On the Solvability of Nonlinear Operator Equations in Normed Spaces (*).

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Sunto. – *Si studia una classe di applicazioni continue negli spazi normati e si dimostrano, per tali applicazioni, proprietà analoghe a quelle della teoria del grado di Leray-Schauder. Si danno esempi di problemi ai limiti per equazioni differenziali ordinarie che possono essere agevolmente trattati usando la teoria sviluppata nel presente lavoro.*

0. – Introduction.

Let E, F be normed spaces and $\Omega \subset E$ be open and bounded. Let $f\colon \bar{\Omega} \to F$ be continuous and such that $f(x) \neq 0$ for any x belonging to the boundary $\partial\Omega$ of Ω. If the equation $f(x) = h(x)$ is solvable whenever $h\colon \bar{\Omega} \to F$ is compact and vanishing on $\partial\Omega$, then we say that f is 0-epi (zero-epi).

The main task of this paper is to develop, with very elementary tools, the theory of 0-epi maps and to show that in many applications it represent a good substitute for degree theory. In facts, 0-epi maps have properties such as existence, boundary dependence, normalization, localization and homotopy invariance analogous to those of degree theory. Moreover, since 0-epi maps may act between different spaces in many cases they can be used to prove existence results for boundary value problems avoiding the tedious procedure of looking for Green functions and transforming these problems into the equivalent integral form. Actually, 0-epi maps may also be viewed as a simple tool which helps the use of Schauder fixed point theorem (in fact, the normalization property is nothing else but a reformulation of this fundamental fixed point theorem) and they can be successfully combined with degree theory to get deeper results.

The plan of this paper is as follows. After introducing the notion of 0-epi maps and giving their main properties (section 1) we proceed (section 2) with the study of 0-epi maps defined on the whole space E (and not merely on the closure of an open bounded subset of E). In this context we get surjectivity results.

The theory of 0-epi maps is then applied (section 3) to nonlinear « abstract » boundary value problems in the context of the so-called alternative methods for boundary value problems.

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The last section is devoted to examples of 0-epi maps and applications to boundary value problems for ordinary differential equations, with a particular attention to the case of boundary value problems depending on a parameter (see Theorem 3.2 and the examples thereafter).

1. – 0-epi maps with bounded domain.

Let E, F be normed spaces and $\Omega \subset E$ be open and bounded. We say that a continuous map $f: \bar{\Omega} \to F$, defined on the closure $\bar{\Omega}$ of Ω, is 0-*admissible* (p-*admissible*, $p \in F$) if $f(x) \neq 0$ ($f(x) \neq p$) for all $x \in \partial\Omega$.

Recall that a continuous map $h: X \to F$, $X \subset E$, is said to be compact if it maps bounded sets into relatively compact subsets of F.

DEFINITION 1.1. – A 0-admissible map $f: \bar{\Omega} \to F$ is said to be 0-epi if for any compact map $h: \bar{\Omega} \to F$ such that $h(x) = 0$ for any $x \in \partial\Omega$ the equation $f(x) = h(x)$ has a solution in Ω.

We point out that when $F = E$ and $f: \bar{\Omega} \to E$ is a compact perturbation of the identity, the above definition agrees with that of essential compact vector field (with respect to Ω) as given by A. GRANAS [4].

A p-admissible map $f: \bar{\Omega} \to F$ is called p-epi, $p \in F$, if the map f-p defined by $(f\text{-}p)(x) = f(x) - p$ is 0-epi. The basic properties of 0-epi maps are analogous to the properties which characterize Brouwer's degree.

EXISTENCE PROPERTY. – *Let $f: \bar{\Omega} \to F$ be p-epi. Then the equation $f(x) = p$ has a solution in Ω.*

NORMALIZATION PROPERTY. – *The inclusion $i: \bar{\Omega} \to E$ is p-epi if and only if $p \in \Omega$.*

PROOF. – (Only if). It is a direct consequence of the definition of p-epi map. (If) It is enough to show that i is 0-epi provided that $0 \in \Omega$. Let $h: E \to E$ be compact and such that $h(x) = 0$ for any $x \notin \Omega$. Since $0 \in \Omega$ the equation $i(x) = h(x)$ has a solution in Ω if and only if the map $h: E \to E$ has a fixed point. The existence of a fixed point for h is ensured by Schauder's fixed point theorem since $\overline{h(E)}$ is compact.

LOCALIZATION PROPERTY. – *Let $f: \bar{\Omega} \to F$ be 0-epi and assume that $f^{-1}(0)$ is contained in an open set $\Omega_1 \subset \Omega$. Then the restriction of f to $\bar{\Omega}_1$, $f|_{\bar{\Omega}_1}: \bar{\Omega}_1 \to F$, is 0-epi.*

PROOF. – Let $h: \bar{\Omega}_1 \to F$ be continuous compact and vanishing on $\partial\Omega_1$. Extend h to a compact map $\hat{h}: \bar{\Omega} \to F$ by putting $\hat{h}(x) = 0$ if $x \in \bar{\Omega} \setminus \Omega_1$. The equation $f(x) = \hat{h}(x)$ has a solution $x_0 \in \Omega$. The condition $f^{-1}(0) \subset \Omega_1$ ensures that $x_0 \in \Omega_1$. Q.E.D.

HOMOTOPY PROPERTY. – *Let* $f\colon \bar{\Omega} \to F$ *be* 0-*epi and let* $h\colon \bar{\Omega}\times[0,1] \to F$ *be compact and such that* $h(x, 0) = 0$ *for any* $x \in \bar{\Omega}$. *Assume that* $f(x) + h(x,t) \neq 0$ *for all* $x \in \partial\Omega$ *and for any* $t \in [0,1]$. *Then* $f(\cdot) + h(\cdot, 1)\colon \bar{\Omega} \to F$ *is* 0-*epi.*

PROOF. – Let $k\colon \bar{\Omega} \to F$ be compact and such that $k(x) = 0$ for all $x \in \partial\Omega$. The set $S = \{x \in \bar{\Omega}\colon f(x) + h(x,t) = k(x)$ for some $t \in [0,1]\}$ is closed since $[0,1]$ is compact. Let $\varphi\colon \bar{\Omega} \to [0,1]$ be a continuous function such that $\varphi(x) = 1$ for every $x \in S$ and $\varphi(x) = 0$ for all $x \in \partial\Omega$. The existence of φ is ensured by Uryshon's Lemma. Consider the equation

$$f(x) = k(x) - h\big(x, \varphi(x)\big)\,.$$

Since the map $\hat{h}\colon \bar{\Omega} \to F$ defined by $\hat{h}(x) = k(x) - h\big(x, \varphi(x)\big)$ is compact and it vanishes on $\partial\Omega$, there exists a solution x_0 of the above equation. Clearly $x_0 \in S$. Hence $\varphi(x_0) = 1$ and

$$f(x_0) + h(x_0, 1) = k(x_0)\,. \qquad \text{Q.E.D.}$$

BOUNDARY DEPENDENCE PROPERTY. – *Let* $f\colon \bar{\Omega} \to F$ *be* 0-*epi and* $k\colon \bar{\Omega} \to F$ *be compact. Assume that* $k(x) = 0$ *for all* $x \in \partial\Omega$. *Then* $f + k\colon \bar{\Omega} \to F$ *is* 0-*epi.*

We shall derive now some consequences of the above properties. Recall that a map $f\colon \bar{\Omega} \to F$ is *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset F$. It is easy to show that if f is proper then $f(\bar{\Omega})$ is closed. Therefore, in this case, $f(\bar{\Omega}) = \overline{f(\Omega)}$.

THEOREM 1.1. – *Let* $f\colon \bar{\Omega} \to F$ *be* 0-*epi and proper. Then* f *maps* Ω *onto a neighborhood of the origin. More precisely if* U_0 *is the connected component of* $F \setminus f(\partial\Omega)$ *containing the origin, then* $U_0 \subset f(\Omega)$.

PROOF. – The set $f(\partial\Omega)$ is closed since f is proper. Hence U_0 is open, which implies that U_0 is also path connected. Let $p \in U_0$ and $\sigma\colon [0,1] \to U_0$ be a continuous map such that $\sigma(0) = 0$ and $\sigma(1) = p$. The homotopy $(x,t) \to f(x) - \sigma(t)$ ensures that $f(\cdot) - \sigma(1)\colon \bar{\Omega} \to F$ is 0-epi, which implies that $p \in f(\Omega)$. Q.E.D.

Observe that if in Theorem 1.1 we assume that $f(\Omega)$ is bounded then $f(\partial\Omega)$ separates the origin from the infinity, i.e. U_0 is bounded.

Let $f\colon \bar{\Omega} \to F$ be continuous, injective and proper. Then $f(\Omega)$ need not be open as it is easily seen by embedding the interval $[0,1] \subset \boldsymbol{R}$ into $\boldsymbol{R}^2$. Neverthless the following result holds.

THEOREM 1.2. – *Let* $f\colon \bar{\Omega} \to F$ *be continuous, injective and proper. Then* $f(\Omega)$ *is open if and only if* f *is* p-*epi for any* $p \in f(\Omega)$.

PROOF. – (If) Assume $p \in f(\Omega)$ and f p-epi. Applying Theorem 1.1 to the map f-p we obtain that f-p maps Ω onto a neighborhood of 0. Thus $f(\Omega)$ is open.

(Only if) Assume that $f(\Omega)$ is open. Since f is injective and proper we have that f is invertible on its image and the inverse is continuous. It is enough to show that if $0 \in f(\Omega)$ then f is 0-epi. Let $h: \bar{\Omega} \to F$ be compact and vanishing on $\partial\Omega$. Define a map $g: F \to F$ by

$$g(y) = \begin{cases} h(f^{-1}(y)) & \text{if } y \in \overline{f(\Omega)} \\ 0 & \text{if } y \notin \overline{f(\Omega)} . \end{cases}$$

Since $\overline{f(\Omega)} = f(\bar{\Omega})$ and $f(\Omega)$ is open we obtain that $\partial f(\Omega) = f(\partial\Omega)$. Hence g vanishes on $\partial f(\Omega)$. This shows that g is continuous. Since $g(F)$ is relatively compact and $g(F) \subset F$ there exists $y_0 \in F$ such that $y_0 = g(y_0)$ (by Schauder Fixed Point Theorem). Clearly $y_0 \in f(\Omega)$ and $x_0 = f^{-1}(y_0)$ is a solution of the equation

$$f(x) = h(x) . \qquad \text{Q.E.D.}$$

The following result is an extension of the Normalization Property.

COROLLARY 1.1. – *Let $f: \bar{\Omega} \to F$ be continuous, injective and proper. Assume that $f(\Omega)$ is open. Then f is p-epi if and only if $p \in f(\Omega)$.*

PROOF. – (Only if) It is a direct consequence of the definitions. (If) It is a consequence of Theorem 1.2. Q.E.D.

Note that if $f: \bar{\Omega} \to F$ is continuous, injective and proper then $f(\bar{\Omega})$ is closed and f is a homeomorphism between $\bar{\Omega}$ and $f(\bar{\Omega})$. On the other hand if $f: \bar{\Omega} \to F$ is a homeomorphism onto $f(\bar{\Omega})$ and $f(\bar{\Omega})$ is closed then f is proper. Hence a homeomorphism f between $\bar{\Omega}$ and $f(\bar{\Omega})$ is proper if and only if $f(\bar{\Omega})$ is closed.

Recall that a set $Q \subset F$ is star-shaped with respect to the origin if $ty \in Q$ for every $y \in Q$ and $t \in [0, 1]$. The following result is a consequence of the Homotopy Property.

THEOREM 1.3. – *Let $f: \bar{\Omega} \to F$ be 0-epi. Assume that there exists a star-shaped (with respect to the origin) subset $Q \subset F$ such that $Q \cap f(\partial\Omega) = \emptyset$. Then the equation $f(x) = h(x)$ has a solution for any compact map $h: \bar{\Omega} \to F$ such that $h(\partial\Omega) \subset Q$. In particular $Q \subset f(\Omega)$.*

PROOF. – Let $x \in \partial\Omega$ and $\lambda \in [0, 1]$. Clearly $\lambda h(x) \in Q$ and $f(x) \notin Q$. Hence $f(x) \neq \lambda h(x)$ for all $x \in \partial\Omega$ and for every $\lambda \in [0, 1]$. Therefore $f - h$ is 0-epi. In particular $f(x) = h(x)$ for some $x \in \Omega$. Let $p \in Q$. By taking as h the constant map $h(x) = p$ for every $x \in \bar{\Omega}$ we obtain that $p \in f(\Omega)$. Thus $Q \subset f(\Omega)$. Q.E.D.

The following two results present possible ways of obtaining 0-epi maps starting from maps having this property.

THEOREM 1.4. – *Let $f: \bar{\Omega} \to F$ be continuous, 0-admissible and proper. Let $\{f_n\}$ be a sequence of 0-epi maps from $\bar{\Omega}$ into F, which converges uniformly to f. Then f is 0-epi.*

PROOF. – Let $h: \bar{\Omega} \to F$ be continuous compact and vanishing on $\partial\Omega$. Let x_n be any solution of the equation $f_n(x) = h(x)$ and put $y_n = f(x_n) - f_n(x_n) = f(x_n) - h(x_n)$ $n = 1, 2, \ldots$. Clearly $y_n \to 0$ as $n \to +\infty$. Since $f - h$ is proper (recall that h is compact) we obtain that $\{x_n\}$ has a cluster point $\bar{x}$. Obviously $f(\bar{x}) = h(\bar{x})$. Q.E.D.

THEOREM 1.5. (Perturbation Theorem for 0-epi maps). – *Let $f: \bar{\Omega} \to F$ be a proper 0-epi map and let $h: \bar{\Omega} \times [-1, 1] \to F$ be compact and such that $h(x, 0) = 0$ for any $x \in \bar{\Omega}$. Then there exists $\varepsilon > 0$ such that $f(\cdot) - h(\cdot, \lambda)$ is 0-epi for every $|\lambda| < \varepsilon$.*

PROOF. – By the Homotopy Property it suffices to show that there exists $\varepsilon > 0$ such that $f(x) \neq h(x, \lambda)$ for all $x \in \partial\Omega$ and $\lambda \in (-\varepsilon, \varepsilon)$. Assume the contrary. Then there exists a sequence $\{(x_n, \lambda_n)\}$ in $\partial\Omega \times [-1, 1]$ such that $\lambda_n \to 0$ and

$$f(x_n) = h(x_n, \lambda_n) .$$

Since h is compact and f is proper the sequence $\{x_n\}$ has a cluster point $x_0 \in \partial\Omega$. It follows that $f(x_0) = h(x_0, 0) = 0$, a contradiction with the 0-admissibility of f. Q.E.D.

The following results exhibit some interesting classes of 0-epi maps (Theorems 1.6-1.9).

Recall that a continuous map $f: \bar{X} \to E$, $X \subset E$, is called a *compact vector field* if the map $h: \bar{X} \to E$ defined by $h(x) = x - f(x)$ is compact.

THEOREM 1.6. – *Let $f: \bar{\Omega} \to E$ be a p-admissible compact vector field defined on the closure $\bar{\Omega}$ of an open bounded set $\Omega \subset E$. If the Leray-Schauder degree* $\deg(f, \Omega, p) \neq 0$ *then f is p-epi.*

PROOF. – Let $h: \bar{\Omega} \to E$ be continuous, compact and vanishing on $\partial\Omega$. Clearly $f - h$ is a compact vector field which coincides with f on $\partial\Omega$. By the boundary dependence property of the Leray-Schauder degree we have $\deg(f - h, \Omega, p) = \deg(f, \Omega, p) \neq 0$. Hence the equation $f(x) - p = h(x)$ has a solution, i.e. f is p-epi. Q.E.D.

The following is an example of a 0-epi map $f: \Omega \to \boldsymbol{R}$, $\Omega \subset \boldsymbol{R}$, with $\deg(f, \Omega, 0) = 0$: $\Omega = (-2, -1) \cup (1, 2)$, $f(x) = x^2 - 2$.

In Theorems 1.7-1.9 below the spaces E, F are assumed to be Banach. This restriction can be removed in many different ways as the comments thereafter show.

THEOREM 1.7. – *Let $L: E \to F$ be bounded, linear and surjective with* $\dim \operatorname{Ker} L = n < +\infty$. *Let $g: \bar{\Omega} \to \boldsymbol{R}^n$ be continuous and such that $g(x) \neq 0$ for any $x \in \partial\Omega \cap$*

$\cap \operatorname{Ker} L$. *If the Brouwer topological degree* $\deg(gJ, J^{-1}(\Omega), 0) \neq 0$ *where* $J\colon \mathbf{R}^n \to E$ *is linear and such that* $\operatorname{Im} J = \operatorname{Ker} L$ *then the map* $M\colon \bar{\Omega} \to F \times \mathbf{R}^n$ *defined by* $M(x) = (Lx, g(x))$ *is* 0-*epi.*

PROOF. – Let $A\colon F \times \mathbf{R}^n \to E$ be defined by $A(y, z) = Sy + Jz$, where $S\colon F \to E$ is any bounded linear right inverse of L (recall that $\dim \operatorname{Ker} L < +\infty$). Observe that A is an isomorphism, thus M is 0-epi if and only if the composition $f = MA\colon A^{-1}(\Omega) \to F \times \mathbf{R}^n$ is 0-epi. Now, $f(y, z) = (L(Sy + Jz), g(Sy + Jz)) = (y, g(Sy + Jz)) = (y, z) - (0, z - g(Sy + Jz))$. Therefore, f is of the form $I - h$, where h maps $A^{-1}(\Omega)$ into the finite dimensional space $\{0\} \times \mathbf{R}^n$. By the definition of the Leray-Schauder degree we get

$$\deg(f, A^{-1}(\Omega), 0) = \deg(f|_{\{0\}\times \mathbf{R}^n}, A^{-1}(\Omega) \cap (\{0\} \times \mathbf{R}^n), 0) = \deg(gJ, J^{-1}(\Omega), 0) \neq 0 .$$

Thus, f is 0-epi (see Theorem 1.6). Q.E.D.

Theorems 1.8-1.9 below are related with boundary value problems « at resonance ».

We recall first that F_0 is a closed n-codimensional subspace of a normed space F if and only if there exists a surjective bounded linear operator $Q\colon F \to \mathbf{R}^n$ such that $F_0 = \operatorname{Ker} Q$ (i.e. F_0 is the intersection of the kernels of n linearly independent bounded functionals). Moreover, Q is unique up to a linear isomorphism of $\mathbf{R}^n$ into itself.

THEOREM 1.8. – *Let* $L\colon E \to F$ *be a bounded Fredholm operator of index* 0 *with* $\dim \operatorname{Ker} L = n$. *Let* $\Omega \subset E$ *be open bounded and let* $h\colon \bar{\Omega} \to F$ *be compact and such that* $h(x) \notin \operatorname{Im} L$ *for any* $x \in \partial\Omega \cap \operatorname{Ker} L$. *Assume that* $\deg(QhJ, J^{-1}(\Omega), 0) \neq 0$ *where* $J\colon \mathbf{R}^n \to E$, $Q\colon F \to \mathbf{R}^n$ *are linear continuous and satisfy* $\operatorname{Im} J = \operatorname{Ker} L$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Then there exists* $\varepsilon > 0$ *such that* $L - \lambda h$ *is* 0-*epi for* $0 < |\lambda| < \varepsilon$.

PROOF. – From Theorem 1.7 it follows that the operator $M\colon \bar{\Omega} \to \operatorname{Im} L \times \mathbf{R}^n$ defined by $M(x) = (Lx, Qh(x))$ is 0-epi. Let $P\colon F \to \operatorname{Im} L$ be any continuous retraction. Since M is proper in $\bar{\Omega}$ we obtain, by Theorem 1.5, that there exists $\varepsilon > 0$ such that the map

$$x \to (Lx - \lambda Ph(x), Qh(x))$$

is 0-epi for any $0 \leqslant |\lambda| < \varepsilon$. Let $k\colon \bar{\Omega} \to F$ be compact and such that $k(x) = 0$ for any $x \in \partial\Omega$. By the Boundary Dependence Property for 0-epi maps the map

$$x \to \Big(Lx - \lambda P\big(h(x) + \lambda^{-1}k(x)\big)\,,\ Q\big(h(x) + \lambda^{-1}k(x)\big)\Big)$$

is 0-epi provided that $0 < |\lambda| < \varepsilon$. Now the result follows from the fact that the system

$$\begin{cases} Lx = \lambda P\big(h(x) + \lambda^{-1}k(x)\big) \\ Q\big(h(x) + \lambda^{-1}k(x)\big) = 0 \end{cases}$$

is equivalent to the equation $Lx - \lambda h(x) = k(x)$. Q.E.D.

THEOREM 1.9. – *Let $L: E \to F$ be a bounded Fredholm operator of index* 0. *Let $h: \bar{\Omega} \to F$ be compact. Assume that there exists a compact linear map $P: E \to F$ such that*

$$\operatorname{Ker}(L+P) = \{0\} \tag{1}$$

$$Lx + Px \neq \lambda\big(h(x) + Px\big) \tag{2}$$

for any $x \in \partial\Omega$ and $\lambda \in [0, 1]$. Then $L - h$ is 0*-epi provided that $0 \in \Omega$.*

PROOF. – We have, by the Fredholm alternative, $\operatorname{ind}(L+P) = \operatorname{ind}(L) = 0$. Therefore by (1) $L+P$ is an isomorphism. Moreover if $0 \in \Omega$, Corollary 1.1 implies that $L+P$ is 0-epi on Ω. The result now follows from the Homotopy Property of 0-epi maps. Q.E.D.

We point out that throughout this paper all the maps are assumed to be continuous. However, in many applications one has to deal with equations of the form $Lx = h(x)$, where $L: E \to F$ is a (not necessarily bounded) Fredholm operator and $h: E \to F$ is a (not necessarily compact) nonlinear operator. Therefore, our methods seem to break down in these cases. Nevertheless, the following considerations are in order.

Let E and F be vector spaces and let $L: E \to F$ be linear. The operator L is said to be *algebraically Fredholm* if $\operatorname{Ker} L$ and $F/\operatorname{Im} L$ are finite dimensional. If, moreover, E and F have normed structures and L has closed image, then L is called a *Fredholm* operator. Finally, we say that a Fredholm operator L is *top-Fredholm* if it is bounded and it admits a right inverse $L^+: \operatorname{Im} L \to E$ (i.e. LL^+ is the inclusion $\operatorname{Im} L \to F$) which is bounded.

PROPOSITION 1.1. – *Let E and F be two vector spaces and let $L: E \to F$ be algebraically Fredholm. Then any normable structure on E induces a unique normable structure on F such that L becomes a top-Fredholm operator. Conversely, any normable structure on F such that* $\operatorname{Im} L$ *is closed induces a unique normable structure on E making L top-Fredholm. Moreover, if a normable structure on E (on F) is complete so is the induced one on F (on E).*

PROOF. – Observe first that given a vector space G and a finite codimensional subspace G_1 of G, then any normable structure on G_1 induces a unique normable structure on G, making G_1 closed. The existence of this structure is evident. The uniqueness is obtained by (topologically) decomposing G as $G = G_1 \oplus G_0$ where G_0 is finite dimensional and by recalling that any finite dimensional vector space admits a unique normable structure.

The above observation and the fact that a finite dimensional subspace of a vector space admits a (topological) direct summand reduces our problem to the very simple case when $L: E \to F$ is an algebraic isomorphism.

Now given a norm $E \to \boldsymbol{R}$ on E (a norm $F \to \boldsymbol{R}$ on F) define a norm on F (on E) by considering the composition $F \xrightarrow{L^{-1}} E \to \boldsymbol{R}$ ($E \xrightarrow{L} F \to \boldsymbol{R}$). With this norm L becomes an isometry. Thus F is Banach if and only if so is E.

The uniqueness of the normable structure depends on the fact that in our reduced problem L and L^{-1} must be bounded (i.e. L is an isomorphism of normable spaces). Q.E.D.

The following situation arises frequently in the applications (see e.g. [1]).

Let E and F be normed spaces, $L\colon E \to F$ be Fredholm and $h\colon E \to F$ be demicontinuous (i.e. continuous from the strong to the weak topology) sending bounded sets into bounded sets. Assume moreover that L admits a compact right inverse $L^+\colon \operatorname{Im} L \to E$. We shall show that under the above assumptions h can be regarded as a compact map from E into the space F endowed with the structure induced by E and L (or any other structure which makes L bounded).

In fact, since $\operatorname{Im} L$ is closed there exists a finite dimensional subspace F_0 of F such that $F = \operatorname{Im} L \oplus F_0$ (topologically). Thus H can be decomposed as $h_1 + h_0$ where $h_1(E) \subset \operatorname{Im} L$ and $h_0(E) \subset F_0$. Since F_0 is finite dimensional it suffices to show that h_1 is compact as a map from E into F_1, where F_1 stands for the space F endowed with the normable structure induced by E via the linear operator L (or any other weaker structure).

The following composition of maps proves our assertion

$$E \xrightarrow{h_1} \operatorname{Im} L \xrightarrow{L^+} E \xrightarrow{L} F_1\,,$$

since $L^+ h_1$ is compact, $L\colon E \to F_1$ is continuous and LL^+ is the inclusion $\operatorname{Im} L \to F_1$.

Let E, F be normed spaces. We recall that any bounded linear operator $L\colon E \to F$ can be uniquely extended to a bounded linear operator $\hat{L}\colon \hat{E} \to \hat{F}$, where $\hat{E}$, $\hat{F}$ stand for the completions of E and F respectively. It is not hard to show that if L is top-Fredholm then so is $\hat{L}$. Moreover, one has $\operatorname{Ker} L = \operatorname{Ker} \hat{L}$, $\widehat{\operatorname{Im} L} = \operatorname{Im} \hat{L}$ and $\operatorname{ind}(L) = \operatorname{ind}(\hat{L})$.

Assume now, as it occurs frequently in applications, that E and F are Banach spaces and $L\colon \mathfrak{D}(L) \to F$ is a Fredholm operator defined on a dense subspace $\mathfrak{D}(L)$ of E. Let $h\colon E \to F$ (or, more generally, $h\colon \overline{U} \to F$, U open in E) be such that $L^+ Ph\colon E \to E$ is compact, where $L^+\colon \operatorname{Im} L \to E$ is any right inverse of L and P is any bounded projection of F onto $\operatorname{Im} L$ (i.e. h is L-compact [3]). We want to show that h can be regarded as a compact map.

Put in F the normable structure which makes $L\colon \mathfrak{D}(L) \to F$ a top-Fredholm operator and extend L to a top-Fredholm operator $\hat{L}\colon E \to \hat{F}$, where $\hat{F}$ stands for the completion of F with respect to the induced structure. Observe now that Ph coincides with the following composition of maps

$$E \xrightarrow{h} F \xrightarrow{P} \operatorname{Im} L \xrightarrow{L^+} E \xrightarrow{\hat{L}} \hat{F}\,.$$

Since L^+Ph is compact, L is continuous and $LL^+Ph: E \to \hat{F}$ coincides with $Ph: E \to \hat{F}$ we obtain that $Ph: E \to \hat{F}$ is compact. Thus $h: E \to \hat{F}$ is also compact since the difference $h - Ph = (I-P)h$ is a finite dimensional map.

We shall describe now another very common situation arising in applications. Let F be a Banach space and let $L: \mathfrak{D}(L) \to F$ be a Fredholm operator defined on a subspace of F. Denote by E the space $\mathfrak{D}(L)$ with the Banach structure which makes L top-Fredholm (or any stronger structure). Assume that the inclusion $E \to F$ is compact. Then the restriction of any continuous map $h: F \to F$ to the subspace $\mathfrak{D}(L)$ can be regarded as a compact map from E into F by considering the composition

$$E \longrightarrow F \xrightarrow{h} F .$$

We close this section with the observation that in Theorems 1.7-1.9 the assumption «E and F are complete» can be removed provided that the linear operator $L: E \to F$ is assumed to have a bounded right inverse.

Note that in the case when E, F are complete then any bounded Fredholm operator $L: E \to F$ has a bounded right inverse, i.e. L is top-Fredholm (to see this apply the Open Mapping Theorem).

2. – 0-epi maps on the whole space.

Let $f: E{:}\to F$ be a continuous map from a normed space E into a normed space F. Given $p \in F$ we say that f is *p-admissible* if $f^{-1}(p)$ is bounded. The map f is *p-epi* (on E) if f is p-epi on any bounded open set $\Omega \supset f^{-1}(p)$. I.e., the restriction $f|_{\overline{\Omega}}$ is *p-epi* (in the former sense) for any bounded open $\Omega \supset f^{-1}(p)$ (or, equivalently, if the equation $f(x) - p = h(x)$ is solvable for any compact h with bounded support).

Notice that in the above definition, in view of the Localization Property for 0-epi maps, we may restrict ourself to sufficiently large open balls centered at the origin.

The plan of this section is as follows.

After a suitable formulation of the Homotopy Property, some facts regarding 0-epi maps defined on the whole space are presented. The proofs are given only when the results are not easy consequences of analogous results previously obtained in the context of 0-epi maps on bounded sets.

HOMOTOPY PROPERTY. – *Let $f: E \to F$ be 0-epi and let $h: E \times [0, 1] \to F$ be compact and such that $h(x, 0) = 0$ for any $x \in E$. If the set*

$$S = \{x \in E: f(x) + h(x, t) = 0 \text{ for some } t \in (0, 1]\} \text{ is bounded},$$

then $f(\cdot) + h(\cdot, 1)$ is 0-epi.

We shall derive now some consequences of the above property. The first one is analogous to Theorem 1.3.

THEOREM 2.1. – *Let $Q \subset F$ be star-shaped with respect to the origin and let $f: E \to F$ be 0-epi. If $f^{-1}(Q)$ is bounded, then the equation $f(x) = h(x)$ has a solution provided that $h: E \to F$ is compact with $\operatorname{Im} h \subset Q$. In particular, $\operatorname{Im} f \supset Q$.*

COROLLARY 2.1. – *Let E and F be Banach spaces and let $L: E \to F$ be bounded and linear. Then L is 0-epi if and only if it is an isomorphism.*

PROOF. – (If) It follows immediately from Corollary 1.1 since L is a continuous, injective, proper and open map.

(Only if) Clearly, L is one-to-one since, being L admissible, $\operatorname{Ker} L$ must be bounded. Since E and F are Banach spaces it remains to show that L is onto. Take $p \in F$ and consider the segment $Q = \{tp: 0 \leqslant t \leqslant 1\}$. The linearity of L (actually, its positive homogeneity) implies that $L^{-1}(Q)$ is bounded. Now, apply Theorem 2.1. Q.E.D.

COROLLARY 2.2. – *Let $f: E \to F$ be 0-epi and such that $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$. Then the equation $f(x) = h(x)$ has a solution for any compact map $h: E \to F$ with bounded image. In particular f is onto.*

PROOF. – Notice that the condition $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$ is equivalent to the fact that the inverse image under f of any bounded subset of F is bounded. Now, take any ball $Q \subset F$, centered at the origin, which contains $\operatorname{Im} h$ and apply Theorem 2.1. Q.E.D.

Observe that under the hypotheses of Corollary 2.2 the map $f + h$ is 0-epi for any compact map $h: E \to F$ having bounded image. In particular, the map f is p-epi for any $p \in F$.

The following theorem, which is a direct consequence of Theorem 1.4, shows how a 0-epi map can be obtained as a uniform limit of 0-epi maps.

THEOREM 2.2. – *Let $f: E \to F$ be 0-admissible and proper on bounded closed sets. Let $\{f_n\}$ be a sequence of 0-epi maps from E into F, converging uniformly to f on bounded subsets of E. If the sets $f_n^{-1}(0)$ are imiformly bounded then f is 0-epi.*

The following results exhibit interesting classes of nonlinear 0-epi maps.

We recall first that a map $f: H \to H$ defined on a Hilbert space H is said to be *monotone* if $(f(x) - f(y), x - y) \geqslant 0$ for all $x, y \in H$, where $(\cdot, \cdot)$ stands for the inner product in H.

THEOREM 2.3. – *Let $f: H \to H$ be a continuous monotone operator which is proper on bounded closed sets. Assume that $(f(x), x) > 0$ jor $\|x\|$ sufficiently large. Then f is 0-epi.*

PROOF. – Given $n \in \boldsymbol{N}$, define $f_n: H \to H$ by $f_n(x) = (1/n)x + f(x)$. By a result of MINTY [6] f_n is a homeomorphism of H onto H. Thus, as a consequence of Corollary 1.1, the map f_n is 0-epi. Now apply Theorem 2.2. Q.E.D.

We give now a characterization of 0-epi maps acting on finite dimensional spaces. Let $f: R^n \to R^m$ be 0-admissible. There exists $r_0 > 0$ such that $f(x) \neq 0$ provided that $\|x\| > r_0$. Therefore given $r > r_0$ we can define $f_r: S^{n-1} \to S^{m-1}$ by $f_r(x) = f(rx)/\|f(rx)\|$ (S^{k-1} denotes the unit sphere or R^k). Let $\hat{f}$ be the homotopy class associated to f_r. This class is clearly independent of $r > r_0$.

THEOREM 2.4. – *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ be 0-admissible. Then f is 0-epi if and only if $\hat{f}$ is nontrivial.*

PROOF. – (See [2], Proposition 6.2.2).

As a consequence of the above theorem we have that there are no 0-epi maps from $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ if $n < m$. It also follows that a 0-admissible map $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is 0-epi if and only if $\deg(f, 0) \neq 0$ (here $\deg(f, 0)$ stands for the Brouwer degree $\deg(f, \Omega, 0)$, where Ω is any bounded open set containing $f^{-1}(0)$).

THEOREM 2.5. – *Let $L: E \to F$ be a bounded Fredholm operator of index 0 from a Banach space E into a Banach space F. Let $h: E \to F$ be compact and odd outside a sufficiently large ball centered at the origin. Then $L + h$ is 0-epi provided $L + h$ is 0-admissible (i.e. $(L + h)^{-1}(0)$ is bounded).*

PROOF. – We have to show that given a compact map $k: E \to F$ with bouoded support the equation $Lx + h(x) = k(x)$ is solvable. Since $\operatorname{ind} L = 0$, there exists a compact operator $K: E \to F$ such that $L + K$ is an isomorphism. The equation $Lx + h(x) = k(x)$ is equivalent to the equation $x = g(x)$, where $g = (L + K)^{-1}(K - h + k)$ Obviously, g is compact and odd outside a sufficiently large ball around the origin. Now, apply the infinite dimensional version of Borsuk Theorem (see A. GRANAS [4]). Q.E.D.

Let $L: E \to F$ be a bounded linear surjective map. Assume that $\dim \operatorname{Ker} L = n < +\infty$. Let $g: E \to \boldsymbol{R}^n$ be continuous. Consider the following problem with nonlinear boundary conditions

$$\begin{cases} Lx = h(x)\,, \\ g(x) = 0\,, \end{cases}$$

where $h: E \to F$ is compact.

The following result which is an easy consequence of Theorem 1.7, is related with problems of this type.

THEOREM 2.6. – *Let $L: E \to F$ be bounded linear and surjective with* $\dim \operatorname{Ker} L = n < +\infty$. *Let $g: E \to \boldsymbol{R}^n$ be continuous and $J: \boldsymbol{R}^n \to E$ be linear with* $\operatorname{Im} J = \operatorname{Ker} L$. *Assume that $g^{-1}(0) \cap \operatorname{Ker} L$ is bounded and* $\deg(gJ, 0) \neq 0$. *Then the map $M: E \to F \times \boldsymbol{R}^n$ defined by $M(x) = (Lx, g(x))$ is 0-epi.*

The following theorem is analogous to a result due to J. L. MAWHIN [5] (see also R. E. GAINES - J. L. MAWHIN [3]).

THEOREM 2.7. – *Let $L: E \to F$ be a bounded Fredholm operator of index* 0 *and let $h: E \to F$ be compact. Assume that the set $S^0 = \operatorname{Ker} L \cap h^{-1}(\operatorname{Im} L)$ is bounded and* $\deg(QhJ, 0) \neq 0$, *where $Q: F \to \boldsymbol{R}^n$ is linear and such that* $\operatorname{Ker} Q = \operatorname{Im} L$, $n = \dim \operatorname{Ker} L$ *and $J: \boldsymbol{R}^n \to E$ is linear with* $\operatorname{Im} J = \operatorname{Ker} L$. *If the set $S^+ = \{x \in E: Lx = \lambda h(x)$ for some $0 < \lambda \leqslant 1\}$ is bounded, then the map $L - h$ is 0-epi.*

PROOF. – Take any open bounded set Ω containing $S^0 \cup S^+$. Theorem 1.7, combined with the Homotopy Property for 0-epi maps, shows that $L - h$ is 0-epi on Ω. The result now follows from the arbitrarity of Ω. Q.E.D.

Notice that the condition « S^0 is bounded » is equivalent to « QhJ is 0-admissible ». Thus, if S^0 is bounded, then $\deg(QhJ, 0)$ is defined and it is different from zero if and only if the map QhJ is 0-epi.

3. – Further examples of 0-epi maps and applications.

In this section some other examples of 0-epi maps with particular concern to ordinary differential operators are given. Few definitions and notations are presented at the beginning.

Given a nonnegative integer k the notation $C^k[a, b]$ stands for the Banach space of all k-times continuously differentiable real functions defined on the compact interval $[a, b]$.

The norm of $x \in C^k[a, b]$ is $\|x\|_k = \sum_{i=0}^{k} \|x^{(i)}\|_0$, where $\|x^{(s)}\|_0 = \max\{|x^{(s)}(t)|: t \in [a, b]\}$

It is well-known that Ascoli's theorem gives the compactness of the inclusion $C^{k+1}[a, b] \to C^k[a, b]$ for any $k \geqslant 0$. This fact allows us to regard any continuous map $f: C^k[a, b] \to C^0[a, b]$ as a compact map from $C^{k+1}[a\ b]$ into $C^0[a\ b]$.

Let E, F and G be Banach spaces with norms $\|\cdot\|_E$, $\|\cdot\|_F$ and $\|\cdot\|_G$ respectively. Let $L: E \to F$ and $B: E \to G$ be bounded linear operators. Assume that the following (boundary value) problem

$$(A) \qquad \begin{cases} Lx = y \\ Bx = 0\,, \end{cases}$$

has a unique solution for any $x \in F$. Then, in the subspace $E_0 = \operatorname{Ker} B$ the norm $\|x\|^0 = \|Lx\|_G$ is equivalent to the norm $\|x\|_E$ (to see this apply the Continuous

Inverse Mapping Theorem) to the identity from $(E_0, \|\cdot\|_E)$ into $(E_0, \|\cdot\|^0)$. Moreover under the additional assumption that $B\colon E \to G$ is onto, the following problem

$$(B) \qquad \begin{cases} Lx = y \\ Bx = z\,, \end{cases}$$

has a unique solution for any couple $(y, z) \in F \times G$ and the norm in E defined by $\|x\|_+ = \|Bx\|_G + \|Lx\|_G$ is equivalent to the norm $\|x\|_E$. By interchanging roles between L and B we may observe that if $L\colon E \to F$ is onto then problem (B) is uniquely solvable for any couple $(y, z) \in F \times G$ if and only if the problem

$$(C) \qquad \begin{cases} Lx = 0 \\ Bx = z\,, \end{cases}$$

is uniquely solvable for any $z \in G$.

To illustrate the above considerations we give some examples of equivalent norms in $C^2[0, 1]$. Namely,

$$(1) \qquad \|x\|^1 = |x(0)| + |x(1)| + \|\ddot{x}\|_0$$

$$(2) \qquad \|x\|^2 = |x(0)| + |\dot{x}(0)| + \|\ddot{x}\|_0$$

$$(3) \qquad \|x\|^3 = |x(0)| + \left|\int_0^1 x(t)\, dt\right| + \|\ddot{x}\|_0\,.$$

To see this consider the following boundary value problems:

$$(1) \qquad \begin{cases} \ddot{x} = y \\ x(0) = a\,, \quad x(1) = b \end{cases}$$

$$(2) \qquad \begin{cases} \ddot{x} = y \\ x(0) = a\,, \quad \dot{x}(0) = b \end{cases}$$

$$(3) \qquad \begin{cases} \ddot{x} = y \\ x(0) = a\,, \quad \displaystyle\int_0^1 x(t)\, dt = b\,. \end{cases}$$

It is perhaps of interest to interpret classical existence results for ordinary differential equations in terms of 0-epi maps.

EXAMPLE 3.1. – Let $L\colon C^1[0, 1] \to C^0[0, 1] \times \boldsymbol{R}$ be the linear isomorphism $Lx(t) = \big(\dot{x}(t), x(0)\big)$ and let $h\colon [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ be a continuous map with bounded image. Define $M\colon C^1[0, 1] \to C^0[0, 1] \times \boldsymbol{R}$ by

$$M(x)(t) = \big(\dot{x}(t) - h(t, x(t)), x(0)\big)\,.$$

Let $\hat{k}\colon C^0[0,1]\to C^0[0,1]$ be the Nemytskij operator associated to h and $k\colon C^1[0,1]\to$ $\to C^0[0,1]\times\boldsymbol{R}$ be the compact continuous map

$$k(x) = (J\hat{k}(x), 0)$$

where J is the linear, compact inclusion of $C^1[0,1]$ into $C^0[0,1]$. Clearly $M = L - k$. Hence M is 0-epi since L is an isomorphism and k is a continuous, compact map with bounded image (see Corollary 2.2).

The role of $\boldsymbol{R}$ in this example is inessential in the sense that it can be replaced by $\boldsymbol{R}^n$ for any natural number $n\geqslant 1$. The other spaces and operators should be changed accordingly.

EXAMPLE 3.2. – Let $C_0^2[0,1]$ be the (closed) subspace of $C^2[0,1]$ of those functions $x(t)$ such that $x(0)=x(1)=0$. Define $f\colon C_0^2[0,1]\to C^0[0,1]$ by $f(x)(t)=$ $\ddot{x}(t)-x^3(t)$. The map f can be regarded as the sum of the linear isomorphism $D^2\colon C_0^2[0,1]\to C^0[0,1]$ defined by $D^2x=\ddot{x}$ and the (nonlinear) compact map g defined by $g(x)(t)=-(x(t))^3$. We shall show that f is 0-epi and such that $\|f(x)\|\to\infty$ as $\|x\|\to\infty$.

Assume that $x\in C^2[0,1]$ is a solution of the boundary value problem

$$\left\{\begin{array}{l}\ddot{x}=\lambda(x^3+y)\\ x(0)=x(1)=0\,,\end{array}\right.\tag{1}$$

where $\lambda\in[0,1]$ and $y\in C^0[0,1]$ are given.

Let $t_0\in(0,1)$ be such that $|x(t_0)|=\|x\|_0=\max\{|x(t)|\colon t\in[0,1]\}$. We have

$$\begin{aligned}0\geqslant\ddot{x}(t_0)\operatorname{sign}x(t_0)=\lambda(x^3(t_0)+y(t_0))\operatorname{sign}x(t_0)\geqslant\lambda(|x^3(t_0)|-|y(t_0)|)=\\ =\lambda(\|x\|_0^3-|y(t_0)|)\geqslant\lambda(\|x\|_0^3-\|y\|_0)\,.\end{aligned}$$

Thus, $\|x\|_0\leqslant\|y\|_0^{\frac{1}{3}}$ (observe that if $\lambda=0$, then $\|x\|_0=0$). Hence, the set $S=\{x\in$ $\in C_0^2[0,1]\colon D^2x+\lambda g(x)=0$ for some $\lambda\in[0,1]\}$ is the singleton $\{0\}$. Thus f is 0-epi and $\|D^2x\|\leqslant\|x\|^3+\|f(x)\|\leqslant 2\|f(x)\|$.

EXAMPLE 3.3. – Let $C_0^2[0,1]$ be as in Example 3.2. For any $x\in C_0^2[0,1]$ define $\|x\|=\|\ddot{x}\|_0$. This is a norm in $C_0^2[0,1]$ and it is equivalent to the norm (induced by $C^2[0,1]$) $\|x\|_2=\|x\|_0+\|\dot{x}\|_0+\|\ddot{x}\|_0$. Let $f\colon C_0^2[0,1]\to C^0[0,1]$ be defined by $f(x)=$ $=\ddot{x}-e^x$. It is easy to see that f is 0-epi. In fact $f=L+h$, where $L\colon C_0^2[0,1]\to$ $\to C^0[0,1]$ is the linear isomorphism $Lx=\ddot{x}$ and h is the compact map $h(x)=-e^x$ (using again the compactness of the inclusion $J\colon C^2[0,1]\to C^0[0,1]$). Therefore it is enough to show that $S=\{x\in C_0^2[0,1]\colon Lx=-\lambda h(x)$ for some $\lambda\in(0,1]\}$ is bounded. This is immediately verified since $\ddot{x}(t)=\lambda e^{x(t)}$ implies (recall that $x(0)=x(1)=0$) $x(t)\leqslant 0$. Therefore $0\leqslant\ddot{x}(t)\leqslant 1$ and $\|x\|=\|\ddot{x}\|_0\leqslant 1$.

The following ordinary differential equation of neutral type arises in quantum mechanics and has been pointed out to us by S. Paveri-Fontana (see problem (D) below).

Let $h\colon [0,1]\times \boldsymbol{R}^2 \to \boldsymbol{R}$ be continuous and such that

$$|h(t, r, s)| \leqslant a + b|r| + c|s|\,,$$

for some $a, b, c \geqslant 0$ and all $(t, r, s) \in [0,1]\times \boldsymbol{R}^2$. Let $\alpha\colon [0,1] \to \boldsymbol{R}$ be continuous and such that $0 \leqslant \alpha(t) \leqslant t$ for all $t \in [0,1]$. Find a C^1 function $x\colon [0,1] \to \boldsymbol{R}$ such that

$$(D) \qquad \begin{cases} \dot{x}(t) = \mu(t)\dot{x}(\alpha(t)) + h\big(t, x(t), x(\alpha(t))\big) \\ x(0) = d \end{cases}$$

where, $\mu\colon [0,1] \to \boldsymbol{R}$ is a given continuous function and $d \in R$.

Example 3.4 below gives a partial answer to this problem. Some preliminaries are needed.

DEFINITION 3.1. – Let $X \subset C^0[0,1]$. We say that a (not necessarily continuous) map $\varphi\colon X \to C^0[0,1]$ is *past-isotonic* if for any $\tau \in [0,1]$ and any pair of functions $x, y \in X$ such that $x(t) \leqslant y(t)$ for all $t \in [0,\tau]$ we have $\varphi(x)(\tau) \leqslant \varphi(y)(\tau)$.

As an example of past-isotonic map take the following. Let $\varphi\colon C^0[0,1] \to C^0[0,1]$ be defined by $\varphi(x)(t) = x(0) + \int_0^t g\big(s, x(s)\big)\, ds$. Then φ is past-isotonic provided that $g\colon [0,1]\times \boldsymbol{R} \to \boldsymbol{R}$ is continuous and non decreasing with respect to the second variable.

Observe also that the composition and the sum of two past-isotonic maps is past-isotonic.

The following result represents a generalization of the well-known Gronwall's Lemma.

LEMMA 3.1. – *Let $\varphi\colon X \to C^0[0,1]$ be past-isotonic. If $x, y \in X$ are such that $x(0) < y(0)$, $x(t) \leqslant \varphi(x)(t)$ and $y(t) > \varphi(y)(t)$ for all $t \in [0,1]$, then $x(t) < y(t)$ for $0 \leqslant t \leqslant 1$.*

PROOF. – Let $\tau = \sup\{s \in [0,1]\colon x(t) < y(t) \text{ for } 0 \leqslant t \leqslant s\}$. Clearly, the set $\{s \in [0,1]\colon x(t) < y(t) \text{ for } 0 \leqslant t \leqslant s\}$ is nonempty since x and y are continuous and $x(0) < y(0)$. Let us show that $\tau = 1$ and $x(1) < y(1)$. We have $x(\tau) \leqslant \varphi(x)(\tau) \leqslant \varphi(y)(\tau) < y(\tau)$. This implies $\tau = 1$, since otherwise we would have $x(\tau) = y(\tau)$.

Q.E.D.

COROLLARY 3.1. – (*Gronwall's Lemma*). *Let $x \in C^0[0,1]$ be such that $x(t) \leqslant a + b\int_0^t x(s)\, ds$, where $b \geqslant 0$. Then $x(t) \leqslant a \exp bt$, $t \in [0,1]$.*

PROOF. – Define the past-isotonic operator $\varphi\colon C^0[0,1]\to C^0[0,1]$ by $\varphi(x)(t) = a + b\int_0^t x(s)\,ds$. Now, given $\varepsilon > 0$, let $y_\varepsilon(t) = (a+\varepsilon)\exp bt$. Clearly,

$$y_\varepsilon(t) = \varepsilon + \varphi(y_\varepsilon)(t) > \varphi(y_\varepsilon)(t) \quad \text{and} \quad y_\varepsilon(0) > a \geqslant x(0)\,.$$

So, by Lemma 3.1 we get $x(t) < y_\varepsilon(t)$ for all $t\in[0,1]$. The arbitrarity of ε shows that $x(t)\leqslant a\exp bt$, $t\in[0,1]$. Q.E.D.

EXAMPLE 3.4. – Assume that in Problem (D) the function $\mu(t)$ satisfies the inequality $|\mu(t)|<1$ for every $t\in[0,1]$. Then the map $M\colon C^1[0,1]\to C^0[0,1]\times\boldsymbol{R}$ defined by

$$M(x)(t) = \Big(\dot{x}(t) - \mu(t)\dot{x}(\alpha(t)) - h\big(t, x(t), x(\alpha(t))\big), \quad x(0)\Big)$$

is 0-epi and such that $\|M(x)\|\to\infty$ as $\|x\|\to\infty$. In particular Problem (D) is solvable for any $d\in\boldsymbol{R}$.

Let $A\colon C^0[0,1]\to C^0[0,1]$ be the linear operator defined by $(Az)(t) = \mu(t)z(\alpha(t))$. It is not difficult to prove that A is bounded and $\|A\| = \max\{|\mu(t)|\colon t\in[0,1]\} = r<1$. This shows that $I-A$, I-the identity, is invertible and $(I-A)^{-1} = \sum_{n=0}^{+\infty} A^n$. Therefore, given $y\in C^0[0,1]$ the function

$$z(t) = \sum_{n=0}^{+\infty}\mu(t)\mu(\alpha(t))\mu\big(\alpha(\alpha(t))\big)\ldots\mu(\alpha^{n-1}(t))\,y(\alpha^n(t))\,.$$

$\big(\alpha^0(t) = t,\ \alpha^m(t) = \alpha(\alpha^{m-1}(t))\big)$ is the unique solution of the functional equation $z(t) - \mu(t)z(\alpha(t)) = y(t)$.

The linear operator $L\colon C^1[0,1]\to C^0[0,1]\times\boldsymbol{R}$ defined by $Lx = ((I-A)Dx, x(0))$, where $Dx = \dot{x}$, is an isomorphism since the problem

$$\begin{cases} \dot{x} = y \\ x(0) = d \end{cases}$$

has a unique solution in $C^1[0,1]$ for any $y\in C^0[0,1]$ and $d\in\boldsymbol{R}$.

Let $k\colon C^1[0,1]\to C^0[0,1]\times\boldsymbol{R}$ be defined by $k(x)(t) = \big(h(t, x(t), x(\alpha(t))), 0\big)$. Clearly, k is compact (since it can be thought as the composition of the compact inclusion $C^1[0,1]\to C^0[0,1]$ with a continuous map and $M = L - k$). M is 0-epi if the set $S = \{x\in C^1[0,1]\colon Lx = \lambda k(x)$, for some $\lambda\in[0,1]\}$ is bounded. Let $x\in S$. Then there exists $\lambda\in[0,1]$ such that

$$\begin{cases} \dot{x}(t) - \mu(t)\dot{x}(\alpha(t)) = \lambda h\big(t, x(t), x(\alpha(t))\big)\,. \\ x(0) = 0\,. \end{cases}$$

Hence

$$\begin{cases} \dot{x}(t) = \lambda \sum_{n=0}^{+\infty} \mu(t)\mu(\alpha(t))\mu\big(\alpha(\alpha(t))\big) \dots \mu(\alpha^{n-1}(t))\, h\big(\alpha^n(t), x(\alpha^n(t)), x(\alpha^{n+1}(t))\big) \\ x(0) = 0 \end{cases}$$

Thus,

$$|x(t)| \leqslant \sum_{n=0}^{+\infty} r^n \int_0^t \big(a + b|x(\alpha^n(s))| + c|x(\alpha^{n+1}(s))|\big)\, ds\,.$$

Now, the operator $\varphi\colon C^0[0,1] \to C^0[0,1]$ defined by

$$\varphi(z)(t) = \sum_{n=0}^{+\infty} r^n \int_0^t \big(a + bz(\alpha^n(s)) + cz(\alpha^{n+1}(s))\big)\, ds$$

is clearly past-isotonic. So, by Lemma 3.1, if $y \in C^0[0,1]$ is such that $y(0) > 0$ (recall that $|x(0)| = 0$ and $y(t) > \varphi(y)(t)$), then we obtain $|x(t)| < y(t)$. A suitable (to our purposes) y is the solution of the following integral equation

$$z(t) = \frac{a+\varepsilon}{1-r} + \frac{1}{1-r}\int_0^t (b+c)\, z(s)\, ds\,,$$

where $\varepsilon > 0$. I.e.,

$$y(t) = \frac{a+\varepsilon}{1-r} \exp \frac{b+c}{1-r} t\,.$$

In fact, since y is increasing and the sequence $\{\alpha^n(t)\}$ is non increasing (recall that $\alpha(t) \leqslant t$), then we have

$$\begin{aligned} y(t) &= \frac{1}{1-r}\left(a + \varepsilon + \int_0^t (b+c)\, y(s)\, ds\right) > \sum_{n=0}^{+\infty} r^n \int_0^t \hbar(a + by(s) + cy(s))\, ds \geqslant \\ &\sum_{n=0}^{+\infty} r^n \int_0^t \big(a + by(\alpha^n(s)) + cy(\alpha^{n+1}(s))\big)\, ds = \varphi(y)(t)\,, \qquad 0 \leqslant t \leqslant 1. \end{aligned}$$

Therefore, by the arbitrarity of ε, we obtain

$$|x(t)| \leqslant \frac{a}{1-r} \exp \frac{(b+c)\, t}{1-r}\,.$$

Thus

$$\|x\|_0 \leqslant \frac{a}{1-r} \exp \frac{b+c}{1-r}\,.$$

On the other hand

$$|\dot{x}(t)| = \Big|\lambda \sum_{n=0}^{+\infty} \mu(t)\mu(\alpha(t)) \dots \mu(\alpha^{n-1}(t))\, h\big(\alpha^n(t), x(\alpha^n(t)), x(\alpha^{n+1}(t))\big)\Big| \leqslant a + (b+c)\|x\|_0 .$$

Thus,

$$\|\dot{x}\|_0 \leqslant \frac{a}{1-r}\left(1 + \frac{b+c}{1-r} \exp \frac{b+c}{1-r}\right).$$

This proves that the set S is bounded (take in $C^1[0,1]$ the norm $\|x\| = \|\dot{x}\|_0 + |x(0)|$ and recall that if $x \in S$ then $x(0) = 0$).

It remains to show that $\|M(x)\| \to +\infty$ as $\|x\| \to +\infty$. Given $x \in C^0[0,1]$ let $Mx = (u, d)$. By slightly modifying the above argument one can show that $\|x\| = \|\dot{x}\|_0 + |d| \leqslant \gamma_1 + \gamma_2\|u\|_0 + \gamma_3|d|$, where $\gamma_1, \gamma_2, \gamma_3$ are suitable positive constants. Thus, $\|x\| \leqslant \gamma_1 + (\gamma_2 + \gamma_3)\|M(x)\|$.

EXAMPLE 3.5 (see [7]). – Let the boundary value problem be given

$$(1) \qquad \begin{cases} \dot{x} + f(t,x) = e(t) \equiv e(t+T) \\ x(0) = x(T) . \end{cases}$$

Assume that $f\colon \boldsymbol{R}\times\boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and periodic of period T with respect to t, $e(t)$ is continuous. We want to show that (1) has a solution provided that there exist positive constants α, K, M such that either one of the following conditions is satisfied

(j) $K\|x\|^{1+\alpha} \leqslant f(t,x)\cdot x, \quad \|x\| \geqslant M,\ t \in [0,T];$

(jj) $-K\|x\|^{1+\alpha} \geqslant f(t,x)\cdot x, \quad \|x\| \geqslant M,\ t \in [0,T].$

Let $E = \{x \in C^1_n[0,T]\colon x(0) = x(T)\}$, $F = C^0_n[0,T]$, $L\colon E \to F$ be defined by $Lx = \dot{x}$. It is easy to see that L is a Fredholm operator of index 0, $\operatorname{Ker} L$ is the n-dimensional space of constant functions and $\operatorname{Im} L$ coincides with the kernel of the linear continuous map $Q\colon F \to \boldsymbol{R}^n$ defined by

$$(2) \qquad Q(y) = \int_0^T y(t)\, dt$$

Let $\hat{h}\colon F \to F$ be defined by $\hat{h}(x)(t) = -f(t, x(t)) + e(t)$ and denote by h the composition of $\hat{h}$ with the compact inclusion $i\colon E \to F$.

On the basis of Theorem 2.7 we have to show that

(i) the set of $v \in \operatorname{Ker} L$ such that $h(v) \in \operatorname{Im} L$ is bounded;

(ii) $\deg(QhJ, 0) \neq 0$ where $J\colon \boldsymbol{R}^n \to E$ is the linear map $J(u) = u$, a constant function;

(iii) the set $S^+ = \{x \in E\colon Lx = \lambda h(x),\ \lambda \in (0,1]\}$ is bounded.

The first property is easily verified. In fact take $v \in \operatorname{Ker} L$. Then $h(v) \in \operatorname{Im} L$ implies

$$\int_0^T f(t, v)\, dt = \int_0^T e(t)\, dt \tag{3}$$

which implies

$$\int_0^T f(t, v) \cdot v\, dt = \int_0^T e(t) \cdot v\, dt\,. \tag{4}$$

Equality (4) is impossible in both cases (j) and (jj) if

$$\|v\| > N = \max\left\{M, \left(\frac{\int_0^T \|e(t)\|\, dt}{K}\right)^{1/\alpha}\right\}.$$

For the second property observe that if $\|u\| \geqslant N + 1$ then

$$QhJ(u) \neq \lambda QhJ(-u)\,, \qquad \lambda \in [0, 1] \tag{5}$$

since $QhJ(u) \cdot u$ is either positive (case j) or negative (case jj) for $\|u\| \geqslant N + 1$. Hence $\deg(QhJ, 0) \neq 0$. To verify the last property let $x(t)$ be a solution of

$$\begin{cases} \dot{x}(t) = -\lambda f(t, x(t)) + \lambda e(t) \\ x(0) = x(T) \end{cases}$$

and put $w(t) = x(t) \cdot x(t)$. There exists $t_0 \in [0, T]$ such that $w(t_0) = \|x(t_0)\|^2 = \|x\|_0^2$. Since $w(0) = w(T)$, $\dot{w}(0) = \dot{w}(T)$ we have $\dot{w}(t_0) = 2x(t_0) \cdot \dot{x}(t_0) = 0$. Hence

$$f(t_0, x(t_0)) \cdot x(t_0) = e(t_0) \cdot x(t_0)$$

which is impossible if

$$\|x(t_0)\| > \max\left\{M, \left(\frac{\|e\|}{K}\right)^{1/\alpha}\right\}.$$

We shall give now some applications of the theory of 0-epi maps to boundary values problems with parameters.

Let E and F be real Banach spaces and let $L\colon E \to F$ be a bounded linear injective operator with $\dim(F/\operatorname{Im} L) = p > 0$. Let $f\colon E \times \boldsymbol{R}^p \to F$ be compact. Consider the equation

$$Lx = f(x, \lambda)\,, \qquad x \in E,\ \lambda \in \boldsymbol{R}^p\,. \tag{1}$$

DEFINITION 3.2. – Any couple (x, λ) with $x \in E$, $\lambda \in \boldsymbol{R}^p$ satisfying (1) is said to be a *solution* of the equation (1). In this context we have the following result.

THEOREM 3.1. – *Let* $L\colon E \to F$ *and* $f\colon E \times \boldsymbol{R}^p \to F$ *be as above and let* $Q\colon F \to \boldsymbol{R}^p$ *be bounded linear and such that* $\operatorname{Ker} Q = \operatorname{Im} L$. *Assume that*

(i) *the set* $N = \{\lambda \in \boldsymbol{R}^p\colon Qf(0, \lambda) = 0\}$ *is bounded,*

(ii) *the set* $S^+ = \{(x, \lambda) \in E \times \boldsymbol{R}^p\colon Lx = \tau f(x, \lambda) \text{ for some } 0 < \tau \leq 1\}$ *is bounded,*

(iii) $\deg(Qf(0, \cdot), 0) \neq 0$, *where the map* $Qf(0, \cdot)\colon \boldsymbol{R}^p \to \boldsymbol{R}^p$ *is defined by* $\lambda \to Qf(0, \lambda)$.

Then the map $\hat{L} - f$ *is* 0*-epi, where* $\hat{L}\colon E \times \boldsymbol{R}^p \to F$ *is defined by* $\hat{L}(x, \lambda) = Lx$.

A more general version of Theorem 3.1 is the following.

THEOREM 3.2. – *Let* $L\colon E \to F$ *be a Fredholm operator with* $\operatorname{ind}(L) = -p$ $(p > 0)$ *and let* $f\colon E \times \boldsymbol{R}^p \to F$ *be compact. Assume that*

(i) *the set* $S^0 = \{(x, \lambda) \in E \times \boldsymbol{R}^p\colon x \in \operatorname{Ker} L,\ f(x, \lambda) \in \operatorname{Im} L\}$ *is bounded,*

(ii) *the set* $S^+ = \{(x, \lambda) \in E \times \boldsymbol{R}^p\colon Lx = \tau f(x, \lambda) \text{ for some } 0 < \tau \leq 1\}$ *is bounded,*

(iii) $\deg(QfJ, 0) \neq 0$, *where* $J\colon \boldsymbol{R}^{q+p} \to E \times \boldsymbol{R}^p$, $Q\colon F \to \boldsymbol{R}^{q+p}$ *are linear continuous and such that* $\operatorname{Im} J = \operatorname{Ker} L \times \boldsymbol{R}^p$, $\operatorname{Ker} Q = \operatorname{Im} L$ *and* $q = \dim \operatorname{Ker} L$.

Then the map $\hat{L} - f$ *is* 0*-epi, where* $\hat{L}\colon E \times \boldsymbol{R}^p \to F$ *is defined by* $\hat{L}(x, \lambda) = Lx$.

PROOF. – Observe that $\hat{L}$ is Fredholm of index zero and apply Theorem 2.7. Q.E.D.

EXAMPLE 3.6. – Consider the problem

$$(1) \qquad \begin{cases} \dot{x} = h(t, x) + \lambda g(t, x) \\ x(0) = x(1) = 0\,, \end{cases}$$

where $h, g\colon [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ are continuous. Assume that there exist positive constants ε, a, b, M such that $|h(t, x)| \leq M$ and $\varepsilon \leq |g(t, x)| \leq a + b|x|$ for all $(t, x) \in [0, 1] \times \boldsymbol{R}$. Then, there exists $x \in C^1[0, 1]$ and $\lambda \in \boldsymbol{R}$ such that (x, λ) is a solution of (1).

We will prove this fact applying Theorem 3.1. To this aim let E be the Banach space $\{x \in C^1[0, 1]\colon x(0) = x(1) = 0\}$ with norm $\|x\| = \|\dot{x}\|_0$ and let F be the Banach space $C^0[0, 1]$. The operator $L\colon E \to F$ defined by $Lx = \dot{x}$ is injective. The map $f\colon E \times \boldsymbol{R} \to F$ defined by $f(x, \lambda)(t) = h\big(t, x(t)\big) + \lambda g\big(t, x(t)\big)$ is compact. We have to find a linear operator $Q\colon F \to \boldsymbol{R}$ such that $\operatorname{Ker} Q = \operatorname{Im} L$. For this observe that given $y \in F$, the problem

$$(2) \qquad \begin{cases} \dot{x} = y \\ x(0) = x(1) = 0\,, \end{cases}$$

is solvable if and only if $\int_0^1 y(t)\, dt = 0$. Thus, Q can be defined by $Qy = \int_0^1 y(t)\, dt$.

Let us show that the set $N = \left\{\lambda \in \boldsymbol{R}\colon \int_0^1 h(t, 0)\, dt + \lambda \int_0^1 g(t, 0)\, dt = 0\right\}$ is bounded. This follows at once from the assumptions on h and g. Actually, if $\lambda \in N$ then $|\lambda| \leqslant M/\varepsilon$. We also have $\deg(Qf(0, \cdot), 0) \neq 0$ since $Qf(0, \lambda) = \int_0^1 h(t, 0)\, dt + \lambda \int_0^1 g(t, 0)\, dt$ is such that $\int_0^1 g(t, 0)\, dt \neq 0$.

It remains to show that the set S^+ is bounded. To this aim let $(x, \lambda) \in S^+$, i.e., (x, λ) is a solution of the following problem

$$\begin{cases} \dot{x} = \tau(h(t, x) + \lambda g(t, x)) \\ x(0) = x(1) = 0\,, \end{cases}$$

for some $0 < \tau \leqslant 1$. Clearly, since $\tau \neq 0$ we must have $\int_0^1 \big(h(t, x(t)) + \lambda g(t, x(t))\big)\, dt = 0$. By the assumptions on h and g it follows that $|\lambda| \leqslant M/\varepsilon$. Since $x(t) = \tau \int_0^t \big(h(s, x(s)) + \lambda g(s, x(s))\big)\, ds$ we have

$$|x(t)| \leqslant \tau M + \tau|\lambda| \int_0^t (a + b|x(s)|) ds \leqslant M + Ma/\varepsilon + (Mb/\varepsilon) \int_0^t |x(s)|\, ds\,.$$

By Gronwall's Lemma we get

$$|x(t)| \leqslant (M + Ma/\varepsilon) \exp(Mbt/\varepsilon) \leqslant (M + Ma/\varepsilon) \exp(Mb/\varepsilon) = k\,.$$

Since $\|x\| = \|\dot{x}\|_0$ we have to evaluate $|\dot{x}(t)|$. Now, $|\dot{x}(t)| \leqslant \tau\big(M + |\lambda|(a + b\|x\|_0)\big) \leqslant M + M/\varepsilon(a + bk) = \hat{k}$. Thus, if $(x, \lambda) \in S^+$, then $\|(x, \lambda)\| = \|\dot{x}\|_0 + |\lambda| \leqslant k + M/\varepsilon$.

Hence problem (1) is solvable.

EXAMPLE 3.7. – Consider the boundary value problem

$$\begin{cases} \ddot{x} = x^3 + \lambda + y \\ x(0) = x(1) = 0 \\ \int_0^1 x(t)\, dt = 0\,, \end{cases} \tag{3}$$

where $y \in C^0[0, 1]$ is given and $\lambda \in \boldsymbol{R}$ is a parameter.

We want to show that there exist $x \in C^2[0, 1]$ and $\lambda \in \boldsymbol{R}$ such that (x, λ) is a solution of (3). Let $E = \left\{(x, \lambda) \in C^2[0, 1] \times \boldsymbol{R}\colon x(0) = x(1) = 0,\ \int_0^1 x(t)\, dt = 0\right\}$ and let $F = C^0[0, 1]$. Define the linear operator $L\colon E \to F$ by $L(x, \lambda)(t) = \ddot{x}(t) - \lambda$. It is easy to see that L is an isomorphism. Define $h\colon E \to F$ by $h(x, \lambda)(t) = x^3(t)$, which is odd and compact. Therefore the map $L - h$ is 0-epi if $(L - h)^{-1}(0)$ is bounded. We shall prove, actually, that $\|Lx - h(x)\| \to +\infty$ as $\|x\| \to +\infty$ (i.e. the inverse image under $L-h$ of any bounded set is bounded) and therefore problem (3) is solvable for any $y \in C^0[0, 1]$.

Let (x, λ) be any solution of (3). Clearly, if $x \neq 0$, the condition $\int_0^1 x(t)\, dt = 0$ implies that there exist t_0 and t_1 belonging to $[0, 1]$ such that $x(t_0) = \max\{x(t): t \in [0, 1]\} > 0$ and $x(t_1) = \min\{x(t): t \in [0, 1]\} < 0$. Clearly,

$$0 \geqslant \ddot{x}(t_0) = x^3(t_0) + \lambda + y(t_0),$$

and

$$0 \leqslant \ddot{x}(t_1) = x^3(t_1) + \lambda + y(t_1).$$

Thus,

$$x^3(t_0) - x^3(t_1) \leqslant y(t_1) - y(t_0) \leqslant 2\|y\|_0 .$$

On the other hand either $x^3(t_0) = \|x\|_0^3$ or $x^3(t_1) = -\|x\|_0^3$ This implies that $\|x\|_0^3 \leqslant x^3(t_0) - x^3(t_1) \leqslant 2\|y\|_0$. Therefore, for any $(x, \lambda) \in E$ we get (by setting $y = Lx - h(x)$) $\|x\|_0^3 \leqslant 2\|Lx - h(x)\|_0$. Now, since L is an isomorphism we can take in E the following norm $\|(x, \lambda)\| = \|L(x, \lambda)\|_0 = \max\{|\ddot{x}(t) - \lambda|: t \in [0, 1]\}$. This norm, by the considerations made at the beginning of this section, is equivalent to the norm $\|(x, \lambda)\| = \|x\|_2 + |\lambda| = \|x\|_0 + \|\dot{x}\|_0 + \|\ddot{x}\|_0 + |\lambda|$. Since, $\ddot{x} - \lambda = x^3 + Lx - h(x)$, we get $|\ddot{x}(t) - \lambda| \leqslant |x(t)|^3 + \|Lx - h(x)\|_0 \leqslant \|x\|^3 + \|Lx - h(x)\|_0$. Thus $\|(x, \lambda)\| = \|\ddot{x} - \lambda\|_0 \leqslant 3\|Lx - h(x)\|_0$. This shows $\|Lx - h(x)\| \to +\infty$ as $\|x\| \to +\infty$.

We close this paper with an example of a nonlinear ordinary differential operator which is admissible but not 0-epi.

EXAMPLE 3.8. – Let $E = \{x \in C^1[0, 1]: x(0) = x(1)\}$. The map $f: E \to C^0[0, 1]$ defined by $f(x)(t) = \dot{x}(t) - x^2(t)$ is admissible (observe that $f(x) = 0$ if and only if $x = 0$) but it is not 0-epi since the boundary value problem

$$\begin{cases} \dot{x}(t) = x^2(t) + \lambda \\ x(0) = x(1) \end{cases}$$

has no solutions for $\lambda > 0$ (recall Theorem 1.5).

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