On the Boundedness and Stability of a Fourth Order Differential Equation (*) (**).

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Summary. — *The aim of this paper is to give sufficient conditions (Theorem 1) for the asymptotic stability (in the large) of the trivial solution $x = 0$ of the differential equation*

$$D(x) = x^{(4)} + f(x, \dot{x}, \ddot{x}, \ddot{x}) = 0.$$

A result (Theorem 2) on the boundedness of the solutions of the differential equation $D(x) = p(t, x, \dot{x}, \ddot{x}, \ddot{x})$ is also established.

1. — Introduction.

In this paper we investigate the stability and boundedness of the solution of the fourth order differential equation

$$(1.1) \quad x^{(4)} + f(x, \dot{x}, \ddot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}, \ddot{x}).$$

The results obtained reduce to those which are less restrictive than established by LALLI and SKRAPEK [4] and EZEILO [1] respectively, for the equations

$$(1.2) \quad x^{(4)} + f_1(\ddot{x})\ddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x, \dot{x}) = p(t),$$

$$(1.3) \quad x^{(4)} + f_1(\ddot{x})\ddot{x} + a_2(\ddot{x}) + g(\dot{x}) + a_4x = p(t).$$

Further they reduce to results which are similar to those obtained by HARROW [3] and EZEILO and TEJUMOLA [2] respectively, for the equations

$$(1.4) \quad x^{(4)} + a\ddot{x} + f_2(\ddot{x}) + g(\dot{x}) + h(x) = p(t),$$

$$(1.5) \quad x^{(4)} + a\ddot{x} + f_2(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \ddot{x}).$$

In what follows, we use the following system which is equivalent to (1.1).

$$(1.6) \quad \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -f(x, y, z, w) + p(t, x, y, z, w).$$

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In (1.6), the functions f and p are continuous for all values of their arguments displayed explicitly. The partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, $\partial f/\partial w$ exist and are continuous for all values of their arguments. Also, the existence and uniqueness of the solutions of (1.6) is assumed.

2. – Theorem 1 and Theorem 2.

THEOREM 1. – Let $(\partial/\partial w)f(0, 0, z, w)|_{w=0} = f_1(z)$ and suppose the following conditions hold:

$$(i) \quad f(0, 0, 0, 0) = 0, \quad f(x, y, 0, w) - f(x, y, 0, 0) - f_1(0)w = 0$$

for all x, y, w and there exist finite positive constants a_1, a_2, a_3 and finite non-negative constants m, n such that $f_1(z) \geq a_1$ for all z ,

$$\frac{f(x, y, z, w) + f(0, y, 0, 0) - f(x, y, 0, 0) - f(0, y, z, 0)}{w} \geq a_1 \quad \text{for } w \neq 0 \text{ and all } x, y, z;$$

$$\frac{f(0, y, z, 0) - f(0, y, 0, 0)}{z} \geq a_2 \quad \text{for } z \neq 0 \text{ and all } y;$$

$$\frac{f(x, y, z, w) - f(x, y, 0, 0) - f_1(z)w}{z} \geq a_2 + m \quad \text{for } z \neq 0 \text{ and all } x, y, w;$$

$$\frac{f(0, y, 0, 0)}{y} \geq a_3 + n \quad \text{for } y \neq 0;$$

$$(ii) \quad [f(x, y, 0, 0) - f(0, y, 0, 0)] \operatorname{sgn} x > 0 \quad \text{for } x \neq 0, \text{ and} \\ [f(x, y, 0, 0) - f(0, y, 0, 0)] \operatorname{sgn} x \rightarrow +\infty \text{ as } |x| \rightarrow +\infty;$$

$$(iii) \quad (\partial/\partial y)[f(x, y, 0, 0) - f(0, y, 0, 0)] = 0, \text{ and there exist finite positive constants } A_4, A_0, A_1 \text{ and a finite constant } A_5 \text{ such that}$$

$$\frac{\partial}{\partial x} f(x, y, 0, 0) \leq A_4 \quad \text{and} \quad \frac{\partial}{\partial y} [f(x, y, 0, 0) - f(0, y, 0, 0)] \leq A_5 \quad \text{for all } x, y,$$

$$\left[a_1 a_2 - \frac{\partial}{\partial y} f(0, y, 0, 0) \right] a_3 - a_1 A_4 f_1(z) \geq A_0 \quad \text{for all } y, z,$$

$$\frac{4a_1 A_0}{a_3} - \left[A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0) + 4a_1 A_5 \right] \geq A_1 \quad \text{for all } x, y;$$

$$(iv) \quad \frac{\partial}{\partial y} f(0, y, 0, 0) - \frac{f(0, y, 0, 0)}{y} \leq \delta_1 < \frac{2A_4 A_0}{a_1 a_3^2} \quad \text{for } y \neq 0;$$

$$(v) \quad \left[\frac{1}{z} \int_0^z f_1(s) ds \right] - f_1(z) \leq \delta_2 < \frac{2A_0}{a_1^2 a_3} \text{ for } z \neq 0;$$

$$(vi) \quad (\partial/\partial y)[f(0, y, z, 0) - f(0, y, 0, 0)] \leq 0 \text{ for all } y, z;$$

$$(vii) \quad [G(x, y, z, w)]^2 \leq (4a_3/A_4)mn, \text{ where}$$

$$G(x, y, z, w) = \frac{f(x, y, z, w) - f(x, y, 0, 0) - f_1(z)w}{z} - a_2 \\ - \int_0^x \frac{1}{y} \frac{\partial}{\partial y} [f(s, y, 0, 0) - f(0, y, 0, 0)] ds - \frac{a_3}{A_4} \frac{\partial}{\partial y} [f(x, y, 0, 0) - f(0, y, 0, 0)]$$

for $y \neq 0, z \neq 0$.

Then every solution of (1.6) with $p(t, x, y, z, w) \equiv 0$ satisfies

$$(2.1) \quad x \rightarrow 0, \quad y \rightarrow 0, \quad z \rightarrow 0, \quad w \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

THEOREM 2. - Suppose conditions (i), (iii), (iv), (v), (vi), (vii) of Theorem 1 hold and suppose further that

$$(a) \quad [f(x, y, 0, 0) - f(0, y, 0, 0)] \operatorname{sgn} x \rightarrow +\infty \text{ as } |x| \rightarrow +\infty;$$

$$(b) \quad \text{the function } p(t, x, y, z, w) \text{ satisfies}$$

$$|p(t, x, y, z, w)| \leq \{A_0 + |y| + |z| + |w|\} \varphi(t)$$

for all t, x, y, z, w , where A_0 is a constant and $\varphi(t) \geq 0$ is a continuous function of t such that

$$\int_0^\infty \varphi(t) dt < \infty.$$

Then there exists a positive constant $K = K(x_0, y_0, z_0, w_0)$ such that the solution $(x(t), y(t), z(t), w(t))$ of (1.6), determined by the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad w(0) = w_0,$$

satisfies

$$(2.2) \quad |x(t)| \leq K, \quad |y(t)| \leq K, \quad |z(t)| \leq K, \quad |w(t)| \leq K.$$

REMARKS.

1) It is easy to verify that the results stated in Theorem 1 and Theorem 2 reduce to results which are slightly less restrictive than those obtained by LALLI and SKRAPEK [4] for equation (1.2). In particular, hypothesis (ii) and (iii) of Theorem 1

and hypothesis (a) of Theorem 2 are weaker restrictions than the corresponding hypotheses in [4]. The results further reduce to those which are less restrictive than obtained by EZEILO [1] for equation (1.3).

2) Theorem 2 is a slight modification of a result given by Ezeilo and Tejumola in [2; Theorem 2].

3. - The function V .

The proofs of (2.1) and (2.2) depend on properties of the function V defined by

$$\begin{aligned}
 (3.1) \quad 2V(x, y, z, w) = & 2d_2 \int_0^x f(s, y, 0, 0) ds - 2d_2 f(0, y, 0, 0) x \\
 & + (a_2 d_2 - A_4 d_1) y^2 + 2 \int_0^y f(0, s, 0, 0) ds \\
 & + 2d_1 \int_0^z f(0, y, s, 0) ds - 2d_1 f(0, y, 0, 0) z \\
 & + 2 \int_0^z f_1(s) s ds - d_2 z^2 + d_1 w^2 \\
 & + 2f(x, y, 0, 0) y - 2f(0, y, 0, 0) y + 2d_1 f(x, y, 0, 0) z \\
 & + 2d_2 y \int_0^z f_1(s) ds + 2d_2 y w + 2zw,
 \end{aligned}$$

where $d_1 = \varepsilon + 1/a_1$, $d_2 = A_4/a_3$ and ε is a positive constant, the value of which is to be defined to advantage later.

Now, let

$$\begin{aligned}
 f_2(y, z) &= f(0, y, z, 0) - f(0, y, 0, 0), \quad h(x, y) = f(x, y, 0, 0) - f(0, y, 0, 0), \\
 (3.2) \quad g(y) &= f(0, y, 0, 0) \quad \text{and} \quad F(z) = \int_0^z f_1(s) ds.
 \end{aligned}$$

Further, let G_1 and F_1 be the continuous functions defined by

$$(3.3) \quad G_1(y) = \begin{cases} \frac{g(y)}{y} & y \neq 0 \\ g'(0) & y = 0 \end{cases} \quad F_1(z) = \begin{cases} \frac{F(z)}{z} & z \neq 0 \\ f_1(0) & z = 0. \end{cases}$$

The following inequalities can be easily checked.

$$(A) \quad \int_0^z f_2(y, s) ds \geq a_2 z^2 \text{ for all } y \text{ and } z;$$

$$(B) \quad d_1 - \frac{f_1(z)}{1} \geq \varepsilon \text{ for all } z,$$

$$d_1 \left[\frac{f(x, y, z, w) + f(0, y, 0, 0) - f(x, y, 0, 0) - f(0, y, z, 0)}{w} \right] - 1 \geq a_1 \varepsilon$$

for $w \neq 0$ and x, y, z ,

$$d_2 - A_4 \frac{y}{g(y)} \geq 0 \quad \text{for } y \neq 0;$$

$$(C) \quad a_2 - d_1 g'(y) - d_2 f_1(z) \geq \frac{A_0}{a_1 a_3} - a_1 a_2 \varepsilon \quad \text{for all } y, z.$$

In (3.1), adding and subtracting $2d_1 g(y)z$ and applying (A), it is not difficult to verify that

$$(3.4) \quad \begin{aligned} 2V(x, y, z, w) \geq & U_1 + U_2 + U_3 + \left\{ d_1 - \frac{1}{F_1(z)} \right\} w^2 \\ & + \frac{1}{F_1(z)} \{w + F(z) + d_2 y F_1(z)\}^2 \\ & + \frac{1}{G_1(y)} \{h(x, y) + g(y) + d_1 z G_1(y)\}^2, \end{aligned}$$

where

$$\begin{aligned} U_1 &= 2d_2 \int_0^x h(s, y) ds - \frac{h^2(x, y)}{G_1(y)}, \\ U_2 &= \{a_2 d_2 - A_4 d_1 - d_2^2 F_1(z)\} y^2 + 2 \int_0^y g(s) ds - yg(y), \\ U_3 &= \{a_2 d_1 - d_2 - d_1^2 G_1(y)\} z^2 + 2 \int_0^z s f_1(s) ds - zF(z). \end{aligned}$$

Now,

$$\begin{aligned} U_1 &\geq 2d_2 \int_0^x h(s, y) ds - \frac{h^2(x, y)}{a_3} \quad \text{by (i),} \\ &= 2 \int_0^x \left[d_2 - \frac{1}{a_3} \frac{\partial}{\partial s} h(s, y) \right] h(s, y) ds \quad \text{since } h(0, y) = 0, \\ &\geq 0 \quad \text{by (ii) and (iii).} \end{aligned}$$

Applying the inequalities (B) and (C) and following the method by Lalli and Skrapek [4] and other it is easily checked that it is possible to select a positive constant D_1 such that

$$(3.5) \quad \begin{aligned} V(x, y, z, w) \geq & D_1(y^2 + z^2 + w^2) \\ & + \frac{1}{2G_1(y)} \{h(x, y) + g(y) + d_1 z G_1(y)\}^2, \end{aligned}$$

where D_1 depends only on the positive constants $a_1, a_2, a_3, A_4, \Delta_0, \delta_1, \delta_2$ and a sufficiently small ε . Further, $V(0, 0, 0, 0) = 0$ and $V(x, y, z, w)$ is positive definite. Also, by (3.5) and (ii) it is clear that $V(x, y, z, w) \rightarrow +\infty$ as $x^2 + y^2 + z^2 + w^2 \rightarrow +\infty$.

A straight forward calculation yields

$$\dot{V} = \dot{V}_1 + \dot{V}_2,$$

where

$$\begin{aligned} \dot{V}_1 = & - \left[d_2 g(y) y - \frac{\partial}{\partial x} f(x, y, 0, 0) y^2 \right] - \left[A_4 d_1 - d_1 \frac{\partial}{\partial x} f(x, y, 0, 0) \right] y z \\ & - d_2 [f(x, y, z, w) - f(x, y, 0, 0) - f_1(z) w] y \\ & + \left[a_2 d_2 + \frac{\partial}{\partial y} h(x, y) \right] y z \\ & - [f(x, y, z, w) - f(x, y, 0, 0) - f_1(z) w] z + d_2 z F(z) \\ & + d_1 \frac{\partial}{\partial y} f(x, y, 0, 0) z^2 + d_1 \int_0^z \frac{\partial}{\partial y} f_2(y, s) s ds + d_2 \int_0^z \frac{\partial}{\partial y} h(s, y) z ds, \end{aligned}$$

and

$$\dot{V}_2 = -d_1 [f(x, y, z, w) + f(0, y, 0, 0) - f(x, y, 0, 0) - f(0, y, z, 0)] w + w^2.$$

Clearly $\dot{V}_2 = 0$ for $w = 0$. For $w \neq 0$,

$$\begin{aligned} \dot{V}_2 = & - \left\{ d_1 \frac{[f(x, y, z, w) + f(0, y, 0, 0) - f(x, y, 0, 0) - f(0, y, z, 0)]}{w} - 1 \right\} w^2 \\ \leq & -a_1 \varepsilon w^2 \text{ by (B).} \end{aligned}$$

For $\dot{V}_1$, four cases are considered.

Case 1. - $y = 0$ and $z = 0$. Then $\dot{V}_1 = 0$.

Case 2. - $y = 0$ and $z \neq 0$. Noting that

$$g'(0) = (\partial/\partial y) f(0, y, 0, 0)|_{y=0} = (\partial/\partial y) f(x, y, 0, 0)|_{y=0}$$

by (iii), it follows that

$$\begin{aligned} \dot{V}_1 = & - \left\{ \left[\frac{f(x, 0, z, w) - f(x, 0, 0, 0) - f_1(z) w}{z} - d_1 g'(0) \right] z^2 - d_2 z F(z) \right\} \\ & + d_1 \int_0^z \frac{\partial}{\partial y} f_2(y, s)|_{y=0} s ds, \end{aligned}$$

$$\begin{aligned}
&\leq -\{a_2 - d_1 g'(0) - d_2 f_1(\theta z)\} z^2 \quad \text{by (i) and (vi) for } 0 < \theta < 1, \\
&\leq -\left[\frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon\right] z^2 \quad \text{by (C),} \\
&\leq -\frac{\Delta_0}{12 a_1 a_3^2} \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}
\end{aligned}$$

Case 3. - $z = 0$ and $y \neq 0$. In this case

$$\begin{aligned}
\dot{V}_1 &= -\left[d_2 \frac{g(y)}{y} - \frac{\partial}{\partial x} f(x, y, 0, 0)\right] y^2 - d_2 [f(x, y, 0, w) - f(x, y, 0, 0) - f_1(0)w] y, \\
&\leq 0 \quad \text{by (i), (iii) and (B).}
\end{aligned}$$

Case 4. - $y \neq 0$ and $z \neq 0$. Here it is easily verified that

$$\begin{aligned}
\dot{V}_1 &\leq -\left[A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0)\right] \left(y + \frac{d_1}{2} z\right)^2 - [d_2 n y^2 + d_2 G(x, y, z, w) y z + m z^2] \\
&\quad - \left\{ \left[a_2 - d_1 g'(y) - \frac{d_1^2}{4} \left(A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0) \right) - d_1 \frac{\partial}{\partial y} h(x, y) \right] z^2 - d_2 z F(z) \right\} \\
&\leq -U_4 - U_5,
\end{aligned}$$

where

$$\begin{aligned}
U_4 &= [d_2 n y^2 + d_2 G(x, y, z, w) y z + m z^2], \\
&\geq 0,
\end{aligned}$$

since $[G(x, y, z, w)]^2 \leq 4mn/d_2$ by (vii).

Now

$$\begin{aligned}
U_5 &= \left\{ \left[a_2 - d_1 g'(y) - \frac{d_1^2}{4} \left(A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0) \right) - d_1 \frac{\partial}{\partial y} h(x, y) \right] z^2 - z F(z) \right\} \\
&\geq \left\{ a_2 - d_1 g'(y) - d_2 f_1(\theta z) - \frac{d_1^2}{4} \left[A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0) \right] - d_1 \frac{\partial}{\partial y} h(x, y) \right\} z^2
\end{aligned}$$

for $0 < \theta < 1$,

$$\begin{aligned}
&\geq \left\{ \frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon - \frac{d_1^2}{4} \left[A_4 - \frac{\partial}{\partial x} f(x, y, 0, 0) + \frac{4}{d_1} A_5 \right] \right\} z^2 \quad \text{by (iii) and (C),} \\
&\geq \left\{ \frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon - \frac{d_1^2}{4} \left(\frac{4a_1 \Delta_0}{a_3} - A_1 \right) \right\} z^2 \quad \text{by (iii),} \\
&= \left\{ \frac{d_1^2 A_1}{4} - \left(a_1 a_2 + \varepsilon \frac{a_1 \Delta_0}{a_3} + \frac{2\Delta_0}{a_3} \right) \varepsilon \right\} z^2, \\
&\geq \frac{d_1^2 A_1}{8} z^2 \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}
\end{aligned}$$

Therefore, combining all four cases and choosing $\varepsilon > 0$ sufficiently small, there exists a positive constant D_2 such that

$$(3.6) \quad \dot{V} \leq -D_2(z^2 + w^2).$$

4. – Proofs of Theorem 1 and Theorem 2.

The function V is positive definite and satisfies

$$V(x, y, z, w) \rightarrow +\infty \quad \text{as} \quad x^2 + y^2 + z^2 + w^2 \rightarrow +\infty,$$

and

$$\dot{V} \leq -D_2(z^2 + w^2),$$

for any solution $(x(t), y(t), z(t), w(t))$ of (1.6). Then the usual Barbashin-type arguments show that

$$x(t) \rightarrow 0, \quad y(t) \rightarrow 0, \quad z(t) \rightarrow 0, \quad w(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

This completes Theorem 1.

The proof of Theorem 2 is almost identical to that given by Ezeilo and Tejumola [2; Theorem 2]. Therefore only a brief sketch of the proof will be given.

Using the V function (3.1) and noting (3.4) and (3.5) it is easily seen that

$$\begin{aligned} V &\geq D_1(y^2 + z^2 + w^2) + d_2 \int_0^x h(s, y) ds - \frac{h^2(x, y)}{2G_1(y)} \\ &\quad + \frac{1}{2G_1(y)} \{h(x, y) + g(y) + d_1 z G_1(y)\}^2. \end{aligned}$$

By (a) of Theorem 2 and by continuity of $h(x, y)$ it is clear that a constant ξ_0 exists such that $h(x, y) \operatorname{sgn} x > 0$ for $|x| > \xi_0$. It follows that

$$\begin{aligned} d_2 \int_0^x h(s, y) ds - \frac{h^2(x, y)}{2G_1(y)} &\geq \frac{1}{2} \left\{ 2d_2 \int_0^x h(s, y) ds - \frac{h^2(x, y)}{a_3} \right\}, \\ &= \frac{1}{2} \int_0^x \left\{ d_2 - \frac{1}{a_3} \frac{\partial}{\partial s} h(s, y) \right\} h(s, y) ds, \\ &\geq -D_3, \end{aligned}$$

for some positive constant D_3 . Therefore,

$$V \geq D_1(y^2 + z^2 + w^2) + \frac{1}{2G_1(y)} \{h(x, y) + g(y) + d_1 z G_1(y)\}^2 - D_3.$$

and moreover by (a) Theorem 2,

$$(4.1) \quad V(x, y, z, w) \rightarrow +\infty \quad \text{as} \quad x^2 + y^2 + z^2 + w^2 \rightarrow +\infty.$$

Also, it is easy to check that

$$\dot{V} \leq -D_2(z^2 + w^2) + D_4(|y| + |z| + |w|)|p(t, x, y, z, w)|,$$

for a positive constant D_4 . By (b) Theorem 2 and since $|y| < 1 + y^2$, $2|y||z| \leq y^2 + z^2$ it follows that there exist positive constants D_5 and D_6 such that

$$\dot{V} \leq D_5[1 + y^2 + z^2 + w^2]\varphi(t),$$

and further that

$$\dot{V} - D_6 V \varphi(t) \leq D_6 \varphi(t).$$

Therefore,

$$(4.2) \quad V(t) \leq \psi^{-1}(t) \left\{ V(0) + D_6 \int_0^t \varphi(s) \psi(s) ds \right\} \quad \text{for } t \geq 0,$$

where $V(0) = V(x_0, y_0, z_0, w_0)$ and $\psi(t) = \exp \left(-D_6 \int_0^t \varphi(s) ds \right)$. Since $\psi^{-1}(t) < \infty$ by (b) Theorem 2, it follows from (4.1) and (4.2) that

$$|x(t)| \leq K, \quad |y(t)| \leq K, \quad |z(t)| \leq K, \quad |w(t)| \leq K,$$

for a positive constant K .

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