives of the Kähler form.

The covariant derivative ∇F of the Kähler form of an almost Hermitian manifold is a covariant tensor of degree 3 which has various symmetry properties. We shall define a finite dimensional vector space W that will consist of those tensors that possess the same symmetries. Then we study the decomposition of W into irreducible components under a certain natural representation of the unitary group.

Let V be a real vector space of dimension $2n$ with an almost complex structure J and a real positive definite inner product $\langle\, ,\rangle$. We assume that J and $\langle\, ,\rangle$ are compatible in the sense that $\langle Jx, Jy\rangle = \langle x, y\rangle$ for $x, y \in V$. Let V^* denote the dual space of V, and consider the space $V^*\otimes V^*\otimes V^*$. This space is naturally isomorphic to the space of all trilinear covariant tensors on V. Let W be the subspace of $V^*\otimes V^*\otimes V^*$ defined by

$$W = \{\alpha \in V^*\otimes V^*\otimes V^* \mid \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, Jy, Jz) \text{ for all } x, y, z \in V\}\,.$$

There is a natural inner product on W given by

$$\langle \alpha, \beta\rangle = \sum_{i,j,k=1}^{2n} \alpha(e_i, e_j, e_k)\beta(e_i, e_j, e_k)\,,$$

where $\{e_1, \ldots, e_{2n}\}$ is an arbitrary orthonormal basis of V. Also, for $\alpha \in W$ let $\bar{\alpha} \in V^*$ be defined by

$$\bar{\alpha}(z) = \sum_{i=1}^{2n} \alpha(e_i, e_i, z)$$

for $z \in V$. We define four subspaces of W as follows:

$$\begin{aligned}
W_1 &= \{\alpha \in W \mid \alpha(x, x, z) = 0 \text{ for all } x, z \in V\}\,,\\
W_2 &= \{\alpha \in W \mid \alpha(x, y, z) + \alpha(z, x, y) + \alpha(y, z, x) = 0 \text{ for all } x, y, z \in V\}\,,\\
W_3 &= \{\alpha \in W \mid \alpha(x, y, z) - \alpha(Jx, Jy, z) = \bar{\alpha}(z) = 0 \text{ for all } x, y, z \in V\}\,,\\
W_4 &= \Big\{\alpha \in W \mid \alpha(x, y, z) = -\frac{1}{2(n-1)}\big(\langle x, y\rangle\bar{\alpha}(z) - \langle x, z\rangle\bar{\alpha}(y)\\
&\qquad - \langle x, Jy\rangle\bar{\alpha}(Jz) + \langle x, Jz\rangle\bar{\alpha}(Jy)\big) \text{ for all } x, y, z \in V\Big\}\,.
\end{aligned}$$

The usual representation of $U(n)$ on V induces a representation of $U(n)$ on W. The next theorem describes the decomposition of this induced representation into irreducible components.

THEOREM 2.1. – We have $W = W_1\oplus W_2\oplus W_3\oplus W_4$. This direct sum is orthogonal, and it is preserved under the induced representation of $U(n)$ on W. The induced representation of $U(n)$ on W_i is irreducible. For $n = 1$, $W = \{0\}$; for $n = 2$, $W_1 = W_3 = \{0\}$, so that $W = W_2 \oplus W_4$. For $n = 2$, W_2 and W_4 are nontrivial, and for $n \geqslant 3$ all of the W_i are nontrivial.

PROOF. – It is not difficult to check that W_1 and W_2 are orthogonal and that

$$W_1 \oplus W_2 = \{\alpha \in W \mid \alpha(x, y, z) + \alpha(Jx, Jy, z) = 0 \text{ for all } x, y, z \in V\}.$$

Thus W_1, W_2, W_3, are mutually orthogonal; furthermore

$$W_1 \oplus W_2 \oplus W_3 = \{\alpha \in W \mid \bar{\alpha} = 0\}.$$

Moreover, it can be verified that

$$W_3 \oplus W_4 = \{\alpha \in W \mid \alpha(x, y, z) - \alpha(Jx, Jy, z) = 0 \text{ for all } x, y, z \in V\}.$$

Hence all of the W_i's are mutually orthogonal, and $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. It is also easy to prove that this decomposition is preserved under the action of $U(n)$.

We now show that the induced representation on each nontrivial W_i is irreducible. First consider the case $n \geqslant 3$. To each component of the induced representation of $U(n)$ on W we assign a $U(n)$ invariant symmetric bilinear form which vanishes precisely on that component. Thus the number of components of the representation of $U(n)$ on W is less than or equal to the dimension of the space of quadratic invariants of the representation of $U(n)$ on W.

We now show that when $n \geqslant 3$ this dimension is equal to 4. We define $\| \;\|^2$, A, B, C by

$$\|\alpha\|^2 = \sum_{i,j,k=1}^{2n} \alpha(e_i, e_j, e_k)^2,$$

$$A(\alpha) = \sum_{i,j,k=1}^{2n} \alpha(e_i, e_j, e_k)\alpha(e_j, e_i, e_k),$$

$$B(\alpha) = \sum_{i,j,k=1}^{2n} \alpha(e_i, e_j, e_k)\alpha(Je_i, Je_j, e_k),$$

$$C(\alpha) = \sum_{i=1}^{2n}\left\{\sum_{j=1}^{2n} \alpha(e_j, e_j, e_i)\right\}^2 = \|\bar{\alpha}\|^2.$$

It is clear that $\| \;\|^2$, A, B, C are quadratic invariants of the induced representation of $U(n)$ on W. That they are linearly independent can be proved directly, or it follows from theorem 5.2. We remark that $\| \;\|^2$, A, and C are invariants of $O(2n)$.

To prove that $\| \;\|^2$, A, B, C span the space of quadratic invariants, we must use Weyl's theorem on invariants of the unitary group. Weyl's theorem is stated for the orthogonal group in a convenient way in [2, p. 76] (see also [18]). We shall need instead Weyl's theorem for the unitary group, but in the form of [2, p. 76]. This can be effected by using Hermitian symmetric bilinear forms instead of real symmetric bilinear forms and after wards taking the real and imaginary parts. According to this theorem every quadratic invariant of W is a linear combination of elementary invariants P_σ of the form

$$P_\sigma(\alpha) = \sum_{i_1,\dots,i_{l+3}=1}^{2n} \sigma(\alpha\otimes\alpha\otimes F\otimes\dots\otimes F)(e_{i_1}, e_{i_1}, \dots, e_{i_{l+3}}, e_{i_{l+3}}),$$

where F is given by $F(x, y) = \langle Jx, y\rangle$ and σ is a permutation of degree $2l+6$. Here l is the number of F's occuring in the tensor product, and without loss of generality we may assume that $0\leqslant l\leqslant 3$. Also $\sigma(\alpha\otimes\alpha\otimes F\otimes\dots\otimes F)$ denotes the obvious action of σ as a permutation of the arguments.

It is not difficult to prove that $P_\sigma(\alpha) = 0$, except when $l = 0$ or 2. In the case $l = 0$, it can be checked that every elementary invariant P_σ is a scalar multiple of $\|\ \|^2$, A, or C. When $l = 2$, every elementary invariant P_σ is a scalar multiple of one of these three, or it is a multiple of B.

Thus the representation of $U(n)$ on W has precisely four components, when $n\geqslant 3$. When $n = 1$ or 2, it is easy to verify that the situation degenerates into that described in the statement of the theorem. Hence the theorem follows.

REMARK. – If $\dim V = 2n$, then $\dim W = 2\dim(W_1\oplus W_2) = 2\dim(W_3\oplus W_4) = 2n^2(n-1)$. Also $\dim W_1 = \frac{1}{3}n(n-1)(n-2)$, $\dim W_2 = \frac{2}{3}n(n-1)(n+1)$, $\dim W_3 = n(n+1)(n-2)$ (for $n\geqslant 2$), and $\dim W_4 = 2n$.

3. – The sixteen classes.

Let M be a C^∞ almost Hermitian manifold with metric $\langle\ ,\ \rangle$, Riemannian connection ∇, and almost complex structure J. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. Then we have $\langle JX, JY\rangle = \langle X, Y\rangle$ for $X, Y\in\mathfrak{X}(M)$. Also, S will denote the *Nijenhuis tensor* of M, that is, $S(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ for $X, Y\in\mathfrak{X}(M)$. The *Kähler form* F is given by $F(X, Y) = \langle JX, Y\rangle$; and the *Lee form* is the 1-form θ defined by $\theta(X) = \big((-1)/(n-1)\big)\,\delta F(JX)$, where δ denotes the coderivative.

For any almost Hermitian manifold there is a representation of $U(n)$ on each tangent space M_m. Put

$$W_m = \{\alpha\in M_m^*\otimes M_m^*\otimes M_m^* \mid \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, Jy, Jz)\}.$$

Then the induced representation of $U(n)$ on W_m has the four components W_{m1}, W_{m2}, W_{m3}, W_{m4}, as described in the previous section. It is possible to form from these four a total of sixteen invariant subspaces of W_m (including $\{0\}$ and W_m).

DEFINITION. – Let U be one of the sixteen invariant subspaces of W. For an almost Hermitian manifold M and $m\in M$, let U_m denote the corresponding subspace of W_m. Then $\mathcal{U}$ will denote the class of all almost Hermitian manifolds M such that $(\nabla F)_m \in U_m$ for all $m \in M$.

Of course in order for this definition to be meaningful, one must show that for any almost Hermitian manifold M, ∇F has all the required symmetries in order that $(\nabla F)_m \in W_m$ for all $m \in M$. It is obvious that $\nabla_X(F)(Y, Z)$ is skew-symmetric in Y and Z. Also we have

$$\begin{aligned}\nabla_X(F)(Y, JY) &= XF(Y, JY) - F(\nabla_X Y, JY) - F(Y, \nabla_X JY)\\ &= X\|Y\|^2 - \langle\nabla_X Y, Y\rangle - \langle JY, \nabla_X JY\rangle\\ &= \tfrac{1}{2}X\|Y\|^2 - \tfrac{1}{2}X\|JY\|^2 = 0\,,\end{aligned}$$

so that $\nabla_X(F)(Y, Z) = -\nabla_X(F)(JY, JZ)$ for $X, Y, Z \in \mathfrak{X}(M)$.

The class corresponding to W_i will be denoted by $\mathcal{W}_i$, and that corresponding to $W_i \oplus W_j$ by $\mathcal{W}_i \oplus \mathcal{W}_j$, etc. Also, $\mathcal{K}$ will correspond to $\{0\}$ and $\mathcal{W}$ to W. Some, but not all, of the classes have been studied. We explain how the classes just introduced coincide with classes studied by various authors:

$\mathcal{K} =$ the class of Kähler manifolds;

$\mathcal{W}_1 = \mathcal{NK} =$ the class of nearly Kähler manifolds (also called almost Tachibana spaces);

$\mathcal{W}_2 = \mathcal{AK} =$ the class of almost Kähler manifolds;

$\mathcal{W}_3 = \mathcal{H} \cap \mathcal{SK} =$ the class of Hermitian semi-Kähler manifolds (also called special Hermitian manifolds);

$\mathcal{W}_4 =$ a class which contains locally conformal Kähler manifolds;

$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK} =$ the class of quasi-Kähler manifolds;

$\mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{H} =$ the class of Hermitian manifolds;

$\mathcal{W}_2 \oplus \mathcal{W}_4 =$ a class which contains locally conformal almost Kähler manifolds;

$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{SK} =$ the class of semi-Kähler manifolds;

$\left.\begin{array}{l}\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_1\\ \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_2\end{array}\right\}$ classes studied by HERVELLA and VIDAL [16], [17];

$\mathcal{W} =$ the class of almost Hermitian manifolds.

We now use the results of section 2 to describe the sixteen classes.

THEOREM 3.1. – The defining relations for each of the sixteen classes (in the case that $\dim M \geqslant 6$) are given in Table I. The case $\dim M = 4$ is treated in Table II.

TABLE I. – *Almost Hermitian manifolds of dimension* $\geqslant 6$.

Class	Defining conditions
$\mathcal{K}$	$\nabla F = 0$
$\mathcal{W}_1 = \mathcal{NK}$	$\nabla_X(F)(X, Y) = 0 \quad (\text{or } 3\,\nabla F = dF)$
$\mathcal{W}_2 = \mathcal{AK}$	$dF = 0$
$\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$	$\delta F = S = 0 \quad (\text{or } \nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = \delta F = 0)$
$\mathcal{W}_4$	$\nabla_X(F)(Y, Z) = \dfrac{-1}{2(n-1)}\{\langle X, Y\rangle\,\delta F(Z) - \langle X, Z\rangle\,\delta F(Y) - \langle X, JY\rangle\,\delta F(JZ) + \langle X, JZ\rangle\,\delta F(JY)\}$
$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK}$	$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)(JY, Z) = 0$
$\mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{H}$	$S = 0 \quad (\text{or } \nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0)$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$\nabla_X(F)(X, Y) - \nabla_{JX}(F)(JX, Y) = \delta F = 0$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$dF = F \wedge \theta \quad \left(\text{or } \underset{XYZ}{\mathfrak{S}}\left\{\nabla_X(F)(Y, Z) - \dfrac{1}{n-1} F(X, Y)\,\delta F(JZ)\right\} = 0\right)$
$\mathcal{W}_1 \oplus \mathcal{W}_4$	$\nabla_X(F)(X, Y) = \dfrac{-1}{2(n-1)}\{\lVert X\rVert^2\,\delta F(Y) - \langle X, Y\rangle\,\delta F(X) - \langle JX, Z\rangle\,\delta F(JX)\}$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$\underset{XYZ}{\mathfrak{S}}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = \delta F = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{SK}$	$\delta F = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)\ (JY, Z) = \dfrac{-1}{n-1}\{\langle X, Y\rangle\,\delta F(Z) - \langle X, Z\rangle\,\delta F(Y) - \langle X, JY\rangle\,\delta F(JZ) + \langle X, JZ\rangle\,\delta F(JY)\}$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_1$	$\nabla_X(F)(X, Y) - \nabla_{JX}(F)(JX, Y) = 0 \quad (\text{or } \langle S(X, Y), X\rangle = 0)$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_2$	$\underset{XYZ}{\mathfrak{S}}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = 0 \quad \left(\text{or } \underset{XYZ}{\mathfrak{S}}\langle S(X, Y), JZ\rangle = 0\right)$
$\mathcal{W}$	No condition

PROOF. – The results of section 2 give immediately the defining relations for the following classes: $\mathcal{K}$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, and $\mathcal{W}$. We now check the remaining classes.

For the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ we observe that the orthogonal projection $S_1 \colon W \to W_1$

is given by

$$S_1(\alpha)(x, y, z) = \tfrac{1}{6} \mathop{\mathfrak{S}}_{x,y,z} \{\alpha(x, y, z) - \alpha(Jx, Jy, z)\}$$

for $x, y, z \in V$, where $\mathfrak{S}$ denotes the cyclic sum. Hence $W_2 \oplus W_3 \oplus W_4 = W_1^{\perp} = \{\alpha \mid S_1(\alpha) = 0\}$. Thus we obtain the defining relation for $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Similar arguments apply to $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ and $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. The defining relations for all the other classes with two summands can be determined by taking the intersections among the classes with three summands.

Also, for the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ one proves directly that the defining relation can be written more simply as $dF = F \wedge \theta$. Finally the alternate conditions for the classes $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ follow from the following identities:

$$\langle S(X, Y), JZ\rangle = \langle \nabla_X(J)Y - \nabla_{JX}(J)JY - \nabla_Y(J)X + \nabla_{JY}(J)JX, Z\rangle$$

and

$$2\langle \nabla_X(J)Y - \nabla_{JX}(J)JY, Z\rangle = \langle S(X, Y), JZ\rangle - \langle S(Y, Z), JX\rangle + \langle S(Z, X), JY\rangle$$

for $X, Y, Z \in \mathfrak{X}(M)$.

TABLE II. - *Almost Hermitian manifolds of dimension* $= 4$.

Class	Defining conditions
$\mathcal{K}$	$\nabla F = 0$
$\mathcal{AK} = \mathcal{W}_2$	$dF = 0$
$\mathcal{H} = \mathcal{W}_4$	$S = 0$
$\mathcal{W}$	No condition

REMARK. - A potentially interesting class of almost Hermitian manifolds are those satisfying $dF^k = 0$ for some k with $1 \leqslant k \leqslant n$. For $k = 1$ this class is $\mathcal{W}_2$, and for $k = n - 1$ it is $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. From the results of section 2 we suspect that for arbitrary k, the class defined by $dF^k = 0$ must coincide with one of the classes given in table I. We now determine this class precisely.

THEOREM 3.2. - Let $\mathcal{AK}^{(k)}$ denote the class of almost Hermitian manifolds for which $dF^k = 0$. Then

$$\mathcal{AK}^{(k)} = \begin{cases} \mathcal{W}_2 & \text{for } 1 \leqslant k \leqslant n-2\,, \\ \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 & \text{for } k = n-1\,, \\ \mathcal{W} & \text{for } k = n\,. \end{cases}$$

PROOF. – That $\mathcal{AK}^{(n)} = \mathcal{W}$ is obvious. Next let $1 \leqslant k \leqslant n-1$, and let $\{E_1, JE_1, \ldots, E_n, JE_n\}$ be an orthonormal basis on an open subset of M, where $M \in \mathcal{AK}^{(k)}$. We compute dF^k and find

$$(3.1) \qquad 0 = dF^k(E_{i_1}, JE_{i_1}, \ldots, E_{i_k}, JE_{i_k}, X) = k! \sum_{j=1}^{k} dF(E_{i_j}, JE_{i_j}, X) .$$

Written out, (3.1) is a system of linear equations:

$$(3.2) \qquad \begin{cases} dF(E_1, JE_1, X) + dF(E_2, JE_2, X) + \ldots + dF(E_k, JE_k, X) = 0 , \\ dF(E_1, JE_1, X) + \ldots + dF(E_{k-1}, JE_{k-1}, X) + dF(E_{k+1}, JE_{k+1}, X) = 0 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \end{cases}$$

In the case when $1 \leqslant k \leqslant n-2$, (3.2) has only the trivial solution. Similarly one shows that $dF(X, Y, Z) = 0$, and so $\mathcal{AK}^{(k)} = \mathcal{W}_2$ for $1 \leqslant k \leqslant n-2$. When $k = n-1$, (3.2) reduces to $\delta F(X) = 0$. Hence the theorem follows.

Also, theorem 3.2 follows from a divisibility theorem for symplecticm anifolds. See [5], [6], [23].

4. – Classes preserved under conformal changes of metric.

In this section we determine which of the sixteen classes are preserved under a conformal change of metric; we assume that the almost complex structure remains unchanged. The 2- and 4-dimensional cases are quite simple, and so we assume that all manifolds have dimension at least 6.

DEFINITION. – Let $\mathcal{U}$ be one of the sixteen classes given in table I. Then $\mathcal{U}^0$ will denote the class of all manifolds locally conformally related to manifolds in $\mathcal{U}$. In other words, $(M, J, \langle\,,\rangle^0) \in \mathcal{U}^0$ if and only if for each $m \in M$ there exists an open neighborhood V of m such that $(V, J, \langle\,,\rangle^0)$ is conformally related to $(V, J, \langle\,,\rangle) \in \mathcal{U}$.

It will be convenient to introduce a tensor field μ which is a conformal invariant for almost Hermitian manifolds. There is an analogy between μ and the Weyl conformal tensor. Just as the Weyl conformal tensor measures the failure of a Riemannian manifold to be conformally flat, the tensor field μ measures the failure of an almost Hermitian manifold to be conformally Kählerian.

DEFINITION. – Let $(M, J, \langle\,,\rangle)$ be an almost Hermitian manifold. Then μ is the tensor field of type (2, 1) given by

$$(4.1) \qquad \langle \mu(X, Y), Z\rangle = \nabla_X(F)(Y, Z) + \frac{1}{2(n-1)} \{\langle X, Y\rangle \delta F(Z) - \langle X, Z\rangle \delta F(Y) - \langle X, JY\rangle \delta F(JZ) + \langle X, JZ\rangle \delta F(JY)\} ,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

LEMMA 4.1. – Let $(M, J, \langle\,,\rangle)$ and $(M, J, \langle\,,\rangle^0)$ be locally conformally related almost Hermitian manifolds. Then the corresponding tensor fields μ and μ^0 satisfy $\mu^0 = \mu$.

PROOF. – There is a well-known formula expressing the Riemannian connection of $\langle\,,\rangle^0$ in terms of the Riemannian connection of $\langle\,,\rangle$:

$$\nabla^0_X Y = \nabla_X Y + X(\sigma) Y + Y(\sigma) X - \langle X, Y\rangle \operatorname{grad}\sigma \tag{4.2}$$

for $X, Y \in \mathfrak{X}(M)$ (where $\langle \operatorname{grad}\sigma, X\rangle = X\sigma$). From (4.2) (see [9], [10]) it follows that $\nabla^0 F^0$ is related to ∇F by the formula

$$\nabla^0_X(F^0)(Y, Z) = e^{2\sigma}\{\nabla_X(F)(Y, Z) - \langle X, Y\rangle JZ(\sigma) + \langle X, Z\rangle JY(\sigma) - \langle X, JY\rangle Z(\sigma) + \langle X, JZ\rangle Y(\sigma)\}\,, \tag{4.3}$$

and that $\delta^0 F^0$ is related to δF by

$$\delta^0 F^0(X) = \delta F(X) + 2(n-1) JX(\sigma)\,, \tag{4.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$. From (4.3) and (4.4) we obtain immediately $\mu = \mu^0$.

Next we prove

THEOREM 4.2. – For any class $\mathcal{U}$ given in table I we have $\mathcal{U}^0 \subseteq \mathcal{W}_4 \oplus \mathcal{U}$. Thus $\mathcal{U} = \mathcal{U}^0$ if and only if $\mathcal{W}_4 \subseteq \mathcal{U}$. Hence the conformally invariant classes are: $\mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}$.

PROOF. – The defining relation for each of the classes mentioned in the statement of the theorem can be rewritten in terms of μ. From table I we have

$M \in \mathcal{W}_4$ if and only if $\mu = 0$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_4$ if and only if $\mu(X, X) = 0$ for all $X \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_2 \oplus \mathcal{W}_4$ if and only if $\underset{XYZ}{\mathfrak{S}} \langle \mu(X, Y), Z\rangle = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\mu(X, Y) - \mu(JX, JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ if and only if $\mu(X, Y) + \mu(JX, JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\mu(X, X) - \mu(JX, JX) = 0$ for all $X \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\underset{XYZ}{\mathfrak{S}} \langle \mu(X, Y) - \mu(JX, JY), Z\rangle = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$.

From these descriptions it is clear that if $\mathcal{U}$ is any one of the eight classes containing $\mathcal{W}_4$, then $(M, J, \langle\,,\rangle) \in \mathcal{U}$ if and only if $(M, J, \langle\,,\rangle^0) \in \mathcal{U}$, where $\langle\,,\rangle$ and $\langle\,,\rangle^0$ are conformally related.

We note the following formulas:

$$(4.5)\qquad \langle\mu(X, Y)-\mu(JX, JY), Z\rangle = \langle\nabla_X(J)Y-\nabla_{JX}(J)(JY), Z\rangle$$
$$= \tfrac{1}{2}\langle S(X, Y), JZ\rangle - \tfrac{1}{2}\langle S(Y, Z), JX\rangle + \tfrac{1}{2}\langle S(Z, X), JY\rangle ,$$

$$(4.6)\qquad \underset{XYZ}{\mathfrak{S}}\langle\mu(X, Y), Z\rangle = dF(X, Y, Z)-(F\wedge\theta)(X, Y, Z) .$$

Actually the form $dF-F\wedge\theta$ depends only on the symplectic structure F (see § 8). This is the conformal torsion introduced by LIBERMANN [23, p. 71]. Libermann also introduced a torsion corresponding to μ. Formula (4.6) shows that μ determines $dF-F\wedge\theta$. The converse, however, is not true. For example if $M\in\mathcal{W}_2$ but $M\notin\mathcal{K}$, then $dF=F\wedge\theta=0$ on M but $\mu\neq 0$. See also [24].

Next suppose $\mathcal{U}\cap\mathcal{W}_4=\mathcal{K}$. We have shown that $\mathcal{U}^0\subseteq\mathcal{U}\oplus\mathcal{W}_4$. The following theorem characterizes those manifolds in $\mathcal{U}\oplus\mathcal{W}_4$ which are contained in $\mathcal{U}^0$.

THEOREM 4.3. – Let $\mathcal{U}$ be one of the sixteen classes and suppose that $\mathcal{U}\cap\mathcal{W}_4=\mathcal{K}$. Let $(M, J, \langle\,,\rangle)\in\mathcal{U}\oplus\mathcal{W}_4$. Then

(i) $(M, J, \langle\,,\rangle)\in\mathcal{U}^0$ (that is, $(M, J, \langle\,,\rangle)$ is locally conformally equivalent to an almost Hermitian manifold in $\mathcal{U}$) if and only if the Lee form θ of $(M, J, \langle\,,\rangle)$ is closed.

(ii) $(M, J, \langle\,,\rangle)$ is globally conformally equivalent to a manifold in $\mathcal{U}$ if and only if the Lee form θ of $(M, J, \langle\,,\rangle)$ is exact.

PROOF. – Suppose $(M, J, \langle\,,\rangle)$ is globally conformally equivalent to a manifold $(M, J, \langle\,,\rangle^0)$ in $\mathcal{U}$. Since $\mathcal{U}\cap\mathcal{W}_4=\mathcal{K}$ we have $\mathcal{U}\subseteq\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$. Thus the Lee form of $(M, J, \langle\,,\rangle^0)$ vanishes. By (4.4) the Lee form of $(M, J, \langle\,,\rangle)$ is exact.

Conversely, let the Lee form θ of $(M, J, \langle\,,\rangle)$ be exact, $\theta=df$. Define a function σ by $\sigma=\frac{1}{2}f$, and put $\langle\,,\rangle^0=e^{2\sigma}\langle\,,\rangle$. From (4.2) we have $\delta^0F^0=0$, and so $(M, J, \langle\,,\rangle^0)\in(\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3)\cap(\mathcal{U}\oplus\mathcal{W}_4)=\mathcal{U}$.

This proves (ii). The proof of (i) is the same except that everything is done locally and the Poincaré lemma is used.

Theorem 4.3 shows, for example, that on any Kähler manifold it is possible to make a conformal change of metric and get a manifold in $\mathcal{W}_4$ (which of course is a complex manifold). In this way one can construct many non-Kähler metrics on manifolds in $\mathcal{W}_4$.

A more interesting example (due to VAISMAN [27]) of a non-Kähler manifold in $\mathcal{W}_4$ is $S^1\times S^{2k+1}$, $k\geqslant 1$. In [27] it is shown that $S^1\times S^{2k+1}$ is locally conformally Kählerian. Of course $S^1\times S^{2k+1}$ cannot be globally conformally Kählerian because it does not have the cohomology of a Kähler manifold.

More precisely, Vaisman has shown that $S^1\times S^{2k+1}$ is locally conformally equivalent to $\boldsymbol{C}^{k+1}-\{0\}$. This shows among other things that all of the Chern numbers of $S^1 \times S^{2k+1}$ are zero (because the Chern classes depend only on the almost complex structure).

As for the other Calabi-Eckmann manifolds $S^{2k+1} \times S^{2l+1}$, their class is given in the following theorem.

THEOREM 4.4. – Let $M = S^{2k+1} \times S^{2l+1}$ and let M be given a complex structure in the standard way. Then assuming $k \leqslant l$ we have:

$$M \in \mathcal{K} \quad \text{if and only if} \quad k = l = 1 .$$

$$M \in \mathcal{W}_4, \; M \notin \mathcal{K} \quad \text{if and only if} \quad k = 1, \; l > 1 .$$

$$M \in \mathcal{W}_3 \oplus \mathcal{W}_4, \; M \notin \mathcal{W}_3 \cup \mathcal{W}_4 \quad \text{if and only if both} \quad k, l > 1 .$$

PROOF. – To prove the last statement one computes the tensor field μ for M. For $k, l > 1$ we have $\mu \neq 0$ so that $M \notin \mathcal{W}_4$ and $\theta \neq 0$ so that $M \notin \mathcal{W}_3$. On the other hand, the almost complex structure of M is integrable and so $M \in \mathcal{W}_3 \oplus \mathcal{W}_4$.

More computations for the manifolds $S^{2k+1} \times S^{2l+1}$ are carried out in the proof of theorem 7.1.

5. – The inclusion relations.

First we determine which of the sixteen classes are preserved under Cartesian products. If $\mathcal{U}$ is one of the sixteen classes let

$$\mathcal{U}^2 = \text{the class of Cartesian products of elements in } \mathcal{U} .$$

The following theorem is easy to verify:

THEOREM 5.1. – We have $\mathcal{U}^2 \subset \mathcal{U}$ provided $\mathcal{U}$ is any one of the following classes: $\mathcal{K}$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}$.

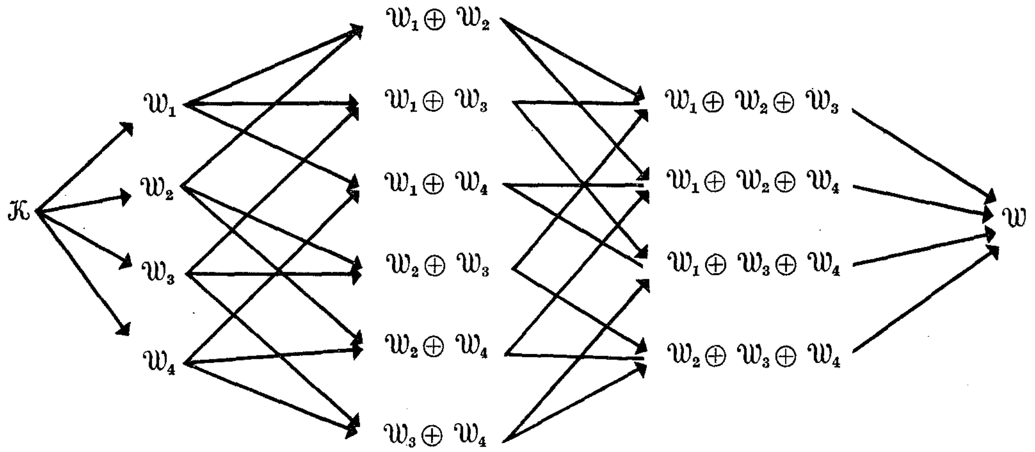
We are now ready to determine completely the inclusion relations between the various classes.

THEOREM 5.2. – All possible inclusion relations are given in table III. All of the inclusions are strict.

PROOF. – That all of the inclusions exist is obvious. We exhibit manifolds to show that each of the inclusions is strict.

First we note that the sphere S^6 is in $\mathcal{W}_1$ but not in $\mathcal{K}$. Also, if $T(M)$ denotes the tangent bundle of a nonflat Riemannian manifold, then $T(M)$ is in $\mathcal{W}_2$ but not in $\mathcal{K}$. Next let M_1 be an ordinary minimal surface in $\boldsymbol{R}^3$. Then $M_1 \times \boldsymbol{R}^4$ has an almost

TABLE III.



Hermitian structure for which $M_1 \times \boldsymbol{R}^4$ is in $\mathcal{W}_3$ but not in $\mathcal{K}$. For all of this see [9], [10], [11]. These three examples, plus Vaisman's example of $S^1 \times S^{2k+1}$, $k \geqslant 1$, show that the following four inclusions are strict: $\mathcal{K} \subset \mathcal{W}_1$, $\mathcal{K} \subset \mathcal{W}_2$, $\mathcal{K} \subset \mathcal{W}_3$, $\mathcal{K} \subset \mathcal{W}_4$.

Next we establish the strictness of the inclusions $\mathcal{W}_i \cup \mathcal{W}_j \subset \mathcal{W}_i \oplus \mathcal{W}_j$. There are two cases to consider.

Case I (i, j, 4 distinct). – Let $M_i \in \mathcal{W}_i - \mathcal{W}_j$, $M_j \in \mathcal{W}_j - \mathcal{W}_i$. Then by theorem 5.1 $(\mathcal{W}_i \oplus \mathcal{W}_j)^2 \subset \mathcal{W}_i \oplus \mathcal{W}_j$ and so $M_i \times M_j \in \mathcal{W}_i \oplus \mathcal{W}_j$. On the other hand it is clear that $M_i \times M_j$ is neither in $\mathcal{W}_i$ nor $\mathcal{W}_j$.

Case II ($i \neq j = 4$). – Let $M_i \in \mathcal{W}_i - \mathcal{K}$. Then we can make a nontrivial change of conformal metric to obtain an almost Hermitian manifold M_i^0 such that $M_i^0 \in \mathcal{W}_i \oplus \mathcal{W}_4$. By equation (4.4), $M_i^0 \notin \mathcal{W}_i$. Also, since $M_i \notin \mathcal{K}$, it follows that $M_i^0 \notin \mathcal{W}_4$. Hence $M_i^0 \notin \mathcal{W}_i \cup \mathcal{W}_4$.

Thus all of the inclusions $\mathcal{W}_i \cup \mathcal{W}_j \subset \mathcal{W}_i \oplus \mathcal{W}_j$ $(i \neq j)$ are strict. In exactly the some way one proves that the inclusions $\mathcal{W}_i \cup \mathcal{W}_j \cup \mathcal{W}_k \subset \mathcal{W}_i \oplus \mathcal{W}_j \oplus \mathcal{W}_k$ (i, j, k distinct) and $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4 \subset \mathcal{W}$ are strict. From this the theorem follows immediately.

In the course of proving theorem 5.2 we have established the following.

COROLLARY 5.3. – The inclusions $\mathcal{K} \subset \mathcal{W}_i$, $\mathcal{W}_i \cup \mathcal{W}_j \subset \mathcal{W}_i \oplus \mathcal{W}_j$, $\mathcal{W}_i \cup \mathcal{W}_j \cup \mathcal{W}_k \subset \mathcal{W}_i \oplus \mathcal{W}_j \oplus \mathcal{W}_k$, and $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4 \subset \mathcal{W}$ are all strict.

6. – Examples of almost Hermitian manifolds illustrating the sixteen types.

There are many non-Kähler almost Hermitian manifolds arising naturally in differential geometry. In section 5 we used a general technique to demonstrate the strictness of all the inclusion relations. We now exhibit some especially interesting almost Hermitian manifolds and show where they fit in.

The class of nearly Kähler manifolds $\mathcal{W}_1 = \mathcal{NK}$. The most well known example in this class is the sphere S^6. S^6 is a 3-symmetric space [12]. Moreover, every 3-symmetric space M has an almost complex structure and a metric such that $M \in \mathcal{W}_1$ [12].

The class of almost Kähler manifolds $\mathcal{W}_2 = \mathcal{AK}$. The tangent bundle $T(M)$ of a Riemannian manifold always has a naturally defined complex structure and metric such that $T(M) \in \mathcal{W}_2$. Moreover, $T(M) \notin \mathcal{K}$ provided M is not flat. Recently THURSTON [26] has given an example of a compact 4-dimensional manifold in $\mathcal{W}_2$ which has no Kähler metric on account of its cohomology.

The class of quasi-Kähler manifolds $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK}$. If M is a compact 3-symmetric space then M is homogeneous [12]. If the isotropy representation is reducible, then M has many metrics which make it a 3-symmetric space. A biinvariant metric is in $\mathcal{W}_1$; any other metric is in $(\mathcal{W}_1 \oplus \mathcal{W}_2) - (\mathcal{W}_1 \cup \mathcal{W}_2)$.

The product of odd dimensional spheres $S^{2k+1} \times S^{2l+1}$ *is in* $\mathcal{W}_3 \oplus \mathcal{W}_4$. See theorem 4.4.

Any complex parallelizable manifold is in $\mathcal{W}_3$. This can be checked directly.

Almost Hermitian manifolds defined by means of vector cross products. In [11] it is shown that any orientable 6-dimensional submanifold of $\boldsymbol{R}^8$ has an almost complex structure. The almost complex structure is constructed by means of one of the two 3-fold vector cross products on $\boldsymbol{R}^8$. When one uses the induced metric a large number of interesting 6-dimensional almost Hermitian manifolds occur. Let M be an almost Hermitian manifold constructed in this way. In [11] the following facts are proved:

(1) $M \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ always.

(2) If $M \subset \boldsymbol{R}^7$, then $M \in \mathcal{K}$ if and only if M is locally flat.

(3) If $M \subset \boldsymbol{R}^7$, then $M \in \mathcal{W}_1$ if and only if M is locally isometric to $\boldsymbol{R}^6$ or to S^6.

(4) If $M \subset \boldsymbol{R}^7$, then $M \in \mathcal{W}_2$ if and only if $M \in \mathcal{K}$.

(5) If $M \subset \boldsymbol{R}^7$, then $M \in \mathcal{W}_1 \oplus \mathcal{W}_2$ if and only if M is locally isometric to $\boldsymbol{R}^6$, S^6, or $S^2 \times \boldsymbol{R}^4$.

(6) If $M \in \mathcal{W}_3$ (that is if M is Hermitian), then M is a minimal submanifold of $\boldsymbol{R}^8$. Furthermore if $M_1 \subset \boldsymbol{R}^3$ is an ordinary minimal surface then $M_1 \times \boldsymbol{R}^4 \in \mathcal{W}_3$.

7. – Invariants involving two derivatives.

Let φ be a function which assigns to each almost Hermitian manifold M a real valued function on M. We call φ a *unitary invariant of order* k, provided that for each $m \in M$ and all normal coordinate systems $(x_1, \dots, x_n)$ at m it is possible to express φ as a polynomial involving a total of k derivatives of the components of the metric tensor and almost complex structure with respect to $(x_1, \ldots, x_n)$. It is required that the polynomial be independent of the choice of normal coordinate system. Unitary invariants of odd order are all zero. In this section we determine all unitary invariants of order 2 (also called linear invariants) for each of the sixteen classes.

For a given class $\mathcal{U}$ of almost Hermitian manifolds of dimension $2n$, let $I_n(\mathcal{U})$ be the space of unitary invariants of order 2. Gilkey [8] has computed $I_n(\mathcal{U})$ when $\mathcal{U} = \mathcal{K}$, the class of Kähler manifolds, or when $\mathcal{U} = \mathcal{W}_3 \oplus \mathcal{W}_4$, the class of Hermitian manifols. First we compute $I_n(\mathcal{W})$. For $M \in \mathcal{W}$ and $m \in M$ let

$$(7.1)\qquad \begin{cases} \|\nabla F\|^2 = \sum_{a,b,c=1}^{2n} \nabla_{e_a}(F)(e_b, e_c)^2, & \|dF\|^2 = \sum_{a,b,c=1}^{2n} dF(e_a, e_b, e_c)^2, \\ \|\delta F\|^2 = \sum_{a=1}^{2n} \delta F(e_a)^2, & \|S\|^2 = \sum_{a,b=1}^{2n} \|S(e_a, e_b)\|^2, \\ \tau = \sum_{a,b=1}^{2n} R(e_a, e_b, e_a, e_b), & \tau^* = \frac{1}{2} \sum_{a,b=1}^{2n} R(e_a, Je_a, e_b, Je_b), \end{cases}$$

where $\{e_1, \ldots, e_{2n}\}$ is an arbitrary orthonormal basis of M_m. Here τ is the scalar curvature of M, and τ^* is the * scalar curvature (see [14]). If $\sigma_1, \ldots, \sigma_k$ are unitary invariants of order 2, we denote by $[\sigma_1, \ldots, \sigma_k]$ all linear combinations of $\sigma_1, \ldots, \sigma_k$ with constant coefficients.

Theorem 7.1. – $I_n(\mathcal{W}) = [\|\nabla F\|^2, \|dF\|^2, \|\delta F\|^2, \|S\|^2, \tau, \tau^*]$ for $n \geq 3$; $I_2(\mathcal{W}) = [\|\nabla F\|^2, \|S\|^2, \tau, \tau^*]$ and $I_1(\mathcal{W}) = [\tau]$.

Proof. – For simplicity we shall treat only the case $n \geq 3$. In the first part of the proof we shall show that the six invariants given by (7.1) span $I_n(\mathcal{W})$. Afterwards it will be demonstrated that no linear relations between the six invariants exist.

First we divide the invariants of order 2 into 3 types. (One checks that these are the only possibilities; this is easy.) The three types are:

Type I. – Invariants involving two first derivatives of the components of the almost complex structure.

Type II. – Invariants involving one second derivative of the components of the almost complex structure.

Type III. – Invariants involving one second derivative of the components of the metric tensor.

Using the power series expansions of [13] one sees that the invariants of each type can be expressed in terms of J, its first and second covariant derivatives, and the curvature tensor. The results of section 2 can be interpreted as the computation of all the invariants of type I. A convenient basis consists of $\|\nabla F\|^2$, $\|dF\|^2$, $\|\delta F\|^2$, and $\|S\|^2$.

Next we determine the invariants of type III. One first observes that because only derivatives of the metric are involved, any invariant of type III is expressible in terms of curvature. Using Weyl's theorem on unitary invariants it follows that the space of invariants of type III is spanned by the elementary invariants of the following kinds:

$$P_\sigma = \sum_{a,b=1}^{2n} \sigma(R)(e_a, e_a, e_b, e_b)\,, \qquad P'_\sigma = \sum_{a,b,c=1}^{2n} \sigma(R\otimes F)(e_a, e_a, e_b, e_b, e_c, e_c)\,,$$

$$P''_\sigma = \sum_{a,b,c=1}^{2n} \sigma(R\otimes F\otimes F)(e_a, e_a, e_b, e_b, e_c, e_c, e_d, e_d)\,,$$

where σ is a permutation of the appropriate degree. See [2, p. 76] and [18]. In fact all invariants of the form P'_σ vanish, and those of the form P_σ and P''_σ are reducible to scalar multiples of τ and τ^*.

Next we show that all invariants of type II can be expressed in terms of those of type I and III. Following the method of computation for the invariants of type III, we find the following invariants (we write a for e_a, a^* for Je_a, etc.):

$$(1)\ \sum_{a,b=1}^{2n} \nabla^2_{ab}(F)(a, b)\,, \qquad (2)\ \sum_{a,b=1}^{2n} \nabla^2_{ab}(F)(a, b^*)\,,$$

$$(3)\ \sum_{a,b=1}^{2n} \nabla^2_{ab}(F)(a^*, b)\,, \qquad (4)\ \sum_{a,b=1}^{2n} \nabla^2_{ab}(F)(a^*, b^*)\,,$$

$$(5)\ \sum_{a,b=1}^{2n} \nabla^2_{aa}(F)(b, b^*)\,, \qquad (6)\ \sum_{a,b=1}^{2n} \nabla^2_{aa^*}(F)(b, b^*)\,.$$

On the other hand, we have the Ricci identity

$$(7.2)\qquad \nabla^2_{WX}(F)(Y, Z) - \nabla^2_{XW}(F)(Y, Z) = F(R_{WX}Y, Z) + F(Y, R_{WX}Z) = -R_{WXYJZ} - R_{WXJYZ}\,,$$

and the identity

$$(7.3)\qquad \nabla^2_{WX}(F)(Y, JY) = -\langle \nabla_W(J)Y, \nabla_X(J)Y\rangle\,,$$

for $W, X, Y, Z \in \mathfrak{X}(M)$. The identity (7.3) is proved by taking the covariant derivative of the equation $\nabla_X(F)(Y, JY) = 0$.

Using (7.2) it follows that the invariants (1), (4), (6) vanish. Furthermore we have $(5) = -\|\nabla F\|^2$ on account of (7.3).

Next let $\delta\theta$ be the coderivative of the Lee form, that is

$$\delta\theta = -\sum_{a=1}^{2n} \nabla_a \theta_a \,. \tag{7.4}$$

From (7.4) and the identity

$$\|dF\|^2 = 3\|\nabla F\|^2 - 6 \sum_{a,b,c=1}^{2n} \nabla_a F_{bc} \nabla_b F_{ac} \tag{7.5}$$

we find that

$$\begin{cases} (2) = -(n-1)\,\delta\theta + \frac{1}{6}\|dF\|^2 - \frac{1}{2}\|\nabla F\|^2 \,, \\ (3) = -(n-1)\,\delta\theta + \|\delta F\|^2 \,. \end{cases} \tag{7.6}$$

On the other hand using (7.2) we have

$$(2) + (3) = \tau - \tau^* \,. \tag{7.7}$$

From (7.6) and (7.7) we eliminate $\delta\theta$ obtaining

$$(2) = -\tfrac{1}{4}\|\nabla F\|^2 + \tfrac{1}{12}\|dF\|^2 - \tfrac{1}{2}\|\delta F\|^2 + \tfrac{1}{2}(\tau - \tau^*) \,,$$

$$(3) = \tfrac{1}{4}\|\nabla F\|^2 - \tfrac{1}{12}\|dF\|^2 + \tfrac{1}{2}\|\delta F\|^2 + \tfrac{1}{2}(\tau - \tau^*) \,.$$

Furthermore, we note that

$$(n-1)\,\delta\theta = -\tfrac{1}{4}\|\nabla F\|^2 + \tfrac{1}{12}\|dF\|^2 + \tfrac{1}{2}\|\delta F\|^2 - \tfrac{1}{2}(\tau - \tau^*) \,. \tag{7.8}$$

Thus we have shown that the six invariants defined by (7.1) span $I_n(\mathcal{W})$. We now show that they are linearly independent in any dimension $2n \geq 6$.

Suppose that for all almost Hermitian manifolds of a given dimension $2n \geq 6$ there is a linear relation

$$A\tau + B\tau^* + C\|\nabla F\|^2 + E\|dF\|^2 + G\|\delta F\|^2 + H\|S\|^2 = 0 \,, \tag{7.9}$$

where A, B, C, E, G, H are constants. By evaluating (7.9) on different almost Hermitian manifolds we show that $A = ... = H = 0$.

First we evaluate (7.9) on CP^n. For CP^n we have $\|\nabla F\|^2 = \|dF\|^2 = \|\delta F\|^2 = \|S\|^2 = 0$ and $\tau = \tau^* \neq 0$. It follows that

$$A = -B \,. \tag{7.10}$$

Next we evaluate (7.9) on $M^{2n} \in \mathcal{W}_1 - \mathcal{K}$; for example we can take $M^{2n} = S^6 \times C^{n-3}$. Then one computes that $\|\nabla F\|^2 = \frac{1}{9}\|dF\|^2 = \frac{1}{16}\|S\|^2 \neq 0$ and $\|\delta F\|^2 = 0$. Also $\tau - \tau^* = \|\nabla F\|^2$ (see [14]). Thus from (7.9) and (7.10) we get

$$A + C + 9E + 16H = 0 \,. \tag{7.11}$$

Continuing, we evaluate (7.9) on $M^{2n} \in \mathcal{W}_2 - \mathcal{K}$; for example we can take $M^{2n} = T(U^n)$ where U is a nonflat Riemannian manifold of dimension n. One checks that $\|S\|^2 = 4\|\nabla F\|^2 \neq 0$ and $\|\delta F\|^2 = \|dF\|^2 = 0$. Furthermore $\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2$ (see [15]). Thus using (7.9) and (7.10) we find

$$-\tfrac{1}{2}A + C + 4H = 0 . \tag{7.12}$$

Similarly we evaluate (7.9) on $M^{2n} \in \mathcal{W}_3 - \mathcal{K}$; for example we can take M^{2n} to be a complex parallelizable manifold which is not Kählerian. For such a manifold we have $\|\delta F\|^2 = \|S\|^2 = 0$, and $\|dF\|^2 = 3\|\nabla F\|^2 \neq 0$. Moreover, from (7.8) and the fact that $\delta\theta = 0$, we obtain $\tau = \tau^*$. Thus from (7.9) and (7.10) it follows that

$$C + 3E = 0 . \tag{7.13}$$

Next we evaluate (7.9) on $S^{2k+1}(r_1) \times S^{2l+1}(r_2)$. Here an (integrable) almost complex structure J on $S^{2k+1}(r_1) \times S^{2l+1}(r_2)$ is given as follows: Let N_1 and N_2 denote the unit outward normals to $S^{2k+1}(r_1)$ and $S^{2l+1}(r_2)$ regarded as hypersurfaces of $\mathbf{C}^{k+1}$ and $\mathbf{C}^{l+1}$, respectively. Let J_1 and J_2 denote the almost complex structures of $\mathbf{C}^{k+1}$ and $\mathbf{C}^{l+1}$, respectively. Then J_1N_1 and J_2N_2 are globally defined vector fields on $S^{2k+1}(r_1)$ and $S^{2l+1}(r_2)$, respectively. Locally, any vector field Z on $S^{2k+1}(r_1) \times S^{2l+1}(r_2)$ can be decomposed as

$$Z = Z_1 + Z_2 + aJ_1N_1 + bJ_2N_2$$

where Z_1 is tangent to $S^{2k+1}(r_1)$, Z_2 is tangent to $S^{2l+1}(r_2)$ and $\langle Z_1, J_1N_1\rangle = \langle Z_2, J_2N_2\rangle = 0$. Then we define an almost complex structure J on $S^{2k+1}(r_1) \times S^{2l+1}(r_2)$ by

$$JZ = J_1Z_1 + J_2Z_2 - bJ_1N_1 + aJ_2N_2 .$$

For this almost complex structure we have

$$\|\nabla F\|^2 = \frac{8k}{r_1^2} + \frac{8l}{r_2^2} \qquad \text{and} \qquad \|\delta F\|^2 = \frac{4k^2}{r_1^2} + \frac{4l^2}{r_2^2} .$$

Furthermore,

$$\tau = \frac{2k(2k+1)}{r_1^2} + \frac{2l(2l+1)}{r_2^2} \qquad \text{and} \qquad \tau^* = \frac{2k}{r_1^2} + \frac{2l}{r_2^2} .$$

Therefore, from (7.9) we have

$$\begin{aligned} 0 &= \frac{1}{r_2^2}\{4k^2(A+G) + 2k(A+B+8C+24E)\} \\ &\quad + \frac{1}{r_1^2}\{4l^2(A+G) + 2l(A+B+8C+24E)\} . \end{aligned} \tag{7.14}$$

Since (7.14) holds for all $r_1, r_2 > 0$ and all nonnegative integers k, l with $k+l=n-1$, we find that

$$A = -G\,, \tag{7.15}$$

$$A+B+8C+24E=0\,. \tag{7.16}$$

(Here (7.16) is already a consequence of (7.10) and (7.13)).

For the final equation we evaluate (7.9) on $M^{2n}\in \mathcal{W}_4 - \mathcal{K}$ for which $\delta\theta \neq 0$. For example if $\langle\,,\rangle$ denotes the standard metric on $\boldsymbol{CP}^n$ we can take $M^{2n} = (\boldsymbol{CP}^n, e^{2\sigma}\langle\,,\rangle)$ where σ is any non constant function. For such a M^{2n} we have $\|S\|^2 = 0$ and $\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2 = (2/(n-1))\|\delta F\|^2 \neq 0$. Also, because of (7.8) $\tau - \tau^* - \|\delta F\|^2 \neq 0$. In view of (7.10), (7.13), and (7.15) we see that (7.9) reduces to

$$A(\tau - \tau^* - \|\delta F\|^2) = 0\,.$$

Since $\tau - \tau^* - \|\delta F\|^2 \neq 0$ we must have

$$A = 0\,. \tag{7.17}$$

Thus we have 6 equations in 6 unknowns: (7.10), (7.11), (7.12), (7.13), (7.15), and (7.17). The unique solution is $A=B=C=E=G=H=0$. This completes the proof.

REMARKS. – In the course of proving theorem 7.1 use was made of the invariant $\delta\theta$. It is expressed in terms of the other invariants by equation (7.8). Another natural invariant that occurs is $\langle \Delta F, F\rangle = \sum_{a=1}^{2n} \Delta F(a, a^*)$. This invariant also can be expressed in terms of the other invariants:

$$\langle \Delta F, F\rangle = \tfrac{1}{3}\|dF\|^2 + 2\|\delta F\|^2\,.$$

Still another important invariant is the trace of the first Chern form, $\sum_{i=1}^{2n} \gamma_1(e_i, Je_i)$. It is easy to see that this must be an invariant of type III, and so it must be a linear combination of τ and τ^*. The exact combination can be determined by evaluating $\sum \gamma_1(e_i, Je_i)$ first on $\boldsymbol{CP}^n$, and then on $S^6\times \boldsymbol{C}^{n-3}$. Using the formula for γ_1 given in [14] for the class $\mathcal{W}_1$, it follows that

$$\sum_{i=1}^{2n} \gamma_1(e_i, Je_i) = \frac{1}{8\pi}(-\tau + 5\tau^*)\,.$$

Finally we determine the unitary invariants of order 2 for each of the 16 classes.

THEOREM 7.2. – Let $n \geq 3$. For each of the 16 classes $\mathcal{U}$, the space $I_n(\mathcal{U})$ is given in table IV. Furthermore the linear relations that describe $I_n(\mathcal{U})$ as a subspace of

TABLE IV. – *Almost Hermitian, manifolds of dimension* $\geqslant 6$.

$\mathcal{U}$	$I_n(\mathcal{U})$	Linear relations among the invariants
$\mathcal{K}$	$[\tau]$	$\tau = \tau^*$, $\|\nabla F\|^2 = \|dF\|^2 = \|\delta F\|^2 = \|S\|^2 = 0$
$\mathcal{W}_1 = \mathcal{NK}$	$[\tau, \tau^*]$	$\tau - \tau^* = \|\nabla F\|^2 = \frac{1}{9}\|dF\|^2 = \frac{1}{16}\|S\|^2$, $\|\delta F\|^2 = 0$
$\mathcal{W}_2 = \mathcal{AK}$	$[\tau, \tau^*]$	$\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2 = -\frac{1}{8}\|S\|^2$, $\|dF\|^2 = \|\delta F\|^2 = 0$
$\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$	$[\tau, \|\nabla F\|^2]$	$\tau = \tau^*$, $\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2$, $\|\delta F\|^2 = \|S\|^2 = 0$
$\mathcal{W}_4$	$[\tau, \tau^*, \|\nabla F\|^2]$	$\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2 = \dfrac{2}{n-1}\|\delta F\|^2$, $\|S\|^2 = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK}$	$[\tau, \tau^*, \|\nabla F\|^2]$	$\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2 + \frac{1}{6}\|dF\|^2$, $\|\nabla F\|^2 = -\frac{1}{3}\|dF\|^2 + \frac{1}{4}\|S\|^2$, $\|\delta F\|^2 = 0$
$\mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{H}$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2]$	$\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2$, $\|S\|^2 = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$[\tau, \tau^*, \|\nabla F\|^2]$	$\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2 + \frac{1}{6}\|dF\|^2$, $\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2 - \frac{1}{8}\|S\|^2$, $\|\delta F\|^2 = 0$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2]$	$\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2 + \frac{1}{4}\|S\|^2$, $\|dF\|^2 = \dfrac{6}{n-1}\|\delta F\|^2$
$\mathcal{W}_1 \oplus \mathcal{W}_4$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2]$	$3\|S\|^2 + 24\|\nabla F\|^2 = 8\|dF\|^2$, $16\|\nabla F\|^2 = \dfrac{32}{n-1}\|\delta F\|^2 + \|S\|^2$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$[\tau, \tau^*, \|\nabla F\|^2]$	$\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2 + \frac{1}{6}\|dF\|^2$, $\|\nabla F\|^2 = \frac{1}{3}\|dF\|^2 + \frac{1}{4}\|S\|^2$, $\|\delta F\|^2 = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{SK}$	$[\tau, \tau^*, \|\nabla F\|^2, \|S\|^2]$	$\tau - \tau^* = -\frac{1}{2}\|\nabla F\|^2 + \frac{1}{6}\|dF\|^2$, $\|\delta F\|^2 = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2, \|S\|^2]$	$3\|S\|^2 - 4\|dF\|^2 + 12(n-1)\|\delta F\|^2 = 12\|\nabla F\|^2$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_1$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2, \|S\|^2]$	$-3\|S\|^2 + 8\|dF\|^2 = 24\|\nabla F\|^2$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_2$	$[\tau, \tau^*, \|\nabla F\|^2, \|\delta F\|^2, \|S\|^2]$	$12\|\nabla F\|^2 = 4\|dF\|^2 + 3\|S\|^2$
$\mathcal{W}$	$[\tau, \tau^*, \|\nabla F\|^2, \|dF\|^2, \|\delta F\|^2, \|S\|^2]$	No condition

$I_n(\mathcal{W})$ are given. The corresponding computations for the case $n=2$ are done in table V.

TABLE V. – *Almost Hermitian manifolds of dimension* = 4.

$\mathcal{U}$	$I_2(\mathcal{U})$	Linear relations among the invariants
$\mathcal{K}$	$[\tau]$	$\tau=\tau^*,\quad \|\nabla F\|^2=\|S\|^2=0$
$\mathcal{W}_2=\mathcal{AK}$	$[\tau,\tau^*]$	$\tau-\tau^*=-\frac{1}{2}\|\nabla F\|^2=-\frac{1}{8}\|S\|^2$
$\mathcal{W}_4=\mathcal{H}$	$[\tau,\tau^*,\|\nabla F\|^2]$	$\|S\|^2=0$
$\mathcal{W}$	$[\tau,\tau^*,\|\nabla F\|^2,\|S\|^2]$	No condition

The method of proof of theorem 7.2 is the same as that of theorem 7.1. We omit the details.

We remark that for $n=2$ we have the following identities, valid for all almost Hermitian manifolds of real dimension 4: $\|\delta F\|^2=\frac{1}{6}\|dF\|^2$ and $\|\nabla F\|^2=\frac{1}{3}\|dF\|^2+$ $+\frac{1}{4}\|S\|^2$.

8. – The four types of almost symplectic manifolds.

In this section we study a generalization of symplectic manifolds using the same philosophy that we used for almost Hermitian manifolds. Instead of studying the representation of $U(n)$ on the space $\mathcal{W}$, however, we decompose the representation of $\mathrm{Sp}(n,\boldsymbol{R})$ on $\Lambda^3(V^*)$. In this manner we show that there are in a natural way four classes of almost symplectic manifolds. Another consequence of these considerations will be (the known fact) that the Lee form can be defined for any symplectic manifold, and is independent of any compatible almost Hermitian structure. See [1], [5], [6], [23].

Let $V=\boldsymbol{R}^{2n}$ be the representation space for the ordinary representation of $\mathrm{Sp}(n,\boldsymbol{R})$. Then there is a natural induced representation of $\mathrm{Sp}(n,\boldsymbol{R})$ on $\Lambda^3(V^*)$. Denote by F the nondegenerate 2-form on V preserved by $\mathrm{Sp}(n,\boldsymbol{R})$. Following [1] we define an isomorphism $\mu\colon V\to V^*$ by $\mu(x)(y)=-F(x,y)$ for $x,y\in V$.

Next we shall define a map $\Lambda^3(V^*)\to V^*$ taking α into $\tilde{\alpha}$. This operation was introduced in [5], [6], [23]. Let $\{x_1,\dots,x_{2n}\}$ be a basis of V, and $\{w_1,\dots,w_{2n}\}$ the corresponding dual basis of V^*. Then

$$\tilde{\alpha}(x)=\frac{1}{2(n-1)}\sum_{i=1}^{2n}\alpha(x,x_i,\mu^{-1}w_i)\,.$$

Here it can be checked that this definition does not depend on the choice of the basis $\{x_1,\dots,x_{2n}\}$. The mapping $\alpha\to\tilde{\alpha}$ is linear. For an alternative definition of $\tilde{\alpha}$

see [5], [6], [23]. Put

$$S_1 = \{\alpha \in V \mid \tilde{\alpha} = 0\},$$

$$S_2 = \{\alpha \in V \mid \alpha = F \wedge \tilde{\alpha}\}.$$

LEMMA 8.1. – We have $\Lambda^3(V^*) = S_1 \oplus S_2$. This decomposition is preserved under the action of $\mathrm{Sp}(n, \boldsymbol{R})$. Furthermore $\mathrm{Sp}(n, \boldsymbol{R})$ acts irreducibly on S_1 and S_2.

PROOF. – It is easily checked that $\Lambda^3(V^*) = S_1 \oplus S_2$. Furthermore it is clear from the definitions that $\mathrm{Sp}(n, \boldsymbol{R})$ preserves S_1 and S_2. That $\mathrm{Sp}(n, \boldsymbol{R})$ acts irreducibly on S_1 and S_2, follows from Lepage's decomposition.

We now define the Lee form of a symplectic manifold.

DEFINITION. – Let (M, F) be an almost symplectic manifold, that is, a differentiable manifold together with a 2-form F that has maximal rank everywhere. The *Lee form* θ of (M, F) is the 1-form θ given by

$$\theta(X) = \frac{1}{2(n-1)} \sum_{i=1}^{2n} dF(X, X_i, \mu^{-1} w_i),$$

for $X \in \mathfrak{X}(M)$. Here $\{X_1, \ldots, X_{2n}\}$ is a local basis of vector fields, and $w_1, \ldots, w_{2n}$ is the dual basis of 1-forms. This definition does not depend on the choice of basis.

REMARK. – It is possible to define a coderivative $\hat{\delta}$ for a symplectic manifold [5], [6]. See also [1], [23]. This coderivative does not depend on any compatible Riemannian structure. It is not hard to check that

$$\theta = (n-1)\,\hat{\delta} F .$$

Following the program of section 3 we can associate with each of the 4 subspaces $\{0\}$, S_1, S_2, $\Lambda^2(V^*)$ a class of almost symplectic manifolds. In analogy with theorem 3.1 we have

THEOREM 8.2. – The defining relation for each of the four classes of almost symplectic bundles is given in Table VI. The inclusion relations between the classes are

$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \begin{array}{c} \subset \mathcal{S}_1 \supset \\ \supset \mathcal{S}_2 \subset \end{array} \mathcal{S}_1 \cup \mathcal{S}_2 \subset \mathcal{S}_1 \oplus \mathcal{S}_2 . \tag{8.1}$$

TABLE VI. – *Almost symplectic manifolds of dimension* $\geqslant 4$.

Class	Defining conditions
$\mathcal{S}$	$dF = 0$
$\mathcal{S}_1$	$\theta = 0$
$\mathcal{S}_1$	$dF = F \wedge \theta$
$\mathcal{S}_1 \oplus \mathcal{S}_2$	No condition

Finally let $\Phi\colon \mathcal{W} \to \mathcal{S}$ be the forgetful function. (Thus Φ applied to an almost Hermitian manifold is the same manifold considered as an almost symplectic manifold.) It i sthen clear that

$$\Phi(\mathcal{W}_2) = \mathcal{S}\,,$$

$$\Phi(\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3) = \mathcal{S}_1\,,$$

$$\Phi(\mathcal{W}_4) = \mathcal{S}_2\,,$$

$$\Phi(\mathcal{W}) = \mathcal{S}_1 \oplus \mathcal{S}_2\,.$$

REFERENCES

[1] A. AVEZ - A. LICHNEROWICZ - A. DIAZ-MIRANDA, *Sur l'algèbre des automorphismes infinitésimaux d'une variété symplectique*, J. Differential Geometry, **9** (1974), pp. 1-40.

[2] M. BERGER - P. GAUDUCHON - E. MAZET, *Le spectre d'une variété riemannienne*, Lecture Notes in Mathematics, vol. 194, Springer-Verlag, Berlin and New York, 1971.

[3] P. DOMBROWSKI, *On the geometry of the tangent bundle*, J. Reine Angew. Math., **210** (1962), pp. 73-88.

[4] H. DONNELLY, *Invariance theory of Hermitian manifolds*, Proc. Amer. Math. Soc., **58** (1976), pp. 229-233.

[5] C. EHRESMANN - P. LIBERMANN, *Sur les formes différentielles extérieures de degré* **2**, C. R. Acad. Sci. Paris, **227** (1948), pp. 420-421.

[6] C. EHRESMANN - P. LIBERMANN, *Le problème d'équivalence des formes extérieures quadratiques*, C. R. Acad. Sci. Paris, **229** (1949), pp. 697-698.

[7] M. FERNÁNDEZ - L. M. HERVELLA, *Un exemple de variété pseudokählérienne*, C. R. Acad. Sci. Paris, 283 (1976), pp. 203-205.

[8] P. GILKEY, *Spectral geometry and the Kähler condition for complex manifolds*, Invent. Math., **26** (1974), pp. 231-258.

[9] A. GRAY, *Minimal varieties and almost Hermitian submanifolds*, Michigan Math. J., **12** (1965), pp. 273-279.

[10] A. GRAY, *Some examples of almost Hermitian manifolds*, Illinois J. Math., **10** (1969), pp. 353-366.

[11] A. GRAY, *Vector cross products on manifolds*, Trans. Amer. Math. Soc., **141** (1969), pp. 465-504.

[12] A. GRAY, *Riemannian manifolds with geodesic symmetries of order 3*, J. Differential Geometry, **7** (1972), pp. 343-369.

[13] A. GRAY, *The volume of a small geodesic ball in a Riemannian manifold*, Michigan Math. J., **20** (1973), pp. 329-344.

[14] A. GRAY, *The structure of nearly Kähler manifolds*, Math. Ann., **223** (1976), pp. 233-248.

[15] A. GRAY, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J., **28** (1976), pp. 601-612.

[16] L. M. HERVELLA - E. VIDAL, *Nouvelles géométries pseudo-kählériennes G_1 et G_2*, C. R. Acad. Sci. Paris, **283** (1976), pp. 115-118.

[17] L. M. HERVELLA - E. VIDAL, *New pseudo-Kähler geometries*, to appear.

[18] N. IWAHORI, *Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$*, J. Math. Soc. Japan, **10** (1958), pp. 145-160.

[19] K. KODAIRA - J. MORROW, *Complex Manifolds*, Holt, Rinehart and Winston, New York, 1971.

[20] S. KOTŌ, *Some theorems on almost Kählerian spaces*, J. Math. Soc. Japan, **12** (1960), pp. 422-433.

[21] H. C. LEE, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math., **65** (1943), pp. 433-438.

[22] T. LEPAGE, *Sur certaines congruences de formes alternées*, Bull. Sci. Roy. Soc. Liège (1946), pp. 21-31.

[23] P. LIBERMANN, *Sur le problème d'équivalence, Thèse, Strasbourg 1953*, Ann. Mat. Pura Appl., **36** (1954), pp. 27-120.

[24] P. LIBERMANN, *Sur la classification des structures hermitiennes*, to appear.

[25] G. PAPY, *Sur le divisibilité des formes alternées*, Bull. Soc. Roy. Sci. Liège (1946), p. 24.

[26] W. P. THURSTON, *Some examples of symplectic manifolds*, Proc. Amer. Math. Soc., **55** (1976), pp. 467-468.

[27] I. VAISMAN, *On locally conformal almost Kähler manifolds*, Israel J. Math., **24** (1976) pp. 338-351.

[28] E. VIDAL ABASCAL, *Algunos resultados sobre variedades casihermiticas*, Actas del V congresso de la agrupación de matematicos de expression latina, Palma, 1978.