# An Identification Problem for an Elliptic Equation in Two Variables (*).

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**Sunto.** – *Si studia il problema inverso di determinare il coefficiente a nell'equazione ellittica in due variabili*

$$(*) \qquad \operatorname{div}(a \operatorname{grad} u) = 0 ,$$

*quando se ne conosce una soluzione u. Si danno un risultato di dipendenza continua di a da u e un metodo di determinazione approssimata di a. Elemento chiave in questi risultati è lo studio di proprietà dei punti critici delle soluzioni u di* (∗).

## Introduction.

In this paper we will consider the elliptic equation

$$(1) \qquad \operatorname{div}\big(a(x) \operatorname{grad} u(x)\big) = 0 , \qquad x \in \Omega ,$$

in which the coefficient $a$ has to be determined when one solution $u$ is known. Here $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$.

This is a nonlinear ill-posed problem of identification.

In recent years, identification problems for elliptic equations have been object of many studies. The results which are of main interest are uniqueness, stability and algorithms.

In MARCELLINI [16], a rather thorough treatment is made in a one-dimensional case. KOHN and VOGELIUS [12], [13], have treated a uniqueness problem raised by CALDERON, [5]. Uniqueness results are also present in a paper by RUNDELL [19]. In RICHTER [17], and ALESSANDRINI [3], certain stability results are given. Various types of algorithms have been devised, and are mainly found in the engineering literature (see e.g. YAKOWITZ-DUCKSTEIN [21] for references). Only few of them, however, take into account the ill-posedness of the problem. Let us mention CHAVENT [6], RICHTER [18], and HOFFMANN-SPREKELS [10].

Here we will discuss about stability and we will propose an algorithm. We must say that our analysis is limited to the two-dimensional case.

(*) Entrata in Redazione il 14 settembre 1985.

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It is a well established matter that the character of the problem depends on the behaviour of the modulus of the gradient of $u$

$$|Du| = (u_{x_1}^2 + u_{x_2}^2)^{\frac{1}{2}} .$$

In fact it is easily seen that (1) can be interpreted as a first order partial differential equation for the unknown coefficient $a$

$$Du \cdot Da + \Delta u \, a = 0 , \quad \text{in } \Omega , \tag{1'}$$

and here the principal part becomes singular at the points where $|Du| = 0$.

A possible strategy is, therefore, the one of setting extra conditions which guarantee that $|Du|$ never vanishes, (see [17], [3]).

For instance in [3], it is treated the problem of determining $a$ when the Green function $G(x, y)$ associated to the elliptic operator in (1) is known for a fixed point $y \in \Omega$. There it is shown that, if $\Omega$ is simply connected, then $|DG(\cdot, y)|$ is always positive. This fact is used to obtain a stability result.

In the present case, however, it would not be satisfactory to restrict the study to the solutions of (1) which do not posses critical points. We will show instead that, in general, critical points of $u$ have a particular structure which is governed by easily detectable properties of the boundary values.

Let us assume that a boundary condition on $u$ is prescribed

$$u = g , \quad \text{on } \partial\Omega , \tag{2}$$

where $g$ is a smooth function which is precisely known. Moreover let us assume that $a$ satisfies the ellipticity condition

$$0 < \lambda_1 \leqslant a(x) \leqslant \lambda_2 , \quad x \in \Omega , \tag{3}$$

and the following regularity hypothesis

$$|Da(x)| \leqslant E , \quad x \in \Omega , \tag{4}$$

where $\lambda_1$, $\lambda_2$, $E$ are fixed positive numbers.

We will prove that, if $g$ has a finite number, $N$, of relative maxima and minima on $\partial\Omega$, then the gradient of $u$ vanishes only at a finite number of interior points, and only with a finite multiplicity. Moreover, the number of critical points and their multiplicities are controlled in terms of $N$ (see Theorem 1.2).

Furthermore we determine a lower bound on $|Du|$ with an explicit non-negative function, which has the same zeros as $|Du|$ (see Theorem 1.3).

These fact are then used to obtain a stability estimate (Theorem 2.1), and in the development of an algorithm.

This algorithm consists of an approximation procedure. We will show that the solution $a_\varepsilon$ of the (well posed) elliptic boundary value problem

$$\left\{\begin{array}{ll} \varepsilon\, \Delta a_\varepsilon + \operatorname{div}\,(a_\varepsilon \operatorname{grad} u) = 0\,, & \text{in} \quad \Omega\,, \\ a_\varepsilon = a\,, & \text{on } \partial\Omega\,, \end{array}\right. \tag{5}$$

converges to $a$ as $\varepsilon \to 0$. Here $u$ is the given solution of (1), (2). Thus an approximate identification is performed solving problem (5) with a suitably chosen value of $\varepsilon$.

It may be useful to make a sketch of the hypotheses and of the main results.

**Data.**

(i) The solution $u$ of (1), (2).

(ii) The boundary values $a|_{\partial\Omega}$ of $a$.

**A priori information.**

(i) The coefficient $a$ satisfies the ellipticity condition: (3).

(ii) The coefficient $a$ has bounded first order derivatives: (4).

(iii) The function $g$, appearing in (2), is $C^2$ smooth and has a finite number, $N$, of relative maxima and minima on $\partial\Omega$.

(iv) $\Omega$ is a simply connected, $C^2$-smooth, bounded domain in $\mathbb{R}^2$.

**Results.**

(i) *Stability.* The mapping

$$L^2(\Omega)\times L^\infty(\partial\Omega) \ni \big(u,\, a|_{\partial\Omega}\big) \to a \in L^\infty_{\text{loc}}(\Omega)$$

is Hölder-continuous (Theorem 2.1).

(ii) *Convergence of the algorithm.* As $\varepsilon$ tends to 0, the solution $a_\varepsilon$ of (5) converges to $a$ in $L^p_{\text{loc}}(\Omega)$ for every $p < \infty$. The rate of convergence is of Hölder type. Moreover the algorithm is stable with respect to $L^2$-perturbations of $u$ (Theorems 3.1, 4.1).

Some comments are in order.

I) We wish to point out that the above stability results depend only on the $L^2$-error on $u$. This is necessary if we are dealing with applications in which only noisy measurements are available. It is also interesting to consider the case in which only a finite number of data is given, and we will show how the above results can be adapted to this case (see Section 5).

II) By the use of the method of characteristics on equation (1′), it can be argued that the coefficient $a$ is uniquely determined when $u$ is given in $\Omega$, and the boundary values of $a$ are prescribed on the so called inflow boundary of $\Omega$ (i.e., the set of points of $\partial\Omega$ on which the inner normal derivative of $u$ is positive) see BONGIORNO-VALENTE [4], and [17]. In this respect, our assumption that all the boundary values of $a$ are given, may seem redundant. However, it must be noticed that, if only an approximation of $u$ is known, then the determination of the inflow boundary is itself an ill-posed problem. For this reason, the present assumption seems to be rather convenient and not too expensive.

III) In some respects the present algorithm has some resemblances with the method of quasi-reversibility, see LATTES-LIONS [14]. Essentially this algorithm is based on an elliptic singular perturbation problem, and, on this field, a wide literature exists. Let us mention LEVINSON [15], KAMIN [11].

The paper is organized as follows.

In *Section 1* the properties of the critical points of $u$ are investigated. Theorem 1.2 gives an evaluation of the number and of the multiplicities of interior critical points of $u$, the solution of (1), (2). In Theorem 1.3 a lower bound on $|Du|$ is proved.

*Section 2* deals with the main a-priori stability result, which is given in Theorem 2.1.

*Section 3* contains the basic result regarding the convergence of the algorithm, Theorem 3.1.

In *Section 4* we describe a modified version of the algorithm, which applies when noisy data are given. The stability of such an algorithm is proved in Theorem 4.1.

The case of discrete data is treated in *Section 5.* An a-priori error estimate is proved in Theorem 5.1. In Theorem 5.2 an ad-hoc adaptation of the algorithm is treated and an error estimate is given.

In *Section 6* we present, some useful regularity estimates for the direct problem (1), (2). These estimates are well-known and are collected here just for the convenience of the reader.

**Notation and definitions.**

1) $B_d(x)$ denotes the disk centered at $x \in \mathbb{R}^2$ with radius $d$.

2) $\Omega_d = \{x \in \Omega\colon d(x, \partial\Omega) > d\}$.

3) $\operatorname{diam}\Omega$ denotes the diameter of $\Omega$.

4) $|\Omega|$ denotes the area of $\Omega$.

5) $\operatorname{osc}_{\partial\Omega} g = \max_{\partial\Omega} g - \min_{\partial\Omega} g$.

6) When no other ground domain is specified: $\|\cdot\|_p$ stands for the $L^p$-norm in $\Omega$, $1 \leqslant p \leqslant +\infty$.

7) In places we will assume that $\Omega$ is $C^2$-smooth. In such a case $\Omega$ satisfies an interior and exterior sphere condition. We define $d_0$ as the largest positive number

such that, for every $x \in \mathbb{R}^2 \setminus \partial\Omega$, there exists a disk of radius $d_0$, which contains $x$ and does not intersect $\partial\Omega$.

8) For the sake of brevity we will use the following convention. Given a function $f$ on a set $A \subset \mathbb{R}^2$, we will refer to the number of maxima (minima) of $f$ as the number of connected components of the set of points of relative maximum (minimum) of $f$ in $A$. Note that if $A$ is a simple closed curve, then $f$ has the same number of maxima and minima.

9) We will denote by $c$ absolute constants, which may be different from line to line. Likewise we will denote by $c_p$ constants depending only on the parameter $p$.

## 1. – Properties of the critical points of $u$.

We start with a theorem due to HARTMAN and WINTNER, the proof of which can be found in [9].

THEOREM 1.1. – Let $u$ be $a$, non-constant, $W^{1,2}_{\text{loc}}(\Omega)$ solution of (1), where $a$ satisfies (3), (4).

For every $x^0 \in \Omega$ there exists and integer $n \geqslant 1$ and a homogeneous harmonic polynomial $H_n$ of degree $n$, such that $u$ satisfies, as $x \to x^0$,

$$\left\{\begin{array}{l} u(x) = u(x^0) + H_n(x - x^0) + o(|x - x^0|^n)\,, \\ Du(x) = DH_n(x - x^0) + o(|x - x^0|^{n-1})\,. \end{array}\right. \tag{1.1}$$

REMARK 1.1. – We list some straightforward consequences of Theorem 1.1, which will be repeatedly used in this section.

(i) The interior critical points are isolated.

(ii) Every interior critical point $x^0$ of $u$ has a finite multiplicity, that is

$$|Du(x)| = O(|x - x^0|^m)$$

where $m = n - 1$, $n$ being the integer given by Theorem 1.1.

(iii) If $x^0$ is an interior critical point of multiplicity $m$, then, in a neighbourhood of $x^0$, the level line $\{x \in \Omega \colon u(x) = u(x^0)\}$ is made of $m + 1$ regular arcs intersecting with equal angles at $x^0$.

THEOREM 1.2. – Let $\Omega$ be a bounded simply connected domain in $\mathbb{R}^2$. Let $a$ satisfy (3), (4). Let $g$ be a continuous function such that $g|_{\partial\Omega}$ has $N$ maxima. Let $u$ be the $W^{1,2}_{\text{loc}}(\Omega)$ solution of (1) with boundary condition (2).

The interior critical points of $u$ are finite in number and, if $m_1, \ldots, m_K$ are their multiplicities, then the following inequality holds

$$\sum_{i=1}^{K} m_i \leqslant N - 1 . \tag{1.2}$$

THEOREM 1.3. – Let the hypotheses of Theorem 1.2 be satisfied. Moreover, let us assume that $\Omega$ is $C^2$-smooth and that $g \in C^2(\bar{\Omega})$.

There exist points $x_1, \ldots, x_K$ in $\Omega$ and positive integers $m_1, \ldots, m_K$ satisfying (1.2), such that for every $x \in \Omega_d$, $d > 0$,

$$|Du(x)| \geqslant C_1 \prod_{i=1}^{K} (|x - x_i|/\text{diam } \Omega)^{m_i} , \tag{1.3}$$

where $C_1$ is a positive constant depending only on $d$, $\Omega$, $\lambda_1$, $E$, $\operatorname*{osc}_{\partial\Omega} g$ and $\|g\|_{C^2(\bar{\Omega})}\}$

It is convenient to split the proof of Theorem 1.2 into a sequence of three lemmas.

LEMMA 1.1. – Let $\Omega$ be bounded and simply connected. Let $u$ be a $W^{1,2}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ solution of (1) where $a$ satisfies (3), (4). If $u$ has an infinite number of interior critical points, then $u|_{\partial\Omega}$ has an infinite number of relative maxima and minima.

PROOF (Outline). – We may distinguish two cases.

(i) There exists a critical point $x^0 \in \Omega$ such that an infinite number of critical points of $u$ is contained in the level line $\{x \in \Omega : u(x) = u(x^0)\}$.

(ii) There exists a sequence $\{x_n\}$ of critical points in $\Omega$, such that for every $n, m, n \neq m, u(x_n) \neq u(x_m)$.

*Case* (i). – Note that $\Omega \setminus \{x \in \Omega : u(x) = u(x^0)\}$ is made of an infinite number of connected components $A_1, A_2, \ldots, A_n, \ldots$ such that, for every $i$, on $\partial A_i \cap \Omega$, $u = u(x^0)$ and, on $A_i$, either $u > u(x^0)$ or $u < u(x^0)$.

*Case* (ii). – By induction we may find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and two sequences of non-empty open subsets of $\Omega$, $\{C_k\}$, $\{B_k\}$ having the following properties

(*a*) $C_k \supset C_{k+1}$, for every $k = 1, 2, \ldots$;

(*b*) $u|_{\partial C_k \cap \Omega} = u(x_{n_k})$, for every $k = 1, 2, \ldots$; and: $x_{n_j} \in C_k$, for every $j \geqslant k + 1$;

(*c*) $B_k \subset C_k \setminus C_{k+1}$, $u|_{\partial B_k \cap \Omega} = u(x_{n_{k+1}})$; and: $u|_{B_k} > u(x_{n_{k+1}})$, or: $u|_{B_k} < u(x_{n_{k+1}})$; for every $k = 1, 2, \ldots$.

Therefore, in both cases (i) and (ii), $u$ has an infinite number of maxima or minima in $\bar{\Omega}$. By the maximum principle all these maxima and minima are attained at the boundary. Thus $u|_{\partial\Omega}$, the trace of $u$, has an infinite number of maxima and minima, in fact, since $\partial\Omega$ is a simple closed curve, $u|_{\partial\Omega}$ has as many maxima as minima.

REMARK 1.2. – By the above lemma, if the hypotheses of Theorem 1.2 are satisfied, then $u$ has only a finite number of interior critical points. Thus, only the estimate (1.2) remains to be proved.

LEMMA 1.2. – Let the hypotheses of Theorem 1.2 be satisfied. If, in addition, we assume that $u$ has a unique critical point $x^0 \in \Omega$, then its multiplicity $m$ satisfies

$$m \leqslant N - 1\,, \tag{1.4}$$

PROOF. – The level line $\{x \in \bar{\Omega}: u(x) = u(x^0)\}$ splits $\bar{\Omega}$ into at least $2(m+1)$ connected components, and, on at least $m+1$ of them, we have

$$u > u(x^0)$$

that is, each of them contains at least one maximum, which, by the maximum principle, is attained at the boundary.

Thus $g$ has at least $m+1$ maxima, or, which is the same, (1.4) holds.

LEMMA 1.3. – Let the hypotheses of Theorem 1.2 be satisfied. Let $x_1, \ldots, x_K$ be the interior critical points of $u$, and let $m_1, \ldots, m_K$ be their respective multiplicities. If $x_1, \ldots, x_K$ all belong to the same connected component of the level line $\{x \in \Omega: u(x) = u(x^0)\}$, then the estimate (1.2) holds.

PROOF. – By induction on the number $K$ of critical points. If $K = 1$ then (1.2) holds by Lemma 1.2.

Let us assume (1.2) true when $K \leqslant \bar{K}$. Let $K = \bar{K} + 1$. Let us denote $t = u(x_1) = \ldots = u(x_{\bar{K}+1})$, and let $\gamma$ be the connected component of $\{x \in \Omega: u(x) = t\}$ containing $x_1, \ldots, x_{\bar{K}+1}$. We can always find a critical point such that there exists only one arc in $\gamma$, connecting it to another critical point. Up to a renumbering, we may denote these two critical points $x_{\bar{K}+1}, x_{\bar{K}}$ respectively, let $\alpha$ be the arc in $\gamma$ connecting them.

Now we see that there exist exactly two regions $A^+, A^-$ in $\Omega$, which are connected components of the level sets $\{x \in \Omega: u(x) > t\}$, $\{x \in \Omega: u(x) < t\}$ respectively, and which verify the condition

$$\partial A^+ \cap \partial A^- = \alpha\,.$$

We may find a simple arc $\beta$ in $A^+ \cup A^- \cup \alpha$, which is a line of steepest descent of $u$, and has endpoints on $\partial A^+ \cap \partial\Omega$, $\partial A^- \cap \partial\Omega$, which we will denote $y^+, y^-$ respectively.

Note that $\beta$ intersects $\alpha$ just once, and, since $\Omega$ is simply connected, it splits $\Omega$ into two simply connected domains $\Omega^1$, $\Omega^2$ such that $x_1, \ldots, x_{\bar{K}} \in \Omega^1$, $x_{\bar{K}+1} \in \Omega^2$. Let $N_1, N_2$ be the numbers of maxima of $u$ on $\partial\Omega^1$, $\partial\Omega^2$ respectively.

A detailed, but simple, inspection shows that $N_1 + N_2$ may take only one of the three values: $N - 1$, $N$, $N + 1$, respectively whether both, one, or none of the following two facts occur:

(i) $y^+$ is a point of relative minimum for $u|_{\partial\Omega}$;

(ii) $y^-$ is a point of relative maximum for $u|_{\partial\Omega}$.

In conclusion, we always get

$$N_1 + N_2 \leqslant N + 1 .$$

Now, by the induction hypothesis,

$$\sum_{i=1}^{\bar{K}} m_i \leqslant N_1 - 1 , \qquad m_{\bar{K}+1} \leqslant N_2 - 1 ,$$

and hence, finally,

$$\sum_{i=1}^{\bar{K}+1} m_i \leqslant N_1 + N_2 - 2 \leqslant N - 1 .$$

PROOF OF THEOREM 1.2. – Given the set

$$S = \bigcup_{i=1}^{K} \{x \in \Omega \colon u(x) = u(x_i)\} ,$$

let $\gamma_1, \ldots, \gamma_l$ be the connected components of $S$ which contain at least one of the critical points. Clearly $l \leqslant K$.

We proceed by induction on $l$. If $l = 1$ then (1.2) holds by Lemma 1.3.

Assume that (1.2) holds when $l \leqslant \bar{l}$. Let $l = \bar{l} + 1$. Up to a renumbering, we choose $\gamma_{\bar{l}+1}$ in such a way that $\gamma_1, \ldots, \gamma_{\bar{l}}$ all lie in the same connected component of $\Omega \setminus \gamma_{\bar{l}+1}$. Let $A$ be such component. Up to a change of sign, setting $t = u|_{\gamma_{\bar{l}+1}}$, we assume $u > t$ on a neighbourhood of $\gamma_{\bar{l}+1} \cap \partial A$ in $A$.

Now it is immediately seen that there exist $\varepsilon > 0$ and a simple arc $\beta$ in $A$, with endpoints on $\partial\Omega$, which is contained in the level line $\{x \in \Omega \colon u(x) = t + \varepsilon\}$ and which separates $\gamma_{\bar{l}+1}$ from $\gamma_1, \ldots, \gamma_{\bar{l}}$. Let $\Omega^1, \Omega^2$ be the components of $\Omega \setminus \beta$, let $\gamma_1, \ldots, \gamma_{\bar{l}} \subset \Omega^1$, and $\gamma_{\bar{l}+1} \subset \Omega^2$. Now, if $N_1, N_2$ are the numbers of maxima of $u|_{\partial\Omega^1}$, $u|_{\partial\Omega^2}$ respectively, then we find that

$$N_1 + N_2 = N + 1 ,$$

in fact $\beta$ is a connected set of points of relative maximum for $\Omega^2$ and of relative minimum for $\Omega^1$.

Therefore, by the induction hypothesis,

$$\sum_{i=1}^{K} m_i = \sum_{x_i \in \Omega^1} m_i + \sum_{x_i \in \Omega^2} m_i \leqslant (N_1 - 1) + (N_2 - 1) = N - 1 .$$

PROOF OF THEOREM 1.3. – Let $x_1, \ldots, x_K$ be the critical points of $u$ in $\Omega$, and let $m_1, \ldots, m_K$ be the respective multiplicities. The following equation can be derived from (1) by a lengthy, but not difficult, computation

$$\Delta \log |Du| = - \operatorname{div} \left(a^{-1}|Du|^{-2}(Da \cdot Du)\, Du\right), \qquad \text{in } \Omega \setminus \{x_1, \ldots, x_K\} .$$

The equation above has to be meant in the weak sense. Note that, by the regularity properties of $u$, see Lemma 6.1, $\log |Du|$ is a $W^{1,p}_{\text{loc}}$ function in $\Omega \setminus \{x_1, \ldots, x_K\}$ for every $p < \infty$ and, by (4), the vector field $a^{-1}|Du|^{-2}(Da \cdot Du)\, Du$ belongs to $L^\infty(\Omega)$.

Let $G(x, y)$ be Green's function for Laplace's equation in $\Omega$, that is

$$\begin{cases} \Delta_x G(x, y) = - \delta(x - y) , & x, y \in \Omega , \\ G(\cdot, y)|_{\partial\Omega} = 0 , & y \in \Omega . \end{cases}$$

Let us define

$$\varphi = \log |Du| + 2\pi \sum_{i=1}^{K} m_i G(\cdot, x_i) , \tag{1.5}$$

note that, in $\Omega \setminus \{x_1, \ldots, x_K\}$,

$$\Delta \varphi = - \operatorname{div} \left(a^{-1}|Du|^{-2}(Da \cdot Du)\, Du\right) , \tag{1.6}$$

moreover, note that, by (1.5), $\varphi$ has finite limit at the points $x_1, \ldots, x_K$. Therefore, since the right hand side of (1.6) is the divergence of a bounded vector field, it turns out that $x_1, \ldots, x_K$ are removable singularities for $\varphi$. That is (1.6) holds in all of $\Omega$ and $\varphi$ is locally bounded in $\Omega$. By (1.5) and by the estimate

$$2\pi G(x, y) \leqslant - \log \left(|x - y|/\operatorname{diam} \Omega\right) ,$$

we get

$$|Du(x)| \geqslant \exp \left\{ \min_{\Omega_d} \varphi \right\} \prod_{i=1}^{K} \left(|x - x_i|/\operatorname{diam} \Omega\right)^{m_i} , \qquad x \in \Omega_d .$$

The rest of the proof consists of the evaluation from below of $\varphi$ on $\Omega_d$. We will make use of equation (1.6) and of the boundary condition

$$\varphi = \log |Du| , \qquad \text{on } \partial\Omega . \tag{1.7}$$

We will apply $L^\infty$-estimates and Harnack's inequality to such linear elliptic equation in divergence form (see GILBARG, TRUDINGER [8]).

Let us denote

$$M = \max_{\bar{\Omega}} \varphi . \tag{1.8}$$

The maximum principle tells us

$$M \leqslant \max_{\partial\Omega} \log |Du| + c|\Omega|^{\frac{1}{2}} E \lambda_1^{-1} . \tag{1.9}$$

Moreover, since $v = M - \varphi$ is a non-negative solution of

$$\Delta v = \operatorname{div}\left(a^{-1}|Du|^{-2}(Da \cdot Du)\, Du\right), \quad \text{in } \Omega ,$$

by Harnack's inequality, we have for every $d \leqslant d_0/2$,

$$\max_{\Omega_d} (M - \varphi) \leqslant \exp\{c|\Omega| d^{-2}\}\left\{\min_{\Omega_d} (M - \varphi) + |\Omega|^{\frac{1}{2}} E \lambda_1^{-1}\right\}, \tag{1.10}$$

(see Remark 1.3 at the end of the proof).

Therefore we get

$$\min_{\Omega_d} \varphi \geqslant M - \exp\{c|\Omega| d^{-2}\}\left\{(M - \max_{\Omega_d} \varphi) + |\Omega|^{\frac{1}{2}} E \lambda_1^{-1}\right\}. \tag{1.11}$$

We will evaluate separately $M$ and $M - \max\{\varphi(x) : x \in \Omega_d\}$.

By (1.5), and since $G(x, y)$ is non-negative we have

$$M \geqslant \max_{\Omega} \log |Du| ,$$

on the other hand, note that

$$\operatorname*{osc}_{\partial\Omega} g \leqslant \tfrac{1}{2} |\partial\Omega| \max_{\Omega} |Du| \leqslant c|\Omega| d_0^{-1} \max_{\Omega} |Du| , \tag{1.12}$$

in fact: $|\partial\Omega| =$ perimeter of $\Omega \leqslant c|\Omega| d_0^{-1}$. Therefore we get

$$M \geqslant \log\left(c d_0 |\Omega|^{-1} \operatorname*{osc}_{\partial\Omega} g\right) . \tag{1.13}$$

Now note that by (1.5), (1.9) we have

$$M - \max_{\Omega_d} \varphi \leqslant \max_{\partial\Omega} \log |Du| + c|\Omega|^{\frac{1}{2}} E \lambda_1^{-1} - \max_{\Omega_d} \log |Du| . \tag{1.14}$$

Let $\bar{x} \in \partial\Omega$ be such that

$$\log |Du(\bar{x})| = \max_{\partial\Omega} \log |Du| ,$$

let $z \in \partial\Omega_d$ be such that $|\bar{x} - z| = d$, hence we get

$$(1.15)\quad \max_{\partial\Omega} \log |Du| - \max_{\Omega_d} \log |Du| \leqslant \log |Du(\bar{x})| - \log |Du(z)| \leqslant \leqslant c_p d^{1-2/p} \|D \log |Du|\|_{L^p(B_d(z))} \leqslant c_p d^{1-2/p} \|D^2 u\|_p (\min_{B_d(z)} |Du|)^{-1} .$$

Here, use has been made of Morrey's inequality ([1]), and $p$ is any fixed number larger than 2. Again by Morrey's inequality

$$\min_{B_d(z)} |Du| \geqslant |Du(\bar{x})| - c_p d^{1-2/p} \|D^2 u\|_p = \|Du\|_\infty - c_p d^{1-2/p} \|D^2 u\|_p .$$

Now, since $\|D^2 u\|_p$ can be bounded in terms of $\|g\|_{C^2(\bar{\Omega})}$ (see Lemma 6.1) we may find $d_1$, $d_1 \leqslant d_0/2$, independent of $u$, such that

$$2c_p d_1^{1-2/p} \|D^2 u\|_p \leqslant \|Du\|_\infty ,$$

namely

$$(1.16)\quad d_1 = \min \{d_0/2, [(c_p d_0 \operatorname*{osc}_{\partial\Omega} g)^{p/(p-2)} |\Omega|^{-(p-1)/(p-2)} (\|D^2 g\|_\infty + \mu \|g\|_\infty)^{-p/(p-2)}]\} ,$$

where $\mu$ is given by (6.5).

Thus, for every $d \leqslant d_1$, we obtain

$$(1.17)\quad \max_{\partial\Omega} \log |Du| - \max_{\Omega_d} \log |Du| \leqslant 2c_p d^{1-2/p} \|D^2 u\|_p \|Du\|_\infty^{-1} \leqslant 1 .$$

Consequently, combining (1.14) and (1.17), we get

$$(1.18)\quad M - \max_{\Omega_d} \varphi \leqslant c|\Omega|^{\frac{1}{2}} E \lambda_1^{-1} + 1 ,$$

and hence, recalling (1.11), (1.13), we obtain

$$(1.19)\quad \min_{\Omega_d} \varphi \geqslant \log (cd_0 |\Omega|^{-1} \operatorname*{osc}_{\partial\Omega} g) - \exp \{c|\Omega| d^{-2}\} (1 + c|\Omega|^{\frac{1}{2}} E \lambda_1^{-1}) ,$$

which implies (1.3) with

$$(1.20)\quad C_1 = cd_0 |\Omega|^{-1} (\operatorname*{osc}_{\partial\Omega} g) \exp \{- (1 + c|\Omega|^{\frac{1}{2}} E \lambda_1^{-1}) \exp [c|\Omega| (d \wedge d_1)^{-2}]\} ,$$

where $d_1$ is defined by (1.16).

REMARK 1.3. – Roughly speaking, the constant appearing in (1.10) is obtained by the repeated use of Harnack's inequality on a chain of disks in $\Omega$ having radii all proportional to $d$. The number of such disks can be estimated as $O(|\Omega|d^{-2})$, if $d \leqslant d_0$. By a more refined construction with disks of variable radii we are led to (1.10) with the constant

$$(d_0 d^{-1})^c \exp\{c|\Omega|d_0^{-2}\}.$$

Hence $C_1$ in (1.20) may be replaced by

$$C_1' = cd_0|\Omega|^{-1}(\operatorname*{osc}_{\partial\Omega} g)\exp\{-(1 + c|\Omega|^{\frac{1}{2}}E\lambda_1^{-1})\cdot(d_0(d\wedge d_1)^{-1})^c \exp[c|\Omega|d_0^{-2}]\}. \tag{1.20'}$$

## 2. – Stability estimates.

THEOREM 2.1. – *Let $\Omega$ be a $C^2$-smooth, simply connected bounded domain. Let $a$, $b$ be functions satisfying* (3), (4), *let $g, k \in C^2(\bar{\Omega})$ and let $u$, $v$ be respectively the $W^{1,2}(\Omega)$ solutions of*

$$\begin{cases} \operatorname{div}(a\,Du) = 0\,, & \text{in } \ \Omega\,,\\ u = g\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

$$\begin{cases} \operatorname{div}(b\,Dv) = 0\,, & \text{in } \ \Omega\,,\\ v = k\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.2}$$

*Let $N$ be any positive integer. If $g$ has at most $N$ maxima and $N$ minima on $\partial\Omega$, then, for every $d, \theta, d > 0$, $0 < \theta < \frac{1}{2}$, the following estimate holds*

$$\|a - b\|_{L^\infty(\Omega_d)} \leqslant C_2\{\max_{\partial\Omega}|a - b| + (\|u - v\|_2|\Omega|^{-\frac{1}{2}})^{\frac{1}{2}-\theta}\}^{1/(2N+1)} \tag{2.3}$$

*here $C_2$ depends only on $d$, $\theta$, $\Omega$, $N$, $\lambda_1$, $\lambda_2$, $E$, $\operatorname*{osc}_{\partial\Omega} g$ and $\|g\|_{C^2(\bar{\Omega})}$, $\|k\|_{C^2(\bar{\Omega})}$.*

The theorem above is a straightforward consequence of the following two lemmas.

LEMMA 2.1. – *Let the hypotheses of Theorem 2.1 be satisfied, then the following estimate holds*

$$\int_\Omega |a - b||Du|^2 \leqslant C_3\Big\{\max_{\partial\Omega}|a - b| + (\|u - v\|_2|\Omega|^{-\frac{1}{2}})^{\frac{1}{2}-\theta}\Big\}. \tag{2.4}$$

*Here $C_3$ depends only on $\theta$, $\Omega$, $\lambda_1$, $\lambda_2$, $E$ and $\|g\|_{C^2(\bar{\Omega})}$, $\|k\|_{C^2(\bar{\Omega})}$.*

LEMMA 2.2. – If $u$ is the solution of (2.1), where $a$, $\Omega$ and $g$ satisfy the hypotheses of Theorem 2.1, then the following weighted interpolation inequality holds

$$\int_{\Omega_d} |\varphi| \leqslant C_4 \|\varphi\|_{L^\infty(\Omega_d)}^{(N-1)/N} \left(\int_\Omega |\varphi||Du|^2\right)^{1/N}. \tag{2.5}$$

Here $\varphi$ is any $L^\infty_{\text{loc}}(\Omega)$ function and $C_4$ is a constant which depends only on $d$, $\Omega$, $N$, $\lambda_1$, $E$, $\operatorname{osc} g$ and $\|g\|_{C^2(\bar\Omega)}$.

PROOF OF THEOREM 2.1. – We may combine (2.4), (2.5) with $\varphi = a - b$, and the interpolation inequality (see ADAMS [1])

$$\|\varphi\|_{L^\infty(\Omega_d)} \leqslant c \cdot \{\|D\varphi\|^{\frac{2}{3}}_{L^\infty(\Omega_d)} \|\varphi\|^{\frac{1}{3}}_{L^1(\Omega_d)} + d_0^{-1}\|\varphi\|_{L^1(\Omega_d)}\}$$

which holds if $d \leqslant d_0/2$. In fact in such a case $\Omega_d$ fulfils an interior sphere condition with spheres of radius $d_0/2$. If $d > d_0/2$, then we trivially majorize: $\|\varphi\|_{L^\infty(\Omega_d)} \leqslant \|\varphi\|_{L^\infty(\Omega_{d_0/2})}$.

Therefore (2.3) holds with

$$C_2 = [[E^{\frac{2}{3}} + \lambda_2^{\frac{2}{3}} |\Omega|^{\frac{2}{3}} d_0^{-2}]\, C_4^{\frac{1}{3}}]^{3N/(2N+1)} C_3^{1/(2N+1)}. \tag{2.6}$$

PROOF OF LEMMA 2.1. – Let us denote $\varphi = a - b$, $\eta = \max_{\partial\Omega} |\varphi|$. By (2.1), (2.2) we get, for every $\zeta \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \varphi\, Du \cdot D\zeta = -\int_\Omega b\, D(u - v) \cdot D\zeta. \tag{2.7}$$

For any $h > 0$, consider

$$\zeta(x) = h^{-1}([\varphi(x) - \eta]^+ \wedge h)\, u(x), \quad x \in \Omega, \tag{2.8}$$

(here: $[t]^+ =$ positive part of $t$, $t \wedge s = \min\{t, s\}$).

Let us denote

$$\Omega(t) = \{x \in \Omega : \varphi(x) > t\}.$$

We obtain

$$\begin{aligned} -\int_{\Omega(\eta)} \varphi |Du|^2 h^{-1}((\varphi - \eta)\wedge h) - \int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}} h^{-1}(\varphi\, D\varphi)\cdot(u\, Du) &= \\ = \int_{\Omega(\eta)} b\, D(u - v)\cdot Du h^{-1}((\varphi - \eta)\wedge h) + h^{-1} \int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}} b\, D(u - v)\cdot D\varphi u&. \end{aligned} \tag{2.9}$$

Let us examine the second integral on the left-hand side, and let us integrate by parts

$$h^{-1}\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}(\varphi\, D\varphi)\cdot(u\, Du)=(4h)^{-1}\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}D\varphi^2\cdot Du^2=$$
$$=(4h)^{-1}\left\{\int_{\partial\Omega(\eta)}\varphi^2\frac{\partial}{\partial \boldsymbol{n}}u^2\,ds-\int_{\partial\Omega(\eta+h)}\varphi^2\frac{\partial}{\partial \boldsymbol{n}}u^2\,ds-\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}\varphi^2\,\Delta u^2\right\}.$$

Here, in the first two integrals, $ds$ denotes the arc-length element, $\boldsymbol{n}$ denotes the outer normal. Now observe that, on $\partial\Omega(t)$, $\varphi=t$, therefore we get

$$h^{-1}\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}(\varphi\, D\varphi)\cdot(u\, Du)=(4h)^{-1}\left\{\eta^2\int_{\Omega(\eta)}\Delta u^2-(\eta+h)^2\int_{\Omega(\eta+h)}\Delta u^2-\int_{\Omega(\eta)/\overline{\Omega(\eta+h)}}\varphi^2\,\Delta u^2\right\}=$$
$$=(4h)^{-1}\left\{-\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}(\varphi^2-\eta^2)\,\Delta u^2-\int_{\Omega(\eta+h)}\left((\eta+h)^2-\eta^2\right)\Delta u^2\right\}.$$

Consequently, we obtain

$$\left|h^{-1}\int_{\Omega(\eta)\setminus\overline{\Omega(\eta+h)}}(\varphi\, D\varphi)\cdot(u\, Du)\right|\leqslant(4h)^{-1}\left((\eta+h)^2-\eta^2\right)\int_{\Omega(\eta)}|\Delta u^2|\leqslant$$
$$\leqslant(\eta+h/2)\int_{\Omega(\eta)}\left(|u||\Delta u|+|Du|^2\right).\tag{2.10}$$

Thus, combining (2.9) and (2.10), we get

$$\int_{\Omega(\eta+h)}\varphi|Du|^2\leqslant(\eta+h/2)\int_{\Omega(\eta)}\left(|u||\Delta u|+|Du|^2\right)+\lambda_2\int_{\Omega(\eta)}|D(u-v)||Du|+$$
$$+\lambda_2 h^{-1}\int_{\Omega(\eta)}|D(u-v)||D\varphi||u|.$$

Note that, replacing $\varphi$ with $-\varphi$ in (2.7) and at all following stages, an analogous inequality is obtained.

Finally we get

$$\int_{|\varphi|>\eta+h}|\varphi||Du|^2\leqslant(\eta+h/2)\int_{|\varphi|>\eta}\left(|u||\Delta u|+|Du|^2\right)+\lambda_2\int_{|\varphi|>\eta}|D(u-v)|\left(|Du|+h^{-1}|D\varphi||u|\right),$$

and, consequently,

$$\int_{\Omega}|\varphi||Du|^2\leqslant(\eta+h/2)\int_{\Omega}|u||\Delta u|+(2\eta+3h/2)\int_{\Omega}|Du|^2+$$
$$+\lambda_2\int_{\Omega}|D(u-v)|\left(|Du|+h^{-1}|D\varphi||u|\right).\tag{2.11}$$

In turn we get, for every $p$, $1 < p < \infty$,

$$\int_\Omega |\varphi||Du|^2 \leqslant (\eta + h)[\|u\|_\infty \|\Delta u\|_1 + 2\|Du\|_2^2] + \\ + \|D(u - v)\|_{p'} \lambda_2[\|Du\|_p + h^{-1}\|D\varphi\|_p \|u\|_\infty], \tag{2.12}$$

and hence, by the regularity estimates of Lemma 6.1 in the Appendix,

$$\int_\Omega |\varphi||Du|^2 \leqslant Q_1(\eta + h) + (Q_2 + Q_3 h^{-1})\|D(u - v)\|_{p'} \tag{2.13}$$

where

$$Q_1 = |\Omega|^{\frac{1}{6}} \lambda_1^{-\frac{3}{2}} \lambda_2^{\frac{1}{2}} E\|Dg\|_2 \|g\|_\infty + 2\lambda_1^{-1}\lambda_2\|Dg\|_2^2, \tag{2.14}$$

$$Q_2 = c_p \lambda_2 |\Omega|^{(p+2)/2p}\{\|D^2 g\|_\infty + \mu\|g\|_\infty\}, \tag{2.15}$$

$$Q_3 = |\Omega|^{1/p}\lambda_2 E\|g\|_\infty. \tag{2.16}$$

Let us now use the interpolation inequality (see ADAMS [1])

$$\|D(u - v)\|_{p'} \leqslant c|\Omega| d_0^{-2}\{\|D^2(u - v)\|_p^\alpha \|u - v\|_2^{1-\alpha} + d_0^{-2/p}\|u - v\|_2\},$$

which holds for $\alpha = 2/(3p - 2)$, $p > 4/3$. We get, by (6.1), (6.3) in the Appendix,

$$\|D(u - v)\|_{p'} \leqslant Q_4(\|u - v\|_2/|\Omega|^{\frac{1}{2}})^{1-\alpha},$$

where

$$Q_4 = c_p d_0^{-2}|\Omega|^{\frac{3}{2} + \alpha(\frac{1}{2} - 1/p)} \cdot \\ \cdot [\|D^2 g\|_\infty + \|D^2 k\|_\infty + (\mu + d_0^{-2/\alpha p}|\Omega|^{-1+1/\alpha p})(\|g\|_\infty + \|k\|_\infty)]^\alpha. \tag{2.17}$$

Therefore

$$\int_\Omega |\varphi||Du|^2 \leqslant Q_1(\eta + h) + (Q_2 + Q_3 h^{-1}) \cdot Q_4(\|u - v\|_2 |\Omega|^{-\frac{1}{2}})^{1-\alpha} \leqslant Q_1\eta + \\ + Q_2 Q_4(\|g\|_\infty + \|k\|_\infty)^{(1-\alpha)/2}(\|u - v\|_2|\Omega|^{-\frac{1}{2}})^{(1-\alpha)/2} + Q_1 h + Q_3 Q_4(\|u - v\|_2|\Omega|^{-\frac{1}{2}})^{1-\alpha} h^{-1}$$

and picking $h = (\|u - v\|_2|\Omega|^{-\frac{1}{2}})^{(1-\alpha)/2}$ and fixing $p$ such that $\alpha = 2\theta$, we obtain (2.4) with

$$C_3 = Q_1 + Q_4\big(Q_2(\|g\|_\infty + \|k\|_\infty)^{(1-\alpha)/2} + Q_3\big). \tag{2.18}$$

PROOF OF LEMMA 2.2. – The hypotheses of Theorem 1.3 are satisfied, thus (1.3), (1.2) hold.

Therefore, for every positive $r$, we have

$$\int_{\Omega_d}|\varphi|\leqslant\int_{\left(\bigcup_{i=1}^{K}B_r(x_i)\right)\cap\Omega_d}|\varphi|+\int_{\Omega_d\setminus\left(\bigcup_{i=1}^{K}B_r(x_i)\right)}|\varphi|\leqslant\pi K r^2\|\varphi\|_{L^\infty(\Omega_d)}+C_1^{-2}(\operatorname{diam}\Omega/r)^{\left(2\sum_{i=1}^{K}m_i\right)}.$$

$$\cdot\chi_{[0,\operatorname{diam}\Omega)}(r)\int_{\Omega_d}|\varphi||Du|^2\leqslant\pi(N-1)r^2\|\varphi\|_{L^\infty(\Omega_d)}+C_1^{-2}(\operatorname{diam}\Omega/r)^{2(N-1)}\int_{\Omega_d}|\varphi||Du|^2 .$$

Now we may minimize this last expression as $r$ ranges over $(0,+\infty)$. Therefore (2.5) follows with

$$C_4=2\big(\pi(N-1)(\operatorname{diam}\Omega)^2\big)^{(N-1)/N}C_1^{-2/N}. \tag{2.19}$$

## 3. – Convergence of the algorithm.

In this section we are concerned with convergence properties of the $W^{1,2}(\Omega)$ solution $a_\varepsilon$ of the elliptic boundary value problem

$$\begin{cases}\varepsilon\,\Delta a_\varepsilon+\operatorname{div}(a_\varepsilon\operatorname{grad}u)=0\,, & \text{in}\quad\Omega\,,\\ a_\varepsilon=a\,, & \text{on}\ \partial\Omega\,.\end{cases} \tag{3.1}$$

We start with a remark.

REMARK 3.1. – The $W^{1,2}(\Omega)$ solution $a_\varepsilon$ of (3.1) exists and is unique. In fact, for every $\varepsilon\neq0$, (3.1) is equivalent to

$$\begin{cases}\operatorname{div}\big(e^{-u/\varepsilon}\operatorname{grad}(e^{u/\varepsilon}a_\varepsilon)\big)=0\,, & \text{in}\quad\Omega\,,\\ e^{u/\varepsilon}a_\varepsilon=e^{u/\varepsilon}a\,, & \text{on}\ \partial\Omega\,.\end{cases} \tag{3.2}$$

Here we are dealing with a divergence structure elliptic equation with pure principal part in the unknown $e^{u/\varepsilon}a_\varepsilon$. The coefficient $e^{-u/\varepsilon}$ is uniformly elliptic and is $W^{2,p}(\Omega)$ – regular (see Lemma 6.1). The boundary value, $e^{u/\varepsilon}a$, is Lipschitz continuous.

It follows, by standard regularity results that $a_\varepsilon$ is a Lipschitz continuous functions in $\bar{\Omega}$. Note also that, if we fix $\varepsilon\neq0$, then the mapping $L^2(\Omega)\ni u\to a_\varepsilon\in L^\infty(\Omega)$ is continuous.

THEOREM 3.1. – Let $\Omega$ be a $C^2$-smooth, simply connected bounded domain in $\mathbb{R}^2$. Let $a$ satisfy (3), (4) and let $u$ be the $W^{1,2}(\Omega)$ solution of (1), (2), where $g\in C^2(\bar{\Omega})$ and $g|_{\partial\Omega}$ has at most $N$ maxima. Let $a_\varepsilon$ be the $W^{1,2}(\Omega)$ solution of (3.1). If $\varepsilon\in(0,1)$, then the following estimate holds for every $q \in [1, \infty)$

$$\left(\int_{\Omega_d} |a - a_\varepsilon|^q\right)^{1/q} \leqslant C_5 \varepsilon^{1/2qN} . \tag{3.3}$$

Here $d > 0$, and $C_5$ is a constant which depends only on $d$, $q$, $\Omega$, $N$, $\lambda_1$, $\lambda_2$, $E$, $\operatorname{osc}_{\partial\Omega} g$ and $\|g\|_{C^2(\bar{\Omega})}$.

The proof of Theorem 3.1 follows from the two lemmas below.

LEMMA 3.1. – Let $a$ satisfy (3), (4). Let $u$ be the solution of (1), (2) with $g \in C^2(\bar{\Omega})$. If $a_\varepsilon$ is the $W^{1,2}(\Omega)$ solution of (3.1) then

$$0 < a_\varepsilon(x) \leqslant C_6 a(x) , \quad x \in \Omega , \tag{3.4}$$

$$\|Da_\varepsilon\|_2 \leqslant C_6(1 + \lambda_1^{-1}\lambda_2) E |\Omega|^{\frac{1}{2}} , \tag{3.5}$$

here

$$C_6 = \exp\{\pi^{\frac{1}{2}} \lambda_1^{-1} E |\Omega|^{\frac{1}{2}}\} . \tag{3.6}$$

LEMMA 3.2. – If the hypotheses of Lemma 3.1 are satisfied, then the following estimate holds for every $\varepsilon \in (0, 1)$

$$\int_\Omega |a - a_\varepsilon| |Du|^2 \leqslant C_7 \varepsilon^{\frac{1}{2}} . \tag{3.7}$$

Here $C_7$ depends only on $\Omega$, $\lambda_1$, $\lambda_2$, $E$ and $\|g\|_{C^2(\bar{\Omega})}$.

PROOF OF THEOREM 3.1. – Let us combine (3.4), (3.7) by the use of Lemma 2.2. We obtain

$$\int_{\Omega_d} |a - a_\varepsilon| \leqslant C_8 \varepsilon^{1/2N}, \tag{3.8}$$

where

$$C_8 = C_4(\lambda_2 C_6)^{(N-1)/N} C_7^{1/N} , \tag{3.9}$$

thus, applying Hölder's inequality and (3.4), the estimate (3.3) follows with

$$C_5 = C_8^{1/q}(\lambda_2 C_6)^{1-1/q} . \tag{3.10}$$

PROOF OF LEMMA 3.1. – By the use of the maximum principle, equation (3.2) yields

$$a_\varepsilon(x) \geqslant \lambda_1 \exp\{(\min u - \max u)/\varepsilon\} > 0 , \quad x \in \Omega . \tag{3.11}$$

Therefore only the upper bound on $a_\varepsilon$ and (3.5) remain to be proved.

Let us assume $a \in C^\infty(\Omega)$, in such a case it turns out that also $u$, $a_\varepsilon \in C^\infty(\Omega)$. Now denoting

$$\varphi = a - a_\varepsilon$$

we have

$$\begin{cases} \varepsilon\, \Delta\varphi + \operatorname{div}(\varphi\, Du) = \varepsilon\, \Delta a\,, & \text{in} \quad \Omega\,, \\ \varphi = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.12}$$

In the sequel we will make use of arguments coming from the method of rearrangements for elliptic equations (see: TALENTI [20]).

Consider the following level sets for the function $\varphi/a$

$$\Omega^-(t) = \{x \in \Omega \colon \varphi(x)/a(x) < -t\}\,. \tag{3.13}$$

Note that, for every $t > 0$,

$$\int_{\Omega^-(t)} \operatorname{div}(\varphi\, Du) = 0\,, \tag{3.14}$$

in fact

$$\operatorname{div}(\varphi\, Du) = \operatorname{div}\big(a(\varphi/a)\, Du\big) = a\, D\big((\varphi/a) + t\big)\cdot Du\,,$$

and, by (1), this last term has zero integral on $\Omega^-(t)$. Therefore for every $t > 0$

$$\int_{\Omega^-(t)} \Delta\varphi = \int_{\Omega^-(t)} \Delta a\,.$$

Note that, by Sard's lemma, for almost every $t$, $\partial\Omega^-(t)$ is a smooth closed curve and coincides with the level line $\{x \in \Omega \colon \varphi/a = -t\}$.

Consequently, for almost every $t > 0$, by an integration by parts we get

$$\int_{\varphi/a = -t} \left[\frac{\varphi}{a}\frac{\partial a}{\partial \boldsymbol{n}} + a\frac{\partial}{\partial \boldsymbol{n}}\left(\frac{\varphi}{a}\right)\right] ds = \int_{\varphi/a = -t} \frac{da}{\partial \boldsymbol{n}}\, ds\,,$$

where $\boldsymbol{n} = |D(\varphi/a)|^{-1} D(\varphi/a)$, and $ds$ is the arc-length element. Consequently

$$\int_{\varphi/a = -t} a \left|D\frac{\varphi}{a}\right| ds = \int_{\varphi/a = -t} (1 + t)\frac{\partial a}{\partial \boldsymbol{n}}\, ds\,. \tag{3.15}$$

Let us define

$$\mu(t) = \text{measure of } \Omega^-(t)\,, \tag{3.16}$$

$$P(t) = \text{perimeter of } \Omega^-(t)\,. \tag{3.17}$$

For almost every $t > 0$ we have

$$\frac{d}{dt}\mu(t) = -\int\limits_{\varphi/a=-t}\left|D\frac{\varphi}{a}\right|^{-1} ds$$

(see [20]).

Now we have, by Hölder's inequality and (3.15),

$$(P(t))^2 \leqslant \int\limits_{\varphi/a=-t}\left|D\frac{\varphi}{a}\right|^{-1} ds \int\limits_{\varphi/a=-t}\left|D\frac{\varphi}{a}\right| ds \leqslant \left(-\frac{d}{dt}\mu(t)\right)\lambda_1^{-1}E(1+t)P(t)\,,$$

therefore

$$P(t) \leqslant \left(-\frac{d}{dt}\mu(t)\right)\lambda_1^{-1}E(1+t)\,.$$

By the isoperimetric inequality, we obtain

$$2\sqrt{\pi}\,(\mu(t))^{\frac{1}{2}} \leqslant \left(-\frac{d}{dt}\mu(t)\right)\lambda_1 E(1+t)\,, \tag{3.18}$$

or, as is the same,

$$\sqrt{\pi}\,\lambda_1 E^{-1}(1+t) \leqslant -\frac{d}{dt}(\mu(t))^{\frac{1}{2}}\,,$$

thus, integrating on both sides,

$$\sqrt{\pi}\,\lambda_1 E^{-1}\log(1+t) \leqslant (\mu(0))^{\frac{1}{2}} - (\mu(t))^{\frac{1}{2}} \leqslant |\Omega|^{\frac{1}{2}} - (\mu(t))^{\frac{1}{2}}\,.$$

Therefore $\mu(t) = 0$, for every $t$ such that

$$(1+t) \leqslant \exp\{\pi^{-\frac{1}{2}}\lambda_1^{-1}E|\Omega|^{\frac{1}{2}}\}\,,$$

that is

$$-\min_{\Omega}(\varphi/a) \leqslant \exp\{\pi^{-\frac{1}{2}}\lambda_1^{-1}E|\Omega|^{\frac{1}{2}}\} - 1\,, \tag{3.19}$$

which, together with (3.11), implies (3.4).

Now note that

$$\int_{\Omega}\operatorname{div}(\varphi\, Du)(\varphi/a) = 0\,,$$

therefore we have

$$\int_{\Omega}\Delta\varphi(\varphi/a) = \int_{\Omega}\Delta a(\varphi/a)\,,$$

now, writing: $\Delta\varphi = \Delta(a(\varphi/a))$, and integrating by parts, we get

$$\int_\Omega a|D(\varphi/a)|^2 + (\varphi/a)\, Da\cdot D(\varphi/a) = \int_\Omega Da\cdot D(\varphi/a)\,,$$

$$\int_\Omega a|D(\varphi/a)|^2 = \int_\Omega Da\cdot D(\varphi/a)\big(1-(\varphi/a)\big) \leqslant E|\Omega|^{\frac12}\big(1-\min_\Omega(\varphi/a)\big)\left(\int_\Omega |D(\varphi/a)|^2\right)^{\frac12},$$

therefore, by (3) and (3.19), we obtain

$$\int_\Omega |D(\varphi/a)|^2 \leqslant E^2\lambda_1^{-1}|\Omega| \exp\{2\pi^{-\frac12}\lambda_1^{-1}E|\Omega|^{\frac12}\}\,.$$

And, finally, we get

$$\|D\varphi\|_2 \leqslant \|a\, D(\varphi/a)\|_2 + \|(\varphi/a)\, Da\|_2 \leqslant E\lambda_1^{-1}\lambda_2|\Omega|^{\frac12} \exp\{\pi^{-\frac12}\lambda_1^{-1}E|\Omega|^{\frac12}\} + \tag{3.20}$$
$$+ E|\Omega|^{\frac12}[\exp\{\pi^{-\frac12}\lambda_1^{-1}E|\Omega|^{\frac12}\} - 1]$$

which yields (3.5).

The proof will be completed once we have removed the $C^\infty$-smoothness hypothesis on $a$.

This may be done by an approximation procedure.

Let $a^n \in C^\infty(\Omega)$, $n = 1, 2, \ldots$, verifying (3), (4) be such that $a^n \to a$ in $W^{1,p}(\Omega)$, $2 < p < \infty$.

Let $u_n$ be the solution of

$$\begin{cases} \operatorname{div}(a^n\, Du_n) = 0\,, & \text{in } \Omega\,, \\ u_n = g\,, & \text{on } \partial\Omega\,, \end{cases} \tag{3.21}$$

and let $a_\varepsilon^n$ be the solution of

$$\begin{cases} \varepsilon\, \Delta a_\varepsilon^n + \operatorname{div}(a_\varepsilon^n\, Du_n) = 0\,, & \text{in } \Omega\,, \\ a_\varepsilon^n = a^n\,, & \text{on } \partial\Omega\,, \end{cases}$$

or, as is the same,

$$\begin{cases} \operatorname{div}\big(e^{-u_n/\varepsilon} D(e^{u_n/\varepsilon} a_\varepsilon^n)\big) = 0\,, & \text{in } \Omega\,, \\ a_\varepsilon^n = a^n\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.22}$$

Obviously (3.4), (3.5) hold, for every $n$ if $a$ is replaced with $a^n$, and $a_\varepsilon$ with $a_\varepsilon^n$. Now note that, by (3.21),

$$u_n \to u \quad \text{in } C^1(\bar\Omega)$$

and, by (3.22),

$$a_\varepsilon^n \to a_\varepsilon \quad \text{in } W^{1,2}(\Omega)$$

thus, passing to the limit, $a_\varepsilon$ verifies (3.4), (3.5).

PROOF OF LEMMA 3.2. – Let $\varphi = a - a_\varepsilon$. By (3.12), we have, for every $\zeta \in W_0^{1,2}(\Omega)$

$$\int_\Omega \varphi\, Du\cdot D\zeta = \varepsilon \int_\Omega Da_\varepsilon \cdot D\zeta\,.$$

Let us pick $\zeta \in W_0^{1,2}$, as follows

$$\zeta(x) = h^{-1}([\varphi(x)]^+ \wedge h)\, u(x)\,, \qquad x\in\Omega,\ h>0\,.$$

We obtain

$$\int_{\varphi>0} \varphi|Du|^2 = -\,h^{-1}\int_{0<\varphi<h} (\varphi\, D\varphi)\cdot(u\, Du) - \varepsilon\int_{0<\varphi<h} Da_\varepsilon\cdot(h^{-1}u\, D\varphi) - \varepsilon\int_{\varphi>0} Da_\varepsilon\cdot Du h^{-1}([\varphi]^+\wedge h)\,.$$

Repeating the argument used during the proof of Lemma 2.1 in order to obtain (2.10) we are led to

$$\left|h^{-1}\int_{0<\varphi<h}(\varphi\, D\varphi)\cdot(u\, Du)\right| \leqslant (h/2)\int_{\varphi>0}|u||\Delta u| + |Du|^2$$

and thus

$$\int_{\varphi>0}\varphi|Du|^2 \leqslant (h/2)\int_{\varphi>0}|u||\Delta u| + |Du|^2 + \varepsilon\left[h^{-1}\int_{\varphi>0}|Da_\varepsilon||D\varphi||u| + \int_{\varphi>0}|Da_\varepsilon||Du|\right].$$

A similar inequality is obtained replacing $\varphi$ with $-\varphi$ from the beginning. We get

$$\int_\Omega|\varphi||Du|^2 \leqslant (h/2)\int_\Omega(|u||\Delta u| + |Du|^2) + \varepsilon[h^{-1}\|Da_\varepsilon\|_2\|D\varphi\|_2\|u\|_\infty + \|Da_\varepsilon\|_2\|Du\|_2]\,. \tag{3.23}$$

Now, recalling an analogous estimate in Lemma 2.2, we have

$$\int_\Omega |u||\Delta u| + |Du|^2 \leqslant Q_1$$

where $Q_1$ is given by (2.14). Therefore by (6.1), (6.2) in the Appendix, and by (3.5), we get

$$\int_\Omega|\varphi||Du|^2 \leqslant hQ_1/2 + \varepsilon\,[h^{-1}C_6^2(1+\lambda_1^{-1}\lambda_2)^2E^2|\Omega|\|g\|_\infty + C_6(1+\lambda_1^{-1}\lambda_2)E|\Omega|^{\frac12}\lambda_1^{-\frac12}\lambda_2^{\frac12}\|Dg\|_2],$$

hence, picking $h=\varepsilon^{\frac12}$, we obtain (3.7) with

$$C_7 = [Q_1/2 + C_6^2(1+\lambda_1^{-1}\lambda_2)^2E|\Omega|\|g\|_\infty + C_6(1+\lambda_1^{-1}\lambda_2)\lambda_1^{-\frac12}\lambda_2^{\frac12}E|\Omega|^{\frac12}\|Dg\|_2]\,. \tag{3.25}$$

## 4. – Approximate identification from noisy data.

Let us assume that only an approximation $\bar{u}$ of $u$ is known in $\Omega$.

Assume also that the following error estimate is given

$$|\Omega|^{-1}\int_\Omega |u-\bar{u}|^2 < \delta^2 . \tag{4.1}$$

We will evaluate the error between $a$ and the solution to a problem like (3.1), in which $u$ is replaced by a suitable smoothing of $\bar{u}$.

To this purpose, let us introduce the following definitions.

Let $\psi$ be a $C_0^\infty(\mathbf{R}^2)$ non-negative function such that $\operatorname{supp}\psi \subseteq \bar{B}_1(0)$; $\int \psi(x)\,dx = 1$.

We define the following smoothing operator over $\Omega$

$$S_h[f](x) = \int_\Omega h^{-2}\psi\big((x-y)/h\big)\big(f(y)-g(y)\big)\,dy + g(x) . \tag{4.2}$$

Here $g$ is the $C^2(\bar{\Omega})$ function which coincides with $u$ on $\partial\Omega$.

For every $\varepsilon, h > 0$, we define $a_\varepsilon^h$ as the solution of the problem

$$\begin{cases} \varepsilon\,\Delta a_\varepsilon^h + \operatorname{div}\big(a_\varepsilon^h\, DS_h[\bar{u}]\big) = 0 , & \text{in } \Omega ,\\ a_\varepsilon^h = a , & \text{on } \partial\Omega . \end{cases} \tag{4.3}$$

The approximate identification algorithm consists in solving problem (4.3) with a suitable choice of the parameters $\varepsilon$, $h$. The following theorem tells how to choose $\varepsilon$ and $h$, is such a way that the error between $a$ and $a_\varepsilon^h$ is small when $\delta$, the $L^2$-error on $u$, is small.

THEOREM 4.1. – Let $\Omega$ be a $C^2$-smooth, simply connected bounded domain in $\mathbf{R}^2$. Let $a$ satisfy (3), (4), let $u$ be the solution of (1), (2) where $g \in C^2(\bar{\Omega})$ and $g|_{\partial\Omega}$ has $N$ maxima.

Let $\bar{u} \in L^2(\Omega)$ satisfy (4.1), and let $a_\varepsilon^h$ be the solution of (4.3), where $S_h$ is given by (4.2). Let $\theta \in (0, \frac{1}{3})$.

There exists a number $\delta_1 > 0$, depending only on $\theta$, $N$, $\Omega$, $\lambda_1$, $\lambda_2$, $E$ and $\|g\|_{C^2(\bar{\Omega})}$, such that if

$$\delta < \delta_1 ,$$

and if we set

$$h = (\operatorname{diam}\Omega)\,\delta^{\frac{1}{3}+\theta/2} ,$$

$$\varepsilon = \delta^{(\frac{1}{3}-\theta)(2N/(2N+1))} ,$$

then $a_\varepsilon^h$ satisfies

$$\int_{\Omega_d} |a - a_\varepsilon^h| \leqslant C_9 \delta^{(\frac{1}{2}-\theta)(1/(2N+1))} . \tag{4.4}$$

Here $C_9$ is a constant which depends only on $d$, $\theta$, $N$, $\Omega$, $\lambda_1$, $\lambda_2$, $E$, $\operatorname{osc} g$ and $\|g\|_{C^2(\bar{\Omega})}$.

REMARK 4.1. – Some extensions of this theorem could be proved with no additional difficulties. For instance, the case in which also the boundary values of $a$ are approximately known could be treated. Moreover, with the due changes, the $L^1$-norm appearing in (4.4), could be replaced by any $L^p$-norm, $p < \infty$.

We premise the proof with two lemmas.

LEMMA 4.1. – Let the hypotheses of Theorem 4.1 be satisfied. Let $v \in C^1(\Omega)$, and let $b_\varepsilon$ be the solution of

$$\begin{cases} \varepsilon \Delta b_\varepsilon + \operatorname{div}(b_\varepsilon Dv) = 0 , & \text{in } \Omega , \\ b_\varepsilon = a , & \text{on } \partial\Omega . \end{cases} \tag{4.5}$$

The following estimate holds

$$\|a_\varepsilon - b_\varepsilon\|_\infty \leqslant (C_6 - 1)\lambda_2 E^{-1} \varepsilon^{-1} \|b_\varepsilon\|_\infty \|D(u - v)\|_\infty , \tag{4.6}$$

where $C_6$ is given by (3.6).

LEMMA 4.2. – Let $u$ be the solution of (1), (2), where $a$ satisfies (3), (4) and $g \in C^2(\bar{\Omega})$.

Let $\bar{u} \in L^2(\Omega)$ satisfy (4.1). The following estimate holds for every $p > 2$

$$\|Du - DS_h[\bar{u}]\|_\infty \leqslant C_{10}\left(\left(\frac{h}{\operatorname{diam}\Omega}\right)^{1-2/p} + \left(\frac{h}{\operatorname{diam}\Omega}\right)^{-2}\delta\right) \tag{4.7}$$

where $C_{10}$ is a constant depending only on $\psi$, $p$, $\Omega$, $E$, $\lambda_1$, $\lambda_2$, and $\|g\|_{C^2(\bar{\Omega})}$.

PROOF OF THEOREM 4.1. – Let us apply Lemma 4.1 with $v = S_h[\bar{u}]$, we obtain

$$\|a_\varepsilon - a_\varepsilon^h\|_\infty \leqslant (C_6 - 1)\lambda_2 E^{-1} \varepsilon^{-1} \|a_\varepsilon^h\|_\infty \|D(u - S_h[\bar{u}])\|_\infty ,$$

and, by Lemma 4.2,

$$\|a_\varepsilon - a_\varepsilon^h\|_\infty \leqslant C_{11} \|a_\varepsilon^h\|_\infty \varepsilon^{-1}\left(\left(\frac{h}{\operatorname{diam}\Omega}\right)^{1-2/p} + \left(\frac{h}{\operatorname{diam}\Omega}\right)^{-2}\delta\right),$$

here $p > 2$, and

$$C_{11} = C_{10}(C_6 - 1)\,\lambda_2 E^{-1}. \tag{4.8}$$

Now fixing $p = 4/\theta + \frac{2}{3}$, we get

$$\|a_\varepsilon - a_\varepsilon^h\|_\infty \leqslant 2C_{11}(\|a_\varepsilon\| + \|a_\varepsilon - a_\varepsilon^h\|_\infty)\,\varepsilon^{-\frac{1}{2}}\delta^{\frac{1}{3}-\theta}.$$

Let us fix $\delta_0 \in (0, 1)$ such that

$$C_{11}\delta_0^{\frac{1}{3}-\theta} \leqslant \tfrac{1}{4}, \tag{4.9}$$

thus, if $\delta \leqslant \delta_0 \varepsilon^{3/(1-3\theta)}$, then we have

$$\|a_\varepsilon - a_\varepsilon^h\|_\infty \leqslant 4C_{11}\|a_\varepsilon\|_\infty \varepsilon^{-1}\delta^{\frac{1}{3}-\theta},$$

and recalling (3.4) we get

$$\|a_\varepsilon - a_\varepsilon^h\|_\infty \leqslant \lambda_2\, C_6\, C_{11}\, \varepsilon^{-1}\delta^{\frac{1}{3}-\theta}. \tag{4.10}$$

Let us now combine (4.10) with (3.8),

$$\int_{\Omega_d} |a_\varepsilon^h - a| \leqslant \lambda_2 |\Omega| C_6\, C_{11}\varepsilon^{-1}\delta^{\frac{1}{3}-\theta} + C_8 \varepsilon^{1/(2N)}, \tag{4.11}$$

and, if we replace

$$\varepsilon = \delta^{(\frac{1}{3}-\theta)2N/(2N+1)},$$

then we get (4.4) with

$$C_9 = \lambda_2 |\Omega| C_6 C_{11} + C_8. \tag{4.12}$$

Finally, recall the condition

$$\delta \leqslant \delta_0 \varepsilon^{3/(1-3\theta)},$$

therefore, the theorem holds with

$$\delta_1 = \delta_0^{1+2N}.$$

PROOF OF LEMMA 4.1. – Let $\varphi = a_\varepsilon - b_\varepsilon$. We obtain

$$\begin{cases} \varepsilon\,\Delta\varphi + \operatorname{div}(\varphi\, Du) = -\operatorname{div}\big(b_\varepsilon D(u - v)\big), & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

We apply again the method used in Lemma 3.1.

By an approximation argument it could be shown once more that we may assume $a, v \in C^\infty(\Omega)$ without loss of generality. In such a case we have also $a_\varepsilon$, $b_\varepsilon$, $\varphi \in C^\infty(\Omega)$.

We denote, for every $t > 0$,

$$\Omega(t) = \{x \in \Omega\colon |\varphi(x)/a(x)| > t\}\,, \qquad \mu(t) = \text{meas}\ \Omega(t)\,,$$
$$P(t) = \text{perimeter of}\ \Omega(t)\,.$$

Observe that, for every $t > 0$, we have

$$\int\limits_{\Omega(t)} \text{div}\,(\varphi\, Du) = 0\,,$$

therefore

$$\int\limits_{\Omega(t)} \Delta\varphi = -\,\varepsilon^{-1}\int\limits_{\Omega(t)} \text{div}\,\big(b_\varepsilon D(u - v)\big)\,,$$

now, setting $\Delta\varphi = \Delta\big(a(\varphi/a)\big)$, and integrating by parts, we get, for almost every $t$,

$$\int\limits_{|\varphi/a|=t} a\, D(\varphi/a)\, ds = -\,t\int\limits_{|\varphi/a|=t} Da\cdot D(\varphi/a)|D(\varphi/a)|^{-1}\, ds - \varepsilon^{-1}\int\limits_{|\varphi/a|=t} b_\varepsilon\, D(u - v)\cdot D(\varphi/a)|D(\varphi/a)|^{-1}\, ds$$

here, and below, integration with respect to the arc-length is understood.

Consequently we get

$$\lambda_1\int\limits_{|\varphi/a|=t} |D(\varphi/a)|\, ds \leqslant \int\limits_{|\varphi/a|=t} |D(\varphi/a)|\, ds \leqslant P(t)[Et + \varepsilon^{-1}\|b_\varepsilon\|_\infty\|D(u - v)\|_\infty]\,.$$

By Hölder's inequality we have

$$P^2(t) \leqslant \int\limits_{|\varphi/a|=t} |D(\varphi/a)|\, ds \int\limits_{|\varphi/a|=t} |D(\varphi/a)|^{-1}\, ds = \int\limits_{|\varphi/a|=t} |D(\varphi/a)|\, ds(-\,\mu'(t))\,,$$

therefore

$$P(t) \leqslant \lambda_1^{-1}(-\,\mu'(t))[Et + \varepsilon^{-1}\|b\|_\infty\|D(u - v)\|_\infty]\,,$$

and, by the isoperimetric inequality

$$\mu(t) \leqslant (4\pi)^{-1} P^2(t)\,,$$

we obtain

$$\frac{d}{dt}\big(\mu(t)\big)^{\frac{1}{2}} \leqslant \lambda_1 \pi^{\frac{1}{2}}[Et + \varepsilon^{-1}\|b_\varepsilon\|_\infty\|D(u - v)\|_\infty]^{-1}\,.$$

An integration yields

$$\mu(t)\leqslant\Big[|\Omega|^{\frac12}+\lambda_1\pi^{\frac12}E^{-1}\Big(\log\big(\varepsilon^{-1}\|b_\varepsilon\|_\infty\|D(u-v)\|_\infty\big)-\log\big(Et+\varepsilon^{-1}\|b_\varepsilon\|_\infty\|D(u-v)\|_\infty\big)\Big)\Big]^2,$$

which implies $\mu(t)=0$ for every $t$ such that

$$t\geqslant\big[\exp\{\lambda_1^{-1}\pi^{-\frac12}|\Omega|^{\frac12}E\}-1\big]\varepsilon^{-1}\|b_\varepsilon\|_\infty\|D(u-v)\|_\infty,$$

and thus (4.6) follows.

PROOF OF LEMMA 4.2. – For every $x\in\Omega$ we have

$$\begin{aligned}|Du(x)-DS_h[\bar u](x)|&\leqslant|Du(x)-DS_h[u](x)|+|DS_h[u-\bar u](x)|=\\&=\Big|\int_{\mathbf{R}^2}h^{-2}\psi((x-y)/h)\big(D(u-g)(x)-D(u-g)(y)\,\chi_\Omega(y)\big)\,dy\Big|+\\&\quad+\Big|\int_{\mathbf{R}^2}D_x\big(h^{-2}\psi((x-y)/h)\big)u(y)-\bar u(y))\,\chi_\Omega(y)\,dy\Big|,\end{aligned}$$

hence, applying Young's inequality for convolutions two times,

$$\begin{aligned}|Du(x)-DS_h[\bar u](x)|&\leqslant\max_{\substack{y\in\Omega\\|y-x|<h}}|D(u-g)(x)-D(u-g)(y)|+\\&\quad+\|D\big(h^{-2}\psi((x-\cdot)/h)\big)\|_{L^2(\mathbf{R}^2)}\|u-\bar u\|_{L^2(\Omega)}\leqslant\\&\quad c_p(\|D^2(u-g)\|_p+d_0^{-1}\|D(u-v)\|_p)\,h^{1-2/p}+\|D\psi\|_{L^2(\mathbf{R}^2)}h^{-2}|\Omega|^{\frac12}\,\delta\,;\end{aligned}$$

here, use has been made of Morrey's inequality.

Hence, by the use of regularity estimates on $u$ (see Lemma 6.1), (4.7) follows with

$$C_{10}=c_p[\|D^2g\|_\infty+\mu\|g\|_\infty](\operatorname{diam}\Omega)+\|D\psi\|_{L^2(\mathbf{R}^2)}(\operatorname{diam}\Omega)^{-1},\tag{4.13}$$

here $\mu$ is the constant defined in (6.5).

## 5. – The discrete data case.

We introduce the following definitions.

A triangulation $\Theta=\{T_1,\ldots,T_M\}$ in $\Omega$, is a set of internally non overlapping triangles $T_1,\ldots,T_M$, whose union covers $\bar\Omega$ and which have the following properties. Every side of each triangle either is a (possibly curved) arc of $\partial\Omega$, or intersects $\partial\Omega$ at most in one vertex. The sides of the second type are straight segments and each of them is a common side to two triangles in $\Theta$.

Two numbers $h$, $\sigma$ are associated to a triangulation $\Theta$: $h$ is defined by

$$h = \max_{i=1,\dots,M} \operatorname{diam} T_i\,,$$

and $\sigma$ is the largest number such that each $T_i$ contains a ball of radius $\sigma h$. We will refer to $h$ as the size of the triangulation $\Theta$.

Let $P_1, \dots, P_L \in \bar{\Omega}$, be the vertices of the triangles in $\Theta$.

We define the finite element interpolation operator $\Pi$ associated to the triangulation $\Theta$, and to the nodes $P_1, \dots, P_L$ as follows.

For every $L$-tuple $v = (v_1, \dots, v_L)$, $\Pi v$ is the continuous function on $\Omega$ which is linear in every triangle $T_1, \dots, T_M$ and, at the points $P_1, \dots, P_L$, attains to the values $v_1, \dots, v_L$, respectively.

THEOREM 5.1. – Let the hypotheses of Theorem 2.1 be satisfied, and let $P_1, \dots, P_L$ be the vertices of a triangulation $\Theta$ as described above. The following estimate holds for every $\theta \in (0, \frac{1}{2})$

$$\begin{aligned}(5.1)\qquad &\|a - b\|_{L^\infty(\Omega_d)} \leqslant \\ &\leqslant C_2 \left\{ \max_{\partial\Omega} |a - b| + \left[ c\sigma^{-1} \left( L^{-1} \sum_{j=1}^{L} \big(u(P_j) - v(P_j)\big)^2 \right)^{\frac{1}{2}} + C_{12} h \right]^{-\theta + \frac{1}{2}} \right\}^{1/(2N+1)}\end{aligned}$$

here $C_2$ is the same constant appearing in (2.3) and $C_{12}$ depends on $\Omega$, $\lambda_1$, $E$ and $\|g\|_{C^2(\bar{\Omega})}$, $\|k\|_{C^2(\bar{\Omega})}$.

Consider the case that measurements $\bar{u}_1, \dots, \bar{u}_L$ of $u$ are made at the points $P_1, \dots, P_L$, and assume that the following error estimate is known

$$L^{-1} \sum_{j=1}^{L} \big(u(P_j) - \bar{u}_j\big)^2 \leqslant \delta^2\,. \tag{5.2}$$

An approximate identification of the coefficient $a$ is performed by determining the solution $\bar{a}_\varepsilon$ of the boundary value problem

$$\begin{cases} \varepsilon\, \Delta \bar{a}_\varepsilon + \operatorname{div}(\bar{a}_\varepsilon\, D\Pi\bar{u}) = 0\,, & \text{in } \ \Omega\,, \\ \bar{a}_\varepsilon = a\,, & \text{on } \partial\Omega\,. \end{cases} \tag{5.3}$$

here $\bar{u} = (\bar{u}_1, \dots, \bar{u}_L)$.

The following theorem yields an error estimate, in terms of $\delta$ and of $h$, the size of the triangulation.

THEOREM 5.2. – Let the hypotheses of Theorem 3.1 be satisfied. Let $\bar{u} = (\bar{u}_1, \dots, \bar{u}_L)$ and $\delta > 0$, be such that (5.2) holds.

Let $\bar{a}_\varepsilon$ be the solution of (5.3).

For every $\alpha \in (0,1)$ these exists a number $\delta_2 > 0$ depending only on $\alpha$, $\sigma$, $\Omega$, $\lambda_1$, $\lambda_2$, $E$ and $\|g\|_{C^2(\bar\Omega)}$ such that if

$$\left((h/\operatorname{diam}\Omega)^{-2}\delta + (h/\operatorname{diam}\Omega)^{1-\alpha}\right) \leqslant \delta_2^{1+2N},$$

and if we choose

$$\varepsilon = \left((h/\operatorname{diam}\Omega)^{-2}\delta + (h/\operatorname{diam}\Omega)^{1-\alpha}\right)^{2N/(2N+1)},$$

then we have

$$\int_{\Omega_d} |a - \bar a_\varepsilon| \leqslant (C_8 + |\Omega| C_6 \delta_2^{-1}) \cdot \left((h/\operatorname{diam}\Omega)^{-2}\delta + (h/\operatorname{diam}\Omega)^{1-\alpha}\right)^{1/(2N+1)}, \tag{5.4}$$

here $C_6$, $C_8$ are the constants appearing in (3.6), (3.9) respectively.

REMARK 5.1. – The estimate (5.4) can be interpreted as follows. If the measurements of $u$ are such that the root-mean-square error is $0(\delta)$ independently of the number of the measurements (i.e. of the size $h$), then it is convenient that the size $h$ is $O(\delta^{1/(3-\alpha)})$, otherwise the precision in the approximation of $a$ by $\bar a_\varepsilon$ may be lost.

This is a typical feature of ill-posed problems with discrete data.

PROOF OF THEOREM 5.1. – Just combine (2.3) with the interpolation inequality

$$|\Omega|^{-\frac12}\|f\|_2 \leqslant c\left\{\sigma^{-1}\left(L^{-1}\sum_{i=1}^{L} f^2(P_i)\right)^{\frac12} + \left(\|Df\|_\infty + d_0^{-1}\|f\|_\infty\right)h\right\}, \tag{5.5}$$

and then make use of (6.4) in Lemma (6.1). The theorem follows with

$$C_{12} = c_0\left[|\Omega|^{\frac12}\left(\|D^2 g\|_\infty + \|D^2 k\|_\infty + \mu(\|g\|_\infty + \|k\|_\infty)\right) + d_0^{-1}\right]. \tag{5.6}$$

Inequality (5.5) follows easily once we have noticed that if $\|Df\|_\infty < \infty$, then $f$ is Lipschitz continuous in $\Omega$ with Lipschitz constant $c(\|Df\|_\infty + d_0^{-1}\|f\|_\infty)$, see, for such type of estimates, CIARLET [7].

PROOF OF THEOREM 5.2. – We may rephrase the proof of Theorem 4.1 replacing the estimate (4.7) of Lemma 4.2 with the finite element interpolation inequality

$$\begin{aligned}\|Du - D\Pi\bar u\|_\infty \leqslant c|\Omega|^{\frac12}(\sigma h)^{-2}\left(L^{-1}\sum_{i=1}^{L}\left(u(P_i) - \bar u_i\right)^2\right)^{\frac12} + \\ + c_p\sigma^{-1}\left(\|D^2u\|_p + d_0^{-1}\|Du\|_p\right)h^{1-2/p},\end{aligned} \tag{5.7}$$

which holds for every $p > 2$.

The theorem follows, picking $p = 2/\alpha$, and

$$\delta_2 = \left[c_p(\operatorname{diam}\Omega)^{-1}\lambda_2 E^{-1}(C_6 - 1)\left(\sigma^{-2} + \sigma^{-1}(\|D^2 g\|_\infty + \mu\|g\|_\infty)\right)\right]. \tag{5.8}$$

The estimate (5.7) can be derived from Morrey's inequality (see ADAMS [1])

$$|f(x) - f(y)| \leqslant c_p(\|Df\|_p + d_0^{-1}\|f\|_p)|x - y|^{1-2/p}\,,$$

which holds for every $x, y \in \Omega$, $p > 2$, and the estimate

$$\|D\Pi v\|_\infty \leqslant c|\Omega|^{\frac{1}{2}}(\sigma h)^{-2}\left(L^{-1}\sum_{i=1}^{L} v_i^2\right)^{\frac{1}{2}}$$

(see CIARLET [7]).

## 6. – Appendix.

LEMMA 6.1. – Let $\Omega$ be a $C^2$-smooth bounded domain in $\mathbb{R}^2$. Let $a$ satisfy (3), (4), let $g \in C^2(\bar{\Omega})$. Let $u$ be the weak solution of (1), (2). Then for every $p < \infty$, $u \in W^{2,p}(\Omega)$, and it satisfies the estimates

$$\|u\|_\infty \leqslant \|g\|_\infty\,, \tag{6.1}$$

$$\|Du\|_2 \leqslant \lambda_1^{-\frac{1}{2}}\lambda_2^{\frac{1}{2}}\|Dg\|_2\,, \tag{6.2}$$

$$\|D^2u\|_p \leqslant c_p|\Omega|^{1/p}\{\|D^2g\|_\infty + \mu\|g\|_\infty\}\,, \tag{6.3}$$

$$\|Du\|_\infty \leqslant c_p|\Omega|^{\frac{1}{2}}\{\|D^2g\|_\infty + \mu\|g\|_\infty\}\,, \tag{6.4}$$

here

$$\mu = (\lambda_1^{-1}E + d_0^{-1})^{(2p-2)/(p-2)}|\Omega|^{1/(p-2)}\,, \qquad p > 2\,. \tag{6.5}$$

PROOF. – The estimate (6.1) is a consequence of maximum principle. The inequality (6.2) follows from the variational principle

$$\int_\Omega a|Du|^2 \leqslant \int_\Omega a|Dv|^2\,,$$

for every $v$ such that $v = u$ on $\partial\Omega$.

Let $w = u - g$, we have

$$\begin{cases} \Delta w = -a^{-1}Da\cdot Dw - \Delta g - a^{-1}Da\cdot Dg\,, & \text{in } \Omega\,.\\ w = 0\,, & \text{on } \partial\Omega\,.\end{cases} \tag{6.6}$$

Now by the $L^p$ regularity estimate by AGMON, DOUGLIS and NIRENBERG, [2], we have

$$\|D^2w\|_p \leqslant c_p\{\|\Delta w\|_p + d_0^{-1}\|Dw\|_p + d_0^{-2}\|w\|_p\}$$

for every $p \in (1, \infty)$. Therefore

$$(6.7)\qquad \|D^2 w\|_p \leqslant c_p\{(E\lambda_1^{-1} + d_0^{-1})|\Omega|^{1/p}\|Dw\|_\infty + \|\Delta g\|_p + \\ + E\lambda_1^{-} |\Omega|^{1/p}\|Dg\|_\infty + d_0^{-2}|\Omega|^{1/p}\|g\|_\infty\}.$$

Here we make use of the interpolation inequality

$$(6.8)\qquad \|Df\|_\infty \leqslant c_p\{\|D^2 f\|_p^\eta \|f\|_\infty^{1-\eta} + d_1^{-1}\|f\|_\infty\},$$

which holds for $\eta = p/(p-2)$ and for every $p > 2$ (see [1]).

We obtain

$$\|D^2 w\|_p \leqslant c_p\{\|\Delta g\|_p + E\lambda_1^{-1}|\Omega|^{1/p}\|Dg\|_\infty + [((E\lambda_1^{-1} + d_0^{-2})|\Omega|^{1/p})^{1/(1-\eta)} + d_0^{-2}|\Omega|^{1/p}]\|g\|_\infty\},$$

and applying (6.8) with $f = g$ (6.3) is obtained. Applying (6.8) with $f = u$ (6.4) follows.

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