An Estimate for the Best Constant in a Sobolev Inequality Involving Three Integral Norms (*) (**).

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Summary. — Let $H^1(R^2)$ denote the Sobolev space of all real valued functions on R^2 which, together with their gradients, are square integrable. Let

$$\lambda \equiv \inf_{\varphi \in H^1(R^2), \varphi \neq 0} \iint_{R^2} |\nabla \varphi|^2 dx dy \left(\iint_{R^2} \varphi^2 dx dy / \iint_{R^2} \varphi^6 dx dy \right)^{1/2} \equiv \inf J_2(\varphi).$$

We show that some minimizing sequence has an H^1 weakly convergent subsequence and that if at least one weak limit is nonzero then there is a function $\varphi_0 \in H^1(R^2)$ such that $J_2(\varphi_0) = \lambda$ and that $\lambda \sim \pi^{4/3}$. We provide as asymptotic formula for φ_0 namely, we show $\varphi_0(r) \approx K_0(r) \cdot [K_0^4(r) + 1]^{-1/2}$ as $r \rightarrow +\infty$, where K_0 is the modified Bessel function of the second kind. An extension to the more general inequality

$$C = \inf \left(\int_{R^m} |\nabla \varphi|^p dx \right)^{\alpha/p} \left(\int_{R^m} |\varphi|^q dx \right)^{\beta/q} \left(\int_{R^m} |\varphi|^r dx \right)^{-1/r}$$

where $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $r^{-1} = \beta q^{-1} + \alpha(p^{-1} - m^{-1})$, $p, q, r > 1$ is briefly discussed.

Introduction.

Let R^m , $m = 1, 2, 3$ denote Euclidean m space. The following inequalities are known to hold:

$$(1.1) \quad J_1(f) \equiv \left(\int_{R^1} (f')^2 dx \right) \left(\int_{R^1} f^2 dx \right)^2 \left(\int_{R^1} f^6 dx \right)^{-1} \geq \pi^2/4$$

for $f, f' \in L^2(R^1)$ and equality holding if and only if $f(x) = \alpha f_0(\beta x + \gamma)$, $\alpha\beta \neq 0$, $f_0(x) = 3^{1/4} [\operatorname{sech}(2x)]^{1/2}$. See [17].

$$(1.2) \quad J_3(\varphi) = \left(\iiint_{R^3} |\nabla \varphi|^2 dx dy dz \right) \left(\iiint_{R^3} \varphi^6 dx dy dz \right)^{-1/3} \geq 3(\pi/2)^{4/3}$$

for $\nabla \varphi = (\varphi_x, \varphi_y, \varphi_z) \in L^2(R^3)$ with equality holding when and only when

$$\varphi(x, y, z) = (A^2 + B^2(x^2 + y^2 + z^2))^{-1/2}.$$

(*) This work was supported in part by NSF grant MCS 78-02729.

(**) Entrata in Redazione il 9 gennaio 1979.

This result is a special case of a more general result due to BLISS [2], an excellent discussion of which has been given by TALENTI [18]. There the inequality (for $\nabla u \in L^p(R^m)$, $m = 2, 3, 4, \dots$ and $1 < p < m$)

$$(1.4) \quad C_{m,p} \leq \left(\int_{R^m} |\nabla u|^p dx \right)^{1/p} \left(\int_{R^m} |u|^q dx \right)^{-1/q} \quad (q^{-1} = p^{-1} - m^{-1}) \equiv J_{m,p}(u)$$

is discussed where the best (largest) constant is given by

$$C_{m,p} = \pi^{1/2} m^{1/2} ((m-p)/(p-1))^{1-1/p} \left\{ \frac{\Gamma(m/p) \Gamma(m+1-m/p)}{\Gamma(m) \Gamma(1+m/2)} \right\}^{1/m}$$

which is taken if and only if

$$u(x) = [a^2 + b^2 |x|^{p/(p-1)}]^{1-m/p} \quad (ab \neq 0).$$

If in (1.4) we make the substitution $u = v^s$, $s > 1$, and apply Hölder's inequality to the numerator, we obtain

$$(1.5) \quad (C_{m,p}/s)^{1/s} \leq \left(\int_{R^m} |\nabla v|^P dx \right)^{\alpha/P} \left(\int_{R^m} |v|^Q dx \right)^{\beta/Q} \left(\int_{R^m} |v|^R dx \right)^{-1/R} \equiv J_{m,P,Q,\alpha}(v)$$

where $\alpha = 1/s$, $R = sq$, $P = p\tau$, $Q = p(s-1)\sigma$, $1/\sigma + 1/\tau = 1$ and

$$(1.5)' \quad \frac{1}{R} = \frac{\beta}{Q} + \alpha \left(\frac{1}{P} - \frac{1}{m} \right) \quad \alpha + \beta = 1.$$

Also, in view of the condition $1 < p < m$, P, Q, α, β satisfy

$$(1.5)'' \quad 1 < \alpha PQ / (\alpha Q + \beta P) < m.$$

(See also FRIEDMAN [5]). In general $(C_{m,p}/s)^{1/s}$ is not the best constant for (1.5). It is, however, a lower bound for

$$\lambda_{m,P,Q,\alpha} \equiv \inf J_{m,P,Q,\alpha}(u)$$

where the infimum is taken over those u 's with $u \in L^q(R^m)$, $|\nabla u| \in L^P(R^m)$. It is not known that $\lambda_{m,P,Q,\alpha}$ is attained nor what its value is as a function of the parameters m, P, Q, α .

Moreover, the preceding infimum is known to be positive even if (1.5)'' fails but $\alpha, \beta > 0$, $P, Q, R > 0$ and (1.5)' holds. See [5]. An example is provided later. In this case a crude lower bound can be found by using the ideas found in [5], for example, to establish that $\lambda_{m,P,Q,\alpha} > 0$.

As an example of (1.5) to whose study we shall devote most of the remainder of this paper, consider the case $m = 2$, $q = 4$, $p = 4/3$ in (1.4). There results

$$2^{-1/2}\pi^{3/4} \leq \left(\iint_{R^2} |\nabla u|^{4/3} dx dy \right)^{3/4} \left(\iint_{R^2} |u|^4 dx dy \right)^{1/4}.$$

Using the substitution $u = v^{3/2}$ and following the same procedure that led to (1.5) we find that

$$(1.6) \quad C \equiv \inf J_2(v) \equiv \inf \iint_{R^2} |\nabla v|^2 dx dy \left(\iint_{R^2} v^2 dx dy / \iint_{R^2} v^6 dx dy \right)^{1/2}$$

satisfies

$$C \geq C_1 = 2\pi^{3/2}/9 \simeq 1.237.$$

Now C_1 cannot possibly be the best constant. In fact starting with the sharp inequality of FEDERER, FLEMING and RICHEL [3, 4], namely

$$(FFR) \quad 4\pi \leq \left(\iint_{R^2} |\nabla \varphi| dx dy \right)^2 \left(\iint_{R^2} \varphi^2 dx dy \right)^{-1}$$

setting successively $\varphi = v^2$ and $\varphi = v^3$ and eliminating $\iint_{R^2} v^4 dx dy$ between the resulting inequalities, we find

$$\frac{2\pi}{3} \equiv C_2 \leq J_2(v)$$

where $C_2 \simeq 2.0944 \dots$. By modifying a simple argument of ROSEN [11], one can easily show that

$$J_2(v) > \pi$$

for all $v \in H^1(R^2)$. To see this, and at the same time to see the intimate connection among (1.1), (1.2), and (1.6), consider those elements φ with $\nabla \varphi \in L^2(R^3)$ of the form

$$(1.7) \quad \varphi(x, y, z) = \psi(x, y) f(z).$$

Now $\nabla \varphi \in L^2(R^3)$ if and only if f and ψ are in $H^1(R^1)$ and $H^1(R^2)$ respectively since

$$(1.7.1) \quad \iiint_{R^3} |\nabla \varphi|^2 dx dy dz = \int_{R^1} f^2 dz \iint_{R^2} |\nabla \varphi|^2 dx dy + \int_{R^1} |f'|^2 dz \iint_{R^2} |\psi|^2 dx dy.$$

By replacing $f(z)$ by $f(\lambda z)$ in (1.7) and (1.7.1) for $\lambda > 0$ and introducing the resulting

expressions for φ and $\nabla\varphi$ into (1.2) one obtains the inequality ($\|\cdot\|$ denotes L^2 norm)

$$\lambda^{-2/3}\|f\|^2\|\nabla\psi\|^2 + \lambda^{4/3}\|f'\|^2\|\psi\|^2 > 3(\pi/2)^{4/3}\|f^3\|^{2/3}\|\psi^3\|^{2/3}.$$

If one minimizes the lefthand side of this latter inequality over λ , one finds that

$$J_1(f)[J_2(\psi)]^2 > \pi^4/4.$$

(The inequality is strict because no function of the form (1.7) can optimize (1.2).) The choice $f(z) = 3^{1/4}(\operatorname{sech} 2z)^{1/2}$ then leads to the estimate claimed. (Rosen obtained the weaker result $J_2(\psi) \geq 2.992 \dots$)

Since $\inf J_1 = 1 \cdot (\pi/2)^2$ and $\inf J_3 = 3(\pi/2)^{4/3}$ it is hard to resist the temptation to conjecture that $\inf J_2 = 2(\pi/2)^{3/2}$. This, as we shall see, is very likely not the case.

The plan of the paper is as follows. In the next section we demonstrate the existence of an element $\psi_0 \in H^1(R^2) = \{\psi \mid \psi, \nabla\psi \in L^2(R^2)\}$ which optimizes (1.6). Although we use direct methods of the calculus of variations for this, standard arguments must be modified to take into account not only the unbounded region of integration but also the fact that J_2 involves three integrals in its definition. The direct method has been used by Rodemich to give a proof of (1.4) but the result does not seem to have appeared in the open literature. The method of Bliss-Talenti does not seem to work for the problem (1.6) because their method seems to depend on the explicit exhibition of a two parameter family of extremals. As we mention below extremals seem to be very difficult to find for this problem. It is also unlikely that Nagy's methods will work here since (1.6) is obtained from a double application of Schwarz's inequality in the (FFR) inequality.

The application of the direct method to (1.6) seems to require an additional hypothesis concerning the minimizing sequence. We require the existence of a minimizing sequence $\{v_n\}_{n=1}^\infty (J(v_n) \rightarrow C)$ with the additional restriction that the sequence has a subsequence with a nonzero weak limit. We are unable to show that some weak limit must be nonzero. We suspect that this is the case, however, because the weak limit may be taken to be the characteristic function of $(0, 0)$ i.e., if v is the weak limit, then either $\|v\|_{H^1} \neq 0$ or $v(x, y) = 0$, $(x, y) \neq (0, 0)$ and $v(0, 0) = 1$. We evaluated $\lim_{\tau \rightarrow 0} J_2(v_n)$ for various sequences $\{v_n\}_{n=1}^\infty$ which converged pointwise to v and were unable to obtain a better upper bound for C than that obtained by evaluating J_2 on the asymptotic solution (3.5) of the Euler equation (1.9). We remark that the (FFR) inequality is sharp and that this is known by evaluating the right hand side on sequence of piecewise linear (in the radial variable) approximations to the characteristic function of the unit disk.

In the third section we record an asymptotic expansion for ψ_0 . It is developed using a technique we have not seen before but which is rather a natural one to use.

Finally, we present some numerical computations using the asymptotic expansion and the shooting method on the variational equation that show that

$$J_2(\psi_0) < 4.6016$$

and

$$J_2(\psi_0) \simeq 4.5981.$$

The conjecture is that $J_2(\psi_0) = \pi^{4/3} \cong 4.60116 \dots$

It will be clear from the context that our techniques will extend to other cases of (1.5). However, we prefer to limit our discussion to (1.6) not only for simplicity and because there is some physical interest in (1.6) [8, 11] but also because this is the case for which we have made numerical calculations and have an asymptotic solution of the variational equation (1.9) below.

As we remarked above, the extremals for (1.6) seem impossible to calculate in closed form. To see why this is so, we record the variational equation for (1.2), (1.6) and (1.1) below:

$$(1.8a) \quad \varphi''(\varrho) + 2\varrho^{-1}\varphi'(\varrho) + \varphi^5(\varrho) = 0 \quad 0 < \varrho < +\infty$$

$$(1.8b) \quad \varphi'(0) = 0, \quad \varphi(+\infty) = 0$$

$$(1.9a) \quad \psi''(r) + r^{-1}\psi'(r) + \psi^5(r) - \psi(r) = 0 \quad 0 < r < +\infty$$

$$(1.9b) \quad \psi'(0) = 0, \quad \psi(+\infty) = 0$$

and

$$(1.10a) \quad f''(x) + f^5(x) - f(x) = 0 \quad 0 < x < \infty$$

$$(1.10b) \quad f'(0) = 0, \quad f(+\infty) = 0.$$

Now (1.10) can be solved by direct quadrature. Equation (1.8) can also be solved by quadrature after making the preliminary change of variables $\varphi = \varrho^{-1/2}v$. See [2, 18] and KAMKE [7]. However (1.9) contains the troublesome terms of both equations' namely $r^{-1}\psi'(r)$ (nonconstant coefficient missing in (1.10)) and an inhomogenous undifferentiated term $\psi^5 - \psi$ (missing in 1.8). Thus (1.9) seems unwilling to yield up its monotone decreasing solutions very readily.

Before closing this section, we wish to make two small observations concerning (1.5), (1.5)', and the (FFR) inequality. The first of these is as follows: If we put $\varphi = \psi^2$ in the (FFR) inequality we find that

$$(1.11) \quad \pi < \left(\iint_{R^2} \psi^2 dx dy \right) \left(\iint_{R^2} |\nabla \psi|^2 dx dy \right) \left(\iint_{R^2} \psi^4 dx dy \right)^{-1}.$$

PAYNE [8] and SERRIN [13] independently gave 2 as a lower bound for the right hand side of this inequality. Secondly, although (1.5) and (1.5)' seem to be less general than (1.4) in the sense that (1.4) yields (1.5), we find that (1.5) and (1.5)' are indeed more general than (1.4). That is, if we set $R = 1/4$, $\alpha/P = \beta/Q = 1/4$ and $P = 2$, we find for $m = 2$ that $\alpha = \beta = 1/2$, $s = 2$ and $q = 2$ which means $p = 1$ in (1.4). This (1.11) is only obtainable from a limiting form of (1.4) although it is a special case of (1.5), (1.5)' *without* the identifications made in passing from (1.4) to (1.5).

2. — Existence of a minimizing function.

In order to demonstrate the existence of a minimizing function we will use, for the most part, well known techniques. However, where arguments must be modified we shall go into greater detail.

THEOREM 2.1. — There exists $\psi \in H^1(R^2)$ with the following properties:

- (i) ψ is positive, radially symmetric, monotone decreasing in the radial variable, and continuous together with its first derivative at $r = 0$;
- (ii) ψ is the weak limit in $H^1(R^2)$ and the strong limit in L_x , $p = 2.6$ of a minimizing sequence $\{\psi_n\}_{n=1}^\infty$ for J_2 ;
- (iii) if $\psi \neq 0$, then ψ minimizes J_2 , i.e. $J_2(\psi) \leq J_2(\varphi)$ for all $\varphi \in H^1(R^2)$, $\varphi \neq 0$; ψ is in $C^2(0, \infty)$ and satisfies (1.9a, b) together with the normalizations

$$\int_0^\infty (\psi')^2 dr = 2 \int_0^\infty \psi^2 r dr = \frac{2}{3} \int_0^\infty \psi^6 r ds.$$

PROOF. — We first prove (ii). Since the continuously differentiable functions with compact support ($C_0^1(R^2)$) are dense in $H^1(R^2)$, we have

$$C = \inf \{J_2(\psi) | \psi \in C_0^1(R^2), \psi \neq 0\}.$$

From the rearrangement theorem (Polya-Szego principle, see [6, 10, 14, 15]), if $\psi \in C_0^1(R^2)$ and $\psi > 0$, and ψ^* denotes the monotone decreasing, radially symmetric (about $(0, 0)$) rearrangement of ψ , then $\psi^* \in C_0^1(R^2)$ and $\iint_{R^2} (\psi^*)^p dx dy = \iint_{R^2} \psi^p dx dy$, $\iint_{R^2} |\nabla \psi^*|^p dx dy \leq \iint_{R^2} |\nabla \psi|^p dx dy$ ($p \geq 1$). Therefore $J_2(\psi^*) \leq J_2(\psi)$. Thus,

$$C = \inf \{J_2(\psi) | \psi \in C_0^1(R^2), \psi > 0, \psi = \psi(r) \text{ and } \psi'(r) \leq 0\}.$$

We observe that if α, β are constants and $\alpha\beta \neq 0$, then

$$\psi_1(x, y) \equiv \alpha\psi(\beta x, \beta y)$$

implies that

$$J_2(\psi_1) = J_2(\psi).$$

We may thus further characterize our minimum problem as

$$(2.2) \quad C = \inf \{J_2(\psi) | \psi = \psi(r), \psi'(r) \leq 0, \psi(r) \geq 0 \text{ and } \psi(r) = 0 \text{ if } r \geq 1\}.$$

For ψ 's in this latter class, after the change of variables $s = r^2$, we may write

$$J_2(\psi) = 4\pi \int_0^1 (\psi'(s))^2 s ds \left(\int_0^1 \psi^2(s) ds \int_0^1 \psi^6(s) ds \right)^{1/2}.$$

The recharacterization in (2.2) involved only scaling on the independent variable. We may do a scaling on the function itself to insure that

$$\| \psi \|_1 = \left(\int_0^1 (\psi'(s))^2 s ds \right)^{1/2} + \left(\int_0^1 \psi^2(s) ds \right)^{1/2} = 1.$$

Thus we may recharacterize C as

$$(2.2)' \quad C = \inf \{ J_2(\psi) | \psi \in C_0^1(R^2), \psi = \psi(r), \psi(r) = 0 \text{ if } r \geq 1, \psi'(r) \leq 0 \text{ and } \|\psi\| = 1 \}.$$

Now let $\{\psi_n\}_{n=1}^\infty$ be a minimizing sequence in this latter class so that $J_2(\psi_n) \rightarrow C$, $\|\psi\|_1 = 1$. It follows from Schwarz's inequality that if $x, y \in (0, 1]$,

$$\begin{aligned} |\psi_n(x) - \psi_n(y)| &\leq |\ln(x/y)|^{1/2} \left(\int_0^1 (\psi_n'(s))^2 s ds \right)^{1/2} \\ &\leq |\ln(x/y)|^{1/2} \|\psi_n\|_1 \end{aligned}$$

so that

$$(2.4) \quad |\psi_n(x) - \psi_n(y)| < |\ln(y/x)|^{1/2} \quad 0 < x, y < 1$$

and in particular

$$(2.5) \quad |\psi_n(x)| < |\ln x|^{1/2} \quad 0 < x \leq 1.$$

It follows from (2.4) and (2.5) that on every compact subset of $(0, 1]$, the minimizing sequence is uniformly bounded and equicontinuous. It is then an easy application of the Arzelà-Ascoli theorem to find a subsequence of the minimizing sequence, we also call $\{\psi_n\}_{n=1}^\infty$, that converges pointwise on $(0, 1]$. If ψ denotes this pointwise limit, then on $(0, 1]$, ψ satisfies (2.4) and (2.5) also. Since $|\ln x|^{1/2}$ is in $L^p((0, 1), dx)$ for every $p \geq 1$, it follows from the dominated convergence theorem that

$$\int_0^1 |\psi_n - \psi|^p dx \rightarrow 0$$

for $p = 2, 6$ in particular. Consequently

$$(2.6) \quad \int_0^1 |\psi_n|^p dx \rightarrow \int_0^1 |\psi|^p dx \quad p = 2, 6.$$

We now show that ψ has a weak derivative. Let $H(0, 1)$ denote the completion of the set of continuously differentiable real valued functions on $[0, 1]$ vanishing at the right endpoint with respect to the norm $\|\cdot\|_1$. Let $L_s^2(0, 1)$ denote the com-

pletion of this set with respect to the norm

$$\|\varphi\|_0 \equiv \left(\int_0^1 \varphi^2(s) s \, ds \right)^{\frac{1}{2}}.$$

On $L_s^2(0, 1)$ define linear functionals

$$F_n(\varphi) = \int_0^1 \psi'_n(s) \varphi(s) s \, ds.$$

From Schwarz's inequality and the normalization of the ψ_n , we have

$$|F_n(\varphi)| \leq \|\varphi\|_0$$

so that these are uniformly bounded continuous linear functionals. After an integration by parts (for $\varphi \in C^1(0, 1)$, $\varphi(1) = 0$) we see that

$$F_n(\varphi) = - \int_0^1 \psi_n(s) [\varphi(s) + s\varphi'(s)] \, ds \rightarrow - \int_0^1 \psi(s) [\varphi(s) + s\varphi'(s)] \, ds \equiv F(\varphi).$$

From the preceding inequality, $F(\varphi)$ is a continuous linear functional on a dense subset of $L_s^2(0, 1)$ and thus on all of $L_s^2(0, 1)$. Thus by the Reisz theorem there is an element $\tilde{\psi} \in L_s^2(0, 1)$ such that

$$(2.7) \quad \int_0^1 \tilde{\psi}(s) \varphi(s) s \, ds = - \int_0^1 \psi(s) [\varphi(s) + s\varphi'(s)] \, ds.$$

If in (2.7) we take a sequence of piecewise C^1 approximations to the characteristic (indicator) function of $[0, s_0]$, $0 < s_0 < \tau$, we deduce in the standard way that $\psi' = \tilde{\psi}$ a.e. Since $\tilde{\psi} \in L_s^2(0, \tau)$ it follows that $\psi \in H(0, 1)$.

Suppose now that $\psi \neq 0$ in $H(0, 1)$, i.e., $\int_0^1 \psi^2 \, ds > 0$. Then we claim $J_2(\psi) = C$. Certainly then $J_2(\psi) \geq C$. We have for any $\varepsilon > 0$ and n sufficiently large,

$$\begin{aligned} \int_0^1 (\psi')^2 s \, ds &= \lim_{n \rightarrow +\infty} \int_0^1 \psi' \psi'_n s \, ds \\ &\leq \varepsilon + \left| \int_0^1 \psi' \psi_n s \, ds \right|. \end{aligned}$$

Thus

$$\begin{aligned}
 \int_0^1 (\psi')^2 s \, ds &\leq \varepsilon + \left(\int_0^1 (\psi')^2 s \, ds \right)^{1/2} \left(\int_0^1 (\psi_n)^2 s \, ds \right)^{1/2} \\
 &\leq \varepsilon + \left(\int_0^1 (\psi')^2 s \, ds \right)^{1/2} (J_2^{\frac{1}{2}}(\psi_n)) \left(\int_0^1 \psi_n^6 \, ds \right)^{1/4} \left(\int_0^1 \psi_n^2 \, ds \right)^{-1/4} \\
 &\leq \varepsilon + \left(\int_0^1 (\psi')^2 s \, ds \right)^{1/2} (C + \varepsilon)^{1/2} \left(\int_0^1 \psi^6 \, ds \right)^{1/4} \left(\int_0^1 \psi^2 \, ds \right)^{-1/4}
 \end{aligned}$$

in view of (2.6) and the fact that $J_2(\psi_n) \rightarrow C$. Letting $\varepsilon \rightarrow 0$ in this latter we obtain (after division by $\left(\int_0^1 (\psi')^2 s \, ds \right)^{1/2} \neq 0$) $J_2(\psi) \leq C$ and hence $J_2(\psi) = C$.

Therefore, we have an element $\psi \in H^1(R^2)$ depending only on $r = (x^2 + y^2)^{1/2}$ and decreasing in r such that for every $\eta \in C^\infty$ with compact support on $(0, \infty)$ and every $\varepsilon > 0$,

$$J_2(\psi + \varepsilon\eta) \geq J_2(\psi) = C.$$

Thus, by the usual arguments, one obtains from the equation $dJ_2(\psi + \varepsilon\eta)/d\varepsilon|_{\varepsilon=0} = 0$, the weak form of variational equation

$$(2.8) \quad \int_0^\infty \psi' \eta' r \, dr + A^2 \int_0^\infty \psi \eta r \, dr - B^2 \int_0^\infty \psi^5 \eta r \, dr = 0$$

where

$$\begin{aligned}
 A^2 &= \frac{1}{2} \int_0^\infty (\psi')^2 r \, dr / \int_0^\infty \psi^2 r \, dr \\
 B^2 &= \frac{2}{3} \int_0^\infty (\psi')^2 r \, dr / \int_0^\infty \psi^6 r \, dr.
 \end{aligned}$$

By scaling in the dependent and independent variable we can arrange things so that $A^2 = B^2 = 1$. By approximating the characteristic function of $[0, r_0]$ with a sequence of $C_0^\infty(R^1)$ functions, we obtain

$$(2.9) \quad \int_0^\infty (\psi')^2 r \, dr = 2 \int_0^\infty \psi^2 r \, dr = \frac{2}{3} \int_0^\infty \psi^6 r \, dr$$

$$(2.10) \quad -r\psi'(r) = \int_0^r [\psi(r) - \psi^5(r)] \, dr.$$

Equation (2.10) in turn implies that ψ'' exists in $(0, \infty)$ and that ψ satisfies (1.9a) there. From (2.10) we see that ψ' is continuous on $(0, \infty)$. It follows from (1.9a) that ψ'' is also continuous on $(0, \infty)$. Clearly (2.9) and the monotonicity of ψ imply

$$(2.11.1) \quad \lim_{r \rightarrow +\infty} r\psi^6(r) = \lim_{r \rightarrow +\infty} r\psi^2(r) = 0,$$

$$(2.11.2) \quad \liminf_{r \rightarrow +\infty} (r(\psi'(r))^2) = 0.$$

Now we see by exactly the same arguments as were used to derive (2.4) and (2.5) that

$$|\psi(r)| \leq |\psi(1)| + |\ln r|^{1/2} \left(\int_0^\infty (\psi')^2 r dr \right)^{1/2}$$

so that for some δ , $0 < \delta < 1$ some constant $A > 0$ and all $r \in (0, \delta)$

$$(2.12) \quad |\psi(r)| < A |\ln r|^{1/2} \quad 0 < r < \delta$$

and consequently from (2.10)

$$(2.13) \quad r|\psi'(r)| < B |\ln r|^5 \quad 0 < r < \delta$$

for some $B > 0$. From (2.12), (2.13) and (1.9a) we can now prove that

$$(2.14a) \quad \psi(0) \equiv \lim_{r \rightarrow 0+} \psi(r)$$

exists and

$$(2.14b) \quad \lim_{r \rightarrow 0+} [\psi(r) - \psi(0)]/r \equiv \psi'(0) = \lim_{r \rightarrow 0+} \psi'(r) = 0.$$

In order to prove this claim, we rewrite (1.9a) as an integral equation. We have, by variation of constants, for any solution of (1.9a) satisfying (2.12) and (2.13), for $0 < r < \delta$ and some constants α, β ,

$$(2.15) \quad \psi(r) = \alpha - \int_0^r \psi(s) [1 - \psi^4(s)] s \ln s ds + \ln r (\beta + \int_0^r \psi(s) [1 - \psi^4(s)] s ds).$$

The integrals on the right are absolutely convergent in view of (2.12). We see from (2.12) that unless $\beta = 0$, (2.15) will stand in contradiction to (2.12) as $r \rightarrow 0+$. It is not hard to see that the second integral on the right of (2.15) is of order $r^2 |\ln r|^6$ as $r \rightarrow 0+$ so that when $\beta = 0$, $\lim_{r \rightarrow 0+} \psi(r) = \alpha \equiv \psi(0)$.

A calculation shows that ($\beta = 0$)

$$\psi'(r) = \frac{1}{r} \int_0^r \psi(s)[1 - \psi^4(s)]s ds$$

so

$$\lim_{r \rightarrow +\infty} \psi'(r) = 0.$$

Also

$$\begin{aligned} (\psi(r) - \alpha)/r &= -\frac{1}{r} \int_0^r \psi(s)[1 - \psi^4(s)]s \ln s ds \\ &\quad + \frac{\ln r}{r} \int_0^r \psi(s)[1 - \psi^4(s)]r ds \end{aligned}$$

so that

$$\lim_{r \rightarrow 0+} (\psi(r) - \alpha)/r = \psi'(0) = 0$$

in view of (2.12) and the fact that the first integral on the right of (2.15) is of order $r^2 |\ln r|^6$ as $r \rightarrow 0+$. This finishes the proof of the theorem.

COROLLARY 2.1. — The initial value $\psi(0)$ of the minimizing function ψ_0 , normalized as in (2.9) must satisfy

$$(2.16) \quad \frac{1}{2} \psi^2(0)(\psi^4(0)/3 - 1) = \int_0^\infty (\psi')^2(r) dr/r.$$

PROOF. — Multiply (1.9) by ψ' and integrate the resulting equation over (δ, R) . There results

$$\frac{1}{2} \{ \psi'^2 + \psi^8/3 - \psi^2 \} \Big|_\delta^R + \int_\delta^R (\psi')^2(r) dr/r = 0.$$

If we let $R \rightarrow +\infty$ and $\delta \rightarrow 0+$ and take account of (2.11) and (2.14a), we obtain 2.16.

REMARK 2.1. $\psi_\pm \equiv \pm 1$ are the only nontrivial stationary solutions of (1.9). If ψ were any other solution of (1.9a) with $\psi'(0) = 0$ and which tended to a nonzero stationary solution and for which $\psi'(r) \rightarrow 0$ as $r \rightarrow +\infty$, then

$$(2.17) \quad \frac{1}{2} \psi^2(0)(\psi^4(0)/3 - 1) + \frac{1}{3} = \int_0^\infty (\psi')^2(r) dr/r.$$

Equations (2.9), (2.16) and (2.17) will be used as checks on the numerical results in the next section.

REMARK 2.2. — The equation (1.9a) cannot have a nontrivial solution on a finite interval $(0, R)$, be normalized as in (2.9) and have support in $[0, R]$. To show this we assume $\psi(R) = 0$ and show that $\psi'(R) = 0$ also. It will then follow from (backward) uniqueness that $\psi \equiv 0$ in $[0, R]$. To see that $\psi'(R) = 0$, multiply (1.9a) by $r\psi'$ and integrate over $[0, R]$. There results

$$r^2(\psi'^2/2 + \psi^6/6 - \psi^2/2)|_0^R = \int_0^R (\frac{1}{6}\psi^6 - \frac{1}{2}\psi^2)r dr = 0.$$

From this and the boundary conditions, $R^2(\psi')^2(R) = 0$.

3. — An asymptotic expansion for ψ .

We consider (1.9a) with a positive, decreasing solution $\psi = \psi(r) \rightarrow 0$ as $r \rightarrow +\infty$. The substitution

$$\varphi(\sigma) = \psi^{-2}(\sqrt{\sigma})$$

reduces (1.9a) to the ordinary differential equation

$$(3.1a) \quad \sigma(2\varphi\varphi'' - 3\varphi'^2) + \varphi^2 + 2\varphi\varphi' = 1$$

with the boundary conditions

$$(3.1b) \quad \lim_{\sigma \rightarrow 0+} \sqrt{\sigma}\varphi(\sigma) = 0$$

$$(3.1c) \quad \lim_{\sigma \rightarrow +\infty} \varphi(\sigma) = +\infty.$$

We look for an asymptotic solution of (1.9a) or (3.1a) modeled on the known solution of (1.10a). We set

$$\varphi(\sigma) = 3^{-1/2} \cosh y(\sigma)$$

and assume $y(\sigma) \rightarrow +\infty$ as $\sigma \rightarrow +\infty$. This leads to

$$(3.2) \quad 2(\sigma y''(\sigma) + y'(\sigma)) - \sigma(y'(\sigma))^2 = -1 \quad (\sigma \rightarrow +\infty).$$

Via the substitution $y'(\sigma) = -2w'(\sigma)/w(\sigma)$, we find that

$$(3.3) \quad \sigma w''(\sigma) + w'(\sigma) - \frac{1}{4}w = 0$$

and $y(\sigma) = \ln(w^{-2}(\sigma))$ so $w^2(\sigma) = e^{-y(\sigma)}$. We want $y \rightarrow +\infty$ as $\sigma \rightarrow +\infty$ so $w \rightarrow 0$

as $\sigma \rightarrow +\infty$. The change of variables $r = \sqrt{\sigma}$ takes (3.3) back to

$$(3.4) \quad w''(r) + \frac{1}{r} w'(r) - w(r) = 0$$

which is Bessel's equation for functions of a purely imaginary argument. The general solution is [19]

$$w(r) = AJ_0(ir) + BK_0(r).$$

Where K_0 is the modified Bessel function of the second kind. Now $J_0(ir) \rightarrow +\infty$ as $r \rightarrow +\infty$ so $A = 0$. We may take $B = 1$ since (3.4) is linear. Thus, we find after returning to original variables

$$(3.5a) \quad \psi_0(r) \sim 3^{1/4} [\operatorname{sech}(-2 \ln K_0(r))]^{1/2} \quad r \rightarrow +\infty$$

$$(3.5b) \quad = (2\sqrt{3})^{1/2} \frac{K_0(r)}{[K_0^4(r) + 1]^{1/2}} \quad r \rightarrow +\infty.$$

Now [19]

$$(3.6) \quad K_0(r) \simeq \left(\frac{\pi}{2}\right)^{1/2} e^{-r} r^{-1/2} (1 - (8r)^{-1} + 9(32r^2)^{-1} \dots) \\ \simeq (\pi/2)^{1/2} e^{-r} r^{-1/2} \quad (r \rightarrow +\infty).$$

Thus, to two terms

$$(3.7) \quad \psi(r) \simeq 3^{1/4} [\operatorname{sech}(\ln r + 2r - \ln(\pi/2))]^{1/2}.$$

Notice that (3.5) is an improvement over what one obtains from (1.9) under the assumption that $\psi \rightarrow 0$ as $r \rightarrow +\infty$. Then to a first approximation, ψ_0 also satisfies (3.4) but now we find $\psi_0(r) \sim CK_0(r)$ as $r \rightarrow +\infty$.

It is possible to write down local (near $\sigma = 0$) solutions of (3.1) but we do not do so since they have shown themselves to be of little use computationally.

4. - Numerical results.

We solved the problem (1.9) numerically by using the « shooting » method with $\psi_0(0)$ as the « shooting » parameter. That is, we considered

$$(4.1) \quad \begin{aligned} y''(r) + y'(r)/r + y^5(r) - y(r) &= 0 & 0 < r < +\infty \\ y'(0) &= 0 & y'(+\infty) &= 0 \\ y(0) &\text{ given.} \end{aligned}$$

We also evaluated, using an adaptive Simpson's rule,

$$(4.3) \quad J_2(r) = 2\pi \int_0^\infty (y')^2 r dr \left(\int_0^\infty y^2 r dr \int_0^\infty y^6 r dr \right)^{1/2}$$

for functions $y_R(r)$ defined by

$$(4.4) \quad y_R(r) = 3^{1/4} [\operatorname{sech}(-2 \ln K_0(r + R))]^{1/2}$$

using the first few terms in the asymptotic expansions of K_0 and varying the parameter R .

The finite difference solutions of (4.1) with step size $h = 5 \cdot 10^{-4}$ gave a value of $J_2(y) \simeq 4.5981$. The translated version of (3.7), namely (4.7) with $R \cong 0.405$ gave $J_2(y_R) \simeq 4.60160$. Since $\pi^{4/3} = 4.60115 \dots$, one is sorely tempted to conjecture that $C = \pi^{4/3}$. We are unable to prove the conjecture because of the inability to find a closed form solution of (1.9).

The recursion scheme used to generate the numerical solution of (4.1) was based perhaps on the most naive difference equation obtainable from (4.1) namely

$$y_{i+2} = (2 - 1/i)y_{i+1} + (1/i - 1)y_i + h^2(y_i - y_i^5)$$

with $y_1 = y_2 = y(0)$. The integrals in (4.3) were evaluated in an equally naive fashion, i.e.

$$\begin{aligned} \int_0^\infty y^p r dr &\approx h^2 \sum_{i=1}^N i y_i^p \equiv I_p \quad (p = 2, 6) \\ \int_0^\infty (y')^2 r dr &\approx \sum_{i=1}^N i (y_{i+1} - y_i)^2 \equiv I_d \\ \int_0^\infty (y')^2 r^{-1} dr &\approx h^{-2} \sum_{i=1}^N (y_{i+1} - y_i)^2 / i \equiv K. \end{aligned}$$

We then expect that

$$C = 2\pi I_d (I_2/I_6)^{1/2}.$$

As a check on the numerical computation, we note from (2.9) and (2.16) that we should expect

$$(4.5a) \quad I_d = 2I_2$$

$$(4.5b) \quad I_6 = 3I_2$$

$$(4.5c) \quad K = \frac{1}{2} y_1^2 (y_1^4/3 - 1) \equiv p_6(y_1).$$

As one can easily verify from Table I, we see that equations (4.5) are satisfied to two or three places. The data indicate that for (1.9) the correct initial value ought to be $\psi(0) = 2$.

TABLE I. - *Summary of finite difference method.*

		$h = 0.002$		$h = 0.0005$	
		Upper	Lower	Upper	Lower
1	y_1	1.99619	1.99623	1.999494	1.999514
2	$p_6(y_1)$	8.6375	8.6387	8.67255	8.67315
3	K_i	8.5530	8.5542	8.65150	8.65210
4	I_2	0.6320	0.6318	0.631256	0.63249
5	I_6	1.8940	1.8940	1.90026	1.90028
6	I_a	1.2646	1.2649	2.16835	1.26847
7	C	4.590	4.5902	4.59794	4.59808
8	$r_{\min.}$	6.290	—	5.000	—
9	r_{zero}	—	6.340	—	4.985

The relations (4.5a, b) can be used to express the constant C in terms of one integral alone. Specifically, we might use the data in Table I, lines 4, 5, 6 to obtain the entries in Table II via

$$(4.6) \quad C \cong (2\pi/\sqrt{3})I_a \cong (4\pi/\sqrt{3})I_2 \cong (4\pi/3\sqrt{3})I_6.$$

TABLE II. - *Computation of C using relations (4.6).*

	Solution	Step Size	$4\pi I_2/\sqrt{3}$	$4\pi I_6/3\sqrt{3}$	$2\pi I_a/\sqrt{3}$
1	Upper	0.01	4.525	4.504	4.542
2	Lower	0.01	4.525	4.504	4.542
3	Upper	0.002	4.5852	4.5803	4.5876
4	Lower	0.002	4.5835	4.5805	4.5886
5	Upper	0.0005	4.58937	4.59558	4.60105
6	Lower	0.0005	4.58823	4.59564	4.60149

Figure 1 is a free hand sketch of the numerical solution for step size $h = 0.002$. It illustrates how elusive the monotone solution of 1.9 is. For $y(0) = 1.99619$, the numerical solution dropped monotonically to a minimum of about $2.4429 \cdot 10^{-3}$ at $r = 6.29$ and then slowly increased again to oscillate about the stationary solution $y(r) \equiv 1$. We call this solution the upper solution although its initial value is smaller than that for the lower solution. With $y(0) = 1.99623$, the numerical solution decreases monotonically, crosses the r axis at about $r = 6.340$ and then drops below

Figure 1

the line $y = -1$ and oscillates about $y(r) \equiv -1$. In order to evaluate the integrals in Table I, we used the upper and lower solutions cut off at the minimum point and the zero respectively.

The oscillation about $y = \pm 1$ is to be expected for solutions which get near the lines $y = \pm 1$. If one writes $y(r) = \pm 1 + \varepsilon z(r)$ then for small ε , z very nearly satisfies $z'' + r^{-1}z' + 4z = 0$ so that $z(r) \approx J_0(2r)$.

In Table III we record the computed values of the ratio (1.6) for several functions which are in some sense approximate solutions of (1.9). The least of these gives the best upper bound for C .

TABLE III. — *Values of $J_2(y)$ for various trial functions $y(r)$ ($\pi^{4/3} \simeq 4.60115\dots$).*

Function	Parameter Value	$J_2(y)$
$[\text{sech}(r)]^{1/2}$	None	4.6670
$[\text{sech}(r)]^{1/2}(1 + Rr^2)^{-1/2}$	$R = 3.5$	4.6028
$[\text{sech}(2(r + R) + \ln(R + r))]^{1/2}$	$R = 0.40498$	4.60160
$[\text{sech}(2(r + R) + \ln(r + R) + 1/4(r + R)^{-1})]^{1/2}$	$R = 0.27600$	4.60771
$[\text{sech}(2r + \ln(r + R))]^{1/2}$	$R = 1.096$	4.6170

Acknowledgements. — The author takes pleasure in thanking several of his colleagues and friends for their advice and encouragement. In particular he wishes to thank M. J. GOLDSTEIN and R. STREIT for their help with the computations; R. K. MILLER

and W. MADYCH for suggestions which improved an earlier version of the proof of Theorem 2.1, and L. E. PAYNE, who introduced him the inequality (1.1) and has maintained a constant interest in these problems over the last several years.

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