Real Hypersurfaces in Quaternionic Projective Space (*).

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Summary. – *This paper is devoted to make a systematic study of real hypersurfaces of quaternionic projective space using focal set theory. We obtain three types of such real hypersurfaces. Two of them are known. Third type is new and in its study the first example of proper quaternion CR-submanifold appears. We study real hypersurfaces with constant principal curvatures and classify such hypersurfaces with at most two distinct principal curvatures. Finally we study the Ricci tensor of a real hypersurface of quaternionic projective space and classify pseudo-Einstein, almost-Einstein and Einstein real hypersurfaces.*

0. – Introduction.

Real hypersurfaces of a Riemannian manifold have been largely studied, especially in the case of a real space form (see [4], [9], [17], ...) and a complex space form (see [5] and its references). However, results for real hypersurfaces of quaternion space forms (and concretely, of quaternionic projective space) are few. In fact, the first paper on this subject is the one of Pak, [18], and we can also cite [19].

In the case of complex projective space, CP^m, Cecil and Ryan, [5], applying focal set theory obtain some examples of real hypersurfaces that can classify attending to the behaviour of their Ricci tensor. They also obtain the non-existence of Einstein real hypersurfaces in complex projective space. In the case of quaternionic projective space this result is not true. This can be seen from [15] because there is a radius such that the corresponding geodesic hypersphere is Einstein.

The purpose of the present paper is to make a systematic study of real hypersurfaces of quaternionic projective space. For this we adapt focal set theory to this space, considering quaternionic projective space embedded in a Euclidean space (see [20] and [21]), embedding whose geodesics are easily expressed.

In § 2, using focal set theory we obtain three examples of real hypersurfaces. Two of them are largely known and are the ones classified by Pak [18], attending to the behaviour of their second fundamental form. Third example is new and § 3 and § 4 are centered on its study.

The examples obtained have constant principal curvatures, thus § 5 is devoted to the study of such real hypersurfaces that we classify if they satisfy an additional

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condition concerning certain distributions over the hypersurface associated to the quaternionic structure of quaternionic projective space (Theorem 5.7).

In § 6 we classify real hypersurfaces with two distinct principal curvatures (Theorem 6.2).

Finally, § 7 is devoted to the study of the Ricci tensor of a real hypersurface in quaternionic projective space. Concretely, after a classification of pseudo-Einstein and almost-Einstein (Definition 7.1) real hypersurfaces we obtain that the only Einstein real hypersurfaces are open subsets of geodesic hyperspheres of a certain radius (the ones mentioned above).

The results obtained along the paper establish clear differences between the complex case (as studied by CECIL and RYAN) and quaternionic case.

1. – Quaternionic projective space. Basic statements.

Let us consider Q^{m+1}, $m\geqslant 2$, Q being the algebra of quaternions with its usual symplectic product $\langle\, ,\rangle$ and let $g_0 = \mathrm{Re}\ \langle\, ,\rangle$ be the usual Euclidean metric on Q^{m+1}. The quaternionic projective space QP^m can be obtained from the unit sphere S^{4m+3} of Q^{m+1} by identifying x to λx, $\lambda\in Q$, $|\lambda| = 1$. Thus S^{4m+3} is a fibre bundle over QP^m with structural group S^3 and projection Π. If $q\in S^{4m+3}$ the horizontal subspace of T_qS^{4m+3} is $T'_q = \{p\in Q^{m+1}/\langle p, q\rangle = 0\}$ and the vertical subspace is spanned by j_1q, j_2q and j_3q, where j_1, j_2, j_3 are the unit quaternions.

We shall consider on QP^m the metric given by $g(X, Y) = g_0(X', Y')$ and the connection $\overline{\nabla}_X Y = \Pi_*(\nabla'_{X'} Y')$ for any $X, Y\in TQP^m$ where $'$ denotes the corresponding horizontal lift and ∇' is the covariant differentiation of S^{4m+3}. Then, it is known that the Sasakian 3-structure of S^{4m+3} induces on QP^m a structure of quaternion Kaehlerian manifold of constant quaternionic sectional curvature 4 (see [13], [14]) (¹). That is, there exists on QP^m a 3-dimensional vector bundle $\hat{V}$ of tensors of type $(1, 1)$ with local basis of almost Hermitian structures $\{J_1, J_2, J_3\}$ satisfying,

$$J_1J_2 = -J_2J_1 = J_3 \tag{1.1}$$

$$\overline{\nabla}_X J_i = q_k(X)J_j - q_j(X)J_k \qquad i = 1, 2, 3 \tag{1.2}$$

$$(dq_i + q_j\wedge q_k)(X, Y) = 4g(X, J_iY)\,, \quad i = 1, 2, 3, \tag{1.3}$$

$$\overline{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{k=1}^{3}\{g(J_kY, Z)J_kX - g(J_kX, Z)J_kY + 2g(X, J_kY)J_kZ\} \tag{1.4}$$

(¹) In this paper we shall consider on QP^m the metric of constant quaternionic sectional curvature 4, and $m\geqslant 2$.

for any $X, Y, Z \in TQP^m$, where $\bar{R}$ denotes the Riemannian curvature tensor of $\bar{\nabla}$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$, q_k, $k = 1, 2, 3$, being local 1-forms on QP^m.

Let $HM(m+1) = \{B \in gl(m+1, Q)/B^c = B^t\}$ where B^c (respectively, B^t) denotes the quaternionic conjugate (respectively, the transpose) of B. Consider on $HM(m+1)$ the metric given by

$$g(A, B) = \tfrac{1}{2}\ \text{trace}\ (AB)\,, \qquad A, B \in HM(m+1)\,. \tag{1.5}$$

In [20] SAKAMOTO proves that $\tilde{\psi}$: $S^{4m+3} \to HM(m+1)$ given by

$$\tilde{\psi}(q) = q^{ct}q\,, \qquad q \in S^{4m+3} \tag{1.6}$$

nduces an immersion Ψ: $QP^m \to HM(m+1)$ satisfying

(A) $\Psi(QP^m) = \{B \in HM(m+1)/B^2 = B,\ \text{trace}\ B = 1\}$;

(B) Ψ is an equivariant full isometric embedding into $\{B \in HM(m+1)/\text{trace}\ B = 1\}$.

Therefore, we can consider QP^m identified to $\Psi(QP^m)$. In the rest of the paper the reader would make the necessary changes of notation when QP^m is identified to $\Psi(QP^m)$. Under this identification the tangent and normal spaces to QP^m at $B \in QP^m$ are

$$T_B QP^m = \{X \in HM(m+1)/XB + BX = X\} \tag{1.7}$$

$$T_B^\perp QP^m = \{Z \in HM(m+1)/ZB = BZ\}\,. \tag{1.8}$$

Denoting by $\tilde{\nabla}$ the connection induced on $QP^m = \Psi(QP^m)$ by the Riemannian one of $HM(m+1)$ and by $\tilde{\sigma}$, $\tilde{\nabla}^\perp$ and $\tilde{A}$ respectively the second fundamental form, the normal connection and the shape operator of $QP^m = \Psi(QP^m)$ in $HM(m+1)$ we get, [8],

$$\tilde{\sigma}(X, Y) = (XY + YX)(I - 2B)\,, \qquad \tilde{A}_Z X = (XZ - ZX)(I - 2B) \tag{1.9}$$

$$\tilde{\sigma}(J_k X, J_k Y) = \tilde{\sigma}(X, Y)\,, \qquad k = 1, 2, 3 \tag{1.10}$$

for any $X, Y \in T_B QP^m$, $Z \in T_B^\perp QP^m$, where I denotes the identity matrix of $HM(m+1)$ and $\{J_1, J_2, J_3\}$ is a local basis of the quaternionic structure of QP^m. Moreover from (1.10),

$$\tilde{\nabla}\tilde{\sigma} = 0 \tag{1.11}$$

that is, the second fundamental form of QP^m in $HM(m+1)$ is parallel.

From (1.4), (1.9), (1.10) and the equation of Gauss it follows

$$(1.12)\quad g(\tilde\sigma(X, Y), \tilde\sigma(V, W)) = 2g(X, Y)g(V, W) + g(X, V)g(Y, W) + g(X, W)g(Y, V) + \sum_{k=1}^{3}\{g(J_kX, V)g(J_kY, W) + g(J_kX, W)g(J_kY, V)\}$$

for any $X, Y, V, W \in T_BQP^m$.

Finally, the geodesic γ of QP^m passing through B and having the direction $X \in T_BQP^m$ is given, [20], by

$$(1.13)\quad \gamma(t) = B + \frac{1}{2}\sin 2tX + \frac{1-\cos 2t}{4}\tilde\sigma(X, X).$$

Let M be a real hypersurface of QP^m and N a unit local normal vector field to M. We shall denote $U_k = -J_kN$, $k = 1, 2, 3$, $D' = Sp\{U_1, U_2, U_3\}$ and D the orthogonal complement of D' in TM and $J_kX = T_kX + f_k(X)N$, $k = 1, 2, 3$, where T_kX is the tangent component of J_kX and $f_k(X) = g(X, U_k)$, $X \in TM$. From (1.2) we have

$$(1.14)\quad \nabla_X U_i = -q_j(X)U_k + q_k(X)U + T_iAX, \quad i = 1, 2, 3$$

∇ being the connection induced on M and A the Weingarten endomorphism of M, (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Moreover, from (1.14), we deduce:

$$(1.15)\quad SX = (4m+7)X + hAX - A^2X - 3\sum_{k=1}^{3} f_k(X)U_k,$$

for any $X \in TM$, where $h = \operatorname{trace} A$ and the Ricci tensor of M is given by $S(X, Y) = g(SX, Y)$, for any $X, Y \in TM$.

From (1.14), the Codazzi equation of M in QP^m is,

$$(1.16)\quad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{k=1}^{3}\{f_k(X)T_kY - f_k(Y)T_kX + 2g(X, T_kY)U_k\},$$

for any $X, Y \in TM$.

2. – Focal sets and tubes in QP^m.

Let M^n be an n-dimensional submanifold of QP^m (2). Let $T^\perp M$ denote the normal bundle of M. For any $(B, \xi) \in T^\perp M$, let $F(B, \xi)$ be the point of QP^m at a distance $\|\xi\|$ along the geodesic of QP^m passing through B with direction ξ.

(2) Submanifolds appearing in this paper are considered to be *connected*. For theory of submanifolds of QP^m see [1], [7], [10], [11], ...

DEFINITION 2.1. – A point $E \in QP^m$ is called a *focal point* of multiplicity $v > 0$ of (M, B) if $E = F(B, \xi)$ for some $\xi \in T_B^\perp M$ and the Jacobian of F has nullity v at (B, ξ).

Let $\xi \in T_B^\perp M$, $\|\xi\| = 1$. From (1.13), we have:

$$F(B, r\xi) = B + \frac{1}{2}\sin 2r\xi + \frac{1-\cos 2r}{4}\tilde{\sigma}(\xi, \xi)\,, \tag{2.1}$$

and taking $-\xi$ if necessary, it is enough to consider $r \in (0, \pi/2]$ in order to compute F_*.

Let $\{X_1, \ldots, X_n, \xi, \eta_1, \eta_2, \ldots, \eta_p\}$, $p = 4m - n - 1$, be an orthonormal basis of $T_{(B,r\xi)}(T^\perp M)$ where $\{X_1, \ldots, X_n\}$ (respectively, $\{\eta_1, \ldots, \eta_p\}$) is an orthonormal basis of $T_B M$ (respectively, of $T_{r\xi}(U(T_B^\perp M))$, $U(T_B^\perp M)$ being the set of unit normal vectors of M at B). Then, from (1.10), (1.11), (1.12) and (1.13) it follows:

PROPOSITION 2.2.

i) $$(F_*)_{(B,r\xi)}(X_i) = \frac{1+\cos 2r}{2}X_i + \frac{1}{2}\sin 2r\tilde{\sigma}(X_i, \xi) - \frac{1}{2}\sin 2rA_\xi X_i + \frac{\cos 2r - 1}{2}\sum_{k=1}^{3}\big(g(J_k\xi, X_i)J_k\xi\big) - \frac{1-\cos 2r}{2}\tilde{\sigma}(A_\xi X_i, \xi)\,, \qquad i = 1, \ldots, n\,.$$

ii) $(F_*)_{(B,r\xi)}(\xi) = \cos 2r\xi + \frac{1}{2}\sin 2r\tilde{\sigma}(\xi, \xi)$.

iii) $(F_*)_{(B,r\xi)}(\eta_j) = \frac{1}{2}\sin 2r\eta_j + \frac{1}{2}(1 - \cos 2r)\tilde{\sigma}(\xi, \eta_j)\,, \quad j = 1, \ldots, p\,.$
Where A_ξ denotes the shape operator of M in QP^m corresponding to ξ.

DEFINITION 2.3. – Let $U(T^\perp M)$ be the unit normal bundle of M and $0 < r \leq \pi/2$. Consider $\varphi_r\colon U(T^\perp M) \to QP^m$ given by $\varphi_r(B, \xi) = F(B, r\xi)$, $\varphi_r(U(T^\perp M))$ is called the *tube of radius r over M*.

Notice that for small enough values of r, $\varphi_r(U(T^\perp M))$ is a real hypersurface of QP^m. If M is an orientable real hypersurface of QP^m we consider $\varphi_r\colon M \to QP^m$ given by $\varphi_r(B) = F(B, N_B)$, $B \in M$, where N is a unit normal vector field to M, for those values of r such that φ_r is an immersion, $\varphi_r M$ is called a *parallel hypersurface* at oriented distance r from M.

Let $\bar{M} = \varphi_r(U(T^\perp M))$ be a real hypersurface of QP^m obtained as the tube of radius r over M. If we denote, for any unit $X \in TM$:

$$\bar{X} = \frac{1+\cos 2r}{2}X + \frac{1}{2}\sin 2r\tilde{\sigma}(X, \xi) - \frac{1}{2}\sin 2rA_\xi X - \frac{1-\cos 2r}{2}\tilde{\sigma}(A_\xi X, \xi) + \sum_{k=1}^{3}\frac{\cos 2r - 1}{2}g(J_k\xi, X)J_k\xi\,, \tag{2.2}$$

and

$$\bar{\eta} = \tfrac{1}{2}\sin 2r\eta + \tfrac{1}{2}(1 - \cos 2r)\tilde{\sigma}(\eta, \xi)\,, \tag{2.3}$$

for any unit $\eta \in T^{\perp}M$ such that $g(\eta, \xi) = 0$, from Proposition 2.2 we obtain that the tangent space to $\bar{M}$ at $\bar{B} = \varphi_r(B, \xi)$ is:

$$(2.4) \qquad T_{\bar{B}}\bar{M} = Sp\{\bar{X}_i/\{X_1, \ldots, X_n\} \text{ is an orthonormal basis of } T_B M\} + \\ + Sp\{\tilde{\eta}_j/\{\eta_1, \ldots, \eta_p\} \text{ is an orthonormal basis of } T_{r\xi}(U(T_B^{\perp}M)).$$

Moreover $\bar{M}$ is an orientable real hypersurface whose unit normal vector field N is given by

$$(2.5) \qquad N_{\bar{B}} = \cos 2r\xi + \tfrac{1}{2} \sin 2r\tilde{\sigma}(\xi, \xi)\,.$$

As a local basis of the quaternionic structure of QP^m, $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ is given by [8]:

$$(2.6) \qquad \tilde{J}_k X = j_k(I - 2C)\,, \qquad k = 1, 2, 3\,,$$

for any $X \in T_C QP^m$, $C \in QP^m$, from (1.9), (2.5) and (2.6), $\tilde{J}_k N_{\bar{B}} = \tilde{J}_k N_B$ and then if $\{J_1, J_2, J_3\}$ is any local basis of $\hat{V}$, we get,

$$(2.7) \qquad J_k N_{\bar{B}} = J_k N_B\,, \qquad k = 1, 2, 3\,.$$

From Proposition 2.2, we can obtain the following

LEMMA 2.4. – *If $J_k\xi$, $k = 1, 2, 3$ are normal to M at B,*

i) $(F_*)_{(B,r\xi)}(X_i) = 0$ *if an only if either $r = \pi/2$ or X_i is an eigenvector of A_ξ with eigenvalue* $\cot r$, $i = 1, \ldots, n$.

ii) $(F_*)_{(B,r\xi)}(\eta_j) = 0$ *if and only if $r = \pi/2$, $j = 1, \ldots, p$.*

iii) $(F_*)_{(B,r\xi)}(W) \neq 0$ *in any other case.*

LEMMA 2.5. – *If $J_k\xi$, $k = 1, 2, 3$ are tangent to M at B,*

i) $(F_*)_{(B,r\xi)}(X) = 0$ *if and only if either $r = \pi/2$ or X is an eigenvector of A_ξ with eigenvalue* $\cot r$.

i) $(F_*)_{(B,r\xi)}(J_k\xi) = 0$ *if and only if either $r = \pi/2$ or $J_k\xi$ is an eigenvector of A_ξ with eigenvalue* $2 \cot 2r$, $k = 1, 2, 3$.

iii) $(F_*)_{(B,r\xi)}(\eta_j) = 0$ *if and only if $r = \pi/2$, $j = 1, \ldots, p$.*

iv) $(F_*)_{(B,r\xi)}(W) \neq 0$ *in any other case, for any unit vector $X \in T_B M$ orthogonal to $Sp\{J_1\xi, J_2\xi, J_3\xi\}$.*

LEMMA 2.6. – *If $J_1\xi$ is normal and $J_2\xi$, $J_3\xi$ are tangent to M at B,*

i) $(F_*)_{(B,r\xi)}(X) = 0$ *if and only if either* $r = \pi/2$ *or* X *is an eigenvector of* A_ξ *with eigenvalue* $\cot r$.

ii) $(F_*)_{(B,r\xi)}(J_k\xi) = 0$ *if and only if either* $r = \pi/2$ *or* $J_k\xi$ *is an eigenvector of* A_ξ *with eigenvalue* $2\cot 2r$, $k = 2, 3$.

iii) $(F_*)_{(B,r\xi)}(\eta_j) = 0$ *if and only if* $r = \pi/2$, $j = 1, \ldots, p$.

iv) $(F_*)_{B(r,\xi)}(W) = 0$ *in any other case, for any unit vector* $X \in T_B M$ *orthogonal to* $J_2\xi$ *and* $J_3\xi$.

Let A_r be the shape operator of $\overline{M}$. Using (1.9), (1.10), (1.11), (1.12), (2.2), (2.3), (2.4) and (2.5) we get,

PROPOSITION 2.7. – *Suppose that* $J_k\xi$, $k = 1, 2, 3$ *are normal to* M *at* B. *If* $\{X_1, \ldots, X_n\}$ *is an orthonormal basis of eigenvectors of* A_ξ *with corresponding eigenvalues* $\lambda_i = \cot\alpha_i$, $i = 1, \ldots, n$, $0 < \alpha_i < \pi$,

1) $A_r(\overline{X}_i) = \cot(\alpha_i - r)\overline{X}_i$, $\quad i = 1, \ldots, n$.

2) $A_r(\overline{J_k\xi}) = -2\cot 2r\overline{J_k\xi}$, $\quad k = 1, 2, 3$.

3) $A_r(\bar\eta_j) = -\cot r\bar\eta_j$, $\quad j = 1, \ldots, p-4$,

$\{\xi, J_1\xi, J_2\xi, J_3\xi, \eta_1, \ldots, \eta_{p-4}\}$ *being an orthonormal basis of* $T_B^\perp M$.

PROPOSITION 2.8. – *Suppose that* $J_k\xi$, $k = 1, 2, 3$ *are eigenvectors of* A_ξ *with corresponding eigenvalues* $2\cot 2\theta_k$, $0 < \theta_k < \pi/2$, $k = 1, 2, 3$. *Let* $\{X_1, \ldots, X_{n-3}, J_1\xi, J_2\xi, J_3\xi\}$ *be an orthonormal basis of eigenvectors of* A_ξ *such that* $A_\xi X_i = \cot\alpha_i$, $0 < \alpha_i < \pi$, $i = 1, \ldots, n-3$. *Then:*

1) $A_r(\overline{X}_i) = \cot(\alpha_i - r)\overline{X}_i$, $\quad i = 1, \ldots, n-3$.

2) $A_r(\overline{J_k\xi}) = 2\cot 2(\theta_k - r)\overline{J_k\xi}$, $\quad k = 1, 2, 3$.

3) $A_r(\bar\eta_j) = -\cot r\bar\eta_j$, $\quad j = 1, \ldots, p$.

PROPOSITION 2.9. – *Suppose that* $J_1\xi$ *is normal to* M *at* B *and* $J_2\xi$, $J_3\xi$ *are eigenvectors of* A_ξ *with corresponding eigenvalues* $2\cot 2\theta_k$, $0 < \theta_k < \pi/2$, $k = 2, 3$. *If* $\{X_1, \ldots, X_{n-2}, J_2\xi, J_3\xi\}$ *is an orthonormal basis of eigenvectors of* A_ξ *such that* $A_\xi X_i = \cot\alpha_i X_i$, $0 < \alpha_i < \pi$, $i = 1, \ldots, n-2$,

1) $A_r(\overline{X}_i) = \cot(\alpha_i - r)\overline{X}_i$, $\quad i = 1, \ldots, n-2$.

2) $A_r(\overline{J_k\xi}) = 2\cot 2(\theta_k - r)\overline{J_k\xi}$, $\quad k = 2, 3$.

3) $A_r(\overline{J_1\xi}) = -2\cot 2r\overline{J_1\xi}$.

4) $A_r(\bar\eta) = -\cot r\bar\eta$,

for any $\eta \in U(T_B^\perp M)$ *orthogonal to* $J_1\xi$.

From Lemmas 2.4, 2.5 and 2.6 and Propositions 2.7, 2.8 and 2.9 the following examples of real hipersurfaces of QP^m with constant principal curvatures are obtained:

EXAMPLE 1. – Let $M = \{B\}$, $B \in QP^m$. Then, the tube of radius r, $0 < r < \pi/2$, over M is a real hypersurface of QP^m with two distinct constant principal curvatures $\lambda = \cot r$ and $\mu = 2\cot 2r$ with respective multiplicities $4m - 4$ and 3. Notice that $\varphi_r(U(T^\perp M))$ is the set of points of QP^m at a distance r from B, that is, the *geodesic hypersphere* of center B and radius r.

As $\varphi_{\pi/2}(U(T^\perp M))$ is a QP^{m-1}, such a geodesic hypersphere can also be considered as a tube of radius $\pi/2 - r$ over a QP^{m-1}.

EXAMPLE 2. – Consider $M = QP^k$, $0 < k < m - 1$, embedded as a quaternionic submanifold of QP^m, [11]. Then if $0 < r < \pi/2$, the tube of radius r over M is a real hypersurface of QP^m with three distinct constant principal curvatures $\lambda_1 = \cot r$, $\lambda_2 = -\tan r$ and $\mu = 2\cot 2r$ with respective multiplicities $4l, 4k$ and $3, l = m - k - 1$.

REMARK. – As any quaternionic submanifold of QP^m, $m \geqslant 2$ is totally geodesic, [11], from Examples 1 and 2 it follows that if $0 < r < \pi/2$, the tube of radius r over a quaternionic submanifold of QP^m is a real hypersurface of QP^m with 2 or 3 distinct constant principal curvatures.

Examples 1 and 2 are the ones studied by PAK in [18].

EXAMPLE 3. – Consider the complex projective space CP^m embedded as a totally geodesic totally complex submanifold of QP^m, [10]. From Lemma 2.6 and Proposition 2.9, we have that if $0 < r < \pi/4$ or $\pi/4 < r < \pi/2$, the tube of radius r over CP^m is a real hypersurface of QP^m with four distinct constant principal curvatures $\lambda_1 = \cot r$, $\lambda_2 = -\tan r$, $\mu_1 = 2\cot 2r$ and $\mu_2 = -2\tan 2r$ with respective multiplicities $2(m-1)$, $2(m-1)$, 1 and 2.

3. – Focal points of CP^m in QP^m.

In this section we exhibit the structure of $\varphi_{\pi/4}(U(T^\perp CP^m))$ in QP^m, CP^m being as in Example 3.

Suppose Q^{m+1} identified to $C^{m+1} \times C^{m+1}$ in the following way: $(z_1, z_2) \in C^{m+1} \times C^{m+1}$ is identified to $z_1 + j_2 z_2 \in Q^{m+1}$. (Considering $C = R + j_1 R$.) Under this identification, the unit sphere S^{2m+1} of C^{m+1} is embedded in the unit sphere S^{4m+3} of QP^m by $i\colon S^{2m+1} \to S^{4m+3}$, $i(z) = (z, 0)$, and the fibration $\Pi\colon S^{4m+3} \to QP^m$ with fibre S^3 induces a fibration $\Pi_1\colon S^{2m+1} \to CP^m$ with fibre S^1 such that the following diagram conmutes:

$$\begin{array}{ccc} S^{2m+1} & \xrightarrow{\ i\ } & S^{4m+3} \\ \big\downarrow \Pi_1 & & \big\downarrow \Pi \\ CP^m & \xrightarrow{\ \hat{\imath}\ } & QP^m \end{array}$$

$\hat{\imath}$ being the standard immersion of CP^m into QP^m (see [10]).

Let $p \in QP^m$ and $\overline{p} \in S^{4m+3}$ such that $\Pi(\overline{p}) = p$. It is well known, [13], that under the above identification if $X = \Pi_*(a, b) \in T_p QP^m$, a local basis of the quaternionic structure of QP^m is given by

$$(3.1)\quad J_1' X = \Pi_*(j_1 a, -j_1 b)\,, \qquad J_2' X = \Pi_*(-b, a)\,, \qquad J_3' X = \Pi_*(-j_1 b, -j_1 a)\,,$$

and

$$(3.2)\quad J_1'(T_x CP^m) = T_x(CP^m_{\sim})\,, \qquad J_2'(T_x CP^m) = J_3'(T_x CP^m) = T_x^\perp CP^m\,, \qquad x \in CP^m\,.$$

The action of S^3 over S^{4m+3} is the one given by $SU(2)$.

Let $x \in CP^m$ and $z \in S^{2m+1}$ such that $\Pi_1(z) = x$. If $y \in T_x^\perp CP^m$, from (3.2), $y = J_2' a$ $a \in T_x CP^m$, that is, $a = (\Pi_{1*})_z w$ with $\langle\!\langle z, w \rangle\!\rangle = 0$, where $\langle\!\langle\, , \rangle\!\rangle$ denotes the usual Hermitian product of C^{m+1}. Then, from Definition 2.1 we have that $\varphi_{\pi/4}(U(T^\perp CP^m)) = \Pi(M')$, where

$$(3.3)\qquad M' = \{(z, w) \in S^{4m+3} / \|z\| = \|w\|,\ \langle\!\langle z, w \rangle\!\rangle = 0\}\,.$$

Clearly, from (3.3), M' is invariant under the action of $SU(2)$ and so, $\Pi(M')$ is a submanifold of QP^m.

THEOREM 3.1. – *Let* $N^* = \Pi(M')$. *Then:*

i) N^* *is a* $(4m-3)$*-dimensional quaternion CR-submanifold of* QP^m, [1].

ii) N^* *is minimal in* QP^m.

iii) *If* D *is the quaternionic distribution and* $D^\perp$ *the totally real distribution of* N^* *we have:*

a) $\sigma(D, D^\perp) = \{0\}$,

b) $\sigma(D^\perp, D^\perp) = \{0\}$,

c) $A^2_{J_k Z} X = X\,, \quad k = 1, 2, 3\,,$

for any $X \in D$, *where* Z *is a unit vector field of* $D^\perp$ *and* σ *is the second fundamental form of* N^* *in* QP^m.

PROOF. – Let $(z, w) \in M'$ and $(a, b) \in T_{(z,w)} M'$. As the equations defining M' are invariant by the fibration Π, the tangent space to the fibre at (z, w) is $Sp\{(-j_1 w, -j_1 z), (j_1 z, -j_1 w), (-w, z)\}$. Moreover, from (3.3),

$$(3.4)\qquad g_0(z, a) = g_0(w, b)\,, \qquad g_0(z, b) + g_0(w, a) = g_0(j_1 z, b) + g_0(j_1 w, a) = 0\,.$$

Thus, $\xi_1' = (w, z)$, $\xi_2' = (-z, w)$, $\xi_3' = (-j_1 w, j_1 z)$ are normal vectors to M' at (z, w) As ξ_1', ξ_2', ξ_3' are orthonormal we deduce that they are a basis of $T^\perp_{(z,w)} M'$ and then,

$$(3.5)\qquad T^\perp_{\Pi(z,w)} N^* = Sp\{\Pi_*(\xi_1') = \xi_1, \Pi_*(\xi_2') = \xi_2, \Pi_*(\xi_3') = \xi_3\}\,.$$

From (3.1), (3.4) and (3.5) we have

$$J'_1\xi_1 = -\xi_3 , \quad J'_2\xi_1 = \xi_2 , \quad J'_3\xi_1 = Z , \tag{3.6}$$

$Z = \Pi_*(-j_1 z, -j_1 w)$ being a unit vector tangent to N^* at $\Pi(z, w)$. This proves i).

On the other hand, having in mind that the fibres of Π are totally geodesic we can conclude:

1) If $(a, b) = (-j_1 z, -j_1 w)$, then $A'_{(w,z)}(j_1 z, j_1 w) = (j_1 w, j_1 z)$ is a vertical vector. Thus $A_{\xi_1} Z = 0$.

2) If (a, b) is orthogonal to $(-j_1 z, -j_1 w)$ and orthogonal to the fibre at (z, w), then

$$A'_{(w,z)}(a, b) = -(b, a) + g_0\big((b, a), (-z, w)\big)(-z, w) + g_0\big((b, a), (-j_1 w, j_1 z)\big)(-j_1 w, j_1 z) , \tag{3.7}$$

$$(A'_{(w,z)})^2(a, b) = (a, b) - g_0\big((b, a), (-z, w)\big)(w, -z) - g_0\big((b, a), (-j_1 w, j_1 z)\big)(j_1 z, -j_1 w) , \tag{3.8}$$

therefore $A^2_{\xi_1}(\Pi_*(a, b)) = \Pi_*(a, b)$, where A_{ξ_1} (respectively, $A'_{(w,z)}$) is the corresponding shape operator of N^* in QP^m (respectively, of M' in S^{4m+3}).

Analogously we can prove that $A^2_{\xi_2}\Pi_*(a, b) = A^2_{\xi_3}\Pi_*(a, b) = \Pi_*(a, b)$ and $A_{\xi_2}Z = A_{\xi_3}Z = 0$, for any $\Pi_*(a, b) \in T_{\Pi(z,w)}N^*, \Pi_*(a, b) \in D$ and we conclude the proof of iii).

Finally, ii) follows from i) and iii) having in mind the properties of quaternion CR-submanifolds, [1].

REMARK. – It is easy to prove that N^* is diffeomorphic to $SU(m+1)/SU(2)\times SU(m-1)$.

We also remark that N^* is the first example of proper quaternion CR-submanifold known until now.

4. – Tubes over a mixed totally geodesic quaternion CR submanifold of codimension three in QP^m.

Theorem 3.1 suggests to study a quaternion CR-submanifold M of QP^m which satisfies,

$$\sigma(D, D^\perp) = \{0\} \tag{4.1}$$

where σ is the second fundamental form of M in QP^m and D, $D^\perp$ are, respectively, the quaternionic distribution and the totally real distribution of M. Such a quaternion CR-submanifold is called *mixed totally geodesic*, [1].

Lemma 4.1. – *Let M be a $(4m-3)$-dimensional mixed totally geodesic quaternion CR-submanifold of QP^m. Then the second fundamental form of M has the same behaviour as the one of N^*.*

Proof. – As M is mixed totally geodesic we have, [1],

$$\|A_{J_k Z}X\|^2 = 1\,, \qquad k=1,2,3 \tag{4.2}$$

$$g(\sigma(X,X), J_k A_{J_k Z} Z) = 0\,, \quad k=1,2,3 \tag{4.3}$$

for any unit vector fields $X \in D$, $Z \in D^\perp$, where $A_{J_k Z}$ is the Weingarten endomorphism corresponding to $J_k Z$. Moreover from (4.1) we can choose an orthonormal basis of eigenvectors of $A_{J_1 Z}$, $\{X_1, \ldots, X_{4m-4}, Z\}$ such that

$$A_{J_1 Z} Z = \alpha_1 Z\,, \quad A_{J_1 Z} X_i = \beta_i X_i\,, \quad i=1,\ldots,4m-4\,. \tag{4.4}$$

Thus from (4.3) and (4.4), $\alpha_1 \beta_i = 0$, $i=1,\ldots,4m-4$. Therefore using (4.2), $\alpha_1 = 0$ and we obtain that $A_{J_1 Z}Z = 0$. Now from (4.2), $\beta_i^2 = 1$, that is, $A^2_{J_1 Z}X = X$ for any $X \in D$. Analogously it can be shown that $A_{J_2 Z}Z = A_{J_3 Z}Z = 0$ and $A^2_{J_2 Z}X = A^2_{J_3 Z}X = X$ for any $X \in D$, which concludes the proof.

Proposition 4.2. – *Let M be a $(4m-3)$-dimensional mixed totally geodesic quaternion CR-submanifold of QP^m. If $0 < r < \Pi/4$, then $\varphi_r(U(T^\perp M))$ is a real hypersurface of QP^m with four distinct constant principal curvatures $2\tan 2r$, $-2\cot 2r$, $\cot(\Pi/4 - r)$ and $-\tan(\Pi/4 - r)$ of respective multiplicities $1, 2, 2(m-1)$ and $2(m-1)$.*

Proof. – Let us consider $B \in M$, $\xi \in U(T_B^\perp M)$. From Theorem 3.1 and Lemma 4.1 we can choose a basis in $T_B M$ of eigenvectors of A_ξ, $\{X_1, \ldots, X_{4m-4}, Z\}$ with $D_B^\perp = Sp\{Z\}$ and such that

$$A_\xi X_i = X_i\,, \quad i=1,\ldots,2(m-1)\,, \qquad A_\xi X_j = -X_j\,, \quad j=2m-1,\ldots,4m-1,\ A_\xi Z = 0\,. \tag{4.5}$$

Thus from Proposition 2.2 (1.10), (1.11), (4.5) and having in mind that $\xi = a_{11}J_1 Z + a_{12}J_2 Z + a_{13}J_3 Z$, $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$, we can deduce

$$(\varphi_r)_{*(B,\xi)}X_i = \cos r(\cos r - \sin r)X_i + \sin r(\cos r - \sin r)\tilde\sigma(X_i, \xi)\,, \quad i=1,\ldots,2m-2 \tag{4.6}$$

$$(\varphi_r)_{*(B,\xi)}X_j = \cos r(\cos r + \sin r)X_j + \sin r(\cos r + \sin r)\tilde\sigma(X_j, \xi)\,, \quad j=2m-1,\ldots,4m-4 \tag{4.7}$$

$$(\varphi_r)_{*(B,\xi)}Z = \cos 2rZ\,. \tag{4.8}$$

Moreover, if $\{\xi, \eta_1, \eta_2\}$ is an orthonormal basis of $T^\perp_B M$ such that $\eta_k = a_{k1}J_1Z + a_{k2}J_2Z + a_{k3}J_3Z$, $k = 2, 3$, $(a_{ij}) \in SO(3)$, then from Proposition 2.2 and (1.10),

$$(\varphi_r)_{*(B,\xi)}\eta_k = \tfrac{1}{2}\sin 2r\eta_k\,, \qquad k = 2, 3\,. \tag{4.9}$$

Consequently from (1.12), (4.6), (4.7), (4.8), (4.9) and as $0 < r < \Pi/4$, we conclude that $\bar{M}_r = \varphi_r(U(T^\perp M))$ is a real hypersurface of QP^m.

Let $\bar{B} = \varphi_r(B, \xi)$, from (2.5) the unit normal vector to $\bar{M}_r$ at $\bar{B}$ is given by

$$N_{\bar{B}} = \cos 2r\xi + \tfrac{1}{2}\sin 2r\tilde{\sigma}(\xi, \xi)\,. \tag{4.10}$$

Using the same notation as in § 2, from (4.5) and (4.10) we have

$$\begin{cases} A_r\bar{X}_i = \cot\left(\dfrac{\Pi}{4} - r\right)\bar{X}_i\,, & i = 1, \ldots, 2m-2 \\ A_r\bar{X}_j = \cot\left(\dfrac{3\Pi}{4} - r\right)\bar{X}_j = -\tan\left(\dfrac{\Pi}{4} - r\right)\bar{X}_j\,, & j = 2m-1, \ldots, 4m-4\,. \end{cases} \tag{4.11}$$

Now let $\alpha(t)$ be a differentiable curve on M such that $\alpha(0) = B$ and $\alpha'(0) = Z$. Then, denoting by $\xi(t)$ the parallel displacement of ξ along α in $T^\perp M$, from (1.10), (1.11), (1.12), (4.5), (4.8) and (4.10) it follows

$$-A_r\bar{Z} = \left(\frac{d}{dt}(N_{\varphi_r(\alpha(t),\xi(t))})|_{t=0}\right)^\top = \frac{1}{2}\sin 2r(-\tilde{A}_{\tilde{\sigma}(\xi,\xi)}Z)^\top = \\ = -2\sin 2rZ = -2\tan 2r\bar{Z} \tag{4.12}$$

where $(\ )^\top$ denotes the corresponding component in T_BM.

Finally, and similarly as (4.12) we obtain

$$-A_r\bar{\eta}_k = \cos 2r\eta_k = 2\cot 2r\bar{\eta}_k\,, \qquad k = 2, 3 \tag{4.13}$$

which concludes the proof.

Remark. – Notice that from Proposition 4.2, $\bar{Z}, \bar{\eta}_2, \bar{\eta}_3$ are eigenvectors and from (2.5) and (2.7) $Sp\{J_1\xi, J_2\xi, J_3\xi\} = S_p\{\bar{Z}, \bar{\eta}_2, \bar{\eta}_3\}$, so we conclude

$$g(A_r D', D) = 0\,, \tag{4.14}$$

where D', D are as in § 1. Thus if we take as a new local basis of $\hat{V}$ $\{J_1^0, J_2^0, J_3^0\}$, where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} J_1^0 \\ J_2^0 \\ J_3^0 \end{pmatrix}$$

then the corresponding $U_k^0 = -J_k^0 N_{\bar{B}}$, $k = 1, 2, 3$, are eigenvectors with principal curvatures $2\tan 2r$, $-2\cot 2r$ and $-2\cot 2r$, respectively.

THEOREM 4.3. - *Let M be a $(4m-3)$-dimensional mixed totally geodesic quaternion CR-submanifold of QP^m. Let $\bar{M}$ be a real hypersurface which lies in a tube of radius r, $0 < r < \Pi/4$, over M. Then $\bar{M}$ lies in a tube of radius $r' = \Pi/4 - r$ over CP^m.*

PROOF. - As $\bar{M} \subseteq \varphi_r(U(T^\perp M))$ for some r, $0 < r < \Pi/4$, from Proposition 4.2 and last Remark we can suppose that there exists an orthonormal basis $\{\bar{X}_1, \ldots, \ldots, \bar{X}_{4m-4}, J_1\bar{N}, J_2\bar{N}, J_3\bar{N}\}$ of $T\bar{M}$ satisfying

$$(4.15)\quad \begin{cases} \bar{A}\bar{X}_i = \cot\left(\dfrac{\Pi}{4} - r\right)\bar{X}_i\,, & i = 1, \ldots, 2m-2 \\[2ex] \bar{A}\bar{X}_j = -\tan\left(\dfrac{\Pi}{4} - r\right)\bar{X}_j\,, & j = 2m-1, \ldots, 4m-4 \\[2ex] \bar{A}J_1\bar{N} = 2\tan 2rJ_1\bar{N}\,, & \bar{A}J_k\bar{N} = -2\cot 2rJ_k\bar{N}\,, \quad k = 2, 3 \end{cases}$$

where $\bar{A}$ is the Weingarten endomorphism of $\bar{M}$ and $\bar{N}$ is a unit normal vector field to $\bar{M}$, for some local basis $\{J_1, J_2, J_3\}$ of $\hat{V}$. Using Lemma 2.5 it follows that $\varphi_{r'}\colon \bar{M} \to QP^m$, $r' = \Pi/4 - r$ has constant rank $2m$ on $\bar{M}$ and then (see [3]) for each point $B \in \bar{M}$, $\varphi_{r'}\bar{M}$ is a $2m$-dimensional submanifold in a neighborhood of $\varphi_r(B)$. Moreover, the distribution $T_0(B) = \{X \in T_B\bar{M}/(\varphi_{r'})_* X = 0\}$ is integrable with $2m$-dimensional leaves on $\bar{M}$. That is, there exists a neighborhood U of B in $\bar{M}$ such that $\varphi_{r'}(U) = V$ is a $2m$-dimensional submanifold of QP^m.

Let V_1 and V_2 be, respectively, the eigenspaces corresponding to the eigenvalues $\cot r'$ and $-\tan r'$. Then from (4.15) as T_0 is integrable we have $g([X, Y], U_k) = g([X, U_1], U_k) = 0$, $k = 2, 3$ and then using (1.14) we can conclude

$$(4.16)\qquad q_2(X) = q_3(X) = 0\,, \quad g(X, J_k Y) = 0\,, \quad k = 2, 3$$

for any $X, Y \in V_1$. By a similar reasoning applied to $\varphi_{\pi/2-r}\colon \bar{M} \to QP^m$ it follows

$$(4.17)\qquad q_2(Z) = q_3(Z) = 0\,, \quad g(Z, J_k W) = 0\,, \quad k = 2, 3$$

for any $Z, W \in V_2$.

Thus, from (4.16), (4.17) and bearing in mind the properties of the quaternionic structure we have

$$(4.18)\qquad J_1 V_1 = V_1\,, \quad J_k V_1 = V_2\,, \quad k = 2, 3\,.$$

Moreover, from (1.16), (4.15) and (4.18) it follows

$$(4.19)\qquad g(\nabla_W X, Z) = 0\,, \quad W, Z \in V_2,\ X \in V_1$$

Also, using (1.14), (1.16) and (4.15) it is easy to see that

$$q_2(U_3) = q_3(U_3) = 2\tan 2r' . \tag{4.20}$$

Now, let $\tilde{V}_1 = Sp\{X + \tan r'\tilde{\sigma}(X, \bar{N}_B)/X \in V_1\}$ and $\tilde{V}_2 = \{Z + \tan r'\tilde{\sigma}(Z, \bar{N}_B)/Z \in V_2\}$. Clearly from (1.12), (2.5), (2.7), (4.15) and Proposition 2.2 it follows

$$T_{\bar{B}}V = \tilde{V}_2 \oplus Sp\{J_2\bar{N}_B, J_3\bar{N}_B\} \tag{4.21}$$

$$T_{\bar{B}}V = \tilde{V}_1 \oplus Sp\{U_1, \cos 2r'\bar{N}_B + \tfrac{1}{2}\sin 2r'\tilde{\sigma}(\bar{N}_B, \bar{N}_B)\}, \quad \bar{B} = \varphi_r(B, \bar{N}_B) \tag{4.22}$$

and consequently from (4.18), (4.21) and (4.22)

$$J_1 T_{\bar{B}}V = T_{\bar{B}}V, \quad J_2 T_{\bar{B}}V = J_3 T_{\bar{B}}V = T_{\bar{B}}^{\perp}V \tag{4.23}$$

that is, $T_{\bar{B}}V$ is a totally complex subspace of $T_{\bar{B}}QP^m$ in the sense of [10].

In the following we shall see that V is a totally geodesic submanifold of QP^m, for which we denote by $\xi^* = \cos 2r\bar{N}_B + \frac{1}{2}\sin 2r\tilde{\sigma}(\bar{N}_B, \bar{N}_B)$ and $X^* = X + \tan r'\tilde{\sigma}(X, \bar{N}_B)$ for any $X \in V_1 \oplus V_2$. From (4.15) and similarly as in Proposition 2.8 it follows

$$A^*_{\xi^*} = 0 \tag{4.24}$$

where A^* is the corresponding Weingarten endomorphism of V in QP^m. Moreover if $Z, W \in V_2$ and σ^* is the second fundamental form of V in QP^m,

$$\begin{aligned}\sigma^*(Z^*, W^*) &= (\tilde{\nabla}_Z W + \tilde{\sigma}(Z, W) - \tan r'\tilde{A}_{\tilde{\sigma}(W,\bar{N}_B)}Z + \tan r'(\tilde{\sigma}(\tilde{\nabla}_Z W, \bar{N}_B) + \\ &+ \tilde{\sigma}(\tilde{\nabla}_Z\bar{N}_B, W)^{\perp}) = (\nabla_Z W + (1 + \tan^2 r')\tilde{\sigma}(Z, W) - 2\tan r' g(W, Z)\bar{N}_B - \\ &- \tan r' g(J_1 W, Z)U_1 + \tan r'\tilde{\sigma}(\nabla_Z W, \bar{N}_B) - \tan^2 r' g(W, Z)\tilde{\sigma}(\bar{N}_B, \bar{N}_B))^{\perp}\end{aligned} \tag{4.25}$$

where $^{\perp}$ denotes the corresponding component in $T_{\bar{B}}^{\perp}V$. Thus, from (2.12) and (4.19) $g(\sigma^*(W^*, Z^*), J_1\bar{N}_B) = 0$. From this, together with (4.23) and (4.24) we obtain

$$\sigma^*(\tilde{V}_2, \tilde{V}_2) = \{0\} . \tag{4.26}$$

As from (1.14) and (4.15) $\sigma^*(Z^*, J_2\bar{N}_B) = (\tilde{\nabla}_Z J_2\bar{N}_B + \tilde{\sigma}(Z, J_2\bar{N}_B))^{\perp} = (q_1(Z)J_3\bar{N}_B + \tan r' J_2 Z - \tilde{\sigma}(J_2 Z, \bar{N}_B))^{\perp} = C^{\perp}$ and as $g(C, X^*) = g(C, J_1\bar{N}_B) = 0$ for any $X \in V_1$, we also conclude

$$\sigma^*(\tilde{V}_2, J_2\bar{N}_B) = \{0\} . \tag{4.27}$$

Analogously,

$$\sigma^*(\tilde{V}_2, J_3\bar{N}_B) = \{0\} . \tag{4.28}$$

Finally, from (1.14), (4.10) and (4.15) it follows

$$\sigma^*(J_3\bar{N}_B, J_2\bar{N}_B) = (q_1(J_3\bar{N}_B)J_3\bar{N}_B)^\perp = 0 \tag{4.29}$$

Therefore, V is a totally geodesic submanifold of QP^m. Thus (V, J_1) is a Kaehler submanifold embedded in QP^m as a totally complex totally geodesic submanifold and then, [10], V is an open set of CP^m. It is clear now that as M is connected, it lies on a tube of radius r' over CP^m.

5. – Real hypersurfaces of QP^m with constant principal curvatures.

Let M be a real hypersurface of QP^m with unit local normal vector field N and U_1, U_2, U_3, D' and D as in § 1.

In this section we shall classify real hypersurfaces with constant principal curvatures satisfying $g(AD, D') = 0$

LEMMA 5.1. – *Let M be a real hypersurface of QP^m. Then $g(AD, D') = 0$ if and only if there exists a local basis of $\hat{V}$, $\{J'_1, J'_2, J'_3\}$ such that the corresponding $U'_k = -J'_kN$, $k = 1, 2, 3$ are principal.*

PROOF. – If $g(AD, D') = 0$, $AD \subseteq D$ and $AD' \subseteq D'$. As A is diagonalizable there exist $X_1, X_2, X_3 \in D'$ such that $D' = Sp\{X_1, X_2, X_3\}$ are eigenvectors of A. Thus there exists $P \in SO(3)$, [14], such that

$$P\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}.$$

If we choose as a new local basis of $\hat{V}$ $\{J'_1, J'_2, J'_3\}$ where

$$P\begin{pmatrix} J_1 \\ J_3 \\ J_2 \end{pmatrix} = \begin{pmatrix} J'_1 \\ J'_2 \\ J'_3 \end{pmatrix}$$

the corresponding $U'_k = -J'_kN$, $k = 1, 2, 3$ are principal.

The converse being trivial, this concludes the proof.

Using (1.16) we obtain, [19],

LEMMA 5.2. – *Let M be a real hypersurface of QP^m. If U_1, U_2 and U_3 are principal with corresponding principal curvatures α_1, α_2 and α_3,*

$$X(\alpha_i) = U_i(\alpha_i)f_i(X) - q_j(U_i)(\alpha_i - \alpha_k)f_k(X) + q_k(U_i)(\alpha_i - \alpha_j)f_j(X),$$

$X \in TM$, (i, j, k) being a cyclic permutation of $(1, 2, 3)$.

Let now M be a real hypersurface of QP^m with constant principal curvatures and such that $g(AD, D') = 0$. From Lemma 5.1 we choose a local basis of $\hat{V}$, $\{J_1, J_2, J_3\}$ such that,

$$AU_1 = \alpha_1 U_1 , \quad AU_2 = \alpha_2 U_2 , \quad AU_3 = \alpha_3 U_3 . \tag{5.1}$$

Now, we distinguish the following cases:

CASE I. – *At least, two among* α_1, α_2 *and* α_3 *are equal.*

We can suppose that $\alpha_1 = \alpha_2$ and as they are constant, we consider

$$\alpha_1 = \alpha_2 = 2 \cot 2r , \quad 0 < r < \frac{\pi}{2} . \tag{5.2}$$

From Lemma 2.5, $\varphi_r \colon M \to QP^m$ has constant rank q, and from the Inverse Function Theorem, [3], for any $B \in M$ there exists a neighborhood U of B in M such that $\varphi_r(U) = V$ is a q-dimensional submanifold embedded into QP^m. Moreover the distribution T_0 given by:

$$T_0 = \{X \in T_B M / (\varphi_r)_* X = 0\} , \tag{5.3}$$

is integrable with $(4m - q - 1)$-dimensional leaves.

Consider the map $\eta \colon U \to T^{\perp} V$ given by

$$\eta(B) = \cos 2r N_B + \tfrac{1}{2} \sin 2r \tilde{\sigma}(N_B, N_B) , \tag{5.4}$$

that is well-defined.

LEMMA 5.3. – *With the above notations and under the above conditions, we have*

i) $\operatorname{rank} \eta_* = 4m - 1$ *if* $\alpha_3 \neq -2 \tan 2r$,

ii) $\operatorname{rank} \eta_* = 4m - 2$ *if* $\alpha_3 = -2 \tan 2r$.

PROOF. – Let $\{X_1, \ldots, X_{4m-4}, U_1, U_2, U_3\}$ be an orthonormal basis of eigenvectors of A such that $AX_i = \gamma_i X_i$, $i = 1, \ldots, 4m - 4$. Thus from (1.12), (5.4) and the formulas of Gauss and Weingarten it follows

$$\eta_*(X_i) = -(\gamma_i \cos 2r + \sin 2r) X_i + (\cos 2r - \sin 2r \gamma_i) \tilde{\sigma}(X_i, N) , \quad i = 1, \ldots, 4m - 4 . \tag{5.5}$$

Clearly, from (1.12) and (5.5) $\eta_*(X_1), \ldots, \eta_*(X_{4m-4})$ are linearly independent. From (1.12), (2.7), (5.1) and the formulas of Gauss and Weingarten

$$\eta_*(U_k) = (-\cos 2r \alpha_k - 2 \sin 2r) U_k , \quad k = 1, 2, 3 . \tag{5.6}$$

Therefore $\eta_*(U_k) = 0$ if and only if $\alpha_k = -2\tan 2r$. As $\alpha_1 = \alpha_2 = 2\cot 2r$, from (5.5) and (5.6) we conclude that rank $\eta_* = 4m - 1$ if and only if $\alpha_3 \neq -2\tan 2r$ and rank $\eta_* = 4m - 2$ if and only if $\alpha_3 = -2\tan 2r$.

LEMMA 5.4. - *If* rank $\eta_* = 4m - 1$, *then* $\alpha_2 = \alpha_3$ *and* M *is an open subset of either*

i) *a geodesic hypersphere, or*

ii) *a tube of radius* r *over* QP^k, $0 < k < m - 1$.

PROOF. - From the definition of η if we denote by Ψ_r the tube of radius r over V, then $\Psi_r(-\eta(B)) = B$. Thus η has a left inverse and as rank $\eta_* = 4m - 1$, η is a diffeomorphism onto an open subset $\eta(U)$ of $U(T^\perp V)$. Then, denoting by $U_{\eta(B)}(T^\perp V)$ the fibre of $U(T^\perp V)$, $\eta(U) \cap U_{\eta(B)}(T^\perp V)$ is open in $U_{\eta(B)}(T^\perp V)$ and contains a basis of $T^\perp_{\varphi_r(B)}V$ spanned by $\{\eta(C)/C \in V_1 \subseteq V\}$. From this and Proposition 2.2 $T^\perp_{\varphi_r(B)}V$ is invariant under J_1 and J_2. But then it must be invariant under J_3 and as $g((\varphi_r)_* U_3, J_3\eta(B)) = (\frac{1}{2}\sin 2r\alpha_3 - \cos 2r) g(U_3, U_3)$, $\alpha_3 = 2\cot 2r$. That is, U is a real hypersurface of QP^m with constant principal curvatures, U_1, U_2, U_3 are principal with the same principal curvature and $\varphi_r(U) = V$ is a quaternionic submanifold of QP^m.

By a similar reasoning as the one used in [5], U lies on the tube of radius r over V.

The proof follows from Examples 2 and 3, the fact that the principal curvatures are constant and the connectedness of M.

LEMMA 5.5. - *If* rank $\eta_* = 4m - 2$, M *lies on a tube of radius* r *over a complex projective space* CP^m, $0 < r < \pi/4$ *or* $\pi/4 < r < \pi/2$.

PROOF. - By the assumption and from Lemma 5.3, $\alpha_3 = -2\tan 2r$, then $r \neq \pi/4$. Now (5.1) can be written as

$$(5.7) \qquad AU_k = 2\cot 2r\, U_k\,, \quad k = 1, 2\,, \qquad AU_3 = -2\tan 2r\, U_3\,, \quad 0 < r < \frac{\pi}{4} \text{ or } \frac{\pi}{4} < r < \frac{\pi}{2}.$$

Let $\gamma_1, \ldots, \gamma_p$ be the distinct principal curvatures corresponding to eigenvectors in D. We distinguish:

CASE I1. - $\gamma_t \neq \cot r$, *for any* $t \in \{1, \ldots, p\}$.

From Lemma 2.5, $\varphi_r(U) = V$ is a $(4m - 3)$-dimensional submanifold of QP^m, and from (2.7), (5.4) and (5.7), $\{\eta(B), J_1N_B, J_2N_B\}$ are orthogonal to V at $\varphi_r(B, N)$. Thus $T^\perp_{\varphi_r(B,N)}V = Sp\{\eta(B), J_1\eta(B), J_2\eta(B)\}$ for any $B \in U$, that is $\varphi_r(U)$ is a $(4m - 3)$-dimensional quaternion CR-submanifold of QP^m.

As in the proof of Proposition 2.8, $\varphi_r(U)$ is mixed totally geodesic and Lemma follows in this case from Theorem 4.3.

CASE I2. – $\gamma_t = \cot r$ *for some* $t \in \{1, \ldots, p\}$.

Suppose $t = 1$ and let V_s be the eigenspace corresponding to the eigenvalue γ_s, $s = 1, \ldots, p$. As T_0 is integrable, from Lemma 2.5 and (5.7) we have for any $X, Y \in V_1$,

$$g([X, Y], U_3) = g([X, U_1], U_3) = g([X, U_2], U_3) = 0 \,, \tag{5.8}$$

and from (1.16),

$$(\cot rI - A)[X, Y] = 2 \sum_{k=1}^{3} g(X, J_k Y) U_k \,, \tag{5.9}$$

I denoting the identity automorphism of TM.

From (1.14), (5.8) and (5.9), $g(X, J_3 Y) = 0$ and $q_1(X) = q_2(X) = 0$, for any X, $Y \in V_1$. That is,

$$J_3 V_1 \subseteq V_1^{\perp} \,, \quad q_1(V_1) = q_2(V_1) = \{0\} \,. \tag{5.10}$$

On the other hand, if we take $X, Y \in \{U_1, U_2, U_3\}$, from (1.14), (1.16) and (5.7),

$$q_1(U_1) = q_2(U_2) = 2 \cot 2r \,. \tag{5.11}$$

From Lemma 5.2 it follows $0 = U_3(2 \cot 2r) = (2 \cot 2r + 2 \tan 2r)\, q_2(U_1)$, $0 = U_3(2 \cot 2r) = -(2 \cot 2r + 2 \tan 2r) q_1(U_2)$, then,

$$q_2(U_1) = q_1(U_2) = 0 \,. \tag{5.12}$$

Taking in (5.9) the scalar product with U_k, $k = 1, 2$, we obtain,

$$g([X, Y], U_k) = 2 \cot r g(X, J_k Y) \,, \quad k = 1, 2 \,, \tag{5.13}$$

for any $X, Y \in V_1$.

Now, from (1.3), (5.10), (5.11), (5.12) and (5.13),

$$\begin{aligned} 4g(X, J_i Y) = (dq_i + q_j \wedge q_k)(X, Y) = - q_i([X, Y]) = \\ = - 4 \cot r \cot 2r g(X, J_i Y) \end{aligned} \tag{5.14}$$

$i = 1, 2$, for any $X, Y \in V_1$, (i, j, k) being a cyclic permutation of $(1, 2, 3)$. Thus,

$$J_1 V_1 \subseteq V_1^{\perp} \,, \quad J_2 V_1 \subseteq V_1^{\perp} \,. \tag{5.15}$$

If $X \in V_1$, from (5.15), there exists $Z \in V_t$, $t \neq 1$ such that $g(X, J_i Z) \neq 0$, $i = 1, 2$. From (1.16) applied to X and Z,

$$(\gamma_t I - A)\nabla_X Z - (\cot r\, I - A)\nabla_Z X = \sum_{k=1}^{3} 2g(X, J_k Z) U_k .$$

Therefore,

$$(5.16)\quad (\gamma_t - 2\cot 2r) g(\nabla_X Z, U_1) - (\cot r - 2\cot 2r) g(U_1, \nabla_Z X) = 2g(X, J_1 Z) .$$

As $g(X, J_1 Z) \neq 0$, from (1.14) and (5.16) we conclude that $\gamma_t = \cot r$ for some $t \neq 1$. Thus $\gamma_1, \ldots, \gamma_p$ are not distinct and the proof follows.

Case II. – $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$.

Let $\gamma_1, \ldots, \gamma_p$ be the distinct principal curvatures corresponding to eigenvectors in D. Denote by $V_1, \ldots, V_p$ the corresponding eigenspaces. From (1.16) it follows:

$$(5.17)\qquad (\alpha_k I - A)\nabla_X U_k - (\gamma_j I - A)\nabla_{U_k} X = -J_k X , \quad k = 1, 2, 3 ,$$

for any $X \in V_j$.

As U_1, U_2 and U_3 are principal, from the formulas of Gauss and Weingarten, $\nabla_{D'} D \subseteq D$. Thus from (1.14) and (5.17) we have

$$(5.18)\qquad q_k(X) = 0 , \quad k = 1, 2, 3 ,$$

for any $X \in D$.

Now, from (1.14) and (5.18), we get,

$$(5.19)\qquad \nabla_X U_k = J_k A X , \quad k = 1, 2, 3, \ X \in D .$$

From Lemma 5.2, $U_j(\alpha_i) = \pm(\alpha_i - \alpha_j) q_k(U_i)$, (i, j, k) being a cyclic permutation of $(1, 2, 3)$ and the fact that $\alpha_i \neq \alpha_j$ if $i \neq j$, $i, j = 1, 2, 3$, gives:

$$(5.20)\qquad q_k(U_i) = 0 , \quad k \neq i, \ k\ i = 1, 2, 3 .$$

Take $X \in V_j$ and suppose that for some $k = 1, 2, 3$, $J_k X$ has components in V_n and V_m, $n, m \neq j$. Let $Y \in V_n$, $\overline{Y} \in V_m$ such that $g(X, J_k Y) \neq 0$ and $g(X, J_k \overline{Y}) \neq 0$. From (1.16) applied respectively to X, Y and X, $\overline{Y}$, it follows that

$$(\gamma_n I - A)\nabla_X Y - (\alpha_j I - A)\nabla_Y X = 2 \sum_{k=1}^{3} g(X, J_k Y) U_k ,$$

and

$$2 \sum_{k=1}^{3} g(X, J_k \overline{Y}) U_k = (\gamma_m I - A)\nabla_X \overline{Y} - (\alpha_j I - A)\nabla_{\overline{Y}} X .$$

Then from (5.19) it follows:

$$(5.21) \qquad (\gamma_n - \alpha_k)\gamma_j + (\gamma_j - \alpha_k)\gamma_n = 2 = (\gamma_m - \alpha_k)\gamma_j + (\gamma_j - \alpha_k)\gamma_m .$$

From (5.21), $(\gamma_n - \gamma_m)\gamma_j + (\gamma_n - \gamma_m)(\gamma_j - \alpha_k) = 0$. If $n \neq m$, $\alpha_k = 2\gamma_j$. From this and (5.21), $2 + 2\gamma_j^2 = 0$. Thus $n = m$, that is,

(*) «If $J_k V_j \subseteq V_j^\perp$ for some $k = 1, 2, 3$, $j \in \{1, \ldots, p\}$, there exists $n \neq j$, $n \in \{1, \ldots, p\}$ such that $J_k V_j \subseteq V_n$».

THEOREM 5.6. – *There exist no real hypersurfaces with constant principal curvatures of QP^m such that U_1, U_2 and U_3 are principal with principal curvatures $\alpha_1, \alpha_2, \alpha_3$ respectively, and $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$.*

PROOF. – Suppose that $\alpha_i = 2 \cot 2r_i$, $0 < r_i < \pi/2$, $r_i \neq r_j$, $i \neq j$, $i, j = 1, 2, 3$. We distinguish two cases:

CASE III. – *$\gamma_t \neq \cot r_k$ for some $t = 1, \ldots, p$ and some $k = 1, 2, 3$.*

Suppose $t = 1$, $k = 1$. Considering $\varphi_{r_1}\colon M \to QP^m$, φ_{r_1} has constant rank q. Thus for any $B \in M$ there exists a neighborhood W_1 of B in M such that $\varphi_{r_1}(W_1)$ is a q-dimensional submanifold of QP^m and the distribution $T_0^1(B) = \{X \in T_B M / (\varphi_{r_1})_* X = 0\}$ is integrable with $(4m - q - 1)$-dimensional leaves in M. Then, from Lemma 2.5 it follows

$$(5.22) \qquad g([X, Y], U_2) = g([X, Y], U_3) = 0 , \quad X, Y \in V_1 ,$$

and from (1.16), $(\gamma_1 I - A)[X, Y] = 2 \sum_{k=1}^{3} g(X, J_k Y) U_k$. Thus, from (5.22) we get,

$$(5.23) \qquad J_k V_1 \subseteq V_1^\perp , \quad k = 2, 3 .$$

But from (5.21), if $J_1 V_1$ had a component in some V_n, $n \in \{1, \ldots, p\}$, $(\gamma_n - \alpha_1)\gamma_1 + (\gamma_1 - \alpha_1)\gamma_n = 2$, and as $\alpha_1 = 2 \cot 2r_1$ and $\gamma_1 = \cot r_1$, $\gamma_n = \cot r_1 = \gamma_1$. Thus

$$(5.24) \qquad J_1 V_1 = V_1 .$$

From (*) and (5.23) there exist $m, n \in \{2, \ldots, p\}$ such that $J_2 V_1 \subseteq V_n$ and $J_3 V_1 \subseteq V_m$ and now $J_1 V_1 = V_1$ implies $J_2 V_1 = J_2 J_1 V_1 = J_3 V_1 \subseteq V_m$, that is, $n = m$. Therefore there exists $n \in \{2, \ldots, p\}$ such that

$$(5.25) \qquad J_2 V_1 \subseteq V_n , \quad J_3 V_1 \subseteq V_n .$$

Finally from (5.21) and (5.25) it follows

$$(5.26)\qquad \begin{aligned}&(\gamma_n-\alpha_2)\cot r_1+(\cot r_1-\alpha_2)\gamma_n=2\\&(\gamma_n-\alpha_3)\cot r_1+(\cot r_1-\alpha_3)\gamma_n=2\,.\end{aligned}$$

From (5.26), $\gamma_n=\cot r_1=\gamma_1$, thus $\gamma_1,\dots,\gamma_p$ cannot be distinct. Thus this case is not possible.

CASE II.2. – $\gamma_t\neq\cot r_k$, $k=1,2,3$ *for any* $t\in\{1,\dots,p\}$.

In this case, from (1.16) if $X,Y\in V_t$ we have

$$(5.27)\qquad (\gamma_t I-A)[X,Y]=2\sum_{k=1}^{3}g(X,J_kY)U_k\,.$$

From (5.27) it follows that $\big(2\gamma_t(\gamma_t-2\cot 2r_k)-2\big)g(X,J_kY)=0$ for any X, $Y\in V_t$, $k=1,2,3$. As $2\gamma_t(\gamma_t-2\cot 2r_k)-2=0$ if and only if either $\gamma_t=\cot r_k$ or $\gamma_t=-\tan r_k$, and $\gamma_t\neq\cot r_k$ and we can also suppose that $\gamma_t\neq-\tan r_k$, $k=1,2,3$ (if not, we apply Case II.1 to the tube of radius $\Pi/2-r_k$), we have

$$(5.28)\qquad J_kV_t\subseteq V_t^{\perp}\,,\quad t=1,\dots,p,\ k=1,2,3$$

and then there exist $n,m,q\in\{2,\dots,p\}$ (we suppose $n=2$, $m=3$, $q=4$) such that

$$(5.29)\qquad J_1V_1=V_2\,,\quad J_2V_1=V_3\,,\quad J_3V_1=V_4\,.$$

From (5.29) having in mind (1.1) we obtain

$$(5.30)\qquad J_2V_2=V_4\,,\quad J_1V_3=V_4\,.$$

From (1.3), (1.14), (5.18) and (5.29)

$$(5.31)\qquad \begin{aligned}4g(X,J_1Y)&=(dq_1+q_2\wedge q_3)(X,Y)=-q_1([X,Y])=\\&=-g([X,Y],U_1)q_1(U_1)=-(\gamma_1+\gamma_2)g(X,J_1Y)q_1(U_1)\end{aligned}$$

for any $X\in V_1$, $Y\in V_2$. Then from (5.29) and (5.31)

$$(5.32)\qquad q_1(U_1)=-\frac{4}{\gamma_1+\gamma_2}\,.$$

It is easy to prove in a similar way that

$$(5.33)\qquad q_2(U_2)=-\frac{4}{\gamma_1+\gamma_3}\,,\quad q_1(U_1)=-\frac{4}{\gamma_3+\gamma_4}\,,\quad q_2(U_2)=-\frac{4}{\gamma_2+\gamma_4}\,.$$

Thus from (5.32) and (5.33) $\gamma_1+\gamma_2=\gamma_3+\gamma_4$ and $\gamma_2+\gamma_4=\gamma_1+\gamma_3$. This implies that $\gamma_1=\gamma_4$ and this case is not possible. This concludes the proof.

From Lemma 5.1, Case I and Case II we obtain the following

THEOREM 5.7. – *Let M be a real hypersurface with constant principal curvatures of QP^m such that $g(AD, D')=0$. Let s be the number of distinct principal curvatures of M. Then $s\in\{2,3,4\}$ and*

i) *If $s=2$, M is an open subset of a geodesic hypersphere.*

ii) *If $s=3$, M is an open subset of a tube of radius r, $0<r<\Pi/2$ over QP^k, $0<k<m-1$.*

iii) *If $s=4$, M is an open subset of a tube of radius r, $0<r<\Pi/4$ or $\Pi/4<r<\Pi/2$ over CP^m.*

6. – Real hypersurfaces of quaternionic projective space with two distinct principal curvatures.

In this section we shall determine those real hypersurfaces of QP^m, $m\geqslant 3$, having two distinct principal curvatures at any point.

THEOREM 6.1. – *Let M be a real hypersurface of QP^m, $m\geqslant 3$, with two distinct principal curvatures, λ and μ, at any point of M. Then there exists a local basis $\{J_1', J_2', J_3'\}$ of $\hat{V}$ such that the corresponding U_1', U_2', U_3' are principal with the same principal curvature.*

PROOF. – Let T_λ and T_μ the eigenspaces corresponding respectively to λ and μ. As $TM=T_\lambda\oplus T_\mu$ we can suppose that

$$U_k=a_kX_k+b_kV_k\,,\qquad k=1,2,3 \tag{6.1}$$

for certain unit vector fields $X_k\in T_\lambda$ and $V_k\in T_\mu$, $k=1,2,3$.

I. – Suppose that $a_k, b_k\neq 0$, $k=1,2,3$ on an open set of M. Let now $\Sigma=\{X\in T_\lambda/g(X,X_k)=0,\ k=1,2,3\}$ and $\Omega=\{V\in T_\mu/g(V,V_k)=0,\ k=1,2,3\}$. As $m\geqslant 3$, either Σ or Ω must have dimension $\geqslant 2$. Suppose dim $\Sigma\geqslant 2$ (if not, we would proceed with Ω). If X, Y are orthonormal vector fields in Σ, from (1.16) we have

$$(\lambda I-A)[X,Y]+X(\lambda)Y-Y(\lambda)X=2\sum_{k=1}^{3}g(X,J_kY)U_k \tag{6.2}$$

and taking the inner product of (6.2) and X (respectively, Y) we obtain $X(\lambda)=Y(\lambda)=0$. That is, (6.2) is

$$(\lambda I-A)[X,Y]=2\sum_{k=1}^{3}g(X,J_kY)U_k\,,\qquad X,Y\in\Sigma\,. \tag{6.3}$$

From (6.3) we obtain the following homogeneous linear system whose variables are $g(X, J_k Y)$, $k = 1, 2, 3$

$$
\begin{aligned}
&0 = a_1 g(X, J_1 Y) + a_2 g(X_1, X_2)\, g(X, J_2 Y) + a_3 g(X_1, X_3) g(X, J_3 Y) \\
(6.4)\qquad &0 = a_1 g(X_1, X_2) g(X, J_1 Y) + a_2 g(X, J_2 Y) + a_3 g(X_2, X_3) g(X, J_3 Y) \\
&0 = a_1 g(X_1, X_3) g(X, J_1 Y) + a_2 g(X_2, X_3) g(X, J_2 Y) + a_3 g(X, J_3 Y) \,..
\end{aligned}
$$

This system has trivial solution if and only if X_1, X_2, X_3 are linearly independent. So we distinguish the following cases:

I.1. - X_1, X_2, X_3 are linearly independent.
From (6.4), $g(X, J_k Y) = 0$, $k = 1, 2, 3$, for any $X, Y \in \Sigma$. Thus

$$(6.5)\qquad J_k X \in T_\mu \oplus Sp\{X_1, X_2, X_3\}\,, \quad k = 1, 2, 3,\ X \in \Sigma\,.$$

On the other hand taking X_k, $k = 1, 2, 3$ and a unit $X \in \Sigma$, from (1.16) we get $(\lambda I - A)[X, X_k] + X_k(\lambda) X = \sum_{k=1}^{3} \{(-f_i(X_k) J_i X - 2g(J_i X, X_k) U_i\}$, $k = 1, 2, 3$. Taking the inner product of this expression and X, $X_k(\lambda) = 0$, $k = 1, 2, 3$, thus the above expression is

$$(6.6)\qquad (\lambda I - A)[X, X_k] = \sum_{k=1}^{3} \{- f_i(X_k) J_i X - 2g(J_i X, X_k) U_i\}\,, \qquad k = 1, 2, 3\,.$$

From (6.6) we obtain

$$
\begin{aligned}
&0 = a_1 g(X_1, J_1 X) + a_2 g(X_1, X_2) g(X_1, J_2 X) + a_3 g(X_1, X_3) g(X_1, J_3 X) \\
(6.7)\qquad &0 = a_1 g(X_1, X_2) g(X_1, J_1 X) + a_2 g(X_1, J_2 X) + a_3 g(X_2, X_3) g(X_1, J_3 X) \\
&0 = a_1 g(X_1, X_3) g(X_1, J_1 X) + a_2 g(X_2, X_3) g(X_1, J_2 X) + a_3 g(X_1, J_3 X)
\end{aligned}
$$

and similar systems changing X_1 by X_2 (respectively, by X_3). As X_1, X_2 and X_3 are linearly independent we have

$$(6.8)\qquad g(J_k X, X_i) = 0\,, \quad i, k = 1, 2, 3$$

and thus $0 = g(J_k X, U_i) = a_i g(J_k X, X_i) + b_i g(J_k X, V_i) = b_i g(J_k X, V_i)$, $k = 1, 2, 3$. From this and (6.5),

$$(6.9)\qquad J_k \Sigma \subseteq \Omega\,, \quad k = 1, 2, 3\,.$$

If V_1, V_2, V_3 were also linearly independent, $J_k \Omega \subseteq \Sigma$, $k = 1, 2, 3$ and then $\dim \Sigma + \dim \Omega$ would be even, but if this is the case, $\dim M = \dim \Sigma + \dim \Omega + 6$

is also even and M cannot be a real hypersurface of QP^m. Thus V_1, V_2 and V_3 must be linearly independent.

From (6.9) there exist two orthonormal vectors V, W in Ω and from (1.16) we have

$$(\mu I - A)[V, W] = 2 \sum_{k=1}^{3} g(V, J_k W) U_k . \tag{6.10}$$

If we apply $(\lambda I - A)$ to (6.10) having in mind that the only principal curvatures of M are λ and μ, it follows $0 = 2 \sum_{i=1}^{3} g(V, J_k W)(\lambda I - A) U_k$ and from (6.1)

$$0 = b_1 g(V, J_1 W) V_1 + b_2 g(V, J_2 W) V_2 + b_3 g(V, J_3 W) V_3 . \tag{6.11}$$

From (6.9) we can choose V and W in $\{J_1X, J_2X, J_3X\}$, $X \in \Sigma$, and from (6.11) we have

$$V_1 = V_2 = V_3 = 0 \tag{6.12}$$

which concludes the proof in this case.

I.2. - X_1, X_2, X_3 are not linearly independent.

If V_1, V_2, V_3 were linearly independent the result would follow as in Case I.1. Thus we suppose that V_1, V_2, V_3 are not linearly independent. Then (6.4) admits a nontrivial solution, that is, for any orthonormal X, $Y \in \Sigma$, $g(X, J_k Y) \neq 0$ for some $k = 1, 2, 3$. Suppose that $k = 1$. Applying $(\mu I - A)$ to (6.3) and having in mind (6.1) we have $0 = \sum_{k=1}^{3} a_k g(X, J_k Y) X_k$, that is,

$$X_1 = \frac{a_2 g(X, J_2 Y)}{a_1 g(X, J_1 Y)} X_2 - \frac{a_3 g(X, J_3 Y)}{a_1 g(X, J_1 Y)} X_3 \tag{6.13}$$

and taking $\overline{U}_1 = U_1 + \big(g(X, J_2 Y)/g(X, J_1 Y)\big) U_2 + \big(g(X, J_3 Y)/g(X, J_1 Y)\big) U_3$ from (6.1) and (6.13) we have

$$A\overline{U}_1 = \mu \overline{U}_1 . \tag{6.14}$$

Taking as a new orthonormal basis of D' $\{U_1', U_2', U_3'\}$, where $U_1' = \overline{U}_1/\|\overline{U}_1\|$ and if $C \in SO(3)$ is the matrix of change of basis,

$$C \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{pmatrix} .$$

is a new local basis of $\hat{V}$ such that the corresponding $U'_k = -\bar{J}_k N$, $k = 1, 2, 3$ verify

$$U'_1 = U'_1\,, \quad U'_2 = a'_2 X'_2 + b'_2 V'_2\,, \quad U'_3 = a'_3 X'_3 + b'_3 V'_3\,, \tag{6.15}$$

where U'_1, V'_2, V'_3 are unit vector fields in T_μ and X'_2, X'_3 unit vector fields in T_λ.

If U'_1, V'_2, V'_3 were linearly independent, the proof would finish as in I.1. Suppose thus that U'_1, V'_2, V'_3 are linearly dependent, $g(U'_1, U'_k) = 0$, $k = 2, 3$, implies that if $b'_k \neq 0$, $k = 2, 3$, $g(U'_1, V'_k) = 0$, $k = 2, 3$ and then $V'_3 = \gamma V'_2$ γ being a real valued function defined on some open subset of M. Moreover if $V'_3 = \gamma V'_2$ from (6.15) it follows that $-(b'_3/b'_2)\gamma U'_2 + U'_3 = a'_3 X'_3 - (b'_3/b'_2)\gamma a'_2 X'_2 \in T_\lambda$, that is, there exists a vector field $\bar{U}_2 \in D' \cap T_\lambda$. (Notice that if $b'_k = 0$ for some $k = 2, 3$, either $U'_2 \in T_\lambda$ or $U'_3 \in T_\lambda$, so we can make the following reasoning also in this case.) Taking as a new orthonormal basis of D' $\{U''_1, U''_2, U''_3\}$ where $U''_1 = U'_1$, $U''_2 = \bar{U}_2/\|\bar{U}_2\|$ and if $C' \in SO(3)$ is the matrix of change of basis,

$$C' \begin{pmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{pmatrix} = \begin{pmatrix} J'_1 \\ J'_2 \\ J'_3 \end{pmatrix}$$

is a new local basis of $\hat{V}$ such that $U''_1 = -J'_1 N \in T_\mu$ and $U''_2 = -J'_2 N \in T_\lambda$. Thus (6.1) can be written as

$$U''_1 = U''_1\,, \quad U''_2 = U''_2\,, \quad U''_3 = aX' + bV' \tag{6.16}$$

where U''_2, $X' \in T_\lambda$ and U''_1, $V' \in T_\mu$ are unit vector fields. From (6.16) we obtain

$$(\lambda I - A)[X, Y] = 2 \sum_{k=1}^{3} g(X, J'_k Y)\, U''_k \tag{6.17}$$

$$(\lambda I - A)[X, X'] = -f_3(X) J'_3 X - 2g(J'_3 X, X')\, U''_3 \tag{6.18}$$

for any orthonormal vectors $X, Y \in \Sigma$. From (6.17) and (6.18) we get either $J'_2\Sigma \subseteq \Sigma^\perp$, $J'_3\Sigma \subseteq \Omega$, or $a = 0$. If $J'_2\Sigma \subseteq \Sigma^\perp$, $J'_3\Sigma \subseteq \Omega$, $\dim \Omega \geqslant 2$ and we also obtain either $J'_1\Omega \subseteq \Omega^\perp$, $J'_3\Omega \subseteq \Sigma$ or $b = 0$. But $\dim M = \dim \Sigma + \dim \Omega + \dim Sp\{U''_1, U''_2, aX', bV'\}$, and then either $a = 0$ or $b = 0$. Suppose that $b = 0$ (the case $a = 0$ is similar) From (6.16) we have

$$AU''_1 = \mu U''_1\,, \quad AU''_2 = \lambda U''_2\,, \quad AU''_3 = \lambda U''_3 \tag{6.19}$$

and from (6.14) it follows

$$J'_2\Sigma \subseteq \Omega\,, \quad J'_3\Sigma \subseteq \Omega\,, \quad J'_1\Omega \subseteq \Sigma\,. \tag{6.20}$$

From (1.1) and (6.20), $J'_2\Sigma = J'_1 J'_3 \Sigma \subseteq J'_1\Omega \subseteq \Sigma$, which contradicts (6.20) and proves the Theorem in this case.

II. – If either $a_k = 0$ or $b_k = 0$ for some $k = 1, 2, 3$, the proof follows from Case I.2.

THEOREM 6.2. – *Let M be a rea hypersurface of QP^m, $m \geqslant 3$. Then M has two distinct principal curvatures if and only if M is locally an open subset in a geodesic hypersphere.*

PROOF. – From Theorem 6.1, U_1, U_2 and U_3 are principal with the same principal curvature α for some local basis of $\hat{V}$. From Lemma 5.2 α is locally constant and we can suppose that $\alpha = 2 \cot 2r$, $0 < r < \pi/2$. If λ denotes the other principal curvature of M and V is the set where $\lambda \neq \alpha$, V is clearly open. Taking the maximum rank between φ and $\varphi_{\Pi/2-r}$, there exists an open subset U of V where φ_r has constant rank q and using the Inverse Function Theorem, $\varphi_r(U)$ is a q-dimensional submanifold of QP^m. Reasoning as in Lemmas 5.3 and 5.4, rank $\eta_* = 4m - 1$ over U and $\varphi_r(U)$ is a quaternionic submanifold of QP^m. The result follows from Examples 1 and 2. The converse is trivial.

7. – On the Ricci tensor of a real hypersurface of QP^m.

DEFINITION 7.1. – Let M be a real hypersurface of QP^m. If the Ricci tensor of M satifies,

$$SX = aX + b \sum_{k=1}^{3} f_k(X)\, U_k \tag{7.1}$$

for any $X \in TM$, where a and b are constant, we shall say that M is a *pseudo-Einstein* real hypersurface of QP^m. If the Ricci tensor of M satisfies,

$$SX = aX + b \sum_{k=1}^{3} f_k(AX)\, U_k \tag{7.2}$$

for any $X \in TM$, where a and b are constant, we shall say that M is an *almost Einstein* real hypersurface of QP^m.

From Definition 7.1 and (1.15) it follows

$$(A^2 - hA + \lambda)X = \theta \sum_{k=1}^{3} f_k(X)\, U_k\,, \qquad X \in TM \tag{7.3}$$

for a pseudo-Einstein real hypersurface and

$$(A^2 - hA + \lambda)X = - b \sum_{k=1}^{3} f_k(AX)\, U_k - 3 \sum_{k=1}^{3} f_k(X)\, U_k\,, \qquad X \in TM \tag{7.4}$$

for an almost-Einstein real hypersurface, where $\lambda = a - (4m + 7)$ and $\theta = -(b + 3)$.

LEMMA 7.2. – *Let M be an almost-Einstein real hypersurface of QP^m, $m \geqslant 2$, or a pseudo-Einstein real hypersurface of QP^m, $m \geqslant 3$. Then there exists a local basis $\{J'_1, J'_2, J'_3\}$ of $\hat{V}$ such that the corresponding $U'_k = - J'_k N$, $k = 1, 2, 3$ are principal and at least two among them have the same principal curvature.*

PROOF. – *Case I.* – Suppose that M is a pseudo-Einstein real hypersurface of QP^m, $m \geqslant 3$. Let $T = A^2 - hA$. From (7.3), we have,

$$TX = (4m + 7 - a)X , \qquad TZ = (4m + 4 - (a + b))Z , \tag{7.5}$$

for any $X \in D$, $Z \in D'$.

Let $\{X_1, \ldots, X_{4m-1}\}$ be an orthonormal basis of TM of eigenvectors of A. From the definition of T, $\{X_1, \ldots, X_{4m-1}\}$ are also eigenvectors of T. Then we have:

a) $b \neq -3$. From (7.5), D and D' are invariant under T and T has $4m + 7 - a$ as an eigenvalue for any vector of D and $4m + 4 - (a + b)$ for any vector of D'. Then there exists an orthonormal basis of D', $\{X_1, X_2, X\}$ such that $TX_k = (4m + 4 - (a + b))X_k$, $k = 1, 2, 3$. Thus $g(AD, D') = 0$ and from Lemma 5.1 there exists a local basis $\{J'_1, J'_2, J'_3\}$ of $\hat{V}$ such that the corresponding $U'_k = - J'_k N$, $k = 1, 2, 3$, are principal. Moreover, in this case, from (7.5) at least two among them have the same principal curvature.

b) If $b = -3$, from (7.5), M has at most two distinct principal curvatures at each point and from Theorem 6.1, there exists a local basis $\{J'_1, J'_2, J'_3\}$ of $\hat{V}$ such that $U'_k = - J'_k N$, $k = 1, 2, 3$ are principal with the same principal curvature.

Case II. – Suppose now that M is an almost-Einstein real hypersurface of QP^m, $m \geqslant 2$. From (7.4) we have

$$(A^2 - hA)X + b \sum_{k=1}^{3} g(AX, U_k)\, U_k = (4m + 7 - a)X , \tag{7.6}$$

$$(A^2 - hA)Z + b \sum_{k=1}^{3} g(AZ, U_k)\, U_k = (4m + 4 - a)Z , \tag{7.7}$$

for any $X \in D$, $Z \in D'$.

From (7.6) we have $g\big((A^2 - (h - b)A)X, Z\big) = 0$ for any $X \in D$, $Z \in D'$ and then, taking the tensor $T = A^2 - (h - b)A$, we obtain

$$g(TD, D') = 0 . \tag{7.8}$$

From (7.7), $TZ = (4m + 4 - a)Z + b(AZ)^{\top}$, for any $Z \in D'$, denoting by $(AZ)^{\top}$ the component of AZ in D. Now, from (7.8),

$$b(AZ)^{\top} = 0 . \tag{7.9}$$

If $b \neq 0$, from (7.9), $g(AD, D') = 0$ and from (7.6), (7.7) and Lemma 5.1, there exists a local basis $\{J'_1, J'_2, J'_3\}$ of $\hat{V}$ such that the corresponding $U'_k = -J'_k N$, $k = 1, 2, 3$ are principal and at least two of them have the same principal curvature.

If $b = 0$, from (7.4), M is pseudo-Einstein and $b \neq -3$. Now the result follows from Case I *a*).

Theorem 7.3. – *Let M be a real hypersurface of QP^m. Then*:

1) *If M is almost-Einstein and $m \geqslant 2$, M is an open subset of one of the following*:

 1.1) *A geodesic hypersphere.*

 1.2) *A tube of radius r over QP^k, $0 < k < m-1$, $0 < r < \pi/2$ and* $\cot^2 r = (4k+2)/(4m-4k-2)$.

 1.3) *A tube of radius r over CP^m, $0 < r < \pi/4$ or $\pi/4 < r < \pi/2$ and* $\cot^2 2r = 1/(m-1)$.

2. – *If M is pseudo-Einstein and $m \geqslant 3$, then M is an open subset of either* 1.1 *or* 1.2.

Proof. – Suppose that M is an almost-Einstein real hypersurface of QP^m. From Lemma 7.2, we can suppose that $U_k = -J_k N$, $k = 1, 2, 3$, $\{J_1, J_2, J_3\}$ being a local basis of $\hat{V}$, are principal (on the open set where $\{J_1, J_2, J_3\}$ are defined) and, at least two among them have the same principal curvature. Thus we suppose,

$$AU_1 = \alpha_1 U_1 , \quad AU_2 = \alpha_1 U_2 , \quad AU_3 = \alpha_3 U_3 . \tag{7.10}$$

We shall distinguish two cases:

Case I. – $\alpha = \alpha_1 = \alpha_3$, then from Lemma 5.2, α is locally constant and from (7.4) as a and b are constant, there exists an open subset U of M such that U is a real hypersurface of QP^m with constant principal curvatures such that $g(AD, D') = 0$. Then, from Theorem 5.7 and having in mind that M is connected, M is an open subset of either a geodesic hypersphere or a tube over a QP^k, $0 < k < m-1$, $0 < r < \pi/2$. Among all these tubes only the one appearing in 1.2) is almost-Einstein.

Case II. – Suppose that $\alpha_1 \neq \alpha_3$ at some point B. Then from the continuity of α_1 and α_3, there exists a neighborhood V of B such that $\alpha_1 \neq \alpha_3$ on V and such that J_1, J_2 and J_3 are defined on V. Suppose that

$$AU_k = \alpha_k U_k , \quad k = 1, 2, 3, \ \alpha_1 = \alpha_2 \neq \alpha_3 \text{ on } V . \tag{7.11}$$

Then from Lemma 5.2,

$$Y(\alpha_1) = Y(\alpha_3) = U_1(\alpha_1) = U_2(\alpha_1) = 0 , \tag{7.12}$$

for any $Y \in D$.

II.*a*) If $\alpha_1 \neq 0$, from (7.5), $q_2(U_3) = q_1(U_3) = 0$, that is, $\nabla_{U_3} U_3 = 0$. As $g(\nabla_X \operatorname{grad} \alpha_3, Y) = g(\nabla_Y \operatorname{grad} \alpha_3, X)$, from (1.14) we have, $X(\beta) f_3(Y) + g(\nabla_X U_3, Y) = Y(\beta) f_3(X) + g(\nabla_Y U_3, X)$, for any $X, Y \in TM$, where $\beta = U_3(\alpha_3)$. Taking $X = U_3$ or $Y = U_3$ it follows,

$$\beta g(\nabla_X U_3, Y) = \beta g(\nabla_Y U_3, X). \tag{7.13}$$

Finally, from (1.14) and (7.13) we have $\beta g\big((J_3 A + A J_3) X, Y\big) = 0$, $q_2(X) = q_1(Y) = 0$, for any $X, Y \in D$. But from the equation of Codazzi $J_3 A + A J_3 \neq 0$ on D. Thus $\beta = 0$. From this and (7.12), α_3 is locally constant on V and we can suppose that is constant on V.

From (7.7) $\alpha_1 \alpha_3 = a - (4m + 4)$, thus α_1 is also constant and then from (7.4) and (7.11), V is a real hypersurface with constant principal curvatures such that U_1, U_2 and U_3 are principal and exactly two among their corresponding principal curvatures are equal. Thus from (7.4) and Theorem 5.7, as M is connected, M must be an open subset of 1.3), because this is the one almost-Einstein tube over CP^m.

II.*b*) If $\alpha_1 = 0$, $\alpha_1 = 2 \tan 2(\pi/4)$. Taking the tube of radius $\pi/4$ over V and denoting by Σ a connected open subset of V over which $\varphi_{\pi/4}$ has maximum rank, we conclude as in § 5 that as $\alpha_3 \neq -2 \tan 2(\pi/4)$, rank $\eta_* = 4m - 1$, that is, $\alpha_1 = \alpha_2 = \alpha_3$ on Σ. But this is a contradiction and we conclude the proof in this case

If M is pseudo-Einstein, the proof is similar.

COROLLARY 7.4. – *M is an Einstein real hypersurface of QP^m if and only if M is an open subset of a geodesic hypersphere of QP^m of radius r, with $\cot^2 r = 1/2m$.*

PROOF. – This is the only Einstein real hypersurface of QP^m appearing in Theorem 7.3.

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