Unified Boundedness, Periodicity, and Stability in Ordinary and Functional Differential Equations (*).

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Summary. – *We discuss a unified theory of periodicity of dissipative ordinary and functional differential equations in terms of uniform boundedness. Sufficient conditions for the uniform boundedness are given by means of Liapunov functionals having a weighted norm as an upper bound. The theory is developed for ordinary differential equations, equations with bounded delay, and equations with infinite delay.*

1. – Introduction.

In the study of existence of periodic solutions of a system of ordinary differential equations

$$x' = F(t, x) \tag{1}$$

the properties of uniform boundedness (UB) and uniform ultimate boundedness (UUB) frequently emerge as central. In fact, when (1) is periodic in t and when solutions are unique, then those boundedness properties, together with asymptotic fixed point theorems show that (1) has a periodic solution.

When (1) is linear and written as

$$x' = A(t)x + p(t) \tag{2}$$

with homogeneous system

$$x' = A(t)x\,, \tag{3}$$

then UB for (3) is equivalent to uniform stability of the zero solution of (3), while UB and UUB for (3) is equivalent to uniform asymptotic stability (UAS) of the zero solution of (3).

It has proved to be fruitful to show boundedness and asymptotic stability for (1) and (3) using Liapunov functions. When (1) and (2) are periodic in t the fol-

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lowing scheme holds. Here, functions denoted by W_i are called wedges and they are continuous, strictly increasing to ∞, and $W_i(0) = 0$. Also, for a continuous function V, by $V'_{(1)}$ we mean

$$\limsup_{h\to 0^+} [V(t+h, x+hF(t,x)) - V(t,x)]/h .$$

I) If solutions of (1) are unique, UB and UUB, then (1) has a periodic solution.

II) If the zero solution of (3) is UAS, then (2) has a globally stable periodic solution.

III) If $F(t, 0) = 0$ and if there is a function $V\colon [0,\infty)\times R^n \to R$ such that

$$W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)\ , \qquad V'_{(1)}(t,x) \leqslant -W_3(|x|)$$

then the zero solution of (1) is UAS.

IV) If there is a function $V\colon [0,\infty)\times R^n \to R$ such that

$$W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)\ , \qquad V'_{(1)}(t,x) \leqslant -W_3(|x|) + M$$

then solutions of (1) are UB and UUB.

Thus, I) and II) indicate conditions under which there are periodic solutions, while III) and IV) give conditions under which I) and II) hold.

For many years investigators have been interested in counterparts of I-IV) for systems of Volterra equations

$$x' = h(t,x) + \int_{-\infty}^{t} q(t, s, x(s))\, ds \tag{4}$$

and

$$x' = A(t)x + \int_{-\infty}^{t} C(t,s)x(s)\, ds + p(t)\ , \tag{5}$$

for systems of functional differential equations with bounded delay

$$x' = F(t, x_t) \tag{6}$$

and

$$x' = A(t)x + \int_{t-r}^{t} C(t,s)x(s)\, ds + p(t)\ , \tag{7}$$

and general systems with infinite delay

$$x' = G(t, x(s); -\infty < s \leqslant t) . \tag{8}$$

Much progress has been made.

Under quite general conditions (I) has been advanced to all of these systems (cf. ARINO-BURTON-HADDOCK [1]).

BURTON [3] has obtained results extending (II) to forms of (5). Here we extend (II) to (7). We, therefore, feel that (I) and (II) are fairly well settled.

Part (III) has been the object of intensive investigation for many years. Using an L^2-norm in the upper bound on V, BURTON [6] extended it to (6) and KATO [20] has discussed such forms extensively. In this paper we extend (III) to general infinite delay equations, again using a type of L^2-norm in the upper wedge on V.

The extension of (IV) has been the most challenging part. Using the L^2-norm again we extend part (IV) to all the systems both with bounded and infinite delay.

In fact, we show that the main theorems of stability by Liapunov's direct method for (1) can be advanced to equations with both finite and infinite delay in a completely unified way. And this provides one vehicle for achieving periodicity results for these systems.

2. - Ordinary differential equations.

This section consists of a fairly concise summary of the technical details of the problems discussed in the introduction which we wish to extend to functional differential equations. We focus on

$$x' = F(t, x) \tag{1}$$

in which $F\colon (-\infty, \infty) \times R^n \to R^n$ is continuous so that for each (t_0, x_0) there is a solution $x(t, t_0, x_0)$ satisfying $x(t_0, t_0, x_0) = x_0$ and (1) for $t_0 \leqslant t \leqslant t_0 + \alpha$ for some $\alpha > 0$; if the solution remains bounded, then $\alpha = \infty$; if F is locally Lipschitz in x then it is unique. Whenever a function x is written without its argument, that argument is t.

DEFINITION 1. - Let $F(t, 0) \equiv 0$. The zero solution of (1) is:

a) *stable* if for each $\varepsilon > 0$ and $t_0 \in R$ there exists $\delta > 0$ such that $[|x_0| < \delta,\ t \geqslant t_0]$ imply $|x(t, t_0, x_0)| < \varepsilon$;

b) *uniformly stable* if it is stable and if δ is independent of t_0;

c) *asymptotically stable* if it is stable and if for each $t_0 \in R$ there is an $\eta(t_0) > 0$ such that $|x_0| < \eta$ implies $x(t, t_0, x_0) \to 0$ as $t \to \infty$;

d) *uniformly asymptotically stable* if it is uniformly stable and if there is an $\eta > 0$ and if for each $\mu > 0$ there is an $S > 0$ such that $[t_0 \in R, |x_0| < \eta, t \geqslant t_0 + S]$ imply $|x(t, t_0, x_0)| < \mu$;

e) *equi-asymptotically stable* if it is stable and if η and S in *d*) depend on t_0.

One defines stability of any solution in a similar fashion.

DEFINITION 2. – Solutions of (1) are *uniform bounded* (UB) if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[t_0 \in R, |x_0| < B_1, t \geqslant t_0]$ imply that $|x(t, t_0, x_0)| < B_2$.

DEFINITION 3. – Solutions of (1) are *uniform ultimate bounded for bound B* for (1) if for each $B_3 > 0$ there exists $K > 0$ such that $[t_0 \in R, |x_0| < B_3, t \geqslant t_0 + K]$ imply that $|x(t, t_0, x_0)| < B$.

Under quite general conditions most of these definitions can be characterized by Liapunov functions. Also, when certain conditions concerning Liapunov functions hold, then $x = 0$ is necessarily a solution. A proof of the next result is found in YOSHIZAWA [26].

THEOREM 1. – Suppose there is an open neighborhood D of $x = 0$ in R^n and a continuous function $V: (-\infty, \infty) \times D \to [0, \infty)$ which is locally Lipschitz in x. Let W_i be wedges.

(*a*) If

$$V(t, 0) = 0\,, \qquad W(|x|) \leqslant V(t, x)\,, \qquad \text{and} \qquad V'_{(1)}(t, x) \leqslant 0$$

then the zero solution of (1) is stable.

(*b*) If

$$W_1(|x|) \leqslant V(t, x) \leqslant W_2(|x|) \qquad \text{and} \qquad V'_{(1)}(t, x) \leqslant 0$$

then the zero solution of (1) is uniformly stable.

(*c*) If F is bounded for x bounded and if

$$V(t, 0) = 0\,, \qquad W_1(|x|) \leqslant V(t, x)\,, \qquad V'_{(1)}(t, x) \leqslant -W_3(|x|)\,,$$

then the zero solution of (1) is equi-asymptotically stable.

(*d*) If

$$W_1(|x|) \leqslant V(t, x) \leqslant W_2(|x|)\,, \qquad V'_{(1)}(t, x) \leqslant -W_3(|x|)$$

then the zero solution of (1) is uniformly asymptotically stable.

(*e*) If $D = R^n$,

$$W_1(|x|) \leqslant V(t, x) \leqslant W_2(|x|)\,, \qquad V'_{(1)}(t, x) \leqslant -W_3(|x|) + M$$

for some $M > 0$, then solutions of (1) are uniform bounded and uniform ultimate bounded for bound B.

Many good examples have been given illustrating each part of this theorem.

The next lemma, found in CRONIN [12], is useful in fixed point theory proving the existence of a periodic solution of (1).

LEMMA 1. – Let $F(t + T, x) = F(t, x)$ for some $T > 0$ and all (t, x) and suppose that solutions of (1) are uniquely determined by (t_0, x_0). Equation (1) has a T-periodic solution if and only if there is a (t_0, x_0) with $x(t_0 + T, t_0, x_0) = x_0$.

The next result was proved by M. L. CARTWRIGHT [11] for second order systems and a proof for n-dimensional systems is found in YOSHIZAWA [26]. It is a simple consequence of Browder's fixed point theorem.

THEOREM 2. – If F is locally Lipschitz in x and periodic in t and if solutions are uniform bounded and uniform ultimate bounded for bound B, then (1) has a T-periodic solution.

Because of the structure of (1), Theorem 2 asks far more than it appears to ask. We restate it with the implications derived from it.

THEOREM 2′. – Let the following conditions hold for (1).

1) For each (t_0, x_0) there is a unique solution $x(t, t_0, x_0)$ of (1) defined on $[t_0, \infty)$.

2) Solutions of (1) are uniform bounded and uniform ultimate bounded for bound B.

3) For each $\nu > 0$ there exists $L > 0$ such that $|x_0| < \nu$ and $t_0 \in R$ imply that $|x'(t, t_0, x_0)| < L$.

4) For each (t_0, x_0), $x(t, t_0, x_0)$ is continuous in x_0.

5) If $x(t)$ is a solution of (1), so is $x(t + T)$.

Under these conditions (1) has a T-periodic solution.

It was shown in ARINO-BURTON-HADDOCK [1] that when these concepts are properly extended to general functional differential equations, then a periodic solution results.

The next theorem is a simple consequence of Floquet theory.

THEOREM 3. – If $A(t + T) = A(t)$, $p(t + T) = p(t)$, and if the zero solution of (3) is uniformly asymptotically stable then (2) has a globally stable T-periodic solution.

Just as in Theorem 2, because of the structure of (3), Theorem 3 asks far more than is at first apparent. We restate it as follows.

THEOREM 3'. – Let the following conditions hold.

1) $A(t+T) = A(t)$ and $p(t+T) = p(t)$ for all t and some $T > 0$.

2) All solutions of (2) are bounded.

3) Each solution of (2) is equi-asymptotically stable.

4) The zero solution of (3) is uniformly asymptotically stable.

Under these conditions, (2) has a globally stable periodic solution.

It is of course, important to distinguish between linear and nonlinear systems even in stability theory. So frequently a poor Liapunov function can give just enough information to supplement the linear theory and give a strong result.

EXAMPLE 1. – Consider the linear scalar equation

$$x'' + a(t)x' + x = p(t)$$

with a and p continuous and T-periodic, $a(t) \geqslant 0$ and $a(t) \not\equiv 0$. Then there is a globally stable T-periodic solution.

PROOF. – Write the equation as the system

$$x' = y$$
$$y' = -x - a(t)y + p(t)$$

and in matrix form

$$X' = A(t)X + P(t) .$$

Define a Liapunov function

$$V(x, y) = x^2 + y^2$$

for the homogeneous system

$$X' = A(t)X$$

and get

$$V'(x, y) = -2a(t)y^2 .$$

It is then evident that solutions are all bounded and converge to the x-axis. And further arguments show that all solutions tend to zero. By Floquet theory they tend to zero exponentially; in fact, the zero solution of the homogeneous equation is uniformly asymptotically stable. Theorem 3 completes the proof.

It would be very difficult to find a V satisfying the conditions of Theorem 1 (d). Our goal is to present parallel theory for general functional differential equations.

3. – Equations with bounded delay.

We turn now to the systems

$$x' = F(t, x_t) \tag{6}$$

and

$$x' = A(t)x + \int_{t-r}^{t} C(t, s)x(s)\, ds + p(t)\,. \tag{7}$$

In (7) the functions A, p, and C are continuous everywhere and $r > 0$. In (6), F is a continuous functional defined as follows. Let $h > 0$ and let C denote the space of continuous functions $\varphi\colon [-h, 0] \to R^n$ with $\|\varphi\| = \sup_{-h \leqslant s \leqslant 0} |\varphi(s)|$ where $|\cdot|$ is a norm on R^n. For any $t_0 \in R$ and any continuous function $x\colon [t_0 - h, t_0 + A] \to R^n$, if $t_0 \leqslant t \leqslant t_0 + A$, then $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $-h \leqslant s \leqslant 0$. The function F is continuous in (t, φ) for $-\infty < t < \infty$ and $\varphi \in C$. Moreover, F takes bounded sets into bounded sets.

To specify a solution of (6) we require a $t_0 \in R$ and a function $\varphi \in C$. We then obtain a solution $x(t_0, \varphi)$ on $[t_0, t_0 + \beta)$ with value $x(t, t_0, \varphi)$ and with $x_{t_0}(t_0, \varphi) = \varphi$. If F is locally Lipschitz in φ, then the solution is unique. If the solution remains bounded, then $\beta = \infty$.

For $V(t, \varphi)$ a continuous scalar functional defined for $t \in R$ and $\varphi \in C$ we define

$$V'_{(6)}(t, x_t(t_0, \varphi)) = \limsup_{\delta \to 0^+} (1/\delta)\{V(t + \delta, x_{t+\delta}(t_0, \varphi)) - V(t, x_t(t_0, \varphi))\}\,.$$

Detailed properties of this derivative are found in YOSHIZAWA [26; pp. 186-189]. Corresponding to Def. 1 for (1), we have the following definition for (6).

DEFINITION 4. – Let $F(t, 0) = 0$. The zero solution of (6) is

(a) *stable* if for each $t_0 \in R$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $[\varphi \in C, \|\varphi\| < \delta, t \geqslant t_0]$ imply that $|x(t, t_0, \varphi)| \leqslant \varepsilon$;

(b) *uniformly stable* if it is stable and if δ is independent of t_0;

(c) *asymptotically stable* if it is stable and if for $t_0 \in R$ there is an $\eta > 0$ such that $[\varphi \in C, \|\varphi\| < \eta]$ imply that $x(t, t_0, \varphi) \to 0$ as $t \to \infty$;

(*d*) *uniformly asymptotically stable* if it is uniformly stable and if there is an $\eta > 0$ and for each $\mu > 0$ there exists $S > 0$ such that $[t_0 \in R, \varphi \in C, \|\varphi\| < \eta, t \geqslant t_0 + S]$ imply that $|x(t, t_0, \varphi)| < \mu$;

(*e*) *equi-asymptotically stable* if it is stable and if η and S in (*d*) depend on t_0.

DEFINITION 5. – Solutions of (6) are *uniform bounded* if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[t_0 \in R, \varphi \in C, \|\varphi\| < B_1, t \geqslant t_0]$ imply that $|x(t, t_0, \varphi)| < B_2$.

DEFINITION 6. – Solutions of (6) are *uniform ultimate bounded for bound B* if for each $B_3 > 0$ there exists $K > 0$ such that $[t_0 \in R, \varphi \in C, \|\varphi\| < B_3, t \geqslant t_0 + K]$ imply that $|x(t, t_0, \varphi)| < B$.

Investigators have given much consideration to the extension of Theorem 1 to (6). The following is one possibility and it gives perfect unity between (1) and (6), being a natural counterpart of Theorem 1.

THEOREM 4. – Let $H > 0$ and let $C_H \subset C$ with $\varphi \in C_H$ if $\|\varphi\| < H$. Suppose $V\colon R \times C_H \to [0, \infty)$ is continuous and locally Lipschitz in φ. Let the W_i be wedges.

(*a*) If

$$V(t, 0) = 0 , \quad W(|\varphi(0)|) \leqslant V(t, \varphi) , \quad V'_{(6)}(t, x_t) \leqslant 0$$

then the zero solution of (6) is stable.

(*b*) If

$$W_1(|\varphi(0)|) \leqslant V(t, \varphi) \leqslant W_2(\|\varphi\|) \quad \text{and} \quad V'_{(6)}(t, x_t) \leqslant 0$$

then the zero solution of (6) is uniformly stable.

(*c*) If $F(t, \varphi)$ is bounded for $\|\varphi\| < H$ and if $V(t, 0) = 0$,

$$W_1(|\varphi(0)|) \leqslant V(t, \varphi) , \quad V'_{(6)}(t, x_t) \leqslant - W_2(|x(t)|)$$

then the zero solution of (6) is equi-asymptotically stable.

(*d*) Let $|||\cdot|||$ be the L^2-norm on C. If

$$W_1(|\varphi(0)|) \leqslant V(t, \varphi) \leqslant W_2(|\varphi(0)|) + W_3(|||\varphi|||)$$

and

$$V'_{(6)}(t, x_t) \leqslant - W_4(|x(t)|)$$

then the zero solution of (6) is UAS.

(*e*) If there is an $M > 0$ with

$$W_1(|\varphi(0)|) \leqslant V(t, \varphi) \leqslant W_2(|\varphi(0)|) + W_3\left[\int_{-h}^{0} W_4(|\varphi(s)|)\, ds\right]$$

and

$$V'_{(6)}(t, x_t) \leqslant -W_4(|\varphi(0)|) + M$$

then solutions of (6) are UB and UUB.

PROOF. – Parts (*a*) and (*b*) are classical. Part (*c*) can be patterned after the proof of Theorem 6.1.3 in BURTON [7; p. 160]. Part (*d*) is a result of BURTON [6]; it has a RAZUMIKHIN counterpart by WEN [24]. Our contribution is (*e*) which we now prove.

Let $B_1 > 0$, $t_0 \geqslant 0$, $\varphi \in C$ with $\|\varphi\| \leqslant B_1$ be given and let $x(t) = x(t, t_0, \varphi)$. Integrate

$$V'(t, x_t) \leqslant -W_4(|x(t)|) + M$$

from $t - h$ to t obtaining

$$\int_{t-h}^{t} W_4(|x(s)|)\, ds \leqslant V(t-h, x_{t-h}) - V(t, x_t) + Mh .$$

Now, consider $V(s) = V(s, x_s)$ on any interval $[t_0, L]$ for any $L \geqslant t_0 + h$. Since $V(s)$ is continuous, it has a maximum at some $\bar{t} \in [t_0, L]$. Suppose $\bar{t} \leqslant t_0 + h$; then

$$V(t) \leqslant V(\bar{t}) \leqslant V(t_0) + M(\bar{t} - t_0) \leqslant W_2(B_1) + W_3(hW_4(B_1)) + Mh$$

and thus

$$|x(t)| \leqslant W_1^{-1}[W_2(B_1) + W_3(hW_4(B_1)) + Mh] \stackrel{\text{def}}{=} B_2^* .$$

If $\bar{t} \in [t_0 + h, L]$, then $V(\bar{t} - h) - V(\bar{t}) \leqslant 0$ so that

$$\int_{\bar{t}-h}^{\bar{t}} W_4(|x(s)|)\, ds \leqslant V(\bar{t} - h) - V(\bar{t}) + Mh \leqslant Mh .$$

We note that for such $\bar{t}$, $V'(\bar{t}) \geqslant 0$ and hence $|x(\bar{t})| \leqslant W_4^{-1}(M)$. Thus,

$$W_1(|x(t)|) \leqslant V(t) \leqslant V(\bar{t}) \leqslant W_2(W_4^{-1}(M)) + W_3(Mh)$$

for $t \in [t_0, L]$ and, therefore,

$$|x(t)| \leqslant W_1^{-1}[W_2(W_4^{-1}(M)) + W_3(Mh)] \stackrel{\text{def}}{=} B_2^{**} .$$

Since L is arbitrary, $B_2 = \max[B_2^*, B_2^{**}]$. This proves the UB.

For the UUB, let $B_3 > 0$ be given and find B_4 such that $[t_0 \geqslant 0, \|\varphi\| \leqslant B_3, t \geqslant t_0]$ imply that $|x(t, t_0, \varphi)| < B_4$. We determine a $U > 0$ so that

$$V'(t, x_t) \leqslant -M < 0 \quad \text{if } |x(t)| \geqslant U .$$

Since

$$0 \leqslant V(t, x_t) \leqslant W_2(B_4) + W_3(hW_4(B_4))$$

there is a sufficiently large integer N such that for any interval $[t, t + Nh]$ with $t \geqslant t_0$, then there is some $\bar{t} \in (t, t + Nh)$ with $|x(\bar{t})| < U$. Now

$$(*) \qquad \int_{t-h}^{t} W_4(|x(s)|)\, ds \leqslant -\int_{t-h}^{t} V'(s)\, ds + Mh = V(t-h) - V(t) + Mh \quad \text{for } t \geqslant t_0 + h .$$

Consider the intervals

$$I_1 = [t_0, t_0 + Nh], \quad I_2 = [t_0 + Nh, t_0 + 2Nh], \quad \dots,$$
$$I_i = [t_0 + (i-1)Nh, t_0 + iNh], \quad \dots$$

and select $t_i \in I_i$ such that $V(t_i)$ is the maximum on I_i. In case $t_i = t_0 + (i-1)Nh$ with $|x(t_i)| > U$ then by the choice of N, there is a first $\bar{t}_i \in [t_0 + (i-1)Nh, t_0 + iNh]$ such that

$$|x(\bar{t}_i)| = U .$$

Now, instead of the above choice for I_i, in this case we pick

$$I_i = [\bar{t}_i, t_0 + iNh]$$

and let

$$V(t_i) = \max_{s \in I_i} V(s) .$$

Therefore, in any case we have

$$|x(t_i)| \leqslant U , \quad i = 1, 2, 3, \dots .$$

Now, consider the intervals

$$L_2 = [t_2 - h, t_2]\,, \qquad L_3 = [t_3 - h, t_3]\,, \qquad \ldots, \qquad L_i = [t_i - h, t_i]\,, \qquad \ldots .$$

For each $i = 2, 3, 4, \ldots$ we have two cases.

Case 1. - $V(t_i) + 1 \geqslant V(s)$ for all $s \in L_i$.

Case 2. - $V(t_i) + 1 < V(s_i)$ for some $s_i \in L_i$.

Note that in Case 2, $s_i \notin I_i$ since $V(t_i)$ is the maximum on I_i. If there is no gap between I_{i-1} and I_i, then $s_i \in I_{i-1}$. If there is a gap and $s_i \in [t_0 + (i-1)Nh, \bar{t}_i]$, then we have $|x(t)| \geqslant U$ and thus $V'(t) \leqslant 0$ on $[t_0 + (i-1)Nh, \bar{t}_i]$. Hence

$$V(t_0 + (i-1)Nh) \geqslant V(s_i) > V(t_i) + 1\,.$$

In either case we have

$$V(t_i) + 1 < V(t_{i-1})$$

since $V(t_{i-1})$ is the maximum on I_{i-1}.

By the boundedness of $V(t)$, there is an integer $N^* > 0$ such that Case 2 holds on no more than N^* consecutive intervals L_i. Thus, on some L_j with $j \leqslant N^*$ we have

$$V(t_j) + 1 \geqslant V(s) \qquad \text{for all } s \in L_j\,.$$

From $(*)$ with $t = t_j$ it follows that

$$\int_{t_j - h}^{t_j} W_4(|x(s)|)\, ds \leqslant V(t_j - h) - V(t_j) + Mh \leqslant 1 + Mh$$

and therefore,

$$V(t_j) \leqslant W_2(U) + W_3(1 + Mh)\,.$$

Let

$$V(\tau) = \max_{s \in I_j} V(s)\,.$$

Then

$$V(\tau) \leqslant V(t_j) + 1\,.$$

Now we claim that

$$V(t) \leqslant W_2(U) + W_3(1 + Mh) + 1 \stackrel{\text{def}}{=} D^* \qquad \text{for all } t \geqslant t_j\,.$$

To see this, let t_p be the first $t > t_j$ with $V(t_p) = D^*$. Then notice that $V(t_p) \geqslant V(t_p - h)$ by $(*)$ with $t = t_p$, yielding

$$\int_{t_p-h}^{t_p} W_4(|x(s)|)\,ds \leqslant Mh$$

and so

$$V(t_p) \leqslant W_2(U) + W_3(Mh) < D^*.$$

Hence, for $t \geqslant t_0 + N^*Nh$ we have

$$V(t) \leqslant D^*$$

or

$$|x(t)| \leqslant W_1^{-1}(D^*) \overset{\text{def}}{=} B.$$

Letting $S = N^*Nh$, we obtain UUB. This completes the proof.

EXAMPLE 2. – Consider the scalar equation

$$x' = -[a + (t \sin t)^2]x(t) + bx(t - r(t)) + \cos t$$

with $a > 0$ and constant, b constant, $r'(t) \leqslant \beta$ for some $M > 0$ and $0 < \beta < 1$. If $|b| < a(1 - \beta)$ and $0 \leqslant r(t) \leqslant \nu$ for some $\nu > 0$ then solutions are uniform bounded and uniform ultimate bounded for bound B.

PROOF. – Define

$$V(t, x_t) = |x(t)| + k\int_{t-r(t)}^{t} |x(s)|\,ds$$

for $k = |b|/(1 - \beta)$. Then

$$\begin{aligned} V'(t, x_t) &\leqslant -[a + (t \sin t)^2]|x| + |b||x(t - r(t))| + 1 + k|x| - k|x(t - r(t))|(1 - r'(t)) \leqslant \\ &\leqslant [-a + k]|x| + |b||x(t - r(t))| + 1 - |b||x(t - r(t))| \leqslant \\ &\leqslant [-a + k]|x| + 1 = -\mu|x| + 1. \end{aligned}$$

Thus, by Theorem 4 (e) the result follows.

REMARK. – Much effort has gone into development of results along the line of (e). The classical result is found in YOSHIZAWA [26; p. 206] which has a very restrictive condition on the wedges.

The following result, patterned after Lemma 1 of Cronin for differential equations, is well known.

THEOREM 5. – Let F be locally Lipschitz in φ and $F(t+T, \varphi) = F(t, \varphi)$ for all $(t, \varphi) \in R \times C$. Equation (6) has a T-periodic solution if and only if there is a $(t_0, \varphi) \in R \times C$ with $x(t+T, t_0, \varphi) = \varphi(t)$ for $t_0 - h \leqslant t \leqslant t_0$.

The next result is a consequence of Horn's fixed point theorem. A proof may be found in HALE and LOPES [17], ARINO-BURTON-HADDOCK [1], or in BURTON [8]. It is the counterpart of Theorem 2.

THEOREM 6. – Let F be locally Lipschitz in φ and $F(t+T, \varphi) = F(t, \varphi)$ for all $(t, \varphi) \in R \times C$. If solutions of (6) are UB and UUB for bound B, then (6) has a T-periodic solution.

We now consider a linear Volterra system

$$x' = A(t)x + \int_{t-r}^{t} C(t, s)x(s)\, ds + p(t) \tag{7}$$

and the unperturbed system

$$x' = A(t)x + \int_{t-r}^{t} C(t, s)x(s)\, ds \tag{9}$$

with A, C, and p continuous, with A and p being T-periodic, and with $C(t+T, s+T) = C(t, s)$.

The next result is the counterpart of Theorem 3 as amplified in Theorem 3′.

THEOREM 7. – Suppose that the zero solution of (9) is UAS and that the solution $x(t, 0, 0)$ of (7) is bounded and equi-asymptotically stable at $t = 0$. Then (7) has a globally stable T-periodic solution.

PROOF. – Since (7) and (9) are linear, the difference of two solutions of (7) is a solution of (9). By the assumption of UAS of (9), all solutions of (7) converge uniformly to a solution φ of (7). We now claim that (7) has at most one bounded solution on $(-\infty, \infty)$. In fact, suppose that there are two distinct bounded solutions of (7) on $(-\infty, \infty)$, say φ and ψ, $\psi \not\equiv \varphi$. Then there must be a $t^* \in R$ and $k > 0$ with $|\psi(t^*) - \varphi(t^*)| = k$. Let $\nu = \sup\limits_{-\infty<t<\infty} |\psi(t) - \varphi(t)|$. Note that $\psi(t) - \varphi(t)$ is a solution of (9), so that by the UAS, for $\varepsilon = k/2$, there is a $J > 0$ such that $\{t_1 \in R, |\psi(t) - \varphi(t)| \leqslant \nu$ on $[t_1 - r, t_1]$, $t \geqslant t_1 + J\}$ imply that $|\psi(t) - \varphi(t)| < \varepsilon = k/2$. If we pick t_1 with $t_1 + J < t^*$, we obtain a contradiction. Thus, (9) has at most one bounded solution on $(-\infty, \infty)$.

Next, we prove that there is a bounded solution of (9) on $(-\infty, \infty)$. Suppose that $x(t) = x(t, 0, 0)$ is the solution of (9) on $[0, \infty)$ with $x(t) = 0$ if $-r \leqslant t \leqslant 0$. Define a sequence of solutions of (9) on $[-nT, \infty)$ by $x_n(t) = x(t+nT, 0, 0)$ for $t \geqslant -nT$, n a positive integer. Now $x(t)$ is bounded on $[0, \infty)$, so there is a constant

$M > 0$ with $|x_n(t)| \leqslant M$ on $[-nT, \infty)$. Therefore, we have

$$|x'(t)| \leqslant |A(t)|M + M\int_{t-r}^{t} |C(t, s)|\, ds + |p(t)| \leqslant B$$

on $[0, \infty)$. Moreover, by our assumptions $x_n(t)$ is also a solution of (9) and

$$|x_n'(t)| \leqslant B \quad \text{on } (-nT, \infty)\,.$$

By the equi-asymptotic stability of $x(t, 0, 0)$, for $\nu = M$ and for each $\varepsilon > 0$, there is a P_1 such that

$$|x(t, 0, 0) - x(t + nT, 0, 0)| < \varepsilon \quad \text{if } t \geqslant P_1 T \text{ and } n \geqslant 1$$

or

$$|x(t + PT, 0, 0) - x(t + PT + nT, 0, 0)| < \varepsilon$$

if $t \geqslant 0$, $P \geqslant P_1$, and $n \geqslant 1$. Thus, if $n, m > P_1$ then

$$|x(t + nT, 0, 0) - x(t + mT, 0, 0)| < \varepsilon \quad \text{for all } t \geqslant 0\,;$$

that is, $\{x_n(t)\}$ is a Cauchy sequence converging uniformly on $[0, \infty)$. Also, for every fixed integer k we have

$$|x(t + (n + k)T, 0, 0) - x(t + (m + k)T, 0, 0)| < \varepsilon$$

on $[-kT, \infty)$ so that $\{x_n(t)\}$ converges uniformly on compact subsets of R to some continuous function $z(t)$. Certainly, $z(t)$ is bounded.

We now show that $z(t)$ satisfies (9) on $(-\infty, \infty)$. Consider the derivative of $z(t)$. We have

$$x_n'(t) = A(t)x_n(t) + \int_{t-r}^{t} C(t, s)x_n(s)\, ds + p(t)$$

for $t > -nT$ and $x_n(t) \equiv 0$ for $-nT - r \leqslant t \leqslant -nT$. Let $[a, b]$ be an arbitrary compact subset of R. Let $\delta > 0$ be given and find N such that $n, m > N$ and $a - r \leqslant$ $\leqslant s \leqslant b$ imply $|x_n(s) - x_m(s)| < \delta$ and $-NT < a$. Then for $n, m \geqslant N$ and $t \in [a, b]$ we have

$$|x_n'(t) - x_m'(t)| \leqslant |A(t)||x_n(t) - x_m(t)| + \int_{t-r}^{t} |C(t, s)||x_n(s) - x_m(s)|\, ds \leqslant$$
$$\leqslant \left(|A(t)| + \int_{t-r}^{t} |C(t, s)|\, ds\right) \delta\,.$$

This shows that $\{x_n'(t)\}$ is a Cauchy sequence converging uniformly on compact subsets of R. As the $x_n'(t)$ are continuous, it follows that the $x_n'(t) \to z'(t)$ on all of R.

Since $|C(t, s)x_n(s)| \leqslant M|C(t, s)|$ for $t \geqslant -nT$, by the Lebesgue dominated convergence theorem, on any fixed interval $[-L, \infty)$, we may take he limit as $n \to \infty$ in

$$x_n'(t) = A(t)x_n(t) + \int_{t-r}^{t} C(t, s)x_n(s)\, ds + p(t)$$

to obtain

$$z'(t) = A(t)z(t) + \int_{t-r}^{t} C(t, z)z(s)\, ds + p(t)\,.$$

That relation holds on every interval $[-L, \infty)$ and hence on $(-\infty, \infty)$. Thus, $z(t)$ is the one and only bounded solution of (9) on $(-\infty, \infty)$. Note that $z(t + T)$ is also a bounded solution and so $z(t) = z(t + T)$. This completes the proof.

4. – Equations with unbounded delay.

In this section we consider a system of functional differential equations

$$x' = G(t, x(s)\,; \alpha \leqslant s \leqslant t)\,, \qquad -\infty \leqslant \alpha \leqslant 0\,. \tag{8}$$

To specify a solution of (8) we require a $t_0 \geqslant \alpha$ and a bounded continuous function $\varphi\colon [\alpha, t_0] \to R^n$; we then obtain a continuous solution $x(t, t_0, \varphi)$ satisfying (8) on an interval $[t_0, t_0 + \beta)$ with $x(t, t_0, \varphi) = \varphi(t)$ for $\alpha \leqslant t \leqslant t_0$. To prove the existence of such a solution we need a bit more than just the requirement that when $\psi\colon [\alpha, \infty) \to R^n$ is continuous, then $G(t, \psi(\cdot))$ is continuous. We also need to ask that if $t \geqslant t_0$, if $\varphi\colon [\alpha, t] \to R^n$, and if $\{\psi_n\}$ is a sequence of bounded continuous functions $\psi_n\colon [\alpha, t] \to R^n$ with $\psi_n \to \varphi$ in the supremum norm, then

$$|G(t, \psi_n(\cdot)) - G(t, \varphi(\cdot))| \to 0 \qquad \text{as } n \to \infty\,.$$

If, in addition, G satisfies a local Lipschitz condition in x, then the solution is uniquely determined by the initial function φ. For details see DRIVER [13] or BURTON [7; Chapter 8].

We will shortly be requiring unbounded initial functions, but so much more is required for existence, uniqueness, and continuity in initial functions that separate treatment is advisable. To make the presentation here parallel that for finite delay equations, for each $t > \alpha$ we consider the function space $C(t)$ with $\varphi \in C(t)$ if $\varphi\colon [\alpha, t] \to R^n$ is bounded and continuous. The norm used is the supremum norm, $\|\cdot\|$. Thus, for any $t_0 > \alpha$, our initial function is some $\varphi \in C(t_0)$ and our definitions of stability and boundedness coincide with the ones for bounded delay. A Liapunov functional is denoted by $V(t, x(\cdot))$. We then have the following result which is the counterpart of Theorem 1 for ordinary differential equations and Theorem 4 for finite delay equations.

THEOREM 8. – Let $H > 0$ and for each $t_0 > \alpha$ let $C_H(t_0) \subset C(t_0)$ with $\varphi \in C_H(t_0)$ if $\|\varphi\| < H$. Suppose that for each $t_0 > \alpha$ the function $V\colon [t_0, \infty) \times C_H(t_0) \to [0, \infty)$ is continuous and locally Lipschitz in φ. Let the W_i be wedges.

(*a*) If

$$V(t, 0) = 0\,, \qquad W(|\varphi(t)|) \leqslant V(t, \varphi(\cdot))\,, \qquad V'_{(8)}(t, x(\cdot)) \leqslant 0$$

then the zero solution of (8) is stable.

(*b*) If

$$W_1(|\varphi(t)|) \leqslant V(t, \varphi(\cdot)) \leqslant W_2(\|\varphi\|)\,, \qquad V'_{(8)}(t, x(\cdot)) \leqslant 0$$

then $x = 0$ is uniformly stable.

(*c*) Let $G(t, x(\cdot))$ be bounded whenever $x \in C_H(t)$. If

$$W_1(|\varphi(t)|) \leqslant V(t, \varphi(\cdot))\,, \qquad V(t, 0) = 0\,,$$

and

$$V'_{(8)}(t, x(\cdot)) \leqslant - W_3(|x(t)|)$$

then $x = 0$ is equi-asymptotically stable.

(*d*) If there is a bounded continuous $\Phi\colon [0, \infty) \to [0, \infty)$ which is $L^1[0, \infty)$ with

$$W_1(|\varphi(t)|) \leqslant V(t, \varphi(\cdot)) \leqslant W_2(|\varphi(t)|) + W_3\Big[\int_\alpha^t \Phi(t-s)\, W_4(|\varphi(s)|)\, ds\Big]$$

and

$$V'_{(8)}(t, x(\cdot)) \leqslant - W_5(|x(t)|)$$

then $x = 0$ is uniformly asymptotically stable.

(*e*) Let $M > 0$, $H = \infty$, Φ be as in (*d*) with $\Phi'(t) \leqslant 0$. If

$$W_1(|\varphi(t)|) \leqslant V(t, \varphi(\cdot)) \leqslant W_2(|\varphi(t)|) + W_3\Big[\int_\alpha^t \Phi(t-s)\, W_4(|\varphi(s)|)\, ds\Big]$$

and

$$V'_{(8)}(t, x(\cdot)) \leqslant - W_4(|x(t)|) + M$$

then solutions of (8) are uniform bounded and uniform ultimate bounded for bound B.

PROOF. – Parts (*a*)-(*c*) are classical (cf. DRIVER [13]). The UB of (*e*) extends work of BURTON-HUANG-MAHFOUD [10]. Our proof of (*d*) requires a lemma from BURTON [6].

LEMMA 2. – Let $\{x_n\}$ be a sequence of continuous functions with continuous derivatives, $x_n\colon [0,1] \to [0,1]$. Let $g\colon [0,\infty) \to [0,\infty)$ be continuous, $g(0)=0$, $g(r)>0$ if $r>0$, and let g be nondecreasing. If there exists $\nu>0$ with $\int_0^1 x_n(t)\,dt \geq \nu$ for all n, then there exists $\beta>0$ with $\int_0^1 g(x_n(t))\,dt \geq \beta$ for all n.

We remark that the lemma is certainly valid if $x_n\colon [a,b] \to [c,d]$ with $\int_a^b x_n(t)\,dt \geq \nu$ implying that $\int_a^b g(x_n(t))\,dt \geq \beta$.

We now prove part (*d*). Let $\varepsilon>0$ be given. If $t_0 \geq \alpha$ and $\varphi \in C_\delta(t_0)$ with $\delta<\varepsilon$, then for $x(t,t_0,\varphi)=x(t)$ we have

$$\begin{aligned} W_1(|x(t)|) &\leq V(t,x(\cdot)) \leq V(t_0,\varphi(\cdot)) \leq W_2(\delta) + W_3\Big(W_4(\delta)\int_\alpha^{t_0}\Phi(t_0-s)\,ds\Big) = \\ &= W_2(\delta) + W_3\Big(W_4(\delta)\int_0^{t_0-\alpha}\Phi(u)\,du\Big) \leq W_2(\delta) + W_3\Big(W_4(\delta)\int_0^{\infty}\Phi(u)\,du\Big) < W_1(\varepsilon) \end{aligned}$$

if δ is small enough. This proves the uniform stability.

For $\varepsilon=H$ find δ of uniform stability and let $\delta=\eta$. Let $\nu>0$ be given. We must find $T>0$ such that $[t_0\geq\alpha,\ \varphi\in C_\eta(t_0),\ t\geq t_0+T]$ imply that $|x(t,t_0,\varphi)|<\nu$.

For this $\nu>0$, find $\theta>0$ such that

$$W_2[W_4^{-1}(\theta/J)] + W_3(2\theta) < W_1(\nu)\,,$$

where we let $\Phi(t)\leq J$ for $t\geq 0$. Now, find $r>1$ with

$$W_4(\varepsilon)\int_r^\infty \Phi(u)\,du < \theta\,.$$

If $\varphi\in C_\eta(t_0)$, $t_0\geq\alpha$, $t\geq t_0+r$, then for $x(t)=x(t,t_0,\varphi)$ we have

$$\begin{aligned} W_1(|x(t)|) \leq V(t,x_t) &\leq W_2(|x(t)|) + W_3\Big[\int_\alpha^t \Phi(t-s)\,W_4(|x(s)|)\,ds\Big] \leq \\ &\leq W_2(|x(t)|) + W_3\Big[\int_\alpha^{t-r}\Phi(t-s)\,W_4(\varepsilon)\,ds + \int_{t-r}^t \Phi(t-s)\,W_4(|x(s)|)\,ds\Big] \leq \\ &\leq W_2(|x(t)|) + W_3\Big(W_4(\varepsilon)\int_r^{t-\alpha}\Phi(u)\,du + \int_{t-r}^t J\,W_4(|x(s)|)\,ds\Big) \leq \\ &\leq W_2(|x(t)|) + W_3\Big(W_4(\varepsilon)\int_r^{\infty}\Phi(u)\,du + J\int_{t-r}^t W_4(|x(s)|)\,ds\Big) \leq \\ &\leq W_2(|x(t)|) + W_3\Big(\theta + J\int_{t-r}^t W_4(|x(s)|\,ds\Big)\,. \end{aligned}$$

We will find a t_1 with $|x(t_1)| < W_4^{-1}(\theta/Jr) \overset{\text{def}}{=} \varepsilon_1$ and $\int_{t_1-r}^{t_1} W_4(|x(s)|)\,ds \leqslant \theta/J$. This will mean that

$$W_1(|x(t)|) \leqslant V(t, x_t) \leqslant V(t_1, x_{t_1}) < W_2(W_4^{-1}(\theta/Jr)) + W_3(2\theta) < W_1(\nu) \quad \text{for } t > t_1 .$$

Because $V'(t, x(\cdot)) \leqslant -W_5(|x|)$, there is a $T_2 \geqslant r$ such that $|x(t)| \geqslant W_4^{-1}(\theta/Jr) = \varepsilon_1$ fails for some value of t on every interval of length T_2. Hence, there exists $\{t_n\} \to \infty$ such that $|x(t_n, t_0, \varphi)| < \varepsilon_1$. In particular, we choose

$$t_n \in [t_0 + (n-1)T_2,\, t_0 + nT_2] \quad \text{for } n = 2, 3, \ldots .$$

The length of the intervals is independent of t_0 and $\varphi \in C_\eta(t_0)$.

Now, consider the sequence of functions $\{x_k(t)\}$ defined by

$$x_k(t) = x(t, t_0, \varphi) \quad \text{for } t_k - r \leqslant t \leqslant t_k .$$

Examine those members satisfying

$$(*) \qquad \int_{t_k-r}^{t_k} J W_4(|x(s)|)\,ds \geqslant \theta$$

so that by Lemma 2 we have

$$\int_{t_k-r}^{t_k} W_5(|x(s)|)\,ds \geqslant \beta$$

for some $\beta > 0$.

Next,

$$V(t_0, \varphi) \leqslant W_2(\varepsilon) + W_3\left[\int_{\alpha}^{t_0} \Phi(t_0 - s)\, W_4(\delta)\,ds\right] \leqslant W_2(\varepsilon) + W_3\left(W_4(\delta)\int_0^{\infty} \Phi(u)\,du\right) \overset{\text{def}}{=} \mu$$

is a positive number independent of $t_0 \geqslant \alpha$ and independent of $\varphi \in C_\eta(t_0)$. Now, for $t > t_{2n}$ we have

$$V'(t, x(\cdot)) \leqslant -W_5(|x|)$$

so that

$$V(t, x(\cdot)) \leqslant V(t_0, \varphi) - \int_{t_0}^{t} W_5(|x(s)|)\,ds \leqslant \mu - \sum_{i=1}^{n} \int_{t_{2i}-r}^{t_{2i}} W_5(|x(s)|)\,ds \leqslant \mu - n\beta < 0$$

if $n \geqslant \mu/\beta$. (Here, we have integrated over alternate intervals to be sure the intervals are disjoint.) Hence, if $n > \mu/\beta$, then t_n fails to exist with $(*)$ holding. We choose n as the smallest integer greater than μ/β and we then have

$$|x(t_n, t_0, \varphi)| < \varepsilon_1$$

and

$$\int_{t_n-r}^{t_n} J W_4(|x(s)|)\, ds < 0 .$$

Thus, if

$$T = 2nT_2$$

then $t > t_0 + T$ implies $|x(t, t_0, \varphi)| < \nu$. This completes the proof of (d).

We now prove (e). Let $B_1 > 0$ be given and suppose $\varphi \in C_{B_1}(t_0)$ for arbitrary $t_0 \geqslant \alpha$. Let $x(t) = x(t, t_0, \varphi)$ and $V(t) = V(t, x(\cdot))$. For $t_0 \leqslant s \leqslant t < \infty$ we have

$$V'(s)\,\Phi(t-s) \leqslant [-W_4(|x(s)|) + M]\,\Phi(t-s)$$

so that

$$\begin{aligned}\int_{t_0}^{t} W_4(|x(s)|)\,\Phi(t-s)\, ds &\leqslant -\int_{t_0}^{t} V'(s)\,\Phi(t-s)\, ds + M\int_{t_1}^{t} \Phi(t-s)\, ds = \\ &= -\Big[V(s)\,\Phi(t-s)\big|_{t_0}^{t} + \int_{t_0}^{t} V(s)\,\Phi'(t-s)\, ds\Big] + M\int_{0}^{t-t_0} \Phi(u)\, du \leqslant \\ &\leqslant -V(t)\,\Phi(0) + V(t_0)\,\Phi(t-t_0) + V(\tau)[\Phi(0) - \Phi(t-t_0)] + MJ\end{aligned}$$

where $t_0 \leqslant \tau \leqslant t$ and $J = \int_0^\infty \Phi(u)\, du$.

Suppose there is a t with $V(t) \geqslant V(s)$ for $t_0 \leqslant s \leqslant t$. Either $t = t_0$ or $V'(t) \geqslant 0$. If $t = t_0$ we have

$$\begin{aligned}W_1(|x(t)|) \leqslant V(t, x(\cdot)) \leqslant V(t_0, \varphi) \leqslant W_2(B_1) + W_3\Big[\int_{\alpha}^{t_0} \Phi(t_0 - s)\, W_4(B_1)\, ds\Big] &\leqslant \\ \leqslant W_2(B_1) + W_3(W_4(B_1)J)&\end{aligned}$$

so that

$$|x(t)| \leqslant W_1^{-1}[W_2(B_1) + W_3(W_4(B_1)J)] \stackrel{\text{def}}{=} B_2^* .$$

If $V'(t) \geqslant 0$ then $|x(t)| \leqslant W_4^{-1}(M)$ and

$$\int_{t_0}^{t} W_4(|x(s)|)\,\Phi(t-s)\, ds \leqslant V(t_0)\,\Phi(t-t_0) + MJ$$

so that at the maximum of V we have

$$\begin{aligned}W_1(|x(t)|) &\leqslant V(t, x(\cdot)) \leqslant W_2(W_4^{-1}(M)) + \\ &+ W_3\Big[\int_{\alpha}^{t_0} \Phi(t-s)\, W_4(|\varphi(s)|)\, ds + V(t_0)\,\Phi(t-t_0) + MJ\Big] \leqslant \\ &\leqslant W_2(W_4^{-1}(M)) + W_3[W_4(B_1)J + \{W_2(B_1) + W_3(W_4(B_1)J)\}\,\Phi(t-t_0) + MJ]\end{aligned}$$

or

$$|x(t)| \leqslant W_1^{-1}\Big[W_2(W_4^{-1}(M)) + W_3[W_4(B_1)J + \\ + \{W_2(B_1) + W_3(W_4(B_1)J)\}\Phi(0) + MJ]\Big] \stackrel{\text{def}}{=} B_2^{**}.$$

We then have

$$|x(t)| \leqslant B_2 = \max[B_2^*, B_2^{**}],$$

and this is uniform boundedness.

To prove the UUB, choose $U > 0$ so that $|x| \geqslant U$ implies $V'(t, x(\cdot)) \leqslant -W_4(|x|) + M < -1$. Let $B_3 > 0$ be given and find B_4 such that $[t_0 \geqslant \alpha, \varphi \in C(t_0), \|\varphi\| \leqslant B_3, t \geqslant t_0]$ imply that $|x(t, t_0, \varphi)| < B_4$. Find $T > \alpha$ with $W_4(B_4)\int_T^\infty \Phi(u)\,du < 1$. For $t_0 \geqslant \alpha$ and $x(t) = x(t, t_0, \varphi)$ with $\|\varphi\| \leqslant B_3$, if $t \geqslant t_0 + T$ and if $\int_0^\infty \Phi(u)\,du = J$, then

$$\begin{aligned}\int_{t-T}^{t} \Phi(t-s)\,W_4(|x(s)|)\,ds &\leqslant -\int_{t-T}^{t} V'(s)\,\Phi(t-s)\,ds + M\int_{t-T}^{t}\Phi(t-s)\,ds = \\ &= -\Big[V(s)\,\Phi(t-s)\big|_{t-T}^{t} + \int_{t-T}^{t} V(s)\,\Phi'(t-s)\,ds\Big] + MJ = \\ &= -V(t)\,\Phi(0) + V(t-T)\,\Phi(T) + V(\tau)[\Phi(0) - \Phi(T)] + MJ\end{aligned}$$

where $V(\tau)$ is the maximum of $V(s)$ on $[t-T, t]$. This yields

$$(*) \qquad \int_{t-T}^{t} \Phi(t-s)\,W_4(|x(s)|)\,ds \leqslant -V(t)\,\Phi(0) + V(\tau)\,\Phi(0) + MJ.$$

Consider the intervals

$$I_1 = [t-T, t], \quad I_2 = [t, t+T], \quad I_3 = [t+T, t+2T], \quad \ldots$$

and select $t_i \in I_i$ such that $V(t_i)$ is the maximum of $V(s)$ on I_i, unless t_i is the left end-point of I_i with $|x(t_i)| > U$; in the exceptional case we determine a point $\bar{t}_i > t_i$ such that $|x(\bar{t}_i)| = U$ and $V'(t) < 0$ on $[t_i, \bar{t}_i]$. To accomplish this we may suppose T is so large that such a $\bar{t}_i$ exists on I_i because

$$(**) \qquad V(t) \leqslant W_2(B_4) + W_3(1 + W_4(B_4)J)$$

and $V'(t) \leqslant -1$ if $|x| \geqslant U$. Now, replace I_i by the interval having $\bar{t}_i$ as its left end-point and $t + (i-1)T$ as its right end-point. Call the exceptional interval I_i again and select t_i in I_i with $V(t_i)$ the maximum on I_i.

Now, consider the intervals

$$L_2 = [t_2 - T, t_2], \quad L_3 = [t_3 - T, t_3], \quad \ldots .$$

Note that when $V(t_i)$ is the maximum on I_i, then $V'(t_i) \geq 0$ so $|x(t_i)| \leq U$. Next, consider each i:

Case 1. – Suppose $V(t_i) + 1 \geq V(s)$ for all $s \in L_i$.

Case 2. – Suppose $V(t_i) + 1 < V(s_i)$ for some $s_i \in L_i$.

Note that in Case 2 we may conclude that $s_i \in I_{i-1}$ and $V(t_i) + 1 < V(t_{i-1})$. (This is true because $V(t_i)$ is maximum on I_i, so $s_i \in I_{i-1}$; in the exceptional case where we selected $\bar{t}_i$, if s_i is in $(t_i, \bar{t}_i)$, (note here that t_i is the original one!) then $s_i = t_i$ also qualifies because V decreases on $(t_i, \bar{t}_i)$ and $t_i \in I_{i-1}$.) This means that if $V_i = \sup_{t \in I_i} V(t)$, then $V_i + 1 < V_{i-1}$. Because of (**), there is an integer P such that Case 2 can hold on no more than P consecutive intervals.

Thus, on some L_j with $j \leq P$ we have $V(t_j) + 1 \geq V(s)$ for all $s \in L_j$. This means that if $V(\tau)$ is the maximum of V on L_j, then $V(\tau) \leq V(t_j) + 1$ so that from (*) we have (for $t = t_j$)

$$\int_{t-T}^{t} \Phi(t-s) W_4(|x(s)|)\, ds \leq -[V(t_j) + 1]\Phi(0) + V(\tau)\Phi(0) + MJ + \Phi(0) \leq MJ + \Phi(0) .$$

Now, recall that $|x(t_j)| \leq U$ so $V(t_j) \leq W_2(U) + W_3(1 + \Phi(0) + MJ)$ and at the maximum of V, $V(\tau)$, we have $V(\tau) \leq V(t_j) + 1$. We claim that

$$V(t) \leq W_2(U) + W_3(1 + \Phi(0) + MJ) + 1$$

for all $t \geq t_j$; to see this, let t_p be the first $t > t_j$ with $V(t_p) = V(\tau)$. Then notice that $V(t_p)$ is the maximum of V on $[t_p - T, t_p]$. For $t = t_p$ this yields

$$\int_{t_p - T}^{t_p} W_4(|x(s)|)\, \Phi(t_p - s)\, ds \leq -V(t_p)\Phi(0) + V(t_p)\Phi(0) + MJ = MJ$$

so that

$$V(t_p) \leq W_2(U) + W_3(1 + MJ) ,$$

as required. Moreover, if there is a $t > t_p$ with $V(t) = \max_{t_p \leq s \leq t} V(s)$, then the same bound holds.

Hence, for $t \geq t_1 + T + PT$ we have

$$W_1(|x|) \leq V(t) \leq W_2(U) + W_3(1 + \Phi(0) + MJ) + 1$$

or

$$|x(t)| \leq W_1^{-1}[W_2(U) + W_3(1 + \Phi(0) + MJ) + 1] \overset{\text{def}}{=} B .$$

This completes the proof of UB and UUB. In fact, this completes the proof of the theorem.

EXAMPLE 4. – Consider the scalar equation

$$x' = -x^3 + \int_{-\infty}^{t} C(t-s)\, v(s, x(s))\, ds + f(t)$$

where C, f, and v are continuous, $C \in L^1[0, \infty)$ and $\Phi(t) = \int_t^\infty |C(u)|du \in L^1[0, \infty)$, $|f(t)| \leq P$, $|v(t, x)| \leq \nu + \beta x^2$ with ν, β, and P positive constants. Then solutions are UB and UUB.

PROOF. – Let $\varphi\colon (-\infty, t_0] \to R$ be a bounded continuous initial function and $x(t) = x(t, t_0, \varphi)$. Define

$$V(t, x(\cdot)) = |x| + \beta \int_{-\infty}^{t} \int_t^\infty |C(u-s)|\, du\, x^2(s)\, ds$$

so that, after a calculation, we find

$$V'(t, x(\cdot)) \leq -|x^3| + C_1 x^2 + C_2 \leq -\delta x^2 + K$$

for some positive numbers δ and K. The result now follows from Theorem 8 (e).

In part (e) of Theorem 8 we can replace $\Phi(t-s)$ by a more general $\Phi(t, s)$ as follows.

THEOREM 8 $(e)'$. – Let V be as in Theorem 8 and suppose there is a continuous scalar function $\Phi(t, s)$ defined for $\alpha \leq s \leq t < \infty$, $\partial\Phi(t, s)/\partial s \geq 0$, $0 \leq \Phi(t, s) \leq J$ for some constant $J > 0$, and $\int_\alpha^t \Phi(t, s)\, ds \leq L$ for all $t \geq \alpha$ and some constant L. Suppose also that for some $M > 0$ we have

$$\text{(i)} \qquad W_1(|x(t)|) \leq V(t, x(\cdot)) \leq W_2(|x(t)|) + W_3\Big[\int_\alpha^t \Phi(t, s)\, W_4(|x(s)|)\, ds\Big]$$

and

$$\text{(ii)} \qquad V'_{(8)}(t, x(\cdot)) \leq -W_4(|x(t)|) + M .$$

Then solutions of (8) are UB and UUB.

Since the proof is so similar to that of Theorem 8 (e) it will not be given here.

As a companion to Theorems 3 and 7 we mention the following result which appears in BURTON [3]. Consider again

$$x' = A(t)x + \int_{-\infty}^{t} C(t, s)x(s)\, ds + p(t) \tag{5}$$

with $A(t + T) = A(t)$, $p(t + T) = p(t)$, and $C(t + T, s + T) = C(t, s)$ for some $T > 0$. This means that if $x(t)$ is a solution so is $x(t + T)$.

THEOREM 9. – Suppose that $\int_{-\infty}^{t} |C(t, s)|\, ds$ is continuous on $(-\infty, \infty)$ and that for each $\delta > 0$ there exists $S > 0$ such that $t - t_1 \geq S$ implies that $\int_{-\infty}^{t_1} |C(t, s)|\, ds \leq \delta$. Suppose also that:

(i) If (5) has a solution satisfying (5) on $(-\infty, \infty)$ which is bounded, then it is uniformly asymptotically stable.

(ii) The solution $x(t, 0, 0)$ of (5) is bounded on $[0, \infty)$ and is equi-asymptotically stable at $t_0 = 0$.

Under these conditions (5) has a T-periodic solution.

Many other results on the existence of periodic solutions of functional differential equations with infinite delay are found in ARINO-BURTON-HADDOCK [1], BURTON ([2], [3], [4], [5], [8]), FURUMOCHI ([14], [15]), LANGENHOP [21], WANG [23], and WU-LI-WANG [25].

5. – Unbounded initial functions.

In this section we consider a system with infinite delay

$$x' = F(t, x(s);\ -\infty < s \leq t)\,, \qquad t \geq 0\,. \tag{10}$$

Such systems have been discussed extensively in recent years and there are several formulations of axioms for the state space. (Cf. HALE and KATO [16], KAMINOGO [18], KAPPEL and SCHAPPACHER [19], and SAWANO [22].) Those treatments tend to ask that exponentially unbounded initial functions be allowed and there would seem to be some shortage of motivation for the unbounded initial functions.

There are at least two urgent reasons for unbounded initial functions. First, they form a foundation for establishing compact subsets of initial functions; and this is essential for the use of many fixed point theorems, particularly in the search for periodic solutions. One needs a weighted norm for compactness, but then one needs continuity in initial functions in the weighted norm which automatically admits unbounded initial functions. The second reason is that when we apply a translation operator (Poincaré map), we can not map a solution back into its initial function set unless the initial function set is unbounded. Details are found in ARINO-BURTON-HADDOCK [1].

But the point which is missed so often in the axiomization process is that the differential equation itself provides the guide for the permissible growth of the initial function. We consider, for example, the scalar linear convolution equation

$$x' = Ax + \int_{-\infty}^{t} C(t-s)x(s)\,ds$$

in which $C \in L^1[0, \infty)$ and continuous. We require a continuous initial function $\varphi\colon (-\infty, t_0] \to R$ such that $\int_{-\infty}^{t_0} C(t-s)\varphi(s)\,ds$ is continuous for $t \geq t_0$; it is then possible to prove that there is a unique solution $x(t, t_0, \varphi)$ satisfying the equation for $t \geq t_0$ and with $x(t, t_0, \varphi) = \varphi(t)$ for $t \leq t_0$: For simplicity we take $t_0 = 0$. Now $\int_{-\infty}^{0} |C(t-s)|\,ds = \int_{t}^{\infty} |C(u)|\,du$ and it is shown in BURTON-GRIMMER [9] that there is a continuous increasing function $g\colon [0, \infty) \to [1, \infty)$ with $g(0) = 1$ and $g(r) \to \infty$ as $r \to \infty$ such that $\int_{0}^{\infty} |C(u)| g(u)\,du < \infty$. If we take $\varphi\colon (-\infty, 0] \to R$ with $|\varphi(t)| \leq \nu g(-t)$, then $\int_{-\infty}^{0} C(t-s)\varphi(s)\,ds$ will be continuous.

To completely develop the material of existence, uniqueness, and continual dependence of solutions on initial conditions using unbounded initial functions requires much space. And this has been done in ARINO-BURTON-HADDOCK [1], BURTON ([4], [8]), HALE and KATO [16], KAPPEL and SCHAPPACHER [19]. Thus, for our purposes here we suppose that there is a continuous decreasing function $g\colon (-\infty, 0] \to [1, \infty)$ satisfying $g(0) = 1$ and $g(r) \to \infty$ as $r \to -\infty$ and we consider the Banach space

$$(X, |\cdot|_g)$$

of continuous functions $\varphi\colon (-\infty, 0] \to R^n$ for which

$$|\varphi|_g = \sup_{-\infty < t \leq 0} |\varphi(t)|/g(t)$$

exists.

DEFINITION 7. – We say that a set $Y \subset X$ is *proper* with respect to (10) if $\varphi \in Y$ implies that there is at least one solution $x(t, 0, \varphi)$ satisfying (10) for $0 \leq t < \beta$ for some $\beta > 0$, $x(t, 0, \varphi) = \varphi(t)$ for $t \leq 0$, and if whenever $x(t, 0, \varphi)$ remains bounded then $\beta = \infty$.

We remark that for the system (4), if h and q are continuous and if $\varphi \in X$ implies $\int_{-\infty}^{0} q(t, s, \varphi(s))\,ds$ is continuous for $t \geq 0$ then X is proper with respect to (4).

DEFINITION 8. – Let Y be proper with respect to (10). Then solutions *depend continuously on initial functions* in Y relative to $|\cdot|_g$ if for each $\varphi \in Y$, for each $J > 0$, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(t, 0, \varphi)$ is defined on $[0, J]$ and if $\psi \in Y$ with $|\varphi - \psi|_g < \delta$, then

$$|x(t, 0, \varphi) - x(t, 0, \psi)| < \varepsilon \quad \text{for } 0 \leqslant t \leqslant J .$$

DEFINITION 8. – Solutions of (10) are *g-uniform bounded* if X is proper with respect to (10) and if for each $B_1 > 0$ there is a $B_2 > 0$ such that $[\varphi \in X, |\varphi|_g \leqslant B_1, t \geqslant 0]$ imply that $|x(t, 0, \varphi)| < B_2$.

DEFINITION 9. – Solutions of (10) are *g-uniform ultimate bounded for bound B* if X is proper with respect to (10) and if for each $B_3 > 0$ there exists $K > 0$ such that $[\varphi \in X, |\varphi|_g < B_3, t \geqslant K]$ imply that $|x(t, 0, \varphi)| < B$.

THEOREM 10. – Let $(X, |\cdot|_g)$ be given and let X be proper with respect to (10). Let $V(t, x(\cdot))$ be continuous and locally Lipschitz in x. Suppose also that there exists $\Phi\colon [0, \infty) \to [0, \infty)$ with $\Phi \in L^1[0, \infty)$ and $\Phi'(t) \leqslant 0$ such that:

$$\text{(i)} \qquad \int_{-\infty}^{0} \Phi(t - s)\, W_4(\nu g(s))\, ds \stackrel{\text{def}}{=} H(t, \nu)$$

is continuous and $\Phi(u)\, W_4(\nu g(-u)) \in L^1[0, \infty)$ for each $\nu > 0$;

$$\text{(ii)} \quad W_1(|x(t)|) \leqslant V(t, x(\cdot)) \leqslant W_2(|x(t)|) + W_3\Big[\int_{-\infty}^{t} \Phi(t - s)\, W_4(|x(s)|)\, ds\Big]$$

whenever $x\colon (-\infty, \infty) \to R^n$ is continuous and x restricted to $(-\infty, 0]$ is in X;

$$\text{(iii)} \qquad V'_{(10)}(t, x(\cdot)) \leqslant -W_4(|x(t)|) + M .$$

Under these conditions solutions of (10) are g-uniform bounded and g-uniform ultimate bounded for bound B.

PROOF. – Given $\varphi \in X$ we consider $x(t) = x(t, 0, \varphi)$ and notice that so long as the solution is defined we have

$$W_1(|x(t)|) \leqslant V(t, x(\cdot)) \leqslant V(0, \varphi(\cdot)) + Mt$$

so that $|x(t)|$ is bounded for t bounded and so $x(t)$ exists on $[0, \infty)$.

Note that by (iii) there is a $U > 0$ such that $V'_{(10)}(t, x(\cdot)) < 0$ for $|x| > U$. Now, let $B_1 > 0$ be given and let $\varphi \in X$ with $|\varphi|_g \leqslant B_1$. Suppose there is a $t > 0$ with $V(t) = \max_{0 \leqslant s \leqslant t} V(s)$, where $V(t) = V(t, x(\cdot))$. Then for $0 \leqslant s \leqslant t$ we have the relation

$$V'(s)\, \Phi(t - s) \leqslant [-W_4(|x(s)|) + M]\, \Phi(t - s)$$

so that

$$\int_0^t \Phi(t-s)\,W_4(|x(s)|)\,ds \leqslant -\int_0^t V'(s)\,\Phi(t-s)\,ds + MJ \leqslant V(0)\,\Phi(0) + MJ \leqslant$$
$$\leqslant \Phi(0)[W_2(B_1) + W_3(H(0, B_1)]) + MJ\,,$$

where $J = \int_0^\infty \Phi(s)\,ds$. Now if $V(t)$ is the maximum on $[0, t]$ with $t > 0$, then $V'(t) \geqslant 0$ so $|x(t)| < U$ and this yields

$$V(t) \leqslant W_2(U) + W_3\Big(H(0, B_1) + \int_0^t \Phi(t-s)\,W_4(|x(s)|)\,ds\Big) \leqslant$$
$$\leqslant W_2(U) + W_3\big(H(0, B_1) + \Phi(0)[W_2(B_1) + W_3(H(0, B_1))] + MJ\big) \stackrel{\text{def}}{=} V_m\,.$$

Thus, for all $t \geqslant 0$ we have

$$W_1(|x(t)|) \leqslant V(t, x(\cdot)) \leqslant V(0) + V_m \leqslant W_2(B_1) + W_3(H(0, B_1)) + V_m \stackrel{\text{def}}{=} W_1(B_2)$$

proving g-uniform boundedness.

To prove the g-uniform ultimate boundedness we first note that

$$H(t, \nu) = \int_{-\infty}^0 \Phi(t-s)\,W_4(\nu g(s))\,ds = \int_t^\infty \Phi(u)\,W_4(\nu g(t-u))\,du \leqslant \int_t^\infty \Phi(u)\,W_4(\nu g(-u))\,du$$

and this tends to zero as $t \to \infty$.

Now, given $B_3 > 0$, if $|\varphi|_g \leqslant B_3$, then there is a B_4 as a bound on $|x(t, 0, \varphi)|$. Then for $T > 0$ and $t > T$ we have

$$\int_{-\infty} \Phi(t-s)\,W_4(|x(s, 0, \varphi)|)\,ds \leqslant \int_{-\infty}^0 \Phi(t-s)\,W_4(B_3 g(s))\,ds +$$
$$+ \int_0^{t-T} \Phi(t-s)\,W_4(B_4)\,ds + \int_{t-T}^t \Phi(t-s)\,W_4(|x(s, 0, \varphi)|)\,ds \leqslant H(t, B_3) + \int_T^\infty \Phi(u)\,W_4(B_4)\,du +$$
$$+ \int_{t-T}^t \Phi(t-s)\,W_4(|x(s, 0, \varphi|)\,ds < 1 + \int_{t-T}^t \Phi(t-s)\,W_4(|x(s, 0, \varphi)|)\,ds$$

if T (and, hence, t) is sufficiently large. The remainder of the proof is identical to that of Theorem 8 (e).

Now, corresponding to Theorem 2′ there is the following result of ARINO-BURTON-HADDOCK [1] whose conditions can frequently be verified by means of our Theorem 10, THEOREM 11. – Let $(X, |\cdot|_g)$ be fixed and suppose that X is proper with respect to (10). Suppose also that the following conditions hold:

1 For each $\varphi \in X$ there is a unique solution $x(t, 0, \varphi)$ of (10) defined on $[0, \infty)$.

2 Solutions of (10 are g-UB and g-UUB for bound B.

3 For each $\nu > 0$ there exists $L > 0$ such that $|\varphi|_g < \nu$ and $t_0 \in R$ imply that $|x'(t, 0, \varphi| < L$.

4 Solutions of (10) are continuous in φ.

5 If $x(t)$ is a solution of (10) so is $x(t+T)$. Under these conditions (10) has a T-periodic solution.

We remark that some really good Liapunov functionals have been constructed for (4), but their application has awaited Theorem 10. The following is an example.

EXAMPLE 4 (Revisited). – Let $f(t+T) = f(t)$ for some $T > 0$ and all t, and $v(s+T, x) = v(s, x)$ for all s and x. Consider the V once more with $\Phi(t-s) = \int_{t-s}^{\infty} |C(u)|\, du$ and $W_4(r) = r^2$. According to BURTON-GRIMMER [9] we can find a function g with $g(t)\int_t^{\infty} |C(u|\, du \in L^1[0, \infty)$. Condition (i) of Theorem 10 is satisfied and the equation of Example 4 has a T-periodic solution by Theorem 11. Continuity of $x(t, 0, \varphi)$ in φ is proven in BURTON [8].

We now present a far more interesting example. To this point we have treated the integral as a perturbation; here, we obtain the boundedness from the integral.

EXAMPLE 5. – Consider the scalar equation

$$x' = \int_{-\infty}^{t} C(t-s)x(s)\, ds + p(t) \tag{11}$$

in which C and p are continuous on $[0, \infty)$ with $\int_0^{\infty} |C(u)|\, du < \infty$, $\int_t^{\infty} |C(u)|\, du \in L^1[0, \infty)$, p bounded, say $|p(t)| \leq P$. Write (11) as

$$x' = -G(0)x + (d/dt)\int_{-\infty}^{t} G(t-s)x(s)\, ds + p(t) \tag{12}$$

where $G'(u) = C(u)$. If $G(0) > 0$, $\int_0^{\infty} |G(u)|\, du < \infty$, $\int_t^{\infty} |G(u)|\, du \in L^1[0, \infty)$, $|G(t)|' \leq 0$, and if $2G(0) > [2G(0) + 1]\int_0^{\infty} |G(u)|\, du + 1$, then solutions of (11) are UB and UUB for bound B.

PROOF. – Construct a functional

$$V(t, x(\cdot)) = \Big(x - \int_{-\infty}^{t} G(t-s)x(s)\, ds\Big)^2 + k\int_{-\infty}^{t} \int_{t-s}^{\infty} |G(u)|\, du\, x^2(s)\, ds\,.$$

Let $x(t) = x(t, 0, \varphi)$ be a solution of (12). Then

$$V'_{(12)}(t, x(\cdot)) = 2\Big(x - \int_{-\infty}^{t} G(t-s)\,x(s)\,ds\Big)\,[-G(0)x + p(t)] + k\int_{0}^{\infty}|G(u)|\,du\,x^2 -$$

$$-k\int_{-\infty}^{t}|G(t-s)|x^2(s)\,ds \leqslant -2G(0)x^2 + G(0)\int_{-\infty}^{t}|G(t-s)|[x^2(s) + x^2(t)]\,ds +$$

$$+\int_{-\infty}^{t}|G(t-s)|[x^2(s) + p^2(t)]\,ds + 2xp(t) + k\int_{0}^{\infty}|G(u)|\,du x^2 - k\int_{-\infty}^{t}|G(t-s)|x^2(s)\,ds \leqslant$$

$$\leqslant\Big[-2G(0) + 1 + (2G(0) + 1)\int_{0}^{\infty}|G(u)|\,du\Big]x^2 + p^2(t)\int_{2}^{\infty}|G(u)|\,du + p^2(t)$$

with $k = G(0) + 1$ so that

$$V'_{(12)}(t, x(\cdot)) \leqslant -\alpha x^2 + M \tag{13}$$

for some $\alpha > 0$ and $M > 0$.

Next, we establish an upper bound on V. We have

$$V(t, x(\cdot)) = x^2 - 2x\int_{-\infty}^{t} G(t-s)x(s)\,ds + \Big[\int_{-\infty}^{t} G(t-s)x(s)\,ds\Big]^2 + k\int_{-\infty}^{t}\int_{t-s}^{\infty}|G(u)|\,du x^2(s)\,ds \leqslant$$

$$\leqslant x^2 + \int_{-\infty}^{t}|G(t-s)|[x^2(s) + x^2(t)]\,ds + \int_{-\infty}^{t}|G(t-s)|\,ds\int_{-\infty}^{t}|G(t-s)||x^2(s)|\,ds +$$

$$+ k\int_{-\infty}^{t}\int_{t-s}^{t}|G(u)|\,du x^2(s)\,ds \leqslant \Big[1 + \int_{0}^{\infty}|G(u)|\,du\Big]x^2 +$$

$$+\int_{-\infty}^{t}\Big[|G(t-s)| + \Big[\int_{0}^{\infty}|G(u)|\,du\Big]|G(t-s)| + k\int_{t-s}^{\infty}|G(u)|\,du\Big]x^2(s)\,ds$$

so that

$$V(t, x(\cdot)) \leqslant Ax^2 + \int_{-\infty}^{t}\Phi(t-s)\,x^2(s)\,ds\,. \tag{14}$$

We seem unable to get a lower wedge for V and so we resort to the following technique. Define a new functional

$$W(t, x(\cdot)) = x^2 + \int_{-\infty}^{t}\int_{t-s}^{\infty}|C(u)|\,du x^2(s)\,ds$$

and compute W' along a solution of (11) (which is also (12)):

$$W'(t, x(\cdot)) = 2x\int_{-\infty}^{t} C(t-s)x(s)\,ds + 2xp(t) + \int_{0}^{\infty}|C(u)|\,du x^2 - \int_{-\infty}^{t}|C(t-s)|x^2(s)\,ds \leqslant$$
$$\leqslant \int_{-\infty}^{t}|C(t-s)|[x^2(s) + x^2(t)]\,ds + 2|x||p(t)| + \int_{0}^{\infty}|C(u)|\,du x^2 - \int_{-\infty}^{t}|C(t-s)|x^2(s)\,ds \leqslant$$
$$\leqslant \Big[2\int_{0}^{\infty}|C(u)|\,du\Big]x^2 + 2|x|P \leqslant Jx^2 + R$$

for some $J > 0$, $R > 0$.

Then define a functional

$$\Gamma(t, x(\cdot)) = V(t, x(\cdot)) + (\alpha/2J)\,W(t, x(\cdot))$$

so that

$$(\alpha/2J)x^2 \leqslant \Gamma(t, x(\cdot)) \tag{15}$$

and

$$\Gamma'(t, x(\cdot)) \leqslant -(\alpha/2)x^2 + (\alpha R/2J) + M\,. \tag{16}$$

Moreover, Γ has an upper bound of the form of (14). Hence, by Theorem 8 solutions are UB and UUB.

COROLLARY. – If, in addition to the conditions of Example 5, $p(t+T) = p(t)$, then solutions are g-UB and g-UUB and there is a T-periodic solution.

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