Two Linear Systems Criteria for Exponential Dichotomy (*).

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Summary. – *Two criteria, one postulating the existence of a not necessarily positive definite Lyapunov function and the other postulating conditional input-output stability, are given, ensuring that a system of linear differential equations has an exponential dichotomy.*

1. – Introduction.

Let $A(t)$ be a locally integrable $n \times n$ matrix function on $[0, \infty)$. Two criteria ensuring that the linear differential equation,

$$\dot{x} = A(t)x\,, \tag{1}$$

be *uniformly asymptotically stable* have been much investigated.

The first criterion postulates the existence of a *Lyapunov function* $x^* H(t) x$, where $H(t)$ is a bounded, continuously differentiable positive definite Hermitian matrix function satisfying

$$\dot{H}(t) + H(t)A(t) + A^*(t)H(t) \leqslant -I \tag{2}$$

(cf., for example, KRASOVSKII [6, p. 59] and Theorems 5 and 6 in BROCKETT [3, p. 202]). This has been generalized in two directions.

In [2] ANDERSON and MOORE have shown that if $A(t)$ is continuous and $C(t)$ is a continuous $l \times n$ matrix function such that $(A(t), C(t))$ is uniformly completely observable then the right hand side of (2) can be replaced by $-C^*(t)C(t)$. In [8] (see also COPPEL [4]) Massera and Schäffer have shown that if we drop the condition that $H(t)$ be positive definite and assume that (1) has bounded growth, then we get a criterion for *exponential dichotomy*. We prove a result which includes both these generalizations as special cases. Our result uses a *generalized observability condition*, introduced by MEGAN [9].

The second criterion postulates *bounded-input bounded-output stability*, i.e. if $u(t)$

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is a function in $\mathfrak{L}^\infty$ then *all* solutions of

$$\dot{x} = A(t)x + u(t) \tag{3}$$

are in $\mathfrak{L}^\infty$ (see PERRON [12]). This has also been generalized in two directions.

In [13] (see also Theorems 3 in [3, p. 197]) Silverman and Anderson have considered systems,

$$\dot{x} = A(t)x + B(t)u\,, \tag{4a}$$

$$y = C(t)x\,, \tag{4b}$$

where $B(t)$, $C(t)$ are $n\times m$, $l\times n$ matrix functions. They have shown that if $A(t)$, $B(t)$, $C(t)$ are bounded, $(A(t), B(t))$ is uniformly completely controllable and $(A(t), C(t))$ is uniformly completely observable then the bounded-input bounded-output stability of (4) is equivalent to the uniform asymptotic stability of (1). In [2] ANDERSON and MOORE weakened the boundedness conditions on $A(t)$, $B(t)$, $C(t)$ and replaced $\mathfrak{L}^\infty$ by the space $\mathcal{M}^2$ of piecewise continuous functions with uniformly bounded square norms on intervals of a fixed finite length. In [1] ANDERSON replaced $\mathcal{M}^2$ by $\mathfrak{L}^2$. In [9] and [10] MEGAN has considered $\mathfrak{L}^p$ inputs and $\mathfrak{L}^q$ outputs and used a generalized observability condition.

In their work [8] MASSERA and SCHÄFFER have considered a quite general class of function spaces. Let $\mathfrak{B}$, $\mathfrak{D}$ be two such function spaces. We say that the pair $(\mathfrak{B}, \mathfrak{D})$ is *admissible* for (3) if whenever $u(t)$ is in $\mathfrak{B}$ there exists *a* solution $x(t)$ of (3) in $\mathfrak{D}$. For appropriate choice of $(\mathfrak{B}, \mathfrak{D})$ this yields a criterion for (1) to have an exponential dichotomy.

We have considered pairs $(\mathfrak{L}^p, \mathcal{M}^q)$, $(\mathcal{M}^p, \mathcal{M}^q)$, $(\mathfrak{L}^p, \mathfrak{L}^q)$ $(1\leqslant p,\ q\leqslant\infty)$ and give criteria that (1) have an exponential dichotomy in terms of the admissibility of these pairs for (4), under the assumption that $(A(t), B(t))$ and $(A(t), C(t))$ satisfy generalized controllability and observability conditions. Our technique is to use a duality argument to reduce the problem to the case considered by Massera and Schäffer. We note that our results give a complete answer to the problem posed by Anderson at the end of [1].

2. – Preliminaries.

For the whole of this paper, $A(t)$, $B(t)$, $C(t)$ are locally integrable $n\times n$, $n\times m$, $l\times n$ matrix functions on $[0, \infty)$.

The system (1) is said to have *bounded growth* (resp. *decay*) if there are constants $M > 0$, $L\geqslant 0$ such that

$$|\varphi(t, \tau)| = |X(t)X^{-1}(\tau)| \leqslant M \exp\left[L|t-\tau|\right]$$

for $0 \leqslant \tau \leqslant t$ (resp. $0 \leqslant t \leqslant \tau$), where $X(t)$ is the fundamental matrix for (1) with $X(0) = I$. [$|\cdot|$ denotes the Euclidean norm when the argument is a vector in n-dimensional Euclidean space E^n and the corresponding operator norm when the argument is a matrix.]

(1) is said to have an *exponential dichotomy* if there exist constants $K > 0$, $\gamma > 0$ and a projection P (i.e. $P^2 = P$) such that

$$|X(t) P X^{-1}(\tau)| \leqslant K \exp[-\gamma(t-\tau)] \quad \text{for } 0 \leqslant \tau \leqslant t$$

and

$$|X(t)(I-P) X^{-1}(\tau)| \leqslant K \exp[-\gamma(\tau - t)] \quad \text{for } 0 \leqslant t \leqslant \tau .$$

If $1 \leqslant p \leqslant \infty$, we say that the pair $(A(t), B(t))$ is *p-uniformly controllable* (cf. [9, p. 126]) if for some $\delta > 0$ there exists $\varrho > 0$ such that for all $t \geqslant 0$ and all ξ in E^n,

$$\int_t^{t+\delta} |B^*(\tau)\varphi^*(t+\delta, \tau)\xi|^p d\tau \geqslant \varrho^{-p}|\xi|^p \qquad (\text{when } p < \infty),$$

$$\operatorname*{ess\,sup}_{t \leqslant \tau \leqslant t+\delta} |B^*(\tau)\varphi^*(t+\delta, \tau)\xi| \geqslant \varrho^{-1}|\xi| \qquad (\text{when } p = \infty),$$

where $*$ denotes the conjugate transpose.

If $1 \leqslant p < q \leqslant \infty$, Hölder's inequality shows that p-uniform controllability implies q-uniform controllability. On the other hand, when $B(t)$ is essentially bounded and (1) has bounded growth the inequality

$$\int_t^{t+\delta} |B^*(\tau)\varphi^*(t+\delta, \tau)\xi|^q d\tau \leqslant (NM \exp[L\delta]|\xi|)^{q-p} \int_t^{t+\delta} |B^*(\tau)\varphi^*(t+\delta, \tau)\xi|^p d\tau ,$$

where $N = \operatorname*{ess\,sup}_{t \geqslant 0} |B(t)|$, shows that q-uniform controllability implies p-uniform controllability for $1 \leqslant p < q < \infty$.

$(A(t), B(t))$ is defined to be *uniformly completely controllable* (cf. [2, p. 400]) if (1) has bounded growth and decay, $(A(t), B(t))$ is 2-uniformly controllable and there exists a constant $\alpha \geqslant 0$ such that

$$\xi^* \int_t^{t+\delta} \varphi(t+\delta, \tau) B(\tau) B^*(\tau) \varphi^*(t+\delta, \tau)\, d\tau \cdot \xi \leqslant \alpha |\xi|^2$$

for all $t \geqslant 0$ and ξ in E^n. From equation (10) in [2, p. 400] it is clear, using the bounded growth and decay, that this last condition can be replaced by $\sup_{t \geqslant 0} \int_t^{t+\delta} |B(\tau)|^2 d\tau < \infty$.

Finally, $(A(t), C(t))$ is said to be *p-uniformly observable* if $(A^*(t), C^*(t))$ is p-uniformly controllable.

3. – Lyapunov function criterion.

The following theorems generalize Theorem 5 in [2, p. 411] with the differences that we restrict ourselves to $[0, \infty)$ and the derivative of the Lyapunov function along a solution satisfies an inequality rather than an equality. They also generalize Propositions 1 and 2 in Lecture 7 of [4] and, in the finite-dimensional case, Theorems 92.*B* and 92.*A* in [8, pp. 324, 321].

A vector function on $[0, \infty)$ is said to be *absolutely continuous* if it is absolutely continuous on every compact subinterval.

THEOREM 3.1. – *If* (1) *has an exponential dichotomy and for some* p $(1 \leqslant p < \infty)$,

$$\sup_{t \geqslant 0} \int_t^{t+1} |C(\tau)|^p d\tau < \infty ,$$

there exists a continuous function $V\colon [0, \infty) \times E^n \to \mathbf{R}$ *with the following properties:*

(i) $V(t, \lambda x) = |\lambda|^p V(t, x)$ *for all* t, x *and real* λ;

(ii) *there exists* $\beta > 0$ *such that*

$$|V(t, x)| \leqslant \beta |x|^p \qquad \text{for all } t,\ x;$$

(iii) *if* $x(t)$ *is a solution of* (1), *then* $V(t, x(t))$ *is absolutely continuous and*

$$\frac{d}{dt} V\big(t, x(t)\big) \leqslant - |C(t) x(t)|^p \qquad \text{a.e.}.$$

We define

$$2^{1-p} V(t, x) = \int_t^\infty |C(\tau) X_1(\tau, t) x|^p d\tau - \int_0^t |C(\tau) X_2(\tau, t) x|^p d\tau ,$$

where

$$X_1(t, \tau) = X(t) P X^{-1}(\tau) , \qquad X_2(t, \tau) = X(t)(I - P) X^{-1}(\tau) .$$

Using Lemma 3.1 in Massera and Schäffer [7, p. 524],

$$2^{1-p} |V(t, x)| \leqslant K^p \Big[\int_t^\infty \exp[- p\gamma(\tau - t)] |C(\tau)|^p d\tau + \int_0^t \exp[- p\gamma(t - \tau)] |C(\tau)|^p d\tau \Big] |x|^p \leqslant$$
$$\leqslant 2^{1-p} \beta |x|^p ,$$

where

$$\beta = 2^p K^p (1 - \exp[-p\gamma])^{-1} \cdot \sup_{t \geq 0} \int_t^{t+1} |C(\tau)|^p d\tau .$$

Moreover, if $x(t) = X(t)x(0)$ is a solution of (1), then

$$2^{1-p} V(t, x(t)) = \int_t^{\infty} |C(\tau) X_1(\tau, \tau) x(\tau)|^p d\tau - \int_0^t |C(\tau) X_2(\tau, \tau) x(\tau)|^p d\tau$$

is absolutely continuous and for almost all t,

$$\frac{d}{dt} V(t, x(t)) = -2^{p-1}[|C(t)X_1(t, t)x(t)|^p + |C(t)X_2(t, t)x(t)|^p] \leq$$
$$\leq -[|C(t)X_1(t, t)x(t)| + |C(t)X_2(t, t)x(t)|]^p \leq -|C(t)x(t)|^p .$$

To prove a converse of this theorem we use the following lemma, which generalizes a result mentioned in the remarks at the end of Section 2 in PALMER [11].

LEMMA 3.1. – *Suppose* (1) *has bounded growth and there exists a continuous function* $V: [0, \infty) \times E^n \to \mathbf{R}$ *with the following properties*:

(i) $a(r) = \sup\{|V(t, x)|: t \geq 0, |x| \leq r\} < \infty$ *for all* $r > 0$;

(ii) *there exists* $\delta > 0$ *such that if* $x(t)$ *is a solution of* (1), $V(t, x(t))$ *is nonincreasing and for all* $t \geq 0$,

$$V(t + \delta, x(t + \delta)) - V(t, x(t)) \leq -b(|x(t + \delta)|) ,$$

where $b(r)$ *is a nonnegative nondecreasing function for* $r > 0$ *with* $b(r) > 0$ *if* r *is large enough.*

Then (1) *has an exponential dichotomy.*

Choose $\Delta > 0$ so that $b(\Delta) > 0$. Suppose $x(t)$ is a solution of (1) such that for some t_0,

$$V(t_0) = V(t_0, x(t_0)) < -a(\Delta) .$$

Then if $t \geq t_0$,

$$-a(|x(t)|) \leq V(t) \leq V(t_0) < -a(\Delta)$$

and so $|x(t)| > \Delta$ if $t \geq t_0$. Given $t \geq t_0$, there exists a positive integer m such that

$t_0 + m\delta \leqslant t < t_0 + (m+1)\delta$. Then

$$\begin{aligned} V(t) - V(t_0) &\leqslant V(t_0 + m\delta) - V(t_0) \\ &= \sum_{p=1}^{m} [V(t_0 + p\delta) - V(t_0 + (p-1)\delta)] \\ &\leqslant - \sum_{p=1}^{m} b(|x(t_0 + p\delta)|) \\ &\leqslant - mb(\Delta) \\ &\leqslant - [\delta^{-1}(t - t_0) - 1] b(\Delta) . \end{aligned}$$

So $V(t) \to -\infty$ as $t \to \infty$. Also $a(|x(t)|) \geqslant - V(t) \to \infty$ and hence $|x(t)| \to \infty$ as $t \to \infty$.

Now let $x(t)$ be a solution of (1) such that $|x(0)| = 1$ and $V(t, x(t)) \geqslant - a(\Delta)$ for all $t \geqslant 0$. Then if $\sigma > 0$, $\sigma x(t)$ is also a solution of (1) and $V(t) = V(t, \sigma x(t)) \geqslant - a(\Delta)$ for all $t \geqslant 0$. [Otherwise $\sigma|x(t)| \to \infty$ as $t \to \infty \Rightarrow |x(t)| \to \infty \Rightarrow V(t, x(t)) \to -\infty$.] For all positive integers m,

$$\begin{aligned} -a(\sigma) &\leqslant - V(0) \\ &\leqslant V(m\delta) - V(0) + a(\Delta) \\ &= \sum_{p=1}^{m} [V(p\delta) - V((p-1)\delta)] + a(\Delta) \\ &\leqslant - \sum_{p=1}^{m} b(\sigma|x(p\delta)|) + a(\Delta) . \end{aligned}$$

So $\sum_{p=1}^{\infty} b(\sigma|x(p\delta)|) < \infty$ and hence $\liminf_{p\to\infty} \sigma|x(p\delta)| \leqslant \Delta$ for all $\sigma > 0$. This implies that $\liminf_{t\to\infty} |x(t)| = 0$.

Let t_m be the least value such that $|x(t_m)| = \exp[-m]$. Then $0 = t_0 < t_1 < \dots$. There exists an integer p such that $t_m + p\delta \leqslant t_{m+1} < t_m + (p+1)\delta$. With $\sigma = \Delta \exp[m+1]$ and $V(t) = V(t, \sigma x(t))$,

$$\begin{aligned} -a(\Delta e) &\leqslant - V(t_m) \\ &\leqslant \sum_{q=1}^{p} [V(t_m + q\delta) - V(t_m + (q-1)\delta)] + a(\Delta) \\ &\leqslant - \sum_{q=1}^{p} b(\sigma|x(t_m + q\delta)|) + a(\Delta) \\ &\leqslant - pb(\sigma|x(t_{m+1})|) + a(\Delta) \\ &= - pb(\Delta) + a(\Delta) . \end{aligned}$$

Hence $t_{m+1}-t_m<(p+1)\delta\leqslant[b(\Delta)^{-1}(a(\Delta e)+a(\Delta))+1]\delta$ for all m. Using the bounded growth, it follows as in [4, p. 62] that there exist constants $K>0$ and $\gamma>0$, depending on M, L, δ and the functions a and b, such that

$$|x(t)|\leqslant K\exp[-\gamma(t-\tau)]|x(\tau)| \quad \text{if } 0\leqslant\tau\leqslant t. \tag{5}$$

Let U_1 be the subspace of E^n consisting of initial values of bounded solutions of (1) and let U_2 be any fixed subspace supplementary to U_1. Then, as in [4, p. 62], we can show that exists $T>0$ such that $V(T, X(T)\xi)<-a(1)$ if $\xi\in U_2$, $|\xi|=1$. This means that $|X(t)\xi|>1$ if $t\geqslant T$.

Consider a particular solution $x(t)=X(t)\xi$ with $\xi\in U_2$, $|\xi|=1$. Since $|x(t)|\to\infty$ there exists a greatest value t_m such that $|x(t_m)|=\exp[m]$. Then $0\leqslant t_0<t_1<\dots$ and $t_0\leqslant T$. Let p be an integer such that $t_m+p\delta\leqslant t_{m+1}<t_m+(p+1)\delta$. With $\sigma=\Delta\exp[-m]$ and $V(t)=V(t,\sigma x(t))$,

$$\begin{aligned}-a(\Delta e)-a(\Delta)&=-a(\sigma|x(t_{m+1})|)-a(\sigma|x(t_m)|)\\&\leqslant V(t_{m+1})-V(t_m)\\&\leqslant\sum_{q=1}^{p}[V(t_m+q\delta)-V(t_m+(q-1)\delta)]\\&\leqslant-\sum_{q=1}^{p}b(\sigma|x(t_m+q\delta)|)\\&\leqslant-pb(\sigma|x(t_m)|)\\&=-pb(\Delta)\,.\end{aligned}$$

So $t_{m+1}-t_m<(p+1)\delta\leqslant[b(\Delta)^{-1}(a(\Delta e)+a(\Delta))+1]\delta$. Using the bounded growth it follows that there exist constants $K>0$, $\gamma>0$ as before such that

$$|x(t)|\leqslant K\exp[-\gamma(\tau-t)]|x(\tau)| \quad \text{if } T\leqslant t\leqslant\tau. \tag{6}$$

(5) and [6), together with the bounded growth, imply that (1) has an exponential dichotomy.

We now state our converse of Theorem 3.1.

THEOREM 3.2. – *Suppose* (1) *has bounded growth and for some* $p, 1\leqslant p<\infty$, $(A(t), C(t))$ *is p-uniformly observable. If* $V\colon[0,\infty)\times E^n\to\mathbf{R}$ *is a continuous function such that*

(i) $\sup\{|V(t,x)|: t\geqslant 0,\ |x|\leqslant r\}<\infty$ *for all* $r>0$,

(ii) *for all solutions* $x(t)$ *of* (1), $V(t, x(t))$ *is absolutely continuous and for almost all* t,

$$\frac{d}{dt} V(t, x(t)) \leqslant - |C(t)x(t)|^p ,$$

then (1) *has an exponential dichotomy.*

The hypotheses imply that if $x(t)$ is a solution of (1) then $V(t, x(t))$ is nonincreasing and moreover,

$$\begin{aligned} V(t+\delta, x(t+\delta)) - V(t, x(t)) &\leqslant - \int_t^{t+\delta} |C(\tau)x(\tau)|^p d\tau \\ &= - \int_t^{t+\delta} |C(\tau)\varphi(\tau, t+\delta)x(t+\delta)|^p d\tau \\ &\leqslant - \varrho^{-p}|x(t+\delta)|^p . \end{aligned}$$

Hence the conditions of Lemma 3.1 are fulfilled with $b(r) = \varrho^{-p} r^p$.

4. – Admissibility criteria.

We introduce the Banach function spaces: $\mathfrak{L}^p$ $(1 \leqslant p < \infty)$, the real p-integrable functions f on $[0, \infty)$ with norm,

$$\|f\|_p = \left[\int_0^\infty |f(t)|^p dt\right]^{1/p} ;$$

$\mathfrak{L}^\infty$, the essentially bounded real measurable functions f with norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \geqslant 0} |f(t)| ;$$

$\mathcal{M}^p$ $(1 \leqslant p < \infty)$, the real measurable functions f such that for some $\delta > 0$,

$$\sup_{t \geqslant 0} \left[\int_t^{t+\delta} |f(\tau)|^p d\tau\right]^{1/p} < \infty ,$$

which we use as norm and denote by $|f|_p$. Finally we define $\mathcal{M}^\infty = \mathfrak{L}^\infty$ and $|\cdot|_\infty = \|\cdot\|_\infty$.

A typical one of our Banach spaces we denote as $\mathfrak{B}$ and its corresponding norm as $\|\cdot\|_{\mathfrak{B}}$. We also consider intersections $\mathfrak{B}_1 \cap \mathfrak{B}_2$ of two of our spaces with norm

$\|\cdot\|_{\mathcal{B}_1} + \|\cdot\|_{\mathcal{B}_2}$ and also *associate spaces* $\mathcal{B}'$ (*cf.* [8, p. 50]). A real measurable function f is in $\mathcal{B}'$ if

$$\sup\left\{\int_0^\infty |f(t)||g(t)|dt \colon g \in \mathcal{B},\ \|g\|_{\mathcal{B}} = 1\right\} < \infty .$$

With this as norm, $\mathcal{B}'$ is then a Banach space. In particular, $(\mathfrak{L}^p)' = \mathfrak{L}^{p'}$ for $1 \leqslant p \leqslant \infty$, where $p^{-1} + p'^{-1} = 1$.

If F is a finite-dimensional Banach space with norm $|\cdot|_F$ we denote by $\mathcal{B}(F)$ the vector space of measurable functions $f\colon [0, \infty) \to F$ such that $|f|_F\colon [0, \infty) \to \boldsymbol{R}$ is in $\mathcal{B}$. Clearly, $\mathcal{B}(F)$ is a Banach space with norm $\||f|_F\|_{\mathcal{B}}$, which we write without ambiguity as $\|f\|_{\mathcal{B}}$.

Let $\mathcal{B}$, $\mathfrak{D}$ be two of our Banach spaces. We say that the pair $(\mathcal{B}, \mathfrak{D})$ is *admissible* for the system (4) if for any input u in $\mathcal{B}(E^m)$ there is at least one output y in $\mathfrak{D}(E^l)$. Let $P\colon E^n \to E^n$ be a projection with range the following subspace,

$$\{\xi \in E^n\colon C(t)X(t)\xi \text{ is in } \mathfrak{D}(E^l)\} .$$

Then we define a linear mapping $\theta\colon \mathcal{B}(E^m) \to \mathfrak{D}(E^l)$ by

$$(\theta u)(t) = C(t)[\tilde{x}(t) - X(t)P\tilde{x}(0)] ,$$

where $\tilde{x}(t)$ is a solution of (4a) such that $C(t)\tilde{x}(t)$ is in $\mathfrak{D}(E^l)$. So $(\theta u)(t) = C(t)x(t)$ where $x(t)$ is the unique solution of (4a) such that $C(t)x(t)$ is in $\mathfrak{D}(E^l)$ and $Px(0) = 0$.

We now prove a useful little lemma.

LEMMA 4.1. – *Suppose* (1) *has bounded decay and that for some* r, $1 \leqslant r \leqslant \infty$, $(A(t), C(t))$ *is r-uniformly observable. If* $x(t)$ *is a solution of* (4a) *then for* $t \geqslant \delta$,

$$\varrho^{-1}|x(t)| \leqslant \left[\int_{t-\delta}^{t} |C(\tau)x(\tau)|^r d\tau\right]^{1/r} + M\exp[L\delta]\left[\int_{t-\delta}^{t} |C(\tau)|^r d\tau\right]^{1/r}\int_{t-\delta}^{t} |B(\tau)\,u(\tau)|d\tau \qquad (r < \infty)$$

$$\varrho^{-1}|x(t)| \leqslant \operatorname*{ess\,sup}_{t-\delta \leqslant \tau \leqslant t} |C(\tau)x(\tau)| + M\exp[L\delta] \operatorname*{ess\,sup}_{t-\delta \leqslant \tau \leqslant t} |C(\tau)|\int_{t-\delta}^{t} |B(\tau)u(\tau)|d\tau \qquad (r = \infty) .$$

In particular if $|C|_r < \infty$, $|B|_{r'} < \infty$ *and* $u \in \mathfrak{L}^r(E^m)$, $x(t)$ *is in* $\mathfrak{L}^r(E^n)$ *if and only if* $C(t)x(t)$ *is in* $\mathfrak{L}^r(E^l)$.

Suppose $r > 0$. For $t \geqslant \delta$,

$$\varrho^{-1}|x(t)| \leqslant \Big[\int_{t-\delta}^{t} |C(\tau)\varphi(\tau, t)x(t)|^r d\tau\Big]^{1/r}$$

$$= \Big[\int_{t-\delta}^{t} \Big|C(\tau)x(\tau) + \int_{\tau}^{t} C(\tau)\varphi(\tau, s)B(s)u(s)\,ds\Big|^r d\tau\Big]^{1/r}$$

by the variation of constants formula

$$\leqslant \Big[\int_{t-\delta}^{t} |C(\tau)x(\tau)|^r d\tau\Big]^{1/r} + \Big[\int_{t-\delta}^{t}\Big|\int_{\tau}^{t} C(\tau)\varphi(\tau, s)B(s)u(s)\,ds\Big|^r d\tau\Big]^{1/r}$$

by Minkowski's inequality.

If $r = \infty$, then for $t \geqslant \delta$,

$$\varrho^{-1}|x(t)| \leqslant \operatorname*{ess\,sup}_{t-\delta\leqslant\tau\leqslant t} |C(\tau)\varphi(\tau, t)x(t)| = \operatorname*{ess\,sup}_{t-\delta\leqslant\tau\leqslant t} \Big|C(\tau)x(\tau) + \int_{\tau}^{t} C(\tau)\varphi(\tau, s)B(s)u(s)\,ds\Big|.$$

The inequalities follow immediately.

Suppose now that $|C|_r < \infty$, $|B|_{r'} < \infty$ and u is in $\mathfrak{L}^r(E^m)$. Then if $C(t)x(t)$ is in $\mathfrak{L}^r(E^l)$ it is clear from the inequalities that $x(t)$ is in $\mathfrak{L}^r(E^n)$.

Suppose now that $x(t)$ is in $\mathfrak{L}^r(E^n)$. If $r = \infty$, then clearly $C(t)x(t)$ is in $\mathfrak{L}^\infty(E^l)$. Suppose $r < \infty$. Then

$$\int_0^\infty |C(t)x(t)|^r dt = \sum_{\nu=0}^{\infty} \int_{\nu\delta}^{(\nu+1)\delta} |C(t)x(t)|^r dt.$$

By the variation of constants formula and bounded decay,

$$|x(t)| \leqslant M \exp[L\delta]\Big[|x(s)| + |B|_{r'}\Big\{\int_t^s |u(\tau)|^r d\tau\Big\}^{1/r}\Big]$$

if $t \leqslant s \leqslant t + \delta$, and so with $N = 2^{r-1}M^r \exp[rL\delta]$,

$$|x(t)|^r \leqslant N\Big[|x(s)|^r + |B|_{r'}^r \int_t^s |u(\tau)|^r d\tau\Big].$$

Hence

$$\int_{\nu\delta}^{(\nu+1)\delta} |C(t)x(t)|^r dt \leqslant N|C|_r^r\Big[|x((\nu+1)\delta)|^r + |B|_{r'}^r \int_{\nu\delta}^{(\nu+1)\delta} |u(\tau)|^r d\tau\Big]$$

$$\leqslant N|C|_r^r\Big[N\Big(\delta^{-1}\int_{(\nu+1)\delta}^{(\nu+2)\delta} |x(s)|^r ds + |B|_{r'}^r \int_{(\nu+1)\delta}^{(\nu+2)\delta} |u(\tau)|^r d\tau\Big) + |B|_{r'}^r \int_{\nu\delta}^{(\nu+1)\delta} |u(\tau)|^r d\tau\Big]$$

and therefore

$$\int_0^\infty |C(t)x(t)|^r dt \leqslant N|C|_r^r\Big[N\delta^{-1}\int_1^\infty |x(t)|^r dt + (N+1)|B|_{r'}^r\int_0^\infty |u(\tau)|^r d\tau\Big] < \infty .$$

Lemma 4.1 enables us to show that the operator θ is bounded in certain circumstances.

LEMMA 4.2. – *Suppose that* (1) *has bounded decay and for some pair* (p, q), $1 < p$, $q \leqslant \infty$, $|B|_{p'} < \infty$, $|C|_q < \infty$ *and* $(A(t), C(t))$ *is q-uniformly observable. Then if* $\mathcal{B} = \mathfrak{L}^p$, $\mathfrak{D} = \mathfrak{L}^q$ *or* $\mathcal{M}^q$ *and* $(\mathcal{B}, \mathfrak{D})$ *is admissible for* (4), θ *is a bounded linear operator.*

If u is in $\mathcal{B}(E^m)$, $(\theta u)(t) = C(t)x(t)$ where $x(t)$ is the unique solution of (4a) such that $C(t)x(t)$ is in $\mathfrak{D}(E^l)$ and $Px(0) = 0$. By Lemma 4.1 with $r = q$,

$$\varrho^{-1}|x(t)| \leqslant \|\theta u\|_{\mathfrak{D}} + M\exp[L\delta]|C|_q|B|_{p'}\|u\|_{\mathcal{B}} \quad \text{if } t \geqslant \delta. \tag{7}$$

Suppose $u_\nu \to u$ is $\mathcal{B}(E^m)$ and $\theta u_\nu \to y$ in $\mathfrak{D}(E^l)$. We can write $(\theta u_\nu)(t) = C(t)x_\nu(t)$, as above. Applying (7) to $u = u_\nu - u_\mu$, it follows that $x_\nu(t)$ is uniformly convergent on $[\delta, \infty)$ to a function $x(t)$. Moreover, by Theorem 31.D in [8, p. 89], $x(t)$ is a solution of (4a) on $[\delta, \infty)$. We extend it to $[0, \infty)$ and the variation of constants formula then implies that $x_\nu(t) \to x(t)$ uniformly on $[0, \delta]$ also. In particular, $Px(0) = 0$.

Finally, for all $t \geqslant 0$,

$$\int_t^{t+\delta} |C(\tau)x(\tau) - y(\tau)|d\tau \leqslant \int_t^{t+\delta} |C(\tau)||x(\tau) - x_\nu(\tau)|d\tau + \int_t^{t+\delta} |(\theta u_\nu)(\tau) - y(\tau)|d\tau$$

$$\to 0 \quad \text{as } \nu \to \infty.$$

Hence $y(t) = C(t)x(t)$ a.e. and so $y = \theta u$. The conclusion of the lemma then follows from the closed graph theorem.

REMARKS. – (i) Note that if, under the conditions of Lemma 4.2, we just assume that for inputs in a dense subset of $\mathcal{B}(E^m)$ there is an output in $\mathfrak{D}(E^l)$ but in addition assume that the operator θ (now only defined on a dense subset of $\mathcal{B}(E^m)$) is bounded (as, for example, in [2]), then the admissibility of $(\mathcal{B}, \mathfrak{D})$ can be deduced by the method used in the proof of the lemma.

(ii) In [1], [2], [3], [9], [10], [13] it is assumed that for all inputs u in $\mathcal{B}(E^m)$ the zero-state output,

$$y(t) = C(t)\int_0^t X(t)X^{-1}(\tau)B(\tau)u(\tau)\,d\tau ,$$

is in $\mathfrak{D}(E^l)$. We show under the conditions of Lemma 4.2 and the additional condition that $(A(t), B(t))$ be p'-uniformly controllable that this implies *all* outputs are in $\mathfrak{D}(E^l)$, when the input u is in $\mathfrak{B}(E^m)$.

Let P and θ be as defined above and suppose u is in $\mathfrak{B}(E^m) = \mathfrak{L}^p(E^m)$. From the definition of θ, it is clear that $y(t) = (\theta u)(t)$. Also if $u(t) = 0$ when $t \geqslant \delta$, $(\theta u)(t) = C(t)x(t)$, where

$$x(t) = \int_0^t X(t)PX^{-1}(\tau)B(\tau)u(\tau)\,d\tau - \int_t^\infty X(t)(I-P)X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\,,$$

since $x(t)$ is a solution of (4a) such that $Px(0) = 0$ and $C(t)x(t)$ is in $\mathfrak{D}(E^l)$ $\left(x(t) = X(t)P\int_0^\delta X^{-1}(\tau)B(\tau)u(\tau)\,d\tau \text{ if } t \geqslant \delta \text{ and } |C|_q < \infty\right)$.

Equating $y(t)$ with $C(t)x(t)$, we get

$$C(t)X(t)(I-P)\int_0^\delta X^{-1}(\tau)B(\tau)u(\tau)\,d\tau = 0$$

for $t \geqslant \delta$. By the observability property of $(A(t), C(t))$, this means that

$$(I-P)\int_0^\delta X^{-1}(\tau)B(\tau)u(\tau)\,d\tau = 0\,. \tag{8}$$

Now, for $\xi \in E^n$, define

$$u(t) = \begin{cases} |B^*(t)X^{*-1}(t)(I-P^*)\xi|^\alpha B^*(t)X^{*-1}(t)(I-P^*)\xi & \text{if } 0 \leqslant t \leqslant \delta, \\ 0 \quad \text{if } t \geqslant \delta, \end{cases}$$

where α is $(p^{-1}p'-1)$ if $p > 1$ and 0 if $p = 1$. Then u is in $\mathfrak{L}^p(E^m)$ and so, by (8),

$$\xi^*(I-P)\int_0^\delta X^{-1}(\tau)B(\tau)u(\tau)\,d\tau = \int_0^\delta |B^*(\tau)X^{*-1}(\tau)(I-P^*)\xi|^{2+\alpha}\,d\tau = 0\,.$$

Using the controllability property of $(A(t), B(t))$, it follows that $(I-P^*)\xi = 0$ for all ξ and hence $P = I$, which is what we want.

We now prove a *duality* result.

LEMMA 4.3. – *Under the assumptions of Lemma* 4.2, $(\mathfrak{D}', \mathfrak{B}')$ *is admissible for the adjoint system,*

$$\dot{z} = -A^*(t)z - C^*(t)v\,, \tag{9a}$$

$$w = -B^*(t)z\,. \tag{9b}$$

Moreover when $\mathfrak{D} = \mathfrak{L}^q$, $(p, q) \neq (1, \infty)$ *and* $(A(t), B(t))$ *is* p'*-uniformly controllable, the associated projection can be taken as* $(I - P^*)$.

The proof is a modification of the proofs of Theorems 54.*E* and 53.*E* in [8, pp. 156, 152].

Let $v \in \mathfrak{D}'(E^l)$ be given and let $z(t)$ be the solution of (9*a*) such that

$$z(0) = \int_0^\infty P^* X^*(t)\, C^*(t)\, v(t)\, dt\,.$$

Suppose $u \in \mathfrak{B}(E^m)$ and $u(t) = 0$ if $t \geqslant s$. Then $(\theta u)(t) = C(t)x(t)$, where $x(t)$ is a solution of (4*a*) such that $Px(0) = 0$. Note that $(I - P)\, X^{-1}(s)\, x(s) = 0$ since $x_\infty(t) = X(t)\, X^{-1}(s)\, x(s)$ is a solution of (1) such that $C(t)\, x_\infty(t)$ is in $\mathfrak{D}(E^l)$ $(x_\infty(t) = x(t)$ for $t \geqslant s$ and $|C|_a < \infty)$.

Now

$$\int_0^s z^*(t)\, B(t)\, u(t)\, dt = z^*(s)\, x(s) - z^*(0)\, x(0) + \int_0^s v^*(t)\, C(t)\, x(t)\, dt$$
$$= z^*(s)\, x(s) + \int_0^s v^*(t)\, C(t)\, x(t)\, dt\,.$$

Since

$$\frac{dt}{d}\, z^*(t)\, x_\infty(t) = -\, v^*(t)\, C(t)\, x_\infty(t)\,,$$

we have

$$z^*(s)\, x(s) = z^*(0)\, X^{-1}(s)\, x(s) - \int_0^s v^*(t)\, C(t)\, x_\infty(t)\, dt$$
$$= \int_0^\infty v^*(t)\, C(t)\, X(t)\, P X^{-1}(s)\, x(s)\, dt - \int_0^s v^*(t)\, C(t)\, x_\infty(t)\, dt$$
$$= \int_0^\infty v^*(t)\, C(t)\, X(t)\, X^{-1}(s)\, x(s)\, dt - \int_0^s v^*(t)\, C(t)\, x_\infty(t)\, dt$$
$$= \int_s^\infty v^*(t)\, C(t)\, x(t)\, dt\,.$$

So

$$\int_0^s z^*(t)\, B(t)\, u(t)\, dt = \int_0^\infty v^*(t)\, C(t)\, x(t)\, dt$$

and hence, using Lemma 4.2 with $\|\theta\|$ as the operator norm of θ,

$$\left|\int_0^s z^*(t)\,B(t)\,u(t)\,dt\right| \leqslant \|\theta\|\,\|v\|_{\mathfrak{D}'}\|u\|_{\mathfrak{B}}\,.$$

Replacing u by

$$u_1(t) = \begin{cases} |u(t)|\,|B^*(t)z(t)|^{-1}B^*(t)z(t) & \text{if } B^*(t)z(t) \neq 0,\\ 0 \quad \text{otherwise}, & \end{cases}$$

we get

$$\int_0^\infty |B^*(t)z(t)|\,|u(t)|\,dt \leqslant \|\theta\|\,\|v\|_{\mathfrak{D}'}\|u\|_{\mathfrak{B}}\,. \tag{10}$$

Finally, if $u \in \mathfrak{B}(E^m)$ is arbitrary, then for all $s \geqslant 0$ with $\chi_{[0,s]}$ as the characteristic function of $[0, s]$,

$$\begin{aligned}\int_0^s |B^*(t)z(t)|\,|u(t)|\,dt &= \int_0^\infty |B^*(t)z(t)|\,|(\chi_{[0,s]}u)(t)|\,dt\\ &\leqslant \|\theta\|\,\|v\|_{\mathfrak{D}'}\|\chi_{[0,s]}u\|_{\mathfrak{B}} \quad \text{by (10)}\\ &\leqslant \|\theta\|\,\|v\|_{\mathfrak{D}'}\|u\|_{\mathfrak{B}}\end{aligned}$$

and so (10) holds for all u in $\mathfrak{B}(E^m)$. Hence $|B^*(t)z(t)|$ is in $\mathfrak{B}'$ and $B^*(t)z(t)$ is in $\mathfrak{B}'(E^m)$.

Now if we put $z(t) = X^{*-1}(t)(I - P^*)\xi$ in the above, we get

$$\int_0^s z^*(t)\,B(t)\,u(t)\,dt = -\,\xi^*x(0)$$

and hence, using (31.5) in [8, p. 87] and Lemma 4.1 with $r = q$,

$$\begin{aligned}\left|\int_0^s z^*(t)\,B(t)\,u(t)\,dt\right| &\leqslant |\xi|\left[|x(\delta)| + \int_0^\delta |B(t)\,u(t)|\,dt\right]\exp\left[\int_0^\delta |A(t)|\,dt\right]\\ &\leqslant |\xi|\exp\left[\int_0^\delta |A(t)|\,dt\right]\left[\varrho\|\theta\| + (\varrho M\exp[L\delta]|C|_q + 1)\,|B|_{p'}\right]\|u\|_{\mathfrak{B}}\,.\end{aligned}$$

It follows as above that $B^*(t)z(t)$ is in $\mathfrak{B}'(E^m) = \mathfrak{L}^{p'}(E^m)$ for all ξ.

Now suppose $\mathfrak{D} = \mathfrak{L}^q$, $(p, q) \neq (1, \infty)$ and $(A(t), B(t))$ is p'-uniformly controllable. Suppose there exists ξ such that $P^*\xi \neq 0$ and $B^*(t)\,X^{*-1}(t)\xi$ is in $\mathfrak{B}'(E^m) =$

$= \mathfrak{L}^{p'}(E^m)$. By the p'-uniform controllability of $(A(t), B(t))$, $z(t) = X^{*-1}(t)\xi$ is in $\mathfrak{L}^{p'}(E^n)$. Similarly, $x(t) = X(t)PP^*\xi$ is in $\mathfrak{L}^q(E^n)$ and also, $x^*(t)z(t) = |P^*\xi|^2$ for all t.

When $q < \infty$, we deduce that $|x(t)|^{-1} \leqslant |P^*\xi|^{-2}|z(t)|$ so that $|x(t)|^{-1}$ is in $\mathfrak{L}^{p'}(E^n)$. But, using the bounded decay of (1), we have

$$|x(t)|^{-1} \leqslant Me^L \int_{t-1}^{t} |x(\tau)|^{-1} d\tau$$

and hence $\inf_{t \geqslant 1} |x(t)| > 0$, contradicting $x \in \mathfrak{L}^q(E^n)$. When $q = \infty$ and $p > 1$, we get $|z(t)| \geqslant \|x\|_\infty^{-1} |P^*\xi|^2$, contradicting $z \in \mathfrak{L}^{p'}(E^n)$. Hence, under our additional conditions, we have proved that $B^*(t)X^{*-1}(t)\xi$ is in $\mathfrak{L}^{p'}(E^m)$ if and only if $P^*\xi = 0$.

We now use our lemmas to prove the following theorem.

THEOREM 4.1. – *Suppose that for some pair* (p, q), *where* $1 \leqslant p$, $q \leqslant \infty$ *but* $(p, q) \neq (1, \infty)$,

(i) (1) *has bounded growth and decay,*

(ii) $|B|_{p'} < \infty$ *and* $(A(t), B(t))$ *is* p'*-uniformly controllable* $(p' = p/(p-1))$,

(iii) $|C|_q < \infty$ *and* $(A(t), C(t))$ *is* q*-uniformly observable, and*

(iv) $(\mathfrak{L}^p, \mathcal{M}^q)$ *is admissible for* (4) *when* $p > 1$, $(\mathfrak{L}^1, \mathfrak{L}^q)$ *is admissible for* (4) *when* $p = 1$.

Then (1) *has an exponential dichotomy with projection* P *having the range*

$$\{\xi \in E^n : C(t)X(t)\xi \text{ is in } \mathfrak{D}(E^l)\},$$

where $\mathfrak{D} = \mathcal{M}^q$ *when* $p > 1$ *and* $\mathfrak{L}^q$ *when* $p = 1$.

Suppose, firstly, that $p > 1$. If u is in $\mathfrak{L}^p(E^m)$ and $x(t)$ is a solution of $(4a)$ such that $C(t)x(t)$ is in $\mathcal{M}^q(E^l)$, then it follows from Lemma 4.1 with $r = q$ that $x(t)$ is in $\mathfrak{L}^\infty(E^n)$. So $(\mathfrak{L}^p, \mathfrak{L}^\infty)$ is admissible for (4) with $l = n$, $C(t) \equiv I$ and the associated projection is P (as defined in the statement of the theorem) since $C(t)X(t)\xi$ is in $\mathcal{M}^q(E^l)$ if and only if $X(t)\xi$ is in $\mathfrak{L}^\infty(E^n)$. Lemma 4.3 then applies to show that $(\mathfrak{L}^1, \mathfrak{L}^{p'})$, and hence $(\mathfrak{L}^1 \cap \mathfrak{L}^{p'}, \mathfrak{L}^{p'})$, is admissible for the adjoint system (9) with $l = n$, $C(t) \equiv I$ and with associated projection $(I - P^*)$. By Lemma 4.1 with $r = p'$, $(\mathfrak{L}^1 \cap \mathfrak{L}^{p'}, \mathfrak{L}^{p'})$ must then be admissible for (9) with $l = m = n$, $C(t) \equiv B(t) \equiv I$ and with projection $(I - P^*)$. It follows from Theorem 64.*B* in [8, *p.* 189] (note that this theorem still holds if instead of assuming that $|A|_1 < \infty$ we only assume that (1) has bounded growth) that

$$\dot{z} = -A^*(t)z \tag{11}$$

has an exponential dichotomy with projection $(I - P^*)$ and hence that (1) has one with projection P.

Suppose now that $(\mathfrak{L}^1, \mathfrak{L}^q)$ is admissible for (4) with $1 \leq q < \infty$. By Lemma 4.3, $(\mathfrak{L}^{q'}, \mathfrak{L}^\infty)$ is admissible for the adjoint system (9) with projection $(I - P^*)$ and so, by the first case, (11) has an exponential dichotomy with projection $(I - P^*)$ and hence (1) has one with projection P.

We now prove a converse theorem.

THEOREM 4.2. – *Suppose* (1) *has an exponential dichotomy with projection P and for some pair (p, q), where $1 \leq p,\ q \leq \infty$, $|B|_{p'} < \infty$ and $|C|_q < \infty$. Then $(\mathcal{M}^p, \mathcal{M}^q)$ is admissible for* (4) *and if $p \leq q$, $(\mathfrak{L}^p, \mathfrak{L}^q)$ is admissible. Also, when $(A(t), C(t))$ is q-uniformly observable, the range of P is $\{\xi \in E^n\colon C(t)X(t)\xi$ is in $\mathfrak{D}(E^l)\}$ where $\mathfrak{D} = \mathcal{M}^q$ or $\mathfrak{L}^q$.*

Let u be in $\mathcal{M}^p(E^m)$. Then it follows from Lemma 3.1 in [7, p. 524] that

$$x(t) = \int_0^t X(t)PX^{-1}(\tau)B(\tau)u(\tau)\,d\tau - \int_t^\infty X(t)(I-P)X^{-1}(\tau)B(\tau)u(\tau)\,d\tau$$

is well-defined, is a solution of (4*a*) and for $t \geq 0$,

$$|x(t)| \leq 2K(1 - \exp[-\gamma])^{-1}|B|_{p'}|u|_p$$

(where we are using 1 instead of δ in $|\cdot|_{p'}, |\cdot|_p$). Hence x is in $\mathfrak{L}^\infty(E^n)$ and $y(t) = C(t)x(t)$ is in $\mathcal{M}^q(E^l)$.

Now suppose $1 \leq p < q < \infty$ and let u be in $\mathfrak{L}^p(E^m)$. Then $x(t)$, defined above, is a solution of (4*a*) and

$$|x(t)| \leq K\int_0^\infty \exp\big[-\gamma|t-\tau|\big]\,|B(\tau)|\,|u(\tau)|\,d\tau\,.$$

Following HARTMAN [5, p. 477], we estimate

$$\Big[\int_0^\infty \exp[-\gamma|t-\tau|]\,|B(\tau)|\,|u(\tau)|\,d\tau\Big]^q$$

$$= \Big[\int_0^\infty \{\exp[-\gamma\alpha|t-\tau|]\,|B(\tau)|\,|u(\tau)|^{1-(p/q)}\}\{\exp[-\gamma\beta|t-\tau|]\,|u(\tau)|^{p/q}\}\,d\tau\Big]^q$$

where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$

$$\leq \Big[\int_0^\infty \exp[-\gamma\alpha q|t-\tau|/(q-1)]\,|B(\tau)|^{q/(q-1)}|u(\tau)|^{(q-p)/(q-1)}\,d\tau\Big]^{q-1}\int_0^\infty \exp[-\gamma\beta q|t-\tau|]\,|u(\tau)|^p\,d\tau$$

$$\leq \Big[\int_0^\infty \exp[-\gamma\alpha p'|t-\tau|]\,|B(\tau)|^{p'}d\tau\Big]^{q(p-1)/p}\Big[\int_0^\infty |u(\tau)|^p d\tau\Big]^{(q-p)/p}\int_0^\infty \exp[-\gamma\beta q|t-\tau|]\,|u(\tau)|^p d\tau$$

if $p > 1$

and

$$\leqslant \operatorname*{ess\,sup}_{\tau\geqslant 0} |B(\tau)|^q\cdot\Big[\int_0^\infty |u(\tau)|\,d\tau\Big]^{q-1}\cdot\int_0^\infty \exp\left[-\gamma\beta q|t-\tau|\right]|u(\tau)|\,d\tau \qquad \text{if } p=1\,.$$

In either case and also for $q=p$ (by letting $q\to p$),

$$|x(t)|^q\leqslant c_p^q|B|_{p'}^q\,\|u\|_p^{q-p}\cdot\int_0^\infty \exp\left[-\gamma\beta q|t-\tau|\right]|u(\tau)|^p\,d\tau\,,$$

where $c_p=K2^{1/p'}(1-\exp[-\gamma\alpha p'])^{-1/p'}$.

Then for $1\leqslant p\leqslant q<\infty$,

$$\int_0^\infty |C(t)x(t)|^q\,dt\leqslant c_p^q|B|_{p'}^q\,\|u\|_p^{q-p}\int_0^\infty |C(t)|^q\int_0^\infty \exp\left[-\gamma\beta q|t-\tau|\right]|u(\tau)|^p\,d\tau\,dt$$

$$=c_p^q|B|_{p'}^q\,\|u\|_p^{q-p}\int_0^\infty |u(\tau)|^p\int_0^\infty \exp\left[-\gamma\beta q|t-\tau|\right]|C(t)|^q\,dt\,d\tau$$

$$\leqslant 2c_p^q(1-\exp[-\gamma\beta q])^{-1}(|B|_{p'}\,\|u\|_p\,|C|_q)^q<\infty\,.$$

Finally, if $(I-P)\xi=0$ then $|X(t)\xi|\leqslant K\exp[-\gamma t]|\xi|$ and it follows that $C(t)X(t)\xi$ is in $\mathfrak{L}^q(E^l)$. On the other hand when $(A(t), C(t))$ is q-uniformly observable, the fact that $C(t)X(t)\xi$ is in $\mathcal{M}^q(E^l)$ implies that $X(t)\xi$ is in $\mathfrak{L}^\infty(E^n)$ and hence $(I-P)\xi=0$. So in this case, the range of P consists exactly of those ξ such that $C(t)X(t)\xi$ is in $\mathfrak{D}^\infty(E^l)$ where $\mathfrak{D}$ is $\mathfrak{L}^q$ or $\mathcal{M}^q$.

Theorems 4.1 and 4.2 generalize Theorem 3 in [13, p. 125], Theorem 3 in [2, p. 408] and the Theorem in [1], with the difference that we are working on $[0,\infty)$. In the finite-dimensional case, they also generalize Theorem 4.5 in [10, p. 193] and Theorem 2.3 in [9, p. 129].

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