Existence of Optimal Controls for Systems Governed by Parabolic Partial Differential Equations with Cauchy Boundary Conditions (*).

D. W. Reid - K. L. Teo (Sydney, Australia)

Summary. – *This paper considers the existence of optimal controls for systems governed by a second order parabolic partial differential equation in divergence form with Cauchy conditions. As preliminary results, theorems concerning the convergence of the sequence of weak solutions corresponding to a sequence of admissible controls are proved. Two general forms of criteria are considered. The first one is taken as a function of the weak solution of the system, and the other is taken as a function of the solution of the system and control. Several theorems and corollaries on the existence of optimal controls are then presented.*

1. – Introduction.

Questions concerning the existence of optimal controls for systems governed by parabolic partial differential equations with first boundary conditions have been substantially studied in references [1], [5], [8], [11] and [12]. Although necessary conditions for optimality for problems of optimal control of systems monitored by parabolic partial differential equations with Cauchy conditions and with bounded measurable controls appearing in the coefficients and forcing term can be found in reference [2], the question concerning the existence of optimal controls has not been investigated. The aim of this paper is to provide a partial answer to this question. More precisely, we can *only* allow the bounded measurable controls to appear in the forcing term and *some* coefficients.

The class of systems considered in this paper is described by linear second order parabolic partial differential equations in divergence form with Cauchy boundary conditions. The controls influence the systems by altering the coefficients. Specifically, if D is a class of admissible controls, then, for each $u \in D$, there are functions $a_j^u: Q \to \boldsymbol{R}^1$ $(j = 1, \dots, n)$, $c^u: Q \to \boldsymbol{R}^1$ and $d^u: Q \to \boldsymbol{R}^1$ where $Q = (0, T] \times \boldsymbol{R}^n$ and T a positive real. For a $u \in D$, the system is

$$S_1 \begin{cases} \varphi_t = (a_{ij}\varphi_{x_i} + a_j^u \varphi)_{x_j} + b_j \varphi_{x_j} + c^u \varphi + d^u & \text{on } Q\,, \quad (1.1) \\ \varphi(0, \cdot) = \varphi_0 & \text{on } \boldsymbol{R}^n\,, \quad (1.2) \end{cases}$$

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where $a_{ij}: Q \to \boldsymbol{R}^1$ $(i, j = 1, \ldots, n)$ and $b_j: Q \to \boldsymbol{R}^1$ $(j = 1, \ldots, n)$ are the coefficients not influenced by the control, while $\varphi_0: \boldsymbol{R}^n \to R^1$ is the initial function.

The conditions on the coefficients, the forcing term, the initial data and the performance criterion will be presented in later sections. The class of admissible controls is defined by

$$D \triangleq \{u: Q \to U: u \text{ is measurable}\} \tag{1.3}$$

where U is a compact convex subset of $\boldsymbol{R}^r$ that contains, without loss of generality, the point 0.

The forcing term and those coefficients that depend upon u are constructed from the corresponding functions $a_j: Q \times U \to \boldsymbol{R}^1$, $(j = 1, \ldots, n)$, $c: Q \times U \to \boldsymbol{R}^1$ and $d: Q \times U \to \boldsymbol{R}^1$ which are given. If $z: Q \times U \to \boldsymbol{R}^1$ then $z^u: Q \to \boldsymbol{R}^1$ such that

$$z^u(t, x) = z\big(t, x, u(t, x)\big) . \tag{1.4}$$

In section 2, the assumptions and notations are presented, and in section 3, the existence of weak solutions for system S_1 and the properties of such weak solutions are investigated. The main results of this paper are given in section 4. Theorem 4.1 states a general result for performance criterion that depend only explicitly of φ. By a linearity assumption, we have Theorem 4.2 where the criterion depends explicitly upon both u and φ.

2. – Basic notation and assumptions.

Before describing the basic assumptions to be imposed upon the coefficients of system S_1, we introduce some useful notations.

Let B be a connected, Lebesgue measurable subset of a finite dimensional Euclidean space. Then $\bar{B}$ denotes the closure of B, ∂B the boundary of B and $|B|$ the Lebesgue of B. For $p \in [1, +\infty)$, $L^p(B)$ is the space of functions $z: B \to \boldsymbol{R}^1$ such that

$$\|z\|_{L^p(B)} \triangleq \|z\|_{p,B} \triangleq \left\{ \int_B |z(x)|^p \, dx \right\}^{1/p}$$

is finite, and $L^\infty(B)$ is the space of those functions such that

$$\|z\|_{L^\infty(B)} \triangleq \|z\|_{\infty,B} \triangleq \operatorname*{ess\,sup}_{x \in B} |z(x)|$$

is finite. If l is an integer, $0 \leqslant l \leqslant +\infty$, then $C^l(B)$ denotes the class of l times continuously differentiable functions from B into $\boldsymbol{R}^1$, and $C_0^l(B)$ is the subset of $C^l(B)$ of functions with compact support in B. Let $W^1(B)$ be the completion of the space

$C_0^\infty(B)$ in the norm

$$\|z\|_{W^1(B)} \triangleq \|z\|_{2,B} + \left(\sum_{i=1}^{n} \|z_{x_i}\|_{2,B}^2\right)^{\frac{1}{2}}.$$

Throughout the rest of this paper, z_{x_i} will denote the distributional derivative of z with respect to x_i.

Let X be a Banach space. Then, for $q \in [1, +\infty)$, $L^q(I, X)$ is the class of functions $z\colon (0, T] \to X$ such that

$$\|z\|_{q,X} \triangleq \left\{\int_0^T \|z(t)\|_X^q \, dt\right\}^{1/q}$$

is finite, and, for $q = +\infty$, $L^\infty(I, X)$ is the class of functions $z\colon (0, T] \to X$ such that

$$\|z\|_{\infty,X} \triangleq \operatorname*{ess\,sup}_{t\in(0,T]} \|z(t)\|_X$$

is finite. When $X = L^p(B)$, $p \in [1, +\infty]$, then $\|\cdot\|_{q,X}$ is written as $\|\cdot\|_{p,q,(0,T]\times B}$, and $L^q(I, L^p(B))$ is written as $L^{p,q}((0, T)\times B)$.

Throughout this paper, the following assumptions are imposed on the coefficients, the forcing term and the initial data of system S_1.

A_1: $a_{ij}\colon Q \to \boldsymbol{R}^1$ $(i, j = 1, \dots, n)$ are continuous on Q and there are positive constants μ_1 and μ_2 such that $\mu_1|z|^2 \leqslant a_{ij}(t, x) z_i z_j \leqslant \mu_2 |z|^2$ for all $z \in \boldsymbol{R}^n$ and for all $(t, x) \in Q$.

A_2: a_j, c, $d\colon Q\times U \to \boldsymbol{R}^1$ $(j = 1, \dots, n)$ are measurable on $Q\times U$, and continuous on U for all $(t, x) \in Q$.

A_3: $b_j \in L^\infty(Q)$, $(j = 1, \dots, n)$,

A_4: $\varphi_0 \in L^2(\boldsymbol{R}^n)$.

A_5: There is a positive constant μ_3 such that for all $u \in D$, $j \in \{1, \dots, n\}$, the norms $\|a_j^u\|_{\infty,Q}$, $\|b_j\|_{\infty,Q}$, $\|c^u\|_{\infty,Q}$, $\|d^n\|_{\infty,Q}$, $\|d^u\|_{2,Q}$ and $\|\varphi_0\|_{2,\boldsymbol{R}^n}$ are less than or equal to μ_3.

A_6: For each $(t, x) \in Q$, the set

$$F(t, x, U) \triangleq \left\{\begin{bmatrix} a_1(t, x, u) \\ \cdots\cdots \\ a_n(t, x, u) \\ c(t, x, u) \\ d(t, x, u)\end{bmatrix} \in \boldsymbol{R}^{n+2}\colon u \in U\right\}$$

is convex. Further, it is assumed that $F(\cdot, \cdot, U)$ is a measurable set-valued function defined on Q.

Since U is compact, it is obvious that if $u \in D$ then the i-th component function of u, u_i, belongs to $L^\infty(Q)$. A sequence $\{u^k\}$ in D is said to converge in the weak * topology to u^0, if and only if $\{u_i^k\}$ converges in the weak * topology of $L^\infty(Q)$ to u_i^0, $i = 1, ..., r$. Since U is compact and convex, it should be noted that if $\{u^k\}$ is a sequence in D, then there is a subsequence $\{u^{k_l}\}$ and a function $u^0 \in D$ such that $\{u^{k_l}\}$ converges in the weak * topology to u^0.

A measurable function $z: Q \times U \to \mathbf{R}^1$ is said to satisfy the linearity condition if

$$z(t, x, u) = \langle z^*(t, x), u\rangle + z^{**}(t, x)$$

for all $(t, x, u) \in Q \times U$, where $z^*: Q \to \mathbf{R}^r$ such that all of its component functions belong to $L^\infty(Q)$, $z^{**}: Q \to \mathbf{R}^1$ belongs to $L^\infty(Q)$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^r$. For a function satisfying the linearity condition $z^{u^k} \to z^{u^0}$ in the weak * topology on $L^\infty(Q)$ for all sequences $\{u^k\}$ in D that converge in the weak * * topology to a $u_0 \in D$.

On occasions, we shall make the assumption

A_6': The functions a_j $(j = 1, ..., n)$, c and d satisfy the linearity condition.

It is obvious that Assumption A_6' implies Assumption A_6. However, this stronger assumption will allow us to consider more general criterion functions. All theorems are proved under the Assumptions A_1 to A_5: If Assumption A_6 is invoked, it will be stated that the theorem is being proved under the Convexity Assumption. However, if Assumption A_6' is required, the thorem will be said to be proved under the Linearity Assumption.

3. – Weak solution of parabolic systems.

In this section, it will be shown that, under assumptions A_1 - A_5, system S_1 admits a unique weak solution. Further, certain properties of weak solutions will also be investigated.

DEFINITION 3.1. – The solution to system S_1 corresponding to a control $u \in D$ is a map $\varphi(u) \in L^{2,\infty}(Q) \cap L^2(I, W^1(\mathbf{R}^n))$ such that

$$(3.1)\qquad \int_Q \varphi(u)\eta_t - a_{ij}\varphi(u)_{x_i}\eta_{x_j} - a_j^u\varphi(u)\eta_{x_j} + b_j\varphi(u)_{x_j}\eta + c^u\varphi(u)\eta + d^u\eta = 0$$

for all $\eta \in C_0^1(Q)$ and

$$(3.2)\qquad \lim_{t\downarrow 0} \int_{\mathbf{R}^n} \varphi(u)(t, x)\eta(x)\, dx = \int_{\mathbf{R}^n} \varphi_0(x)\eta(x)\, dx$$

for all $\eta \in C_0^1(\mathbf{R}^n)$.

An the left hand side of (3.1) is a commonly used expression, the following notation is introduced.

DEFINITION 3.2. – If $\eta \in C^1(Q)$, $\varphi \in L^{2,\infty}(Q) \cap L^2(I, W^1(\boldsymbol{R}^n))$ and $u \in D$, then

$$\mathcal{L}^u(\varphi, \eta) = \int_Q \varphi\eta_t - a_{ij}\varphi_{x_i}\eta_{u_j} - a_j^u \varphi\eta_{x_i} + b_j\varphi_{x_i}\eta + c^u\varphi\eta + d^u\eta \tag{3.3a}$$

and

$$\mathcal{L}^u(\varphi, \eta)|_s^t = \int_s^t \int_{\boldsymbol{R}^n} \varphi\eta_t - a_{ij}\varphi_{x_i}\eta_{x_i} - a_j^u\varphi\eta_{x_i} + b_j\varphi_{x_i}\eta + c^u\varphi\eta + d^u\eta \ . \tag{3.3b}$$

THEOREM 3.3. – For each $u \in D$, there is a unique solution $\varphi(u)$ to system S_1. Further there is a constant c_1 that depends only upon T, n, μ_1, μ_2, and μ_3 such that

$$\|\varphi(u)\|_{2,\infty,Q}^2 + \sum_{i=1}^n \|\varphi(u)\|_{2,2,Q}^2 \leqslant c_1(\|d^u\|_{2,Q}^2 + \|\varphi_0\|_{2,\boldsymbol{R}^n}^2) \tag{3.4}$$

and

$$\|\varphi(u)\|_{2,2,Q}^2 \leqslant Tc_1(\|d^u\|_{2,Q}^2 + \|\varphi_0\|_{2,\boldsymbol{R}^n}^2) \ . \tag{3.5}$$

Further $\int_{\boldsymbol{R}^n} \varphi(u)(\sigma, x)\eta(x)\,dx \to \int_{\boldsymbol{R}^n} \varphi_0(x)\eta(x)\,dx$ uniformly with respect to $u \in D$ as $\sigma \to 0$ for all $\eta \in C_0^1(\boldsymbol{R}^n)$.

PROOF. – Let $O_k = \{x \in \boldsymbol{R}^n \colon |x| < k\}$ and $\theta_k = (0, T] \times O_k$, for each positive integer k. For each $u \in D$, let us consider the first boundary problem

$$S_{2,k}^u \begin{cases} \varphi_t = (a_{ji}\varphi_{x_i} + a_j^u)_{x_j} + b_j\varphi_{x_j} + c^u\varphi + d^u & \text{on } \theta_k \\ \varphi(0, \cdot) = \varphi_0 & \text{on } O_k \\ \varphi(t, x) = 0 & \text{on } [0, T] \times \partial O_k . \end{cases}$$

By Theorem 1 ([4], p. 634), we note that, for each positive integer k, there exists a unique weak solution $\varphi_k(u)$ to the problem $S_{2,k}^u$. Further by letting $\zeta = 1$, $s = 0$ and μ, p and $q = \infty$ in Lemma 1 ([4], p. 623), it follows that

$$\|\varphi_k(u)\|_{2,\infty,\theta_k}^2 + \sum_{i=1}^n \|\varphi(u)_{x_i}\|_{2,2,\theta_k}^2 \leqslant c_2 e^{\beta T}(\|d^u\|_{2,\theta_k}^2 + \|\varphi_0\|_{2,O_k}^2) \tag{3.6}$$

where β is a positive constant that depends only upon n, μ_1, μ_2 and μ_3 and c_2 is a positive constant that depends only upon T, n, μ_1, μ_2 and μ_3. Let $c_3 = c_2 e^{\beta T}$, $\varphi_k(u)(t, x)$ be zero for $(t, x) \notin \theta$, and let

$$c_4 = c_3(\mu_3 + \|\varphi_0\|_{2,\boldsymbol{R}^n}^2) \ . \tag{3.7}$$

Recalling that $\|d^u\|_{2,Q} \leq \mu_3$, it follows from (3,6) that

$$\|\varphi_k(u)\|^2_{2,\infty,Q} + \sum_{i=1} \|\varphi_k(u)_{x_i}\|^2_{2,2,Q} \leq c_4 , \tag{3.8}$$

from which we deduce that

$$\|\varphi_k(u)\|_{2,2,Q} \leq T^{\frac{1}{2}} c_4^{\frac{1}{2}}$$

and

$$\|\varphi_k(u)_{x_i}\|_{2,2,Q} \leq c_4^{\frac{1}{2}} , \qquad (i = 1, \ldots, n).$$

As $T^{\frac{1}{2}} c_4^{\frac{1}{2}}$ and $c_4^{\frac{1}{2}}$ are independent of k, there is a subsequence of the vectors $(\varphi_k(u), (\varphi_k(u))_{x_1}, \ldots, (\varphi_k(u))_{x_n}$, which is again indexed by k, and a vector $(\varphi(u), \varphi^1(u), \ldots, \varphi^n(u))$ such that $\varphi_k(u) \to \varphi(u)$, $(\varphi_k(u))_{x_i} \to \varphi^i(u)$ $(i = 1, \ldots, n)$ weakly in $L^{2,2}(Q)$ as $k \to \infty$. It is easily shown that $\varphi^i(u)$ is the distributional derivative of $\varphi(u)$ with respect to x_i and we will use $\varphi(u)_{x_i}$ instead of $\varphi^i(u)$. Clearly, we have

$$\|\varphi(u)\|_{2,2,Q} \leq T^{\frac{1}{2}} c_4^{\frac{1}{2}} \tag{3.9}$$

$$\|\varphi(u)_{x_i}\|_{2,2,Q} \leq c_4^{\frac{1}{2}} \qquad (i = 1, \ldots, n) \tag{3.10}$$

and thus $\varphi(u) \in L^2(I, W^1(\boldsymbol{R}^n))$. Moreover, it follows from (3.8) and lemma 2 ([4], p. 633) that

$$\|\varphi(u)\|_{2,\infty,Q} \leq c_4^{\frac{1}{2}} \tag{3.11}$$

and thus $\varphi(u) \in L^{2,\infty}(Q) \cap L^2(I, W^1(\boldsymbol{R}^n))$.

For any positive integer k, it follows from Equation 2.2 ([4], p. 622) that $\varphi_k(u)$ satisfies the following expression

$$\int_{\boldsymbol{R}^n} \varphi_0(x)\eta(0, x)\, dx + \mathfrak{L}^u(\varphi_k(u), \eta) = 0$$

for all $\eta \in C^1(\bar{Q})$ with compact support in $[0, T) \times O_k$. Let η be an arbitrary but fixed element in $C^1(\bar{Q})$ with compact support in $[0, T) \times Q$. Clearly, k can then be chosen sufficiently large so that the compact support of η is in $[0, T) \times O_k$. Thus, by letting $k \to \infty$ through an appropriate subsequence, it follows from the weak convergence in $L^2(I, W^1(\boldsymbol{R}))$ of the sequence $\{\varphi_k(u)\}$ to $\varphi(u)$ that

$$\int_{\boldsymbol{R}^n} \varphi_0(x)\eta(0, x)\, dx + \mathfrak{L}^u(\varphi(u), \eta) = 0 . \tag{3.12}$$

Note that (3.12) holds for any $\eta \in C^1(\bar{Q})$ with compact support in $[0, T) \times \boldsymbol{R}^n$.

Next, if η is an arbitrary element in $C_0^1(Q)$, then it follows readily from (3.12) that

$$\mathfrak{L}^u(\varphi(u), \eta) = 0 \tag{3.13}$$

Thus, in view of Definition 3.1, $\varphi(u)$ will be a weak solution of system S_1 corresponding with the control u if it also satisfies condition (3.2).

By Equation 1.3 ([4], p. 619), we can infer that

$$\int_{\mathbf{R}^n} \varphi(u)(\sigma, x)\eta(\sigma, x)\, dx + \mathfrak{L}^u(\varphi(u), \eta)|_\sigma^T = 0 \tag{3.14}$$

for all $\eta \in C^1(\bar{Q})$ with compact support in $[0, T) \times \mathbf{R}^n$.

Let $\eta \in C_0^1(\mathbf{R}^n)$, let Ξ be the compact support of η and let $\eta_1 \in C^1([0, T])$ such that $\eta_1(t) = 1$ for $t \in [0, T/4]$ and $\eta_1(t) = 0$ for $t \in [3T/4, T]$. Further, let $\eta_2(t, x) = \eta_1(t)\eta(x)$. Then $\eta_2 \in C^1(\bar{Q})$, equals zero for all $(t, x) \in [0, 3T/4] \times \Xi$ and

$$\eta_2(t, x) = \eta(x), \text{ for } (t, x) \in [0, T/4] \times \mathbf{R}^n. \tag{3.15}$$

For all $\sigma \in [0, T/4]$, it follows from (3.12), (3.14), (3.15), (3.9) and (3.10) that

$$\begin{aligned}
&\left| \int_{\mathbf{R}^n} \varphi(u)(\sigma, x)\eta(x)\, dx - \int_{\mathbf{R}^n} \varphi_0(x)\eta(x)\, dx \right| \\
&= |\mathfrak{L}^u(\varphi(u), \eta_2)|_\sigma^T - \mathfrak{L}^u(\varphi(u), \eta_2)| \\
&= |\mathfrak{L}^u(\varphi(u), \eta_2)|_0^\sigma| \\
&= \left| \int_0^\sigma \int_{\mathbf{R}^n} \varphi(u)\eta_t - a_{ij}\varphi(x)_{x_i}\eta_{x_j} - a_j^u\varphi(u)\eta_{x_j} + b_j\varphi(u)_{x_j} + c^u\varphi(u)\eta + d^u\eta \right| \\
&\leqslant \int_0^\sigma \int_{\mathbf{R}^n} |\varphi(u)\eta_t| + \int_0^\sigma \int_{\mathbf{R}^n} |a_{ij}\eta_{x_i}\varphi_{x_j}(u)| + \int_0^\sigma \int_{\mathbf{R}^n} |a_j^u\eta_{x_j}\varphi(u)| \\
&\quad + \int_0^\sigma \int_{\mathbf{R}^n} |b_i\eta\varphi(u)_{x_j}| + \int_0 \int_{\mathbf{R}^n} |c^u\eta\varphi(u)| + \int_0^\sigma \int_{\mathbf{R}^n} |d^u\eta| \\
&\leqslant \|\eta_t\|_{2,(0,\sigma)\times\Xi} \cdot \|\varphi(u)\|_{2,Q} + \sum_{i,j=1}^n \|a_{ij}\eta_{x_j}\|_{2,(0,\sigma)\times\Xi} \cdot \|\varphi(u)_{x_i}\|_{2,Q} \\
&\quad + \sum_{j=1}^n \|a_j^u\eta_{x_j}\|_{2,(0,\sigma)\times\Xi}\|\varphi(u)\|_{2,Q} + \sum_{j=1}^n \|b_j\eta\|_{2,(0,\sigma)\times\Xi}\|\varphi(u)_{x_j}\|_{2,Q} \\
&\quad + \|c^u\eta\|_{2,(0,\sigma)\times\Xi}\|\varphi(u)\|_{2,Q} + \|d^u\eta\|_{1,\Xi} \\
&\leqslant \|\eta_t\|_{\infty,Q}^{\frac{1}{2}}\|\varphi(u)\|_{2,Q}\sigma^{\frac{1}{2}}|\Xi|^{\frac{1}{2}} + \sum_{i,j=1}^n \|a_{ij}\|_{\infty,Q}^{\frac{1}{2}}\|\eta_{x_j}\|_{\infty,Q}^{\frac{1}{2}}\|\varphi(u)_{x_j}\|_{2,Q}\sigma^{\frac{1}{2}}|\Xi|^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^n \|a_j^u\|_{\infty,Q}^{\frac{1}{2}}\|\eta_{x_j}\|_{\infty,Q}^{\frac{1}{2}}\|\varphi(u)\|_{2,Q}\sigma^{\frac{1}{2}}|\Xi|^{\frac{1}{2}} + \sum_{i=1}^n \|b_j\|_{\infty,Q}^{\frac{1}{2}}\|\eta\|_{\infty,Q}^{\frac{1}{2}}\|\varphi(u)_{x_j}\|_{2,Q}\sigma^{\frac{1}{2}}|\Xi|^{\frac{1}{2}} \\
&\quad + \|c^u\|_{\infty,Q}^{\frac{1}{2}}\|\eta\|_{\infty,Q}^{\frac{1}{2}}\|\varphi(u)\|_{2,Q}\sigma^{\frac{1}{2}}|\Xi|^{\frac{1}{2}} + \|d^u\|_{\infty,Q}\|\eta\|_{\infty,Q}\sigma|\Xi|
\end{aligned}$$

$$\leqslant \|\eta_t\|^{\frac{1}{2}}_{\infty,Q} T^{\frac{1}{2}} c_4^{\frac{1}{2}} \sigma^{\frac{1}{2}} |\Xi| + \sqrt{2}\, n^2 \sqrt{\mu_2} \|n_{x_j}\|^{\frac{1}{2}}_{\infty,Q} c_4^{\frac{1}{2}} \sigma^{\frac{1}{2}} |\Xi|^{\frac{1}{2}} + n\mu_3^{\frac{1}{2}} \|\eta_{x_j}\|^{\frac{1}{2}}_{\infty,Q} T^{\frac{1}{2}} c_4^{\frac{1}{2}} \sigma^{\frac{1}{2}} |\Xi|^{\frac{1}{2}} +$$
$$+ n\mu_3^{\frac{1}{2}} \|\eta\|^{\frac{1}{2}}_{\infty,Q} c_4^{\frac{1}{2}} \sigma^{\frac{1}{2}} |\Xi|^{\frac{1}{2}} + \mu_3^{\frac{1}{2}} \|\eta\|^{\frac{1}{2}}_{\infty,Q} T^{\frac{1}{2}} c_4^{\frac{1}{2}} \sigma^{\frac{1}{2}} |\Xi|^{\frac{1}{2}} + \mu_3 \|\eta\|_{\infty,Q} \sigma |\Xi|$$
$$\leqslant C_1 \sigma^{\frac{1}{2}} + C_2 \sigma$$

where the positive constants C_1 and C_2 are independent of σ and u. Thus, $\int_{R^n} \varphi(u) \cdot (\sigma, x) \eta(x)\, dx$ converges uniformly with respect to $u \in D$ to $\int_{R^n} \varphi_0(x) \eta(x)\, dx$ as $\sigma \to 0$ for any $\eta \in C_0^1(\boldsymbol{R}^n)$.

This proves the last assertion of the theorem. In particular, the convergence holds for each $u \in D$. Thus, condition (2.3) is satisfied. Therefore, for each $u \in D$, $\varphi(u)$ is a weak solution of system S_2.

The uniqueness of the solution to system S_1 corresponding to $u \in D$ follows from Theorem 2 ([4], p. 639). The inequalities (3.4) and (3.5) follow from (3.7), (3.9), (3.10) and (3.11), and letting $c_1 = (n+1)c_3$. This completes the proof.

Throughout the remainder of this paper, $\varphi(u)$ will denote the solution of system S_1 corresponding to $u \in D$.

In reference **3**, it has been shown that if $\varphi(u) \in L^{2,\infty}(Q) \cap L^2(I, W^1(\boldsymbol{R}^n))$ and satisfies expression (3.1) then there exists a representation for $\varphi(u)$ which is continuous in Q. We will therefore always assume that $\varphi(u)$ is continuous in Q.

We shall be concerned with sequences in D. Basically, we shall show that if $\{u_k\}$ is a sequence in D, there is a $u_0 \in D$ and a subsequence of $\{u_k\}$, which is again denoted by the original sequence, such that $\varphi(u_k) \to \varphi(u_0)$ in certain topologies. The relevant theorems are given in Theorem 3.7 and Theorem 3.8. To prove these theorems, the following lemmas are required.

Lemma 3.4. – $\{\varphi(u) : u \in D\}$ is an equicontinuous family of functions that is uniformly bounded on any subset of Q that has a positive distance from the set $\{0\} \times \boldsymbol{R}^n$.

Proof. – Let K be any subset of Q that has a positive distance ϱ_0 from $\{0\} \times \boldsymbol{R}^n$, let $\varrho = \sqrt{\varrho_0}/3$ and let $u \in D$. For an arbitrary but fixed $(\bar{t}, \bar{x}) \in K$, we denote by $G(\varrho) = (\bar{t} - 9\varrho^2, \bar{t}] \times \{y \in \boldsymbol{R}^n : |y - \bar{x}| < 3\varrho/2\}$. Since $\varphi(u) \in L^{2,\infty}(G(\varrho)) \cap L^2\big((\bar{t} - 9\varrho^2, \bar{t}], W^1(\{y \in \boldsymbol{R}^n : |y - \bar{x}| < 3\varrho/2\})\big)$ and satisfies the expression (3.1) in $G(\varrho)$, it follows from Theorem 2 ([3], p. 98) that there exists a positive constant c_5 depending only upon ϱ, n, μ_1, μ_2 and μ_3 such that

$$|\varphi(u)(t, x)| \leqslant c_5 \big(\varrho^{-(n+2)/2} \|\varphi(u)\|_{2,2,G(\varrho)} + \varrho \|d^u\|_{\infty,Q}\big)$$

for all $(t, x) \in G(\varrho)$.

In particular,

$$|\varphi(u)(\bar{t}, \bar{x})| \leqslant c_5 \big(\varrho^{-(n+2)/2} \|\varphi(u)\|_{2,2,G(\varrho)} + \varrho \|d^u\|_{\infty,Q}\big)\,.$$

Since $(\bar{t}, \bar{x})$ is an arbitrary element in K and the constant c_5 is independent of $(\bar{t}, \bar{x})$, it follows that

$$|\varphi(u)(t, x)| \leqslant c_5(\varrho^{-(n+2)/2}\|\varphi(u)\|_{2,2,G(\varrho)} + \varrho\|d^u\|_{\infty,Q}) \tag{3.16}$$

for all $(t, x) \in K$.

In view of Theorem 3.3 and the fact that $\|d^u\|_{\infty,Q} \leqslant \mu_3\|d^u\|_{2,Q} \leqslant \mu_3$ and $\|\varphi_0\|_{2,R^n} \leqslant \mu_3$, we deduce readily from (3.16) that

$$|\varphi(u)(t, x)| \leqslant c_5(\varrho^{-(n+2)/2}\sqrt{2}\, T^{\frac{1}{2}} c_1^{\frac{1}{2}} + \varrho)\, \mu_3 \tag{3.17}$$

for all $(t, x) \in K$.

Thus, $\{\varphi(u)\colon u \in D\}$ is uniformly bounded on any subset of Q that has a positive distance from $\{0\} \times \boldsymbol{R}^n$.

Let $(t, x) \in Q$ and $O = (t/2, T] \times B$ where

$$B \triangleq \{z \in \boldsymbol{R}^n\colon |z_i - x_i| < 2 \quad \text{for all } i = 1, \ldots, n\}.$$

By virtue of (3.17), there is a positive constant c_6 that depends only upon t, T, n, μ_1, μ_2 and μ_3 such that

$$|\varphi(u)(t', x')| \leqslant c_6$$

for all $(t', x') \in O$ and for all $u \in D$. On $\boldsymbol{R}^{1+n}$ define the pseudo norm $|\cdot|_\Delta$ such that

$$|(s, z)|_\Delta = \begin{cases} \max\{z_i^2, -s/4\} & s \leqslant 0\,. \\ \infty & s > 0\,. \end{cases}$$

Since $\varphi(u)$ is a weak solution of (3.1) on O, it follows from Theorem 4 ([3], pp. 110-111) that there are positive constants c_7 and α that depend only upon 0, n, μ_1, μ_2 and μ_3 such that

$$|\varphi(u)(t'', x'') - \varphi(u)(t', x')| \leqslant c_7(c_6 + \mu_3)\{|(t'' - t', x'' - x')|_\Delta/R\}^\alpha$$

for all (t', x'), $(t'', x'') \in O$, $t'' \leqslant t'$, where R is the minimum of 1 and the pseudo distance to ∂O from (t', x'). From this relationship it can easily be shown that if $t', x') \in Q$, and $|(t', x') - (t, x)| < \min(1/4, t/4)$, then

$$|\varphi(u)(t', x') - \varphi(u)(t, x)| \leqslant c_7(c_6 + \mu_3)\{|(t', x') - (t, x)|/4R\}^\gamma$$

where $R = \min\{1, t/16\}$, and thus that $\{\varphi(u), u \in D\}$ is equicontinuous at (t, x). This completes the proof.

Using Lemma 3.3 and the Ascoli-Arzela Theorem, we can easily obtain the following result.

LEMMA 3.5. - Let $\{u_k\}$ be a sequence in D. Then, there exist a subsequence of $\{\varphi(u_k)\}$ such that $\varphi(u_k)$ converges pointwise on Q and uniformly on ny compact subset of Q.

LEMMA 3.6. - Under the Convexity Assumption, if $\{u_k\}$ is a sequence in D then there is a subsequence $\{u_{k_l}\}$, and a $u_0 \in D$ such that $a_i^{u_k} \to a_i^{u_{0l}}$ $(i = 1, \ldots, n)$, ${}^{u_{k_l}} \to c^{u_0}$ and $d^{u_{0l}} \to d^{u_0}$ in the weak $*$ topology on $L^\infty(Q)$ as $l \to \infty$.

PROOF. - As $\|a_i^{u_k}\|_{\infty,Q}$ $(i = 1, \ldots, n)$, $\|c^{u_k}\|_{\infty,Q}$ and $\|d^{u_k}\|_{\infty,Q}$ are bounded by μ_3 for all positive integers k, there exist a_i^*, $(i = 1, \ldots, n)$, c^* and d^*, each belonging to $L^\infty(Q)$, and a subsequence of $\{u_k\}$, which is again indexed by k, such that

$$a_i^{u_k} \to a_i^*\,, \qquad (i = 1, \ldots, n)\,, \qquad c^{u_k} \to c^*\,, \qquad d^{u_k} \to d^*$$

in the weak $*$ topology on $L^\infty(Q)$ as $k \to \infty$.

Next, we shall show that there exists a $u_0 \in D$ such that $a_i^*(t, x) = a_i(t, x, u_0(t, x))$, $(i = 1, \ldots, n)$, $c^*(t, x) = c(t, x, u_0)t, x))$ and $d^*(t, x) = d(t, x, u_0(t, x))$. However this will follow directly from condition A_6 and Theorem 3′ ([7], p. 281) if we can show that

$$y(t, x) \triangleq \begin{bmatrix} a_1^*(t, x) \\ \cdots \\ a_n^*(t, x) \\ c^*(t, x) \\ d^*(t, x) \end{bmatrix} \in F(t, x, U) \triangleq \left\{ \begin{bmatrix} a_1(t, x, u) \\ \cdots \\ a_n(t, x, u) \\ c(t, x, u) \\ d(t, x, u) \end{bmatrix} : u \in U \right\}$$

almost everywhere on Q.

The proof is by contradiction. Suppose it were false. Then, there exists a measurable set $E_0 \subset Q$ so that $|E_0| > 0$ and

$$y(t, x) \notin F(t, x, U) \tag{3.18}$$

for all $(t, x) \in E_0$.

Since $|E_0| > 0$ it follows, respectively, from Theorem 1 (PLIS [9], p. 858) and the Lusin Theorem that we can choose a measurable subset $E_1 \subset Q$, $0 < |E_1| < |E_0|/2$ such that the measurable set-valued function $f(\cdot, \cdot, U)$ and the measurable function y are continuous on $Q \setminus E_1$. Let $E_2 \triangleq E_0 \cap \{Q \setminus E_1\}$.

Since $0 < |E_1| < |E_0|/2$, it follows that $E_2 \subset E_0$ and $|E_2| > 0$. Further, $F(\cdot, \cdot, U)$ and y are continuous on E_2, upon which (3.18) holds.

Let $(t_0, x_0) \in E_2$ and $\varepsilon > 0$. By the continuity of $F(\cdot, \cdot, U)$ at (t_0, x_0), there is an $\delta = \delta(\varepsilon) > 0$ such that

$$F(t, x, U) \subseteq F^\varepsilon(t_0, x_0, U) \tag{3.19}$$

whenever $|(t, x) - (t_0, x_0)| < \delta$, where $F^\varepsilon(t_0, x_0, U)$ is the closed ε neighbourhood of $F(t_0, x_0, U)$.

Let

$$M_\delta = \{(t, x) \in E_2 \colon |(t, x) - (t_0, x_0)| < \delta\}$$

and $\{B_m\}$ be a sequence of subsets of M_δ that contain (t_0, x_0) such that $|B_m| \to 0$ as $m \to \infty$. Further, let

$$y_k(t, x) = \begin{Bmatrix} a_1^{u_k}(t, x) \\ \cdots \\ a_n^{u_k}(t, x) \\ c^{u_k}(t, x) \\ d^{u_k}(t, x) \end{Bmatrix}. \tag{3.20}$$

Since, for all $(t, x) \in M_\delta$,

$$y_k(t, x) \in F(t, x, U) \subseteq F^\varepsilon(t_0, x_0, U)$$

and since, by Assumption A_6, $F^\varepsilon(t_0, x_0, U)$ is closed and convex, it follows that

$$\frac{1}{|B_m|} \int_{B_m} y_k(t, x)\, dx\, dt \in F^\varepsilon(t_0, x_0, U)$$

for all k. Further, as $F^\varepsilon(t_0, x_0, U)$ is closed, it is clear that

$$\lim_{k\to\infty} \frac{1}{|B_m|} \int_{B_m} y_k(t, x)\, dx\, dt \in F^\varepsilon(t_0, x_0, U)\,. \tag{3.21}$$

On the other hand, we note that y_k converges to y in the weak $*$ topology and that $|B_m| < \infty$ for all $m = 1, 2, \ldots$. Thus,

$$\lim_{k\to\infty} \frac{1}{|B_m|} \int_{B_m} y_k(t, x)\, dx\, dt = \frac{1}{|B_m|} \int_{B_m} y(t, x)\, dx\, dt\,.$$

Therefore, it follows from (3.20) that

$$\frac{1}{|B_m|} \int_{B_m} y(t, x)\, dx\, dt \in F^\varepsilon(t_0, x_0, U)\,. \tag{3.22}$$

Again, by using the closure of the set $F^\varepsilon(t_0, x_0, U)$ and the continuity of y at (t_0, x_0), it follows from (3.22) that

$$y(t_0, x_0) = \lim_{m\to\infty} \frac{1}{|B_m|} \int_{B_m} y(t, x)\, dx\, dt \in F^\varepsilon(t_0, x_0, U)\,. \tag{3.23}$$

As (3.22) holds for all $\varepsilon > 0$ and $F(t_0, x_0, U)$ is closed, we obtain that

$$y(t_0, x_0) \in F(t_0, x_0, U)\,.$$

But $(t_0, x_0) \in E_2$, and so from (3.17), we have that

$$y(t_0, x_0) \notin F(t_0, x_0, U)\,.$$

This is a contradiction and thus the proof is complete.

THEOREM 3.7. – Under the Convexity Assumption, if $\{u_k\}$ is a sequence in D then there exists a $u_0 \in D$ and a subsequence $\{u_{k_l}\}$ such that $\{\varphi(u_{k_l})\}$ converges to $\varphi(u_0)$ weakly in $L^2(Q)$, pointwise on Q and uniformly on compact subsets of Q, whereas $\{\varphi(u_{k_l})_{x_i}\}$ converges to $\varphi(u_0)_{x_i}$ $(i = 1, \ldots, n)$ weakly in $L^2(Q)$.

PROOF. – By Theorem (3.3), there is a positive constant c_1 independent of k such that

$$\|\varphi(u_k)\|_{2,2,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} T^{\frac{1}{2}} \mu_3 \tag{3.24}$$

$$\|\varphi(u_k)_{x_i}\|_{2,2,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} \mu_3 \qquad (i = 1, \ldots, n). \tag{3.25}$$

Thus, there exists a subsequence of $\{u_k\}$, which is also indexed by k, and functions φ and φ_i, $(i = 1, \ldots, n)$, each belonging to $L^2(Q)$, such that $\{\varphi(u_k)\}$ and $\{\varphi(u_k)_{x_i}\}$, $(i = 1, \ldots, n)$, converge, respectively, to φ and φ_i, $(i = 1, \ldots, n)$, weakly in $L^2(Q)$ as $k \to \infty$. From (3.24) and (3.25), we have that

$$\|\varphi\|_{2,2,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} T^{\frac{1}{2}} \mu_3 \tag{3.26}$$

$$\|\varphi_i\|_{2,2,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} \mu_3\,. \tag{3.27}$$

Using the definition of the distributional derivative of $\varphi(u_k)$, we have that, for any $\eta \in C_0^1(Q)$,

$$\int_Q \varphi(u_k)(t, x)\, \eta_{x_i}(t, x)\, dx\, dt = -\int_Q \varphi(u_k)_{x_i}(t, x)\, \eta(t, x)\, dt\, dx \tag{3.28}$$

By taking the limit as $k \to \infty$ of (3.28) and noting that both η_{x_i} and η belong to $L^2(Q)$, it follows from the fact that $\varphi(u_k) \to \varphi$ weakly in $L^2(Q)$ and that $\varphi(u_k)_{x_i} \to \varphi_i$ weakly in $L^2(Q)$ that

$$\int_Q \varphi(t, x)\,\eta_{x_i}(t, x)\,dx\,dt = -\int_Q \varphi_i(t, x)\,\eta(t, x)\,dx\,dt \tag{3.29}$$

for any $\eta \in C_0^1(Q)$, and thus $\varphi_i = \varphi_{x_i}$.

By Theorem 3.3, it is clear from assumption A_5 that

$$\|\varphi(u_k)\|_{2,\infty,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} \mu_3$$

for all k. Since c_1 is independent of k, it follows from Lemma 3 ([4], p. 633) that

$$\|\varphi\|_{2,\infty,Q} \leqslant \sqrt{2}\, c_1^{\frac{1}{2}} \mu_3 . \tag{3.30}$$

The inequalities (3.25), (3.26) and (3.30) imply that $\varphi \in L^{\infty,2}(Q) \cap L^2(I, W^1(\boldsymbol{R}^n))$.

Lemma 3.5 shows that there is a further subsequence of $\{u_k\}$, which is denoted by $\{u_{k_l}\}$, such that $\{\varphi(u_{k_l})\}$ converges to $\hat{\varphi}$ pointwise on Q and uniformly on any compact subset of Q. However, since $\{\varphi(u_{k_l})\}$ is a subsequence of $\{\varphi(u_k)\}$ and since $\varphi(u_k)$ converges to φ weakly in $L^2(Q)$ as $k \to \infty$, it is clear that $\varphi(u_{k_l})$ also converges to φ weakly in $L^2(Q)$ as $k \to \infty$.

We shall show that $\hat{\varphi}(x, t) = \varphi(x, t)$ for almost all $(x, t) \in Q$. Suppose it were false. Then, there exists a measurable subset $E_0 \subset Q$ with positive measure such that $\hat{\varphi}(x, t) \neq \varphi(x, t)$ for all $(x, t) \in E_0$. Let E_1 be a measurable subset of E_0 such that $0 < |E_1| < \infty$. Then, $\hat{\varphi}(x, t) \neq \varphi(x, t)$ for all $(x, t) \in E_1$. Recall that $\varphi(u_{k_l})$ converges to $\hat{\varphi}$ pointwise on Q and that $\varphi(u_{k_l})$ converges to φ weakly in $L^2(Q)$ as $l \to \infty$. Thus, in particular, $\varphi(u_{k_l})$ converges to $\hat{\varphi}$ pointwise on E_1 and $\varphi(u_{k_l})$ converges to φ weakly in $L^2(E_1)$ as $l \to \infty$. However, in view of the inequality (3.24) and the fact that $\varphi(u_{k_l})$ converges to $\hat{\varphi}$ pointwise on E_1, it follows from Theorem 13.44 ([6], p. 207) that $\varphi(u_{k_l})$ also converges to $\hat{\varphi}$ weakly in $L^2(E_1)$ as $l \to \infty$. By the uniqueness of the weak limit, we observe readily that $\hat{\varphi}(x, t) = \varphi(x, t)$ for almost all $(x, t) \in E_1$. This is a contradiction. Thus, we conclude that $\hat{\varphi}(x, t) = \varphi(x, t)$ for almost all $(x, t) \in Q$.

By Lemma 3.6, there is a further subsequence of $\{u_{k_l}\}$ which will be indexed by k and a $u_0 \in D$ such that $a_j^{u_k} \to a_i^{u_0}$, $(i = 1, \ldots, n)$, $c^{n_0} \to c^{x_0}$ and $d^{u_k} \to d^{u_0}$ as $k \to \infty$ in the weak $*$ topology on $L^\infty(Q)$. It will be shown that $\varphi = \varphi(u_0)$.

Let $\eta \in C_0^1(Q)$, and Ξ be the compact support of η. Since $\varphi(u_k)$ is the solution of system S_1 corresponding to u_k, then we readily observe from Definition 3.1 and 3.2 that

$$\mathfrak{L}^{u_k}(\varphi(u_k), \eta) = 0 . \tag{3.31}$$

Note that $\varphi(u_k)_{x_i} \to \varphi_{x_i}$ $(i = 1, \ldots, n)$ weakly in $L^2(Q)$ as $n \to \infty$ and $a_{ij}\eta_{x_j}$, $b_j\eta$ belong to $L^2(Q)$. Thus,

$$\int_Q a_{ij}\eta_{x_j}\varphi(u_k)_{x_j} \to \int_Q a_{ij}\eta_{x_j}\varphi_{x_i} \qquad \int_Q b_j\eta\varphi(u_k)_{x_j} \to \int_Q b\ \eta\varphi_{x_i}$$

as $k \to \infty$. Since $d^{u_k} \to d^{u_0}$ in the weak $*$ topology on $L^\infty(Q)$ and $\eta \in L^1(Q)$, it follows that

$$\int_Q d^{u_k}\eta \to \int_Q d^{u_0}\eta$$

as $k \to \infty$. Recall that $\eta \in C_0^1(Q)$ with compact support Ξ. Thus, we observe readily that $a_j^{u_0}\eta_{x_j} \to a_j^{u_0}\eta_{x_j}$ and $c^{u_k}\eta \to c^{u_0}\eta$ weakly in $L^2(Q)$ as $k \to \infty$ and the norms $\|a_j^{u_k}\eta_{u_j}\|_{2,Q}$ and $\|c^{u_k}\eta\|_{2,Q}$ are bounded independently of k. Further $\varphi(u_k) \to \varphi$ almost everywhere (in fact uniformly) on Ξ and $\varphi(u_k)$ is bounded independently of k on Ξ. Thus, it can be easily shown that

$$\int_Q a_j^{u_k}\eta_{x_j}\varphi(u_k) \to \int_Q a_j^{u_0}\eta_{x_j}\varphi, \qquad (j = 1, \dots, n),$$

and

$$\int_Q c^{u_k}\eta\varphi(u_k) \to \int_Q c^{u_0}\eta\varphi$$

as $k \to \infty$.

Next, we recall that $\varphi(u_k) \to \varphi$ weakly in $L^2(Q)$ and $\eta_t \in L^2(Q)$. Thus,

$$\int_Q \varphi(u_k)\,\eta_t \to \int \varphi\eta_t$$

as $k \to \infty$. Putting these results together with (3.31), it follows that

$$\mathfrak{L}^{u_0}(\varphi, \eta) = 0 \tag{3.32}$$

for all $\eta \in C_0^1(Q)$.

Let $\eta \in C_0^1(\boldsymbol{R}^n)$, and define $\eta^* : \bar{Q} \to \boldsymbol{R}^1$ such that $\eta^*(t, x) = \eta(x)\eta_1(t)$ where $\eta_1 \in C^1([0, T])$, $\eta_1(t) = 1$ if $t \in [0, T/4]$ and $\eta_1(t) = 0$ if $t \in [3T/4, T]$. In view of (3.32), it follows from Equation 1.3 ([4], p. 619) that if $\sigma \in [0, T/4]$ then

$$\int_{\boldsymbol{R}^n} \varphi(\sigma, x)\,\eta(x)\,dx = -\,\mathfrak{L}^{u_0}(\varphi, \eta^*)|_\sigma^T .$$

Similarly from (3.31)

$$\int_{\boldsymbol{R}^n} \varphi(u_k)(\sigma, x)\,\eta(x)\,dx = -\,\mathfrak{L}^{u_k}(\varphi(u_k), \eta^*)|_\sigma^T .$$

Using the same type of limits to deduce (3.32) from (3.31), it follows that

$$\mathfrak{L}^{u_k}(\varphi(u_k), \eta^*)|_\sigma^T \to \mathfrak{L}^{u_0}(\varphi, \eta^*)|_\sigma^T$$

as $k \to \infty$. Thus

$$\begin{aligned}
&\lim_{\sigma\downarrow 0} \int_{R^n} \varphi(\sigma, x)\eta(x)\,dx \\
&= -\lim_{\sigma\downarrow 0} \mathcal{L}^{u_0}(\varphi, \eta^*)|_\sigma^T \\
&= -\lim_{\sigma\uparrow 0}\ \lim_{k\to\infty} \mathcal{L}^{u_k}(\varphi(u_k), \eta^*)|_\sigma^T \\
&= \lim_{\sigma\downarrow 0}\ \lim_{k\to\infty} \int_{R^n} \varphi(u_k)(\sigma, x)\eta(x)\,dx\,.
\end{aligned}$$

By the last statement of Theorem 3.3, we recall that

$$\int_{R^n} \varphi(u_k)(\sigma, x)\,\eta(x)\,dx \to \int_{R^n} \varphi_0(x)\,\eta(x)\,dx$$

uniformly with respect to k as $\sigma\downarrow 0$. Thus

$$\lim_{\sigma\downarrow 0} \int_{R^n} \varphi(\sigma, x)\eta(x)\,dx = \int_{R^n} \varphi_0(x)\eta(x)\,dx$$

for all $\eta \in C_0^1(\boldsymbol{R}^n)$.

Therefore, φ is a weak solution of system S_1 corresponding to u_0. However, by Theorem 3.3, $\varphi(u_0)$ is its unique weak solution. This implies that $\varphi = \varphi(u_0)$ and hence the proof is complete.

As a special case, Theorem 3.7 is true under the linearity assumption. However, in this particular case, stronger results are possible. More precisely, for any sequence $\{u_k\} \subset D$, we can find a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ such that not only $\varphi(u_{k_l}\}$ converges to $\varphi(u_0)$ in the sense given in Theorem 3.7 as $l \to \infty$ but also u_k converges in the weak $*$ $*$ topology to u_0. This extra conclusion allows us to include u in criteria explicitly in Theorem 4.5 and Corollary 4.6.

For ease in future references, the corresponding result of Theorem 3.7 with convexity assumption replaced by linearity assumption is presented as follows:

THEOREM 3.8. – Under the linearity assumption, if $\{u_k\}$ is a sequence in D then there is a convergent subsequence $\{u_{k_l}\}$ in the weak $*$ topology with limit $u_0 \in D$ such that $\{\varphi(u_{k_l})\}$ converges to $\varphi(u_0)$ as $l \to \infty$ weakly in $L^2(Q)$, pointwise on Q and uniformly on compact subsets of Q, while $\{\varphi(u_{k_l})_{x_i}\}$ $(i = 1, \dots, n)$ converges, respectively, to $\varphi(u_0)_{x_i}$ $(i = 1, \dots, n)$, weakly in $L^2(Q)$ as $l \to \infty$.

PROOF. – As $\{u_i\}$ is a sequence in D, there is a subsequence, which is also indexed by k, that converges in the weak $*$ topology to a $u_0 \in D$. As in Theorem 3.7, there is a further subsequence of $\{u_k\}$, again indexed by k, such that $\{\varphi(u_k)\}$ converges to a function φ weakly in $L^2(Q)$, pointwise on Q and uniformly on compact subsets of Q, $\{\varphi(u_k)_{x_i}\}$ converge, respectively, weakly in $L^2(Q)$ to φ_{x_i}, $(i = 1, \ldots, n)$, and $\varphi \in L^{2,\infty}(Q) \cap \cap L^2(I, W^1(\boldsymbol{R}^n))$. The facts that $a_i^{u_k} \to a_i^{u_0}$, $(i = 1, \ldots, n)$, $c^{u_k} \to c^{u_0}$ and $d^{u_k} \to d^{u_0}$ follows directly from Assumption A_6'. The rest of the proof is exactly the same as for Theorem 3.7.

4. – The existence of optimal controls.

The discussion so far has been concerned with the properties of solutions of system S_1. To consider an optimal choice of $u \in D$, a performance criterion needs to be specified. In this section, a series of criteria will be presented and for each one it will be shown that an optimal control exists.

The first three cases will be criteria of the form, minimise a function $J: D \to \boldsymbol{R}^1$ where

$$J(u) = f(\varphi(u)) . \tag{4.1}$$

Let $D_1 = \{\varphi(u): u \in D\}$. The function $f: D_1 \to \boldsymbol{R}^1$ will be assumed to be bounded and satisfy a semicontinuity condition. The general result is stated in Theorem 4.1 with particular cases stated as corollaries. Note that if $J(D) = \{x \in \boldsymbol{R}^1: J(u) = x$ for some $u \in D\}$ then an optimal control exists if and only if there is a $u \in D$ such that $J(u) = \inf J(D)$. The existence of such $u \in D$ can be proved by Theorem 3.7 or Theorem 3.8.

Theorem 4.1. – Under the Convexity Assumption, let $f: D_1 \to \boldsymbol{R}^1$ satisfy the following conditions

(i) f is bounded upon D_1, and

(ii) for any sequence $\{\varphi_k\}$ in D_1 that converges to a $\varphi \in D_1$ either weakly in $L^2(Q)$ or pointwise on Q, there is a subsequence of $\{f(\varphi_k)\}$ that converges and

$$f(\varphi) \leq \lim_{l \to \infty} f(\varphi_{k_l}) . \tag{4.2}$$

Then, there is an optimal control to the problem of minimising $J(u) = f(\varphi(u))$ on D subject to system S_1.

Proof. – Since D is non empty and f is bounded upon D, it follows that $J(D)$ is a non empty bounded subset of $\boldsymbol{R}^1$, Thus, $\inf J(D)$ exists and there is a sequence $\{u_k\}$ in D such that

$$\lim_{k \to \infty} J(u_k) = \inf J(D) . \tag{4.3}$$

By Theorem 3.7, there is a subsequence of $\{u_k\}$, which is also indexed by k and a $u_0 \in D$ such that $\varphi(u_k) \to \varphi(u_0)$ both pointwise on Q and weakly in $L^2(Q)$. Thus, by (ii), there is a further subsequence of $\{u_k\}$, again indexed by k, such that

$$f(\varphi(u_0)) \leqslant \lim_{k\to\infty} f(\varphi(u_k)) . \tag{4.4}$$

Therefore, it follows from (4.1) and (4.4) that

$$J(u_0) = f(\varphi(u_0)) \leqslant \lim_{k\to\infty} f(\varphi(u_k)) = \lim_{k\to\infty} J(u_k) .$$

Combining this with (4.3), we observe readily that

$$J(u_0) \leqslant \inf J(D) .$$

As $J(u_0) \in J(D)$, this means $J(u_0) = \inf J(D)$ and thus proves the theorem.

COROLLARY 4.4. – Under the Convexity Assumption, let π be a measure on $\boldsymbol{R}^n$, $H: \boldsymbol{R}^n \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$ be measurable on $\boldsymbol{R}^n \times \boldsymbol{R}^1$, such that $H(x, \cdot)$ is continuous on $\boldsymbol{R}^1$ for each $x \in \boldsymbol{R}^n$ and satisfy the following inequality

$$|H(x, z)| \leqslant |p_1(x)| + |z||p_2(x)| \tag{4.5}$$

where p_1 and p_2 are integrable, with respect to π, functions from $\boldsymbol{R}^n \to \boldsymbol{R}^1$. Then, there is an optimal control to the problem of minimising over D

$$J(u) = \int_{\boldsymbol{R}^n} H(x, \varphi(u)(T, x))\, \pi(dx)$$

subject to system S_1.

PROOF. – Let $f: D_1 \to \boldsymbol{R}^1$ be such that

$$f(\varphi) = \int_{\boldsymbol{R}^n} H(x, \varphi(T, x))\, \pi(dx) .$$

If conditions (i) and (ii) of Theorem 4.1 hold, then the theorem is proved.

By (4.5), for all $x \in \boldsymbol{R}^n$,

$$|H(x, \varphi(T, x))| \leqslant |p_1(x)| + |\varphi(T, x)||p_2(x)| . \tag{4.6}$$

By Lemma 3.4, there is a positive constant C such that, for all $u \in D$ and for all $x \in \boldsymbol{R}^n$, $|\varphi(u)(T, x)| \leqslant C$. Thus, from (4.6) we have that

$$\int_{\boldsymbol{R}^n} |H(x, \varphi(u)(T, x))|\, \pi(dx) \leqslant \int_{\boldsymbol{R}^n} |p_1(x)| \pi(dx) + C \int_{\boldsymbol{R}^n} |p_2(x)| \pi(dx) .$$

Since p_1 and p_2 are integrable with respect to π, the right hand side is a finite bound independent of $u \in D$. Thus, f is bounded on D_1. This implies the condition (i) of Theorem 4.1.

Let $\{\varphi_k\}$ be a sequence in D_1 that converges pointwise to $\varphi_0 \in D_1$: By the continuity of H,

$$H(\cdot, \varphi_k(T, \cdot)) \to H(\cdot, \varphi_0(T, \cdot)) \quad \text{pointwise on } \boldsymbol{R}^n.$$

Recall that $|\varphi(u)(T, x)| \leqslant C$ for all $u \in D$ and for all $x \in \boldsymbol{R}^n$. Thus, it follows from (4.6) and the definition of D_1 that

$$|H(x, \varphi_k(T, x))| \leqslant |p_1(x)| + C|p_2(x)|$$

for all positive integers k and for all $x \in \boldsymbol{R}^n$. Therefore, by virtue of the Dominated Convergence Theorem, we deduce that

$$\lim_{k\to\infty} f(\varphi_k) = \lim_{k\to\infty} \int_{\boldsymbol{R}^n} H(x, \varphi_k(T, x))\, \pi(dx) = \int_{\boldsymbol{R}^n} H(x, \varphi_0(T, x)(\pi(dx) = f(\varphi_0)\,.$$

This, in turn, implies the condition (ii) of Theorem 4.1 and hence the proof is complete.

Corollary 4.3. – Under the Convexity Assumption, let $H: Q \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$ be measurable and satisfy the following conditions.

(i) H is convex on $\boldsymbol{R}^1$ for each $(t, x) \in Q$,

(ii) H is continuous on $Q \times R^1$,

(iii) $H(\cdot, \cdot, 0) \in L^1(Q)$, and

(iv) there exists a constant $\gamma \geqslant 0$, a $p_1 \in L^2(Q)$ and a measurable function $p: Q \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$ such that

$$|p(t, x, z)| \leqslant |p_1(t, x)| + \gamma |z| \tag{4.7}$$

for all $(t, x, z) \in Q \times \boldsymbol{R}^1$, and

$$p(t, x, z_1)(z_2 - z_1) \leqslant H(t, x, z_1) - H(t, x, z_2) \leqslant p(t, x, z_2)(z_1 - z_2) \tag{4.8}$$

for all $z_1 z_2 \in \boldsymbol{R}^1$ and for all $(t, x) \in Q$.

Then there is an optimal control for the problem of minimising

$$J(u) = \int_Q H(t, x, \varphi(u)(t, x))\, dx\, dt$$

over D subject to system S_1.

PROOF. – Let $f: D_1 \to \boldsymbol{R}^1$ be such that $f(\varphi) = \int_Q H(t, x, \varphi(t, x))\, dx\, dt$. By Theorem 4.1, it is required only to show that conditions (i) and (ii) of Theorem 4.1 hold.

Let $\varphi \in D_1$. Since $H(\cdot, \cdot, 0) \in L^1(Q)$, $p(\cdot, \cdot, 0) \in L^2(Q)$ and $\varphi \in L^2(Q)$, it follows from (4.8) that

$$\varphi(t, x) + H(t, x, 0) \leqslant H(t, x, \varphi(t, x)) \leqslant H(t, x, 0) + p(t, x, 0) + p(t, x, 0)\varphi(t, x)$$

for all $(t, x) \in Q$.

Thus by the definition of $f(\varphi)$ and by using 4.7, we deduce readily that

$$|f(\varphi)| \leqslant 2\|H(\cdot, \cdot, 0)\|_{1,Q} + \|p(\cdot, \cdot, 0)\|_{2,Q}\|\varphi\|_{2,Q} + \|p_1\|_{2,0}\|\varphi\|_{2,Q} + \gamma\|\varphi\|_{2,Q}$$

for all $\varphi \in D_1$, where

$$\|z(\cdot, \cdot, 0)\|_{p,Q} \triangleq \left\{\int_Q |z(t, x, 0)|^p\, dx\, dt\right\}^{1/p}$$

for all p, $1 \leqslant p < \infty$.

However, $\|\varphi\|_{2,Q} \leqslant 2T^{\frac{1}{2}} c_1^{\frac{1}{2}} \mu_3$. Thus, f is bounded on D_1. This, in turn, implies the condition (i) of Theorem 4.1.

Let $\{\varphi_k\}$ be a sequence in D_1 that converges to φ^* weakly in $L^2(Q)$. By the Banach Saks Theorem ([10], p. 80) there is a subsequence of $\{\varphi_k\}$, which is also indexed by k, such that $1/\nu \sum_{k=1}^{\nu} \varphi_k \to \varphi^*$ strongly in $L^2(Q)$, as $\nu \to \infty$. Let $\varphi^\nu = 1/\nu \sum_{k=1}^{\nu} \varphi_k$. Because H is convex on $\boldsymbol{R}^1$, for each ν,

$$\frac{1}{\nu}\sum_{k=1}^{\nu} f(\varphi_k) = \frac{1}{\nu}\sum_{k=1}^{\nu} \int_Q H(t, x, \varphi_k(t, x))\, dx\, dt \geqslant \int_Q H(t, x, \varphi^\nu(t, x))\, dx\, dt\,.$$

Thus

$$\lim_{\nu\to\infty} \frac{1}{\nu}\sum_{k=1}^{\nu} f(\varphi_k) \geqslant \lim_{\nu\to\infty} \int_Q H(t, x, \varphi^\nu(t, x))\, dx\, dt\,. \tag{4.9}$$

As $\{f(\varphi_k)\}$ is a bounded sequence in $\boldsymbol{R}^1$, there is a subsequence, again indexed by k, such that $\{f(\varphi_k)\}$ converges to a limit. However, it is well-known that if a sequence of reals converges to a limit then the sequence of the averages also converges to the same limit. Thus, $\lim_{k\to\infty} f(\varphi_k) = \lim_{\nu\to\infty} (1/\nu) \sum_{k=1}^{\nu} f(\varphi_k)$. Combining this relation with (4.9), it follows that

$$\lim_{k\to\infty} f(\varphi_k) \geqslant \lim_{\nu\to\infty} \int_Q H(t, x, \varphi^\nu(t, x))\, dx\, dt\,. \tag{4.10}$$

Letting $z_1 = \varphi^*(t, x)$ and $z_2 = \varphi^\nu(t, x)$ in (4.8), integrating over Q and letting $\nu \to \infty$, we have

$$\text{(4.11)}\quad \lim_{\nu\to\infty} \int_Q p(t, x, \varphi^*(t, x))(\varphi^\nu(t, x) - \varphi^*(t, x))\, dx\, dt \leq \int_Q H(t, x, \varphi^*(t, x))\, dx\, dt - \lim_{\nu\to\infty} \int_Q H(t, x, \varphi^\nu(t, x))\, dx\, dt \leq \leq \lim_{\nu\to\infty} \int_Q p(t, x, \varphi^\nu(t, x))(\varphi^*(t, x) - \varphi^\nu(t, x))\, dx\, dt\,.$$

However, as $\varphi^\nu \to \varphi^*$ strongly in $L^2(Q)$,

$$\Big|\int_Q p(t, x, \varphi^*(t, x))(\varphi^\nu(t, x) - \varphi^*(t, x))\, dx\, dt\Big| \leq \|p(\cdot, \cdot, \varphi^*(\cdot, \cdot))\|_{2,Q} \|\varphi^\nu - \varphi^*\|_{2,Q} \leq (\|p_1\|_{2,Q} + \gamma \|\varphi^*\|_{2,Q}) \|\varphi^\nu - \varphi^*\|_{2,Q} \to 0 \quad \text{as } \nu \to \infty.$$

Thus,

$$\text{(4.12)}\quad \lim_{\nu\to\infty} \int_Q p(t, x, \varphi^*(t, x))(\varphi^\nu(t, x) - \varphi^*(t, x))\, dx\, dt = 0\,.$$

Next, using the fact that $\|\varphi^\nu\|_{2,Q} \leq \sqrt{2}\, T^{\frac12} c_1^{\frac12} \mu_3$ instead of $\|\varphi^*\|_{2,Q} \leq \sqrt{2}\, T^{\frac12} c_1^{\frac12} \mu_3$, we deduce readily from an argument similar to that given for expression (4.12) that

$$\text{(4.13)}\quad \lim_{\nu\to\infty} \int_Q p(t, x, \varphi^\nu(t, x))(\varphi^*(t, x) - \varphi^\nu(t, x))\, dx\, dt = 0\,.$$

Combining (4.11), (4.12) and (4.13), we have that

$$\lim_{\nu\to\infty} \int_Q H(t, x, \varphi^\nu(t, x))\, dx\, dt = \int_Q H(t, x, \varphi^*(t, x))\, dx\, dt = f(\varphi^*)\,.$$

Thus, by virtue of (4.10), it follows readily that

$$\lim_{k\to\infty} f(\varphi_k) \geq f(\varphi^*)\,.$$

This, in turn, implies the condition (ii) of Theorem 4.1. Thus, the proof is complete.

The technique of Corollary 4.2 fails for Corollary 4.3 because $\varphi(u)$ is bounded only on sets that have a positive distance from $\{0\} \times \boldsymbol{R}^n$. However when $\varphi_0 = 0$ a stronger result is available for Lemma 3.4, namely the set $\{\varphi(u) : u \in D\}$ is uniformly bounded upon $\bar{Q}$. This result, which was given in Corollary 3.2 ([4], p. 643), will be used later to prove the Corollary 4.4 below. Note that if $\varphi_0 \in C_0^2(\boldsymbol{R}^n)$ then the system can be transformed into the one with zero initial condition by considering

$$\Phi(t, x) = \varphi(u)(t, x) - \varphi_0(t, x) \in \bar{Q}\ .$$

COROLLARY 4.4. – Under the Convexity Assumption, let $\varphi_0 = 0$ and let $H: Q \times \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$ be measurable such that

(i) H is continuous on $\boldsymbol{R}^1$ for each $(t, x) \in Q$.

(ii) $H(t, x, z) \leqslant p_1(t, x) + |z||p_2(t, x)|$ for all $(t, x, z) \in Q \times \boldsymbol{R}^1$, where $p_1,\ p_2 \in L^1(Q)$.

Then, there is an optimal control to the problem of minimising over D

$$J(u) = \int_Q H\big(t, x, \varphi(u)(t, x)\big)\, dx\, dt$$

subject to the system S_1.

PROOF. – Let $f: D_1 \to \boldsymbol{R}^1$ be defined by $f(\varphi) = \int_Q H\big(t, x, \varphi(t, x)\big)\, dx\, dt$. Since $\varphi_0 \equiv 0$ on $\boldsymbol{R}^n$, by Corollary 3.2 ([4], p. 643) and the definition of D_1, there is a positive constant c such that

$$|\varphi(t, x)| \leqslant c\mu_3$$

for all $\varphi \in D_1$ and for all $(t, x) \in \bar{Q}$.

Thus, it follows from the definition of $f(\varphi)$ and condition (ii) that

$$|f(\varphi)| \leqslant \|p_1\|_{1,Q} + c\mu_3 \|p_2\|_{1,Q}\ .$$

for all $\varphi \in D_1$.

This, in turn, implies that f is bounded on D_1 and hence the condition (i) of Theorem 4.1 is satisfied.

Next, using an argument similar to that given for the corresponding part of Corollary 4.2, we can easily show that the condition (ii) of Theorem 4.1 is also satisfied. Thus, it follows from Theorem 4.1 that there is an optimal control.

THEOREM 4.5. – Under the Linearity Assumption, let $f: D_1 \times D \to R^1$ be measurable such that

(i) f is bounded on $D_1 \times D$ and

(ii) for any sequence $\{u_k\}$ that converges in the weak $*$ topology to a $u_0 \in D$ such that $\{\varphi(u_k)\}$ converges weakly in $L^2(Q)$, or pointwise on Q, to $\varphi(u_0)$, there is a subsequence $\{u_{k_l}\}$ such that $\{f(\varphi(u_{k_l}), u_{k_l})\}$ converges and

$$f(\varphi(u_0), u_0) \leqslant \lim_{l\to\infty} f\big(\varphi(u_{k_l}), u_{k_l}\big)\ .$$

Then, there is an optimal control for the problem of minimising over D

$$J(u) = f(\varphi(u), u)$$

subject to the system S_1.

Proof. – By precisely the same argument as that given for (4.3), we note that if $\inf J(D)$ exists then there is a sequence $\{u_k\} \subset D$ such that $\lim_{k\to\infty} J(u_k) = \inf J(D)$. Since $\{u_k\} \subset D$, by the Linearity assumption it follows from Theorem 3.8 that there is a subsequence, which is also indexed by k, and a $u_0 \in D$ such that $u_k \to u_0$ in the weak $*$ topology and $\varphi(u_k) \to \varphi(u_0)$ in the weak topology of $L^2(Q)$ and pointwise on Q. By condition (ii) there is a further subsequence, also indexed by k, such that

$$f(\varphi(u_0), u_0) \leqslant \lim_{k\to\infty} f(\varphi(u_k), u_k) ,$$

and hence

$$J(u_0) \leqslant \lim_{k\to\infty} J(u_k) = \inf J(D) .$$

Thus,

$$J(u_0) = \inf J(D) .$$

This implies that u_0 is an optimal control and the proof is complete.

Corollary 4.6. – Under the Linearity Assumption, let $H: Q \times \boldsymbol{R}^1 \times U \to \boldsymbol{R}^1$ be a measurable function such that

(i) H is convex on $\boldsymbol{R}^1$ for each $(t, x, u) \in Q \times U$.

(ii) $H(\cdot, \cdot, 0, 0) \in L^2(Q)$,

(iii) there is a function $p_1 \in L^2(Q)$ such that

$$-p_1(t, x)(\varphi_1 - \varphi_2) \leqslant H(t, x, \varphi_1, u) - H(t, x, \varphi_2, u) \leqslant p_1(t, x)(\varphi_1 - \varphi_2)$$

almost everywhere on Q for all $\varphi_1, \varphi_2 \in \boldsymbol{R}^1$, $u \in U$, and

(iv) there is an r-dimensional vector-valued function p_2 with its components belonging to $L^1(Q)$ such that

$$\langle -p_2(t, x), (u_1 - u_2)\rangle \leqslant H(t, x, \varphi, u_1) - H(t, x, \varphi, u_2) \leqslant \langle p_2(t, x), (u_1 - u_2)\rangle$$

almost everwhere on Q for all $\varphi \in \boldsymbol{R}^1$, $u_1, u_2 \in U$. Then, there is an optimal control to the problem of minimising over D,

$$J(u) = \int_Q H(t, x, \varphi(u)(t, x), u(t, x))\, dx\, dt ,$$

subject to the system S_1.

PROOF. – Let $f: D_1 \times D \to \boldsymbol{R}^2$ be defined by

$$f(\varphi, u) = \int_Q H(t, x, \varphi(t, x), u(t, x))\, dx\, dt\,.$$

By (iii) and (iv), we deduce that

$$H(t, x, 0, 0) - p_1(t, x)\varphi(t, x) - \langle p_2(t, x), u(t, x)\rangle \leqslant H(t, x, \varphi(t, x), u(t, x)) \leqslant H(t, x, 0, 0) + p_1(t, x)\varphi(t, x) + \langle p_2(t, x), u(t, x)\rangle$$

for almost all $(t, x) \in Q$, for all $\varphi \in D_1$ and for all $u \in D$. Thus, it follows that

$$|f(\varphi, u)| \leqslant 2\|H(\cdot, \cdot, 0, 0)\|_{1,Q} + 2(\|p_1\|_{2,Q})(\|\varphi\|_{2,Q}) + 2(\|p_2\|_{1,Q})(\|u\|_{\infty,Q})$$

and hence f is bounded upon $D_1 \times D$. This implies the condition (i) of Theorem 4.5.

Let $\{u_k\}$ be a sequence in D. Then, it follows from Theorem 3.8 that there is a subsequence, also indexed by k, and a $u_0 \in D$ such that u_k converges in the weak $*$ $*$ topology to u_0 and $\varphi(u_k)$ converges weakly in $L^2(Q)$ to $\varphi(u_0)$.

For each positive integer ν, let

$$\varphi^\nu = \frac{1}{\nu}\sum_{k=1}^{\nu} \varphi(u_k)\,. \tag{4.14}$$

Since $\varphi(u_k) \to \varphi(u_3)$ weakly in $L^2(Q)$ as $k \to \infty$, it can be easily shown that $\varphi^\nu \to \varphi(u_0)$ weakly in $L^2(Q)$ and $\varphi^\nu - \varphi(u_\nu) \to 0$ weakly in $L^2(Q)$ as $\nu \to \infty$.

Let $k \in [1, \nu]$ be an arbitrary integer. Now, in view of condition (iii), we have

$$\begin{aligned} - p_1(t, x)(\varphi(u_k)(t, x) - \varphi(u_\nu)(t, x)) &\\ \leqslant H(t, x, \varphi(u_k)(t, x), u_\nu(t, x)) - H(t, x, \varphi(u_\nu)(t, x), u_\nu(t, x)) &\\ \leqslant p_1(t, x)(\varphi(u_k)(t, x) - \varphi(u_a)(t, x)) & \end{aligned}$$

almost everywhere on Q. Summing over k, dividing by ν, using (4.14), integrating over Q, and then using the definition of J, we obtain that

$$\begin{aligned} -\int_Q p_1(t, x)(\varphi^\nu(t, x) - \varphi(u_\nu)(t, x))\, dx\, dt &\\ \leqslant \int_Q \frac{1}{\nu}\sum_{k=1}^{\nu} H(t, x, \varphi(u_k)(t, x), u_\nu(t, x))\, dx\, dt - J(u_\nu) &\\ \leqslant \int_Q p_1(t, x)(\varphi^\nu(t, x) - \varphi(u_\nu)(t, x))\, dx\, dt\,. & \end{aligned} \tag{4.15}$$

As $\varphi^\nu - \varphi(u_\nu) \to 0$ weakly in $L^2(Q)$ as $\nu \to \infty$ and $p_1 \in L^2(Q)$, it follows that

$$\int_Q p_1(t, x)\big(\varphi^\nu(t, x) - \varphi(u_\nu)(t, x)\big)\, dx\, dt \to 0 \qquad \text{as } \nu \to \infty.$$

Thus, by taking the limit as $\nu \to \infty$ in (4.15), we obtain that

$$\lim_{\nu\to\infty} \int_Q \frac{1}{\nu} \sum_{k=1}^{\nu} H\big(t, x, \varphi(u_k)(t, x), u_\nu(t, x)\big)\, dx\, dt = \lim_{\nu\to\infty} J(u_\nu)\,. \tag{4.16}$$

By the convexity of H, we have that

$$\frac{1}{\nu}\sum_{k=1}^{\nu} H\big(t, x, \varphi(u_k)(t, x), u_\nu(t, x)\big) \geqslant H\big(t, x, \varphi^\nu(t, x), u_\nu(t, x)\big)\,.$$

Consequently, it follows from (4.16) that

$$\lim_{\nu\to\infty} \int_Q H\big(t, x, \varphi^\nu(t, x), u_\nu(t, x)\big)\, dx\, dt \leqslant \lim_{\nu\to\infty} J(u_\nu)\,. \tag{4.17}$$

Further, by virtue of condition (iii) with $u = u_\nu(t, x)$, $\varphi_1 = \varphi^\nu(t, x)$ and $\varphi_2 = \varphi(u_0)\cdot$ $\cdot(t, x)$, we deduce that

$$\begin{aligned} -\int_Q p_1(t, x)\big(\varphi^\nu(t, x) - \varphi(u_0)(t, x)\big)\, dx\, dt & \\ \leqslant \int_Q H\big(t, x, \varphi^\nu(t, x), u_\nu(t, x)\big)\, dx\, dt - \int_Q H\big(t, x, \varphi(u_0)(t, x), u_\nu(t, x)\big)\, dx\, dt & \\ \leqslant \int_Q p_1(t, x)\big(\varphi^\nu(t, x) - \varphi(u_0)(t, x)\big)\, dx\, dt\,. & \end{aligned} \tag{4.18}$$

Again, as $\varphi^\nu \to \varphi(u_0)$ weakly in $L^2(Q)$ and $p_1 \in L^2(Q)$, it follows from taking $\nu \to \infty$ in inequality (4.18) that

$$\lim_{\nu\to\infty} \int_Q H\big(t, x, \varphi^\nu(t, x), u_\nu(t, x)\big)\, dx\, dt = \lim_{\nu\to\infty} \int_Q H\big(t, x, \varphi(u_0)(t, x), u_\nu(t, x)\big)\, dx\, dt\,. \tag{4.19}$$

The inequality (iv) will be used to evaluate the right hand side of (4.19). For this, let $u_1 = u_\nu(t, x)$, $u_2 = u_0(t, x)$ and $\varphi = \varphi(u_0)(t, x)$, it follows that

$$\begin{aligned} -\int_Q \langle p_2(t, x), u_\nu(t, x) - u_0(t, x)\rangle\, dx\, dt & \\ \leqslant \int_Q H\big(t, x, \varphi(u_0)(t, x), u_\nu(t, x)\big)\, dx\, dt - J(u_0) & \\ \leqslant \int_Q \langle p_2(t, x), u_\nu(t, x) - u_0(t, x)\rangle\, dx\, dt\,. & \end{aligned} \tag{4.20}$$

Since $u_\nu \to u_0$ in the weak $*$ topology and the components of p_2 are in $L^1(Q)$, we observe readily that

$$\lim_{\nu\to\infty}\int_Q \langle p_t(t,x), u_\nu(t,x) - u_0(t,x)\rangle\, dx\, dt = 0 .$$

Thus, we deduce easily from taking the limit of inequality (3.20) with respect to ν that

$$\lim_{\nu\to\infty}\int_Q H(t, x, \varphi_0(u_0)(t,x), u_\nu(t,x))\, dx\, dt = J(u_0) . \tag{4.21}$$

Combining (4.17), (4.19) and (4.21), we conclude that

$$J(u_0) \leqslant \lim_{\nu\to\infty} J(u_\nu) = \lim_{k\to\infty} J(u_k) .$$

Thus, condition (ii) of Theorem 4.5 is also satisfied and hence the conclusion follows immediately from that theorem. This completes the proof.

As was stated in the introduction, this paper only provides a partial answer to the question of the existence of optimal controls of systems governed by linear second order parabolic partial differential equation with Cauchy conditions. The difficulties lie in the proof of Theorems 3.7 and 3.8. Because $\varphi(u_k)_{x_i} \to \varphi_{x_i}$ $(i = 1, \ldots, n)$ only weakly in $L^2(Q)$, we do not have the convergence of

$$\left\{\int_Q a_{ij}(t,x)\,\varphi(u_k)_{x_i}(t,x)\,\eta_{x_j}(t,x)\, dx\, dt\right\} \qquad (i, j = 1, \ldots, n)$$

or of

$$\left\{\int_Q b_j(t,x)\,\varphi(u_k)_{x_i}(t,x)\,\eta(t,x)\, dx\, dt\right\} \qquad (i = 1, \ldots, n)$$

if $a_{ij}(i, j = 1, \ldots, n)$, or $b_j(j = 1, \ldots, n)$ also depend upon u.

It should be noted that in Section 8 of [13], examples are exhibited where the a_{ij} depends upon the control and there is no optimal control. The important case of allowing the b_j's to depend upon u is left as an open question.

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