Linear Boundary Value Problems for Systems of Ordinary Differential Equations on Noncompact Intervals (*).

II. - Stability and Bounded Perturbations.

M. CECCHI (Firenze) (**), M. MARINI (Firenze) (**) - P. L. ZEZZA (Siena) (***)

Sunto. - In questa nota si esaminano la completa continuità e il rango dell'operatore integrale associato al sistema

$$\begin{aligned}\dot{x} - A(t)x &= f(t, x) \\ Tx &= r\end{aligned}$$

al variare del tipo di stabilità della parte lineare.

Introduction.

In [3] we proved the existence of a bounded solution for the linear boundary value problem

$$(0.1) \quad \dot{x} - A(t)x = f(t, x)$$

$$(0.2) \quad Tx = r$$

under the hypotheses, among others, that the associated linear system

$$(0.3) \quad \dot{y} - A(t)y = 0$$

is stable and that f is such that

« there are two functions $p(t), q(t) \in C([a, b], R)$, nonnegative, integrable on $[a, b]$ and such that

$$(0.4) \quad \int_a^b p(s) ds = F < +\infty, \quad \int_a^b q(s) ds = A < +\infty$$

$$(0.5) \quad \|X^{-1}(t)f(t, u)\| < p(t)\|u\| + q(t)$$

where $X^{-1}(t)$ is the inverse of any fundamental matrix of (0.3). »

These last conditions are not easily verifiable because they concern simultaneously the behaviour both of $X(t)$ and of the forcing term $f(t, x)$.

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(**) Istituto di Matematica Applicata, Facoltà di Ingegneria, Università di Firenze.

(***) Istituto di Matematica, Università di Siena.

In this paper we shall give some criteria for the existence of a solution of (0.1)–(0.2) under hypotheses dealing with $X(t)$ and $f(t, x)$ separately.

In this way we obtain conditions that are more easily verifiable; moreover some results of this paper can be used to solve some problems that can't be attacked with the methods exposed in [3].

As in [4] the results here exposed can be extended to the nonlinear boundary value problem

$$(0.6) \quad \dot{x} - A(t)x = f(t, x)$$

$$(0.7) \quad Tx = h(x).$$

For an ample bibliography on this subject see [3], [4], [7].

§ 1. – Let $C = C([a, b], R^n)$ be the locally convex space of continuous functions from $[a, b]$ into R^n , and let

$$BC = \left\{ x \in C \text{ such that } \sup_{t \in [a, b]} \|x(t)\| < +\infty \right\};$$

BC is a Banach space with respect to the norm

$$\|x\| = \sup_{t \in [a, b]} \|x(t)\| \quad (-\infty < a < b \leq +\infty).$$

Let us consider the linear boundary value problem:

$$(1.1) \quad \dot{x} = A(t)x + f(t, x)$$

$$r \in R^m \quad (m \leq n)$$

$$(1.2) \quad Tx = r$$

where:

- a) $A(t)$ is a real, continuous $n \times n$ matrix, defined on $[a, b]$ and such that the linear system associated to (1.1)

$$(1.3) \quad \dot{y} - A(t)y = 0$$

is stable, i.e. the space D of all BC -solutions of (1.3) has dimension n ;

- b) $f(t, u) \in C([a, b] \times R^n, R^n)$;

- c) T is a linear continuous operator

$$T: \text{dom } T \subset BC \rightarrow R^m \quad m \leq n$$

with $D \subset \text{dom } T$, $T(D) = R^m$.

REMARK. — These conditions assure that the linear problem associated to (1.1)-(1.2) for $f(t, x) = 0$ has a solution for every $r \in R^m$.

Let L be the linear operator

$$L: \text{dom } L \subset BC \rightarrow C \times R^m$$

$$L: x \rightarrow (\dot{x} - A(t)x, Tx)$$

where

$$\text{dom } L = \{x \in \text{dom } T \text{ such that } x \in C^1\} = C^1 \cap \text{dom } T$$

and let N be the operator

$$N: \text{dom } N \subset BC \rightarrow C \times R^m$$

$$N: x \rightarrow (f(\cdot, x(\cdot)), r).$$

REMARK. — Because of the hypothesis of continuity on f , $\text{dom } N = BC$. Reasoning as in [3] and using the results in [11] we can affirm that the problem (1.1)-(1.2) is equivalent to

$$(1.4) \quad \begin{cases} x = Mx \\ x \in \mathcal{A} \end{cases}$$

with

$$\mathcal{A} = \{x \in BC \text{ such that } Nx \in \text{Im } L\} = N^{-1}(\text{Im } L)$$

$$(1.5) \quad M: x \rightarrow Px + K_P Nx$$

where P is the topological projection defined in [3] and K_P is the operator

$$(1.6) \quad \begin{aligned} K_P: C \times R^m &\rightarrow \text{dom } L \\ K_P(b(t), r) &\rightarrow X(t)\tilde{c} + X(t) \int_a^t X^{-1}(s)b(s) ds \end{aligned}$$

with $\tilde{c}$ a vector of R^n depending on $b(t), r$ and T , $X(t)$ is a fundamental matrix of the linear system (1.3).

It is clear that the existence of a solution for the problem (1.1)-(1.2), that is of fixed points for the operator M , depends especially on the set $\mathcal{A}$. If $\mathcal{A}$ is a subset of BC_i , with

$$BC_i = \left\{ x \in BC \text{ such that } \exists \lim_{t \rightarrow b} x(t) = l_x \right\} (\|l_x\| < +\infty)$$

we can use a compactness theorem for the search of fixed points, whereas if $\mathcal{A} \not\subset BC_i$ we have to apply, as in [3], a diagonal process. The fact that $\mathcal{A}$ may or may not

be a subset of BC_i depends both on the behaviour of the solutions of the linear system (1.3) and on the forcing term $f(t, x)$.

Let $X(t)$ be any fundamental matrix of the system (1.3), then

COROLLARY 1.1. — The following statements are equivalent:

- i) the limit of each solution of (1.1) for $t \rightarrow b$ exists and it is finite;
- ii) the limits

$$\lim_{t \rightarrow b} X(t) \quad \lim_{t \rightarrow b} X(t) \int_a^t X^{-1}(s) f(s, x(s)) ds$$

where $x(t)$ is a solution of (1.1), exist and are finite.

PROOF.

ii) $\rightarrow$ i), obvious.

i) $\rightarrow$ ii), let $x(t)$ be a solution of (1.1), such that $x(a) = 0$; then by assumptions the limit of $x(t)$, i.e.

$$\lim_{t \rightarrow b} X(t) \int_a^t X^{-1}(s) f(s, x(s)) ds$$

exists.

By subtraction there exists also the limit of $X(t)$. $\square$

We note that the property ii) of Corollary 1.1 is clearly independent from the choice of the fundamental matrix of (1.3).

We give the following:

DEFINITION 1.1. — The Banach space B is said to be admissible for the system (1.1) if every solution $x = x(t)$ is such that $x \in B$.

It is then clear that the stability of the linear system (1.1) is a necessary condition for the admissibility of BC_i for (1.1). If BC_i is admissible for (1.1) and if $BC_i \subset \text{dom } T$, then the operator M , given in (1.5), takes values in BC_i . In this case we can use the following compactness theorem (see [1]).

THEOREM 1.1. — An operator

$$K: Y \rightarrow BC_i$$

Y Banach space, is completely continuous if and only if for every bounded set $\Theta \subset Y$, $K(\Theta)$ is:

- 1) equibounded;

- 2) equicontinuous;
- 3) uniformly convergent in the sense that

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall t > \delta(\varepsilon),$$

$$\forall g \in K(\Theta) \Rightarrow \|g(t) - l_g\| \leq \varepsilon.$$

In order to prove the existence of a solution for equation (1.4), we have only to obtain an a priori bound and to use a fixed point theorem.

The problem of the search for a solution of equation (1.4) becomes then the one of finding conditions assuring the admissibility of BC_t for (1.1), the equiboundedness, the equicontinuity and the uniform convergence of $K_P N(\Theta)$, Θ bounded set of BC .

§ 2. – In this section we recall some results concerning the stability of the linear system (1.3). For a wider exposition see [2], [5], [6], [8].

LEMMA 2.1. – If the system (1.3) is stable, then it is strongly stable if and only if

$$\liminf_{t \rightarrow +\infty} \int_a^t \text{tr } A(s) ds > -\infty.$$

COROLLARY 2.1. – If $\int_a^{+\infty} \|A(t)\| dt < +\infty$, then (1.3) is strongly stable and moreover $\lim_{t \rightarrow +\infty} X(t) = V$, where V is a real non singular matrix and $X(t)$ is any fundamental matrix of (1.3).

If $\int_a^{+\infty} \|A(t)\| dt < +\infty$, it follows immediatly from Corollary 2.1 that $\lim_{t \rightarrow +\infty} X^{-1}(t) = W$, where W is a real nonsingular matrix, and that system (1.3) is neither asymptotically stable nor asymptotically uniformly stable.

Let us now consider the nonlinear system

$$(1.1) \quad \dot{x} = A(t)x + f(t, x)$$

where f is a continuous function,

$$f: [a, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We shall give some criteria for the asymptotic behaviour of the solutions of (1.1) depending on the kind of stability of the linear part and on the kind of nonlinearity of f (see [6]).

THEOREM 2.1. — If the system

$$(1.3) \quad \dot{y} = A(t)y$$

is uniformly and asymptotically stable and if the following conditions hold:

$$(2.1) \quad \begin{aligned} & f(t, u) \in C([a, +\infty) \times R^n, R^n), \text{ and there are two functions} \\ & p(t), q(t) \in C([a, +\infty), R) \text{ non negative, integrable such that} \end{aligned}$$

$$\int_a^{+\infty} p(t) dt < +\infty, \quad \int_a^{+\infty} q(t) dt < +\infty$$

and

$$\|f(t, u)\| \leq p(t)\|u\| + q(t)$$

then every solution of (1.1) tends to zero.

COROLLARY 2.2. — If the system (1.3) is uniformly stable, then all solutions of (1.1) are bounded if $f(t, u)$ satisfies the hypotheses of Theorem 2.1.

THEOREM 2.2. — If the system (1.3) is integrally stable, i.e. for every fundamental matrix $X(t)$ of (1.3)

$$\exists h > 0 \text{ s.t. } \int_a^t \|X(t)X^{-1}(s)\| ds \leq h \quad \text{for } t \geq a,$$

and if the following conditions holds:

$$(2.2) \quad \begin{aligned} & f(t, u) \in C([a, +\infty) \times R^n, R^n), \text{ and there are a nonnegative constant } \gamma \text{ and} \\ & \text{a real positive function } \lambda = \lambda(t) \text{ which tends to zero such that} \end{aligned}$$

$$\|f(t, u)\| \leq \gamma\|u\| + \lambda(t), \quad h\gamma < 1$$

then every solution of (1.1) tends to zero.

COROLLARY 2.3. — If the system (1.3) is asymptotically-uniformly stable, and if $f(t, u)$ satisfies condition (2.1) or (2.2) then every solution of (1.1) tends to zero.

§ 3. — We can now show some different ways of approach to the problem (1.1)-(1.2).

THEOREM 3.1. – Suppose that for the system (1.1)-(1.2) the following hypotheses hold:

- a) $A(t)$ is a continuous real $n \times n$ matrix defined on $[a, b)$ ($-\infty < a < b \leq +\infty$) such that the space D of the solutions of the linear system (1.3) is contained in BC_i ;
- b) let $f \in C([a, b) \times R^n, R^n)$ be such that

$$\|X^{-1}(t)f(t, u)\| \leq p(t)\|u\| + q(t)$$

where $p(t), q(t)$ are non negative real continuous functions, defined on $[a, b)$ such that

$$\int_a^b p(t) dt < +\infty, \quad \int_a^b q(t) dt < +\infty$$

and $X(t)$ is any fundamental matrix of (1.3);

- c) T is a linear bounded operator such that $\text{dom } T = BC_i$, $\text{Im } T = R^m$ ($m \leq n$) and $TX(t)$ has rank (maximum) m .

Under these hypotheses BC_i is admissible for (1.1), the operator M , given in (1.5), is defined on BC , its image is contained in BC_i and it is completely continuous.

PROOF. – See [3].

This theorem, as we said in the introduction, is hardly applicable because the hypotheses concern simultaneously the behaviour both of the solutions of the linear system (1.3) and of the forcing term $f(t, x)$.

We give now some solutions of the problem when the hypotheses on the linear part and the nonlinear one are separated.

THEOREM 3.2. – Suppose that for system (1.1)-(1.2), the following hypotheses hold:

- a) $A(t)$ is a continuous real $n \times n$ matrix, defined on $[a, b)$ ($-\infty < a < b = +\infty$) such that the linear system (1.3) is asymptotically and uniformly stable;
- b) Let $f \in C([a, b) \times R^n, R^n)$ such that

$$\|f(t, u)\| \leq p(t)\|u\| + q(t)$$

where $p(t)$ and $q(t)$ are nonnegative real continuous functions, defined on $[a, b)$ such that

$$\int_a^b p(t) dt < +\infty, \quad \int_a^b q(t) dt < +\infty;$$

c) T is a linear bounded operator such that $\text{dom } T \supset BC_0$, where

$$BC_0 = \left\{ x \in BC_1 \text{ such that } \lim_{t \rightarrow +\infty} x(t) = 0 \right\},$$

$\text{Im } T = R^m$ ($m \leq n$), $TX(t)$ has rank (maximum) m , where $X(t)$ is any fundamental matrix of (1.3).

Under these hypotheses BC_0 is admissible for (1.1), the operator M , given in (1.5), is defined on BC , its image is contained in BC_0 , and it is completely continuous.

PROOF. — For Theorem 2.1 every solution of (1.1) tends to zero: BC_0 is then admissible for (1.1).

From Theorem 2.1 we get that $\forall g \in BC$

$$(3.1) \quad \int_a^t X(t)X^{-1}(s)f(s, g(s)) \, ds \in BC_0.$$

Therefore, recalling the expression of the operator M , we have

$$\text{dom } M = BC, \quad \text{Im } M \subset BC_0.$$

Let us now prove that the operator M is completely continuous.

P is completely continuous because it is a topological projection onto a finite dimensional space, then it will be enough show that the operator $K_P N$ is completely continuous. The continuity of this operator follows immediately from (1.6) because of the hypothesis of uniform stability and applying the Lebesgue dominated convergence Theorem.

Let Θ be a bounded set $\Theta \subset BC$:

$$x \in \Theta \Rightarrow \|x\| \leq \mu,$$

we prove that the set $K_P N(\Theta)$ is equibounded, equicontinuous and uniformly convergent (see Theorem 1.1).

The equiboundedness is obvious.

For the equicontinuity it is enough to show that, considering (1.6), if $t_1, t_2 \in [a, b]$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta \Rightarrow$

$$(3.2) \quad \left\| \int_a^{t_2} X(t_2)X^{-1}(s)f(s, x(s)) \, ds - \int_a^{t_1} X(t_1)X^{-1}(s)f(s, x(s)) \, ds \right\| < \varepsilon$$

where $x \in \Theta$.

Let $t_2 > t_1$. For the uniform stability of (1.3), $\exists k \in R^+$ such that

$$\|X(t)X^{-1}(s)\| \leq k, \quad a \leq s \leq t < b$$

We have then:

$$\begin{aligned}
 & \left\| \int_a^{t_2} X(t_2) X^{-1}(s) f(s, x(s)) ds - \int_a^{t_1} X(t_1) X^{-1}(s) f(s, x(s)) ds \right\| \leq \\
 & \leq \|X(t_2) - X(t_1)\| \left\| \int_a^{t_1} X^{-1}(s) f(s, x(s)) ds \right\| + \\
 & + \left\| \int_{t_1}^{t_2} X(t_2) X^{-1}(s) f(s, x(s)) ds \right\| \leq \\
 & \leq \|X(t_2) - X(t_1)\| \left\| \int_a^{t_1} X^{-1}(s) f(s, x(s)) ds \right\| + \\
 & + k \int_{t_1}^{t_2} \|f(s, x(s))\| ds \leq \\
 & \leq \|X(t_2) - X(t_1)\| \left\| \int_a^{t_1} X^{-1}(s) f(s, x(s)) ds \right\| + k \left(\mu \int_{t_1}^{t_2} p(s) ds + \int_{t_1}^{t_2} q(s) ds \right)
 \end{aligned}$$

and for $t_1 \rightarrow t_2$ we get (3.2). The uniform convergence of $K_p N(\Theta)$ follows immediately from (3.1).

REMARK 3.1. - This last theorem can be applied in some cases when Theorem 3.1 fails.

EXAMPLE. - Consider the scalar equation

$$(3.3) \quad \dot{x} + \frac{1}{t} x = \frac{1}{t^2} \quad t \in [1, +\infty)$$

we have:

$$X(t) = 1/t, \quad X(t) X^{-1}(s) = s/t \leq 1, \quad 1 \leq s \leq t < +\infty.$$

The linear equation associated to (3.3) is asymptotically and uniformly stable and it is possible to apply Theorem 3.2. On the contrary we can't use Theorem 3.1 because we have $X^{-1}f(t, x) = 1/t$ which hasn't a converging integral on $[1, +\infty)$.

THEOREM 3.3. - Suppose that, for the system (1.1)-(1.2), the following hypotheses hold:

- a) $A(t)$ is a continuous real $n \times n$ matrix, defined on $[a, b)$, $(-\infty < a < b = +\infty)$ such that the linear system (1.3) is asymptotically-uniformly stable, i.e. $\exists h > 0$, $\exists \alpha > 0$ such that for every fundamental matrix $X(t)$ of (1.3) we have

$$\|X(t) X^{-1}(s)\| \leq h \exp(-\alpha(t-s)), \quad a \leq s \leq t < b = +\infty$$

- b) Let $f(t, u) \in C([a, b] \times R^n, R^n)$ such that there are a nonnegative constant γ and a real positive function $\lambda = \lambda(t)$ which tends to zero so that

$$\|f(t, u)\| \leq \gamma \|u\| + \lambda(t) \quad \frac{h\gamma}{\alpha} < 1$$

- c) T is a linear bounded operator such that $\text{dom } T \supset BC_0$, $\text{Im } T = R^m$ ($m \leq n$) and $TX(t)$ has rank (maximum) m .

Under these hypotheses BC_0 is admissible for (1.1) and the operator M , given in (1.5), is defined on BC , its image is contained in BC_0 and it is completely continuous.

PROOF. — The admissibility follows immediately from Corollary 2.3. From b) and from Theorem 2.2 it is easy to check that M is defined on BC and that it takes values in BC_0 . As concern the complete continuity of M , it can be obtained with slight modifications from the proof of the preceding theorem. We prove only the equicontinuity. From (1.6) will be enough to prove that, chosen $t_1, t_2 \in [a, b]$, one gets

$$\lim_{t_2 \rightarrow t_1} \left\| \int_a^{t_2} X(t_2) X^{-1}(s) f(s, x(s)) ds - \int_a^{t_1} X(t_1) X^{-1}(s) f(s, x(s)) ds \right\| = 0$$

where $x \in \Theta \subset BC$, Θ bounded.

Let $t_2 > t_1$. We have

$$\begin{aligned} & \left\| \int_a^{t_2} X(t_2) X^{-1}(s) f(s, x(s)) ds - \int_a^{t_1} X(t_1) X^{-1}(s) f(s, x(s)) ds \right\| \leq \\ & \leq \|X(t_2) - X(t_1)\| \left\| \int_a^{t_1} X^{-1}(s) f(s, x(s)) ds \right\| + \left\| \int_{t_1}^{t_2} X(t_2) X^{-1}(s) f(s, x(s)) ds \right\| \leq \\ & \leq \|X(t_2) - X(t_1)\| \left\| \int_a^{t_1} X^{-1}(s) f(s, x(s)) ds \right\| + \int_{t_1}^{t_2} h(\gamma \|x\| + \lambda) \exp(-\alpha(t-s)) ds \end{aligned}$$

and taking the limit for $t_2 \rightarrow t_1$ the statement follows. $\square$

We note that similar results can be obtained from Theorem 3.3 replacing condition b) with the condition b) of Theorem 3.2 (see Corollary 2.3) or supposing that the linear system is integrally stable.

REMARK 3.2. — For the scalar equation

$$(3.4) \quad \dot{x} + x = \exp(-t)$$

we can't use Theorem 3.1 because we have

$$X^{-1}(t)f(t, x) = 1.$$

Nevertheless we can use Theorem 3.3 choosing $h=1, \alpha=1, \gamma=0$.

REMARK 3.3. - Theorem 3.3 can be applied to the equation

$$\dot{x} + x = \frac{1}{2} \log(1+x^2)$$

because $\gamma = \frac{1}{2}$ and $h=1$. Nevertheless it is not possible to utilize Theorem 3.2, because condition b) fails.

Viceversa Theorem 3.2 can be applied to the equation

$$\dot{x} + (1+t)^{-1}x = \exp(-t), \quad 0 \leq t < +\infty$$

but Theorem 3.3 can't because the equation is asymptotically and uniformly stable but it is not asymptotically uniformly stable. In fact we have

$$X(t) = (t+1)^{-1}, \quad X(t)X^{-1}(s) = \frac{s+1}{t+1} \leq 1 \quad 0 \leq s \leq t < +\infty.$$

REMARK 3.4. - If the system (1.3) is asymptotically-uniformly stable, the operator L is onto $BC \times R^m$ and then the operator M becomes the operator $\mathcal{M}$ introduced by J. MAWHIN ([10]).

We have stated theorems which guarantee the existence of solutions in BC_0 by adding an a priori bound. We now show how to solve the problem in BC_1 , imposing separated hypotheses on the linear part and on the forcing term.

COROLLARY 3.3. - Suppose that, for the system (1.1)-(1.2) the following hypotheses hold:

- a) Let $A(t)$ be a real, continuous, $n \times n$ matrix defined on $[a, b)$ ($-\infty < a < b = +\infty$) such that

$$\int_a^b \|A(t)\| dt < +\infty$$

- b) Let $f(t, u)$ be such that verifies condition b) of Theorem 3.2.
c) T is a bounded linear operator such that $\text{dom } T \supset BC_1$, $\text{Im } T = R^m$ ($m \leq n$), $TX(t)$ has rank (maximum) m , where $X(t)$ is any fundamental matrix of (1.3).

Under such hypotheses BC_i is admissible for (1.1), the operator M , given in (1.5), is defined on BC , its image is contained in BC_i and it is completely continuous.

PROOF. — From Corollary 2.1 we infer that the linear system (1.3) is strongly stable and that for every fundamental matrix $X(t)$ of (1.3) we have

$$\lim_{t \rightarrow b} X(t) = V, \quad \lim_{t \rightarrow b} X^{-1}(t) = W$$

(V, W non singular matrices).

Therefore condition $b)$ of Theorem 3.2 implies the condition $b)$ of Theorem 3.1. The statement follows from Theorem 3.1. $\square$

REMARK 3.5. — Theorems 3.2 and 3.3 hold also for $b < +\infty$. In this case we must modify the hypotheses on the linear system (1.3), we shall speak of « stability for $t \rightarrow b$ » instead of « stability in the future ».

We conclude recalling that the results here obtained can be easily extended to the nonlinear boundary value problem

$$\begin{cases} \dot{x} - A(t)x = f(t, x) \\ Tx = h(x) \end{cases}$$

where h is a continuous function

$$h: \text{dom } T \subset BC \rightarrow R^m, \quad (m \leq n)$$

such that

$$\|h(u)\| \leq h_1 \|u\| + h_2, \quad h_1, h_2 \in R^+$$

as it had been done in [4] for some less general cases.

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