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Pseudodistances and Pseudometrics on Real and Complex Manifolds (*).

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Summary. – *In this paper we study the relationships between a class of distances and infinitesimal metrics on real and complex manifolds and their behavior under differentiable and holomorphic mappings. Some application to Riemannian and Finsler geometry are given and also new proofs and generalizations of some results of Royden, Harris and Reiffen on Kobayashi and Carathéodory metrics on complex manifolds are obtained. In particular we prove that on every complex manifold (finite or infinite-dimensional) the Kobayashi distance is the integrated form of the corresponding infinitesimal metric.*

0. – Introduction.

In this work we investigate a class of pseudodistances and pseudometrics, which we call admissible, on real or complex manifold modelled on open sets of a locally convex vector topological space, including Riemannian and Finsler metrics on real manifolds and Kobayashi-type and Carathéodory pseudodistances and pseudometrics on complex manifolds.

After defining the derivative of an admissible pseudodistance and the (upper and lower) integrated forms of an admissible pseudometric we prove that for every admissible pseudodistance d on M the integrated form of its derivative is the inner pseudodistance associated to d.

Next we give a characterization of the derivative of a pseudodistance and of the lower integrated form of a pseudometric by means of an extremality property. We investigate the behavior of admissible pseudodistances and pseudometrics under differentiable mappings; some applications to Finsler geometry are given.

We apply these results to the study of Carathéodory and Kobayashi-type pseudodistances and pseudometrics on complex manifolds. Among the Kobayashi-type pseudometrics and pseudodistances the Kobayashi and Hahn pseudometrics and pseudodistances are included.

The main result is that every Kobayashi-type pseudodistance is the integrated form of the corresponding pseudometric and symmetrically, the Carathéodory pseudometric is the derivative of the corresponding pseudodistance, improving and unifying particular results of Royden and Harris for the Kobayashi pseudometric and of Reiffen and Harris for the Carathéodory pseudometric.

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It is known that there are manifolds for which the Carathéodory pseudodistance is not inner, and therefore it is not the integrated form of the relative pseudometric. We give here the example of a domain for which the Kobayashi pseudometric is not the derivative of the corresponding pseudodistance.

We investigate the integrability of Kobayashi-type pseudometrics along curves on manifolds with countable base modelled on a separable Fréchet space.

We prove similar results for pseudometrics and pseudodistances arising in the study of projective mappings on manifolds endowed with an affine connection.

Finally we prove that a lower semicontinuous convex positive admissible metric on a finite dimensional connected real manifold is the derivative of its integrated form extending a result of Busemann and Mayer.

§ 1. – Let E be a Hausdorff locally convex topological vector space and let M be a connected manifold modelled on open domains of E.

For every x in M a *local coordinate system at* x is by definition a pair (U, φ), where U is an open neighbourhood of x in M and φ a differentiable (holomorphic) homeomorphic mapping between U and an open convex balanced neighbourhood of the origin B in E such that $\varphi(x) = 0$.

We call *E-pseudodistance*, or *admissible pseudodistance* on M a mapping

$$d\colon M\times M \to \boldsymbol{R}_+ = [0, +\infty)$$

which is a pseudodistance on M (that is a mapping which is symmetric, satisfies the triangle inequality and verifies $d(x, x) = 0$ for every x in M) such that for every x in M there exist a local coordinate system (U, φ) at x, an open neighbourhood V of x in U and a positive constant C such that

$$d(y, z) \leqslant Cp\big(\varphi(y) - \varphi(z)\big)\,, \quad y \in V\,, \quad z \in V\,,$$

where p is the Minkowsky functional associated to $B = \varphi(U)$.

We call *E-pseudometric*, or *admissible pseudometric* on M a mapping

$$F\colon TM \to \boldsymbol{R}_+\,,$$

where TM is the (real) tangent fiber bundle to M, satisfying

$$F(t\xi) = |t|F(\xi)\,, \quad t \in \boldsymbol{R}\,, \quad \xi \in TM\,,$$

and such that, for every x in M, there exist (U, φ) at x, V and C as before such that

$$F(\xi) \leqslant Cp\big(D\varphi(\xi)\big)\,, \quad \xi \in TV\,,$$

where $D\varphi$ is the differential of φ and the tangent space at every poinf of $B = \varphi(U)$ is naturally identified with E.

No assumption is made on the regularity of F.

We call a curve $u\colon [a, b] \to E$ *absolutely continuous* if for every seminorm p in E and for every $\varepsilon > 0$ there exist $\delta > 0$ such that if $(t_1, s_1), \dots, (t_n, s_n)$ is a family of pairwise disjoint intervals contained in $[a, b]$ for which $\sum (t_i - s_i) < \delta$, then $\sum p\big(u(t_i) - u(s_i)\big) < \varepsilon$.

For the general case we call a curve $u\colon [a, b] \to M$ *absolutely continuous* if for every t in $[a, b]$ and every local coordinate system (U, φ) at $u(t)$ the curve $\varphi \circ u|_{\varphi^{-1}(U)}\colon \varphi^{-1}(U) \to E$ is absolutely continuous in the above sense. When $M = E$ the two definitions of absolute continuity coincide by the compactness of the interval $[a, b]$.

For every manifold M an *admissible curve* in M is, by definition, a curve

$$u\colon [a, b] \to M$$

which satisfies the following conditions:

a) the curve u is absolutely continuous;

b) the curve u is almost everywhere derivable in the interval $[a, b]$.

For example, every piecewise C^1 curve is an admissible curve.

REMARK. – If E is a reflexive Banach space condition *a*) implies condition *b*), but there are isometries from the unit interval to the Banach space $L^1(0, 1)$ which are not derivable at any point (for details see [4] 2.9.22, 2.9.23).

Now we define the integrated forms of E-pseudometrics.

Let F be an E-pseudometric on M. For x and y in M let

$$d^*(x, y) = d^*(F)(x, y) = \inf\left\{ {}^{*}\!\!\int_a^b F\big(Du(t)\big)\, dt \,\Big|\, u\colon [a, b] \to M \text{ admissible},\ u(a) = x,\ u(b) = y \right\}$$

$$d_*(x, y) = d_*(F)(x, y) = \inf\left\{ {}_{*}\!\!\int_a^b F\big(Du(t)\big)\, dt \,\Big|\, u\colon [a, b] \to M \text{ admissible},\ u(a) = x,\ u(b) = y \right\}$$

where $\overset{*}{\int}$ and $\underset{*}{\int}$ stand respectively for the upper and lower integral.

The functions $d^*(F)$ and $d_*(F)$ are E-pseudometrics and will be called respectively the *upper* and *lower integrated form* of F.

An E-pseudometric F is said to be *weakly integrable* if $d^*(F)$ and $d_*(F)$ are equal. In this case we put $d(F) = d^*(F) = d_*(F)$ and call it the integrated form of F.

The E-pseudometric F is said to be *strongly integrable* if for every admissible curve $u\colon [a, b] \to M$ the function $F\big(D(u(t))\big)$ is Lebesgue integrable on the interval $[a, b]$.

Clearly every strongly integrable E-pseudometric is weakly integrable.

We introduce an equivalence relation on the space of all E-pseudometrics. We call two E-pseudometrics F and G *equivalent* if for every admissible curve $u\colon [a, b] \to M$ the functions $F(u(t))$ and $G(u(t))$ are almost everywhere equal on $[a, b]$. Clearly that is an equivalence relation on E-pseudometrics.

It is clear that the upper (lower) integrated forms of equivalent pseudometrics coincide.

To define the derivative on an E-pseudodistance we need some result from the theory of length of curves in pseudometric spaces.

Let X be a set and let d be a pseudodistance on X. A mappings defined in a closed interval $[a, b] \subset \mathbf{R}$ into X is said to be a curve. For a curve $u\colon [a, b] \to X$ the *length* of u is the number

$$L(u) = \sup \{\sum d(u(t_{i+1}), u(t_i)) | a = t_1 < \ldots < t_n = b\} .$$

The curve $u\colon [a, b] \to X$ is absolutely continuous (with respect to d) if for every $\varepsilon > 0$ there exist $\delta > 0$ such that if $(t_1, s_1), \ldots, (t_n, s_n)$ is a family of pairwise disjoint intervals contained in $[a, b]$ for which if $\sum (t_i - s_i) < \delta$ then $\sum d(u(t_i), u(s_i)) < \varepsilon$. For such a curve the following theorem holds:

THEOREM 1.1. - *Let $u\colon [a, b] \to X$ be an absolutely continuous curve. For every t in $[a, b]$ put*

$$D^*(t) = \limsup_{h \to 0} d(u(t + h), u(t))/|h| ,$$

$$D_*(t) = \liminf_{h \to 0} d(u(t + h), u(t))/|h| .$$

Then the functions $D^*(t)$ and $D_*(t)$ coincide almost everywhere in $[a, b]$; they are Lebesgue integrable; furthermore $L(u) < +\infty$, and

$$L(u) = \int_a^b D^*(t)\,dt = \int_a^b D_*(t)\,dt .$$

The above definition of absolute continuity and the proof of this theorem are in [16] § 13 where only the metric case is considered, but the proof does holds also in this less restrictive case.

Now we define the derivative of every E-pseudometric up to equivalence.

Let d be an E-pseudometric on M. For every $\xi \in TM$ define

$$F^*(\xi) = F^*(d)(\xi) = \limsup_{h \to 0} d(u(h), u(0))/|h| ,$$

$$F_*(\xi) = F_*(d)(\xi) = \liminf_{h \to 0} d(u(h), u(0))/|h| ,$$

where u is a curve in M defined in a neighbourhood of 0 in $\mathbf{R}$, which is derivable at 0 and such that $u'(0) = \xi$. The definition does not depend on the choice of the curve u. Indeed let u_1 and u_2 be two curves with $u_1'(0) = u_2'(0) = \xi$. Put $x = u_1(0) = u_2(0)$. Let (U, φ) be a local coordinate system at x. Since the pseudometric d is admissible there exist an open neighbourhood V of x in U and a positive constant C such that for every y and z in V

$$d(y, z) \leqslant Cp\big(\varphi(y) - \varphi(z)\big) .$$

Then

$$\begin{aligned}\limsup_{h\to 0} d(u_1(h), u_1(0))/|h| &\leqslant \limsup_{h\to 0} \Big(d\big(u_1(h), u_2(h)\big) + d\big(u_2(h), u_2(0)\big)\Big)/|h| \leqslant \\ &\leqslant \lim_{h\to 0} Cp\Big(\varphi\big(u_1(h)\big) - \varphi\big(u_2(h)\big)\Big)/|h| + \limsup_{h\to 0} d\big(u_2(h), u_2(0)\big)/|h| = \\ &= \limsup_{h\to 0} d\big(u_2(h), u_2(0)\big)/|h| .\end{aligned}$$

Interchanging u_1 with u_2 we obtain the opposite inequality. By a symilar argument we have

$$\liminf_{h\to 0} d\big(u_1(h), u_1(0)\big)/|h| = \liminf_{h\to 0} \Big(d\big(u_2(h), u_2(0)\big)\Big)/|h| .$$

Hence the functions F^* and F_* are well defined on the tangent bundle to M and it is easy to show that they are admissible pseudometrics, The following theorem holds:

THEOREM 1.2. – *For every E-pseudodistance d the E-pseudometrics F^* and F_* defined above are equivalent. For every admissible curve u: $[a, b] \to M$ the function $F^*\big(D(u(t))\big)$ $\Big($and hence $F_*\big(D(u(t))\big)\Big)$ is Lebesgue measurable and*

$$L(u) = \int_a^b F^*(Du(t))\, dt = \int_a^b F_*(Du(t))\, dt .$$

where the length is computed with respect to the pseudodistance d.

PROOF. – Let u: $[a, b] \to M$ be an admissible curve. By the compactness of the interval $[a, b]$, the curve u is absolutely continuous with respect to the pseudometric d. With the notations of theorem 1 we have $F^*(Du(t)) = D^*u(t)$ and $F_*(Du(t)) = D_*u(t)$. The assertion follows from theorem 1.

We call the equivalence class of the E-pseudometric F^* (or F_*) the *derivative* of the E-pseudodistance d and write $F(d)$ for it. By abuse of language we call derivative of the E-pseudodistance d any E-pseudometric which is in the equivalence class of F^*.

Let d be an E-pseudodistance on M; for x and y in M define

$$d^i(x, y) = \inf \{L(u) | u\colon [a, b] \to M \text{ admissible, } u(a) = x,\ u(b) = y\} .$$

Since M is connected, every pair of points in M can be joined by an admissible curve.

A straightforward argument shows that d^i is an admissible pseudodistance on M. It is called the *inner distance associated to* d and the distance d is said to be *inner* if $d = d^i$. The teminology is consistent since for every E-pseudometric we have $(d^i)^i = d^i$.

Theorem 1.2 implies the following theorem:

THEOREM 1.3. – *For every admissible pseudodistance d on the connected manifold M the derivative $F(d)$ of d is strongly integrable and its integrated form coincides with the inner distance associated to d.*

COROLLARY 1.1. – *An admissible pseudodistance on M is the integrated form of its derivative if, and only if, it is inner.*

Let denote by $\mathfrak{D}(M)$ and by $\mathcal{M}(M)$ respectively the set of the E-pseudodistances and the E-pseudometrics on the connected manifold M.

For d and h in $\mathfrak{D}(M)$ set by definition $d \leqslant h$ if $d(x, y) \leqslant h(x, y)$ for every choice of x and y in M. This relation is an order relation on $\mathfrak{D}(M)$.

For F and G in $\mathcal{M}(M)$ set by definition $F \leqslant G$ if for every admissible curve $u\colon [a, b] \to M$, then $F(Du(t)) \leqslant G(Du(t))$ for almost every t in the interval $[a, b]$. This relation is a pre-order relation on the space $\mathcal{M}(M)$ and we have $F \leqslant G$ and $G \leqslant F$ if, and only if, the E-pseudometrics F and G are equivalent.

With respect to these order relations we have the following characterization of the derivative of any E-pseudodistance and of the (lower) integrated form of any E-pseudometric:

THEOREM 1.4. – *For* $d \in \mathfrak{D}(M)$ *and* $F \in \mathcal{M}(M)$,

(1) $F(d) = \min \{G \in \mathcal{M}(M) | d_*(G) \geqslant d\}$,

(2) $d_*(F) = \max \{h \in \mathfrak{D}(M) | F(h) \leqslant F\}$.

PROOF. – By theorem 3 the integrated form of the derivative of d is greater than d. Let G be an E-pseudometric such that $d_*(G) \geqslant d$. We have to prove that $G \geqslant F(d)$. Let $u\colon [a, b] \to M$ be an admissible curve in M. Let $f\colon [a, b] \to \mathbf{R}$ be a Lebesgue measurable function such that $f(t) \leqslant G(Du(t))$ for every $t \in [a, b]$, and

$$\int_a^b f(t)\, dt = {}_*\!\!\int_a^b G(Du(t))\, dt .$$

Then for every t and s in $[a, b]$ with $t < s$ we have

$$\int_t^s f(t)\,dt = \int_{*t}^s G(Du(t))\,dt\,.$$

Indeed we have

$$\int_a^t f(t)\,dt \leqslant \int_{*a}^t G(Du(t))\,dt\,, \qquad \int_t^s f(t)\,dt \leqslant \int_{*t}^s G(Du(t))\,dt\,, \qquad \int_s^b f(t)\,dt \leqslant \int_{*s}^b G(Du(t))\,dt\,,$$

$$\int_a^t f(t)\,dt + \int_t^s f(t)\,dt + \int_s^b f(t)\,dt = \int_a^b f(t)\,dt =$$

$$= \int_{*a}^b G\big(D(u(t))\big)\,dt = \int_{*a}^t G\big(D(u(t))\big)\,dt + \int_{*t}^s G\big(D(u(t))\big)\,dt + \int_{*t}^b G\big(D(u(t))\big)\,dt\,,$$

and the latter conditions holds only if

$$\int_t^s f(t)\,dt = \int_{*t}^s G(Du(t))\,dt\,.$$

Then, at almost every Lebesgue point t of the function f we have

$$F(d)(u(t)) = \lim_{h\to 0} d(u(t+h), u(t))/|h| \leqslant \lim_{h\leftarrow 0} d^*(G)(u(t)+h), u(t))/|h| \leqslant$$

$$\leqslant \lim_{h\to 0} \frac{1}{h} \int_{*t}^{t+h} G\big(D(u(s))\big)\,ds = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f(s)\,ds = f(t) \leqslant G(Du(t))\,.$$

Hence $F(d) \leqslant G$ and (1) follows.

Consider now the pseudometric F. By (1) and theorem 1.3, the derivative of the pseudodistance $d_*(F)$ is less than F. Let h be an E-pseudodistance such that $F(h) \leqslant F$. We have to prove that $h \leqslant d_*(F)$. Let x and y be arbitrary in M; let $u\colon [a, b] \to M$ be an admissible curve such that $u(a) = x$ and $u(b) = y$. Since by theorem 3

$$h(x, y) \leqslant \int_a^b F(h)(Du(t))\,dt \leqslant \int_{*a}^b F(Du(t))\,dt\,,$$

the fact that the curve u is arbitrary we have $h(x, y) \leqslant d_*(F)(x, y)$, and (2) follows.

§ 2. – In this section we describe the behavior of pseudodistances and pseudometrics under differentiable mappings.

Fix two manifolds M and N (not necessarily modelled on the same topological vector space) and a family $\mathcal{F}$ of differentiable mappings from M to N.

Consider an admissible pseudometric F on M weakly integrable and let $d = d(F)$. Suppose, with the notations of section 1, that for every $\xi \in TM$ we have $F_*(d)(\xi) \leqslant F(\xi)$.

We observe that by theorem 1.3, for every weakly integrable pseudometric on M there exists an equivalent pseudometric which enjoys this property.

For every x and y in N set

$$\delta_{\mathcal{F}}(x, y) = \inf \{d(x', y')\} ,$$

where the greatest lower bound is taken over all pairs x' and y' in N for which there exists $f \in \mathcal{F}$ such that $f(x') = x$ and $f(y') = y$. If such x', y' and f cannot be found then put $\delta_{\mathcal{F}}(x, y) = +\infty$. The function $\delta_{\mathcal{F}}$ is symmetric but in general does not satisfies the triangle inequality. Hence, for every x and y in M, set

$$d_{\mathcal{F}}(x, y) = \inf \{\sum \delta_{\mathcal{F}}(p_{i+1}, p_i) | x = p_1, \ldots, p_n = y\} .$$

For $\xi \in TM$ put

$$F_{\mathcal{F}}(\xi) = \inf \{F(\eta)\} ,$$

where the greatest lower bound is taken over all $\eta \in TN$ for which there exists $f \in \mathcal{F}$ such that $Df(\eta) = \xi$. If such ξ and f cannot be found then put $F_{\mathcal{F}}(\xi) = +\infty$.

It is not difficult to prove that if $d_{\mathcal{F}}(x, y)$ and $F_{\mathcal{F}}(\xi)$ are finite for every choice of x and y in M and ξ in TM then they are respectively a pseudodistance and a pseudometric on N.

THEOREM 2.1. – *If the pseudometric $F_{\mathcal{F}}$ and the pseudodistance $d_{\mathcal{F}}$ are admissible then the pseudometric $F_{\mathcal{F}}$ is weakly integrable and the pseudodistance $d_{\mathcal{F}}$ is its integrated form.*

PROOF. – Let h^* and h_* be respectively the upper and lower integrated forms of $F_{\mathcal{F}}$. We shall prove that $h^* \leqslant d_{\mathcal{F}}$ and $d_{\mathcal{F}} \leqslant h_*$.

To establish the first inequality it suffices to prove that $h^* \leqslant \delta_{\mathcal{F}}$. Let x and y in N be arbitrary. We can suppose that $\delta_{\mathcal{F}}(x, y) < +\infty$. Let $\varepsilon > 0$. By definition of $d_{\mathcal{F}}$ there exist x' and y' in M and $f \in \mathcal{F}$ such that

$$d(x', y') < d_{\mathcal{F}}(x, y) + \varepsilon .$$

Since d is the integrated form of F, there exists an admissible curve $u\colon [a, b] \to M$ joining x' and y' such that

$$\int_a^{*b} F(Du(t))\, dt < d(x', y') + \varepsilon .$$

We have

$$h^*(x, y) \leqslant \int_a^{*b} F_{\mathcal{F}}(D(f\circ u)(t))\, dt \leqslant \int_a^{*b} F(Du(t))\, dt < d(x', y') + \varepsilon < d_{\mathcal{F}}(x, y) + 2\varepsilon .$$

The fact that $\varepsilon > 0$ is arbitrary implies $h^*(x, y) \leqslant d_{\mathcal{F}}(x, y)$.

Let $G = F_*(d_{\mathcal{F}})$. In order to prove that $d_{\mathcal{F}} \leqslant h_*$, by (2) of theorem 4 of section 1, it suffices to show that for every $\xi \in TN$ we have $G(\xi) \leqslant F_{\mathcal{F}}(\xi)$.

Let $\xi \in TN$ and let $\varepsilon > 0$ be arbitrary. By definition of $F_{\mathcal{F}}$ there exists $\eta \in TM$ and $f \in \mathcal{F}$ such that $D(f)(\eta) = \xi$ and $F(\eta) < F_{\mathcal{F}}(\xi) + \varepsilon$.

Let u be a C^1 curve in M defined in a neighbourhood of 0 such that $Du(0) = \eta$. Then

$$G(\xi) = \liminf_{h\to 0} d_{\mathcal{F}}\big(f(u(t+h)), f(u(t))\big)/|h| \leqslant \liminf_{h\to 0} d(u(t+h), u(t))/|h| = \\ = F_*(d)(\eta) \leqslant F(\eta) < F_{\mathcal{F}}(\xi) + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, the assertion follows.

EXAMPLE. – If $\mathcal{F} = \{f\}$, where $f\colon M \to N$ is a surjective differentiable mapping such that for every $x \in M$ there exist $x' \in N$ such that $f(x') = x$ and the mapping f is open in a neighbourhood of x', then the hypotheses of theorem 1 are satisfied. In this case our theorem 1 improves a result of [14].

Let M, N and $\mathcal{F}$ be as before. Consider an admissible pseudodistance d on N and let $F = F_*(d)$.

For every x and y in M put

$$d_{\mathcal{F}}(x, y) = \sup\big\{d(f(x), f(y)) \,|\, f \in \mathcal{F}\big\},$$

and for every $\xi \in TM$

$$F_{\mathcal{F}}(\xi) = \sup\big\{F(Df(\xi)) \,|\, f \in \mathcal{F}\big\}.$$

Also in this case it is not difficult to prove that, if $d_{\mathcal{F}}(x, y)$ and $F_{\mathcal{F}}(\xi)$ are finite, then they are respectively a pseudodistance and a pseudometric on N.

THEOREM 2.2. – *If the pseudometric $F^{\mathcal{F}}$ and the pseudodistance $d^{\mathcal{F}}$ are admissible then the pseudometric $F^{\mathcal{F}}$ is the derivative of the pseudodistance $d^{\mathcal{F}}$. Thus $F^{\mathcal{F}}$ is strongly integrable and its integrated form is the inner pseudodistance associated to $d^{\mathcal{F}}$.*

PROOF. – Let $G = F_*(d^{\mathcal{F}})$. First we show that for every $\xi \in TM$ we have $F^{\mathcal{F}}(\xi) \leqslant G(\xi)$. Let $\xi \in TM$ and let $\varepsilon > 0$ be arbitrary. By definition of $F^{\mathcal{F}}$ there exists $f \in \mathcal{F}$ such that $F(Df(\xi)) > F^{\mathcal{F}}(\xi) - \varepsilon$.

Let u be a C^1 curve in M defined in a neighbourhood of 0 such that $Du(0) = \xi$. Then

$$G(\xi) = \liminf_{h \to 0} d^{\mathcal{F}}(u(t+h), u(t))/|h| \geqslant \liminf_{h \to 0} d\big(f(u(t+h)), f(u(t))\big)/|h| = \\ = F\big(Df(\xi)\big) > F^{\mathcal{F}}\{u(\xi)) - \varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary the assertion follows. Let h be the lower integrated forms of $F^{\mathcal{F}}$.

In order to prove that $F^{\mathcal{F}} \geqslant G$, by (1) of theorem 4 of section, it suffices to show that for every choice of x and y in M we have $h(x, y) \geqslant d^{\mathcal{F}}(x, y)$.

Let x and y in M be arbitrary and let $\varepsilon > 0$. By definition of $d_{\mathcal{F}}$ there exist $f \in \mathcal{F}$ such that

$$d\big(f(x), f(y)\big) > d_{\mathcal{F}}(x, y) - \varepsilon \,.$$

Since h is the lower integrated form of $F^{\mathcal{F}}$ there exists an admissible curve $u: [a, b] \to M$ joining x and y such that

$$h(x, y) > {}^{*}\!\!\int_a^b F^{\mathcal{F}}\big(Du(t)\big)\, dt - \varepsilon \,.$$

We have

$$h(x, y) > {}^{*}\!\!\int_a^b F^{\mathcal{F}}\big(Du(t)\big)\, dt - \varepsilon \geqslant \int_a^b F\big(D(f \circ u)(t)\big)\, dt - \varepsilon \geqslant d\big(f(x), f(y)\big) - \varepsilon \geqslant d^{\mathcal{F}}(x, y) - 2\varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary we have $h(x, y) \geqslant d^{\mathcal{F}}(x, y)$, and the assertion follows.

EXAMPLE. – Let E be topological vector space and let $V \subset E$ be a linear subspace with the topology induced by E. Let M be a manifold modelled on E and let N be a submanifold of M modelled on V. Let d be an admissible pseudodistance on M. Let F be the derivative of d, G be the restriction of F to N and h be the restriction of the pseudometric d to N. Then the integrated form of G on N coincides with the inner distance associated to h. Indeed, we have $h = d_{\mathcal{F}}$ and $G = F_{\mathcal{F}}$, where $\mathcal{F}$ is the family containing only the inclusion map $i: N \to M$. Since the hypotheses of theorem 2.2 in this case are clearly satisfied, then the assertion follows.

§ 3. – In this section we shall deal with the Kobayashi, Hahn and Carathéodory pseudodistances and pseudometrics on complex manifolds.

We recall some definitions. Let Δ be the open unit disc in $\boldsymbol{C}$. Consider the

Poincaré metric

$$\langle v\rangle_z = |v|/(1-|z|^2)\,, \quad z\in \Delta\,, \quad v\in \boldsymbol{C}\,,$$

and the corrisponding distance

$$\omega(z, w) = \tfrac{1}{2}\log\big((1+|\alpha(z, w)|)/(1-|\alpha(z, w)|)\big)\,,$$

where

$$\alpha(z, w) = (z-w)/(1-z\overline{w})\,.$$

Let M be a complex manifold.

The Carathéodory pseudodistance c_M is defined for every x and y in M by

$$c_M(x, y) = \sup\big\{\omega(f(x), f(y))\,|\,f\in \mathrm{Hol}\,(M, \Delta)\big\}\,,$$

and the relative pseudometric γ_M by

$$\gamma_M(\xi) = \sup\big\{\langle Df(\xi)\rangle_{f(x)}\,|\,f\in \mathrm{Hol}\,(M, \Delta)\big\}$$

for every ξ in the tangent space at x in M.

Let P be a family of holomorphic mappings from Δ to M containin all injective holomorphic mappings.

For $\mathcal{F} = \mathrm{Hol}\,(\Delta, M)$, the family of all holomorphic mappings from Δ to M, the function $d_{\mathcal{F}}$ and $F_{\mathcal{F}}$ are respectively the Kobayashi pseudodistance k_M and pseudometric $\varkappa_M$ on M; for $\mathcal{F}$ the family of all injective holomorphic mappings, the function $d_{\mathcal{F}}$ and $F_{\mathcal{F}}$ are respectively the Hahn pseudodistance h_M and pseudometric η_M.

Let $\mathcal{F}\subset \mathrm{Hol}\,(\Delta, M)$ be an arbitrary family. Since the family $\mathcal{F}$ contains all holomorphic injective mappings then

$$k_M \leqslant d_{\mathcal{F}} \leqslant h_M$$

and

$$\varkappa_M \leqslant F_{\mathcal{F}} \leqslant \eta_M\,.$$

We claim that $d_{\mathcal{F}}$ is an admissible pseudodistance and that $F_{\mathcal{F}}$ is an admissible pseudometric. By the previous inequalities it is enough to prove this fact for the Hahn pseudodistance and the Hahn pseudometric respectively. In this case our claim follows from the following proposition:

PROPOSITION 3.1. – *Let $D\subset E$ be a convex balanced neighbourhood of the origin in E and let p be its Minkowsky functional. Then*

(1) $\eta_D(x, v)\leqslant p(v)/(1-p(x))$.

(2) $h_D(x, y)\leqslant p(x-y)/(1-\max\,(p(x), p(y))$.

PROOF. – For $x \in E$ and $r > 0$ put $B(x, r) = \{y \in E | p(y - x) < r\}$ and $B(r) = B(0, r)$. By an Hahn-Banach type argument, as in [5] (IV.1.8 and V.1.5) we have for every $x \in B(r)$ and $v \in E$

$$\eta_{B(r)}(0, v) = r^{-1}p(v) ,$$
$$h_{B(r)}(0, x) = \omega(0, r^{-1}p(x)) .$$

Let $x \in D$. Then $D' = B(x, 1 - p(x)) \subset D$, and therefore, for every $v \in E$ we have

$$\eta_D(x, v) \leqslant \eta_{D'}(x, v) = \eta_{B(1-p(x))}(x, v) = p(v)/(1 - p(x)) ,$$

and (1) follows.

Let x and y be in D and let S be the segment joining x to y. If $a = \max(p(x), p(y))$, then, by the convexity of D, for every $z \in S$ we have $B(z, 1 - a) \subset D$. Fix $\varepsilon > 0$. Let $n \in N$ be so large that

$$h_{B(a)}(0, (y - x)/n) < (1 + \varepsilon)p((y - x)/n)/(1 - a) .$$

It is always possible to find such an n, since the derivative of the function $\omega(0, t)$ at $t = 0$ is equal to 1.

For $k = 0, \dots, n$ put $z^k = x + k(y - x)/n$. Then we have

$$h_D(x, y) \leqslant \overline{U}\, h_D(z^k, z^{k+1}) \leqslant \overline{U}\, h_{B(z^k, a)}(z^k, z^{k+1}) = \overline{U}\, h_{B(a)}(0, (y - x)/n) =$$
$$= \overline{U}(1 + \varepsilon)p((y - x)/n)/(1 - a) = (1 + \varepsilon)p(y - x)/(1 - a) .$$

Since $\varepsilon > 0$ is arbitrary the proof is complete.

By theorem 2.1 it follows that for every family $\mathcal{F} \subset \mathrm{Hol}(\Delta, M)$ containing all injective holomorphic mappings the pseudodistance $d_{\mathcal{F}}$ is the integrated form of the pseudometric $F_{\mathcal{F}}$.

Choosing $\mathcal{F} = \mathrm{Hol}(\Delta, M)$ (resp. $\mathcal{F} = \{f \in \mathrm{Hol}(\Delta, M) | f \text{ injective}\}$ we obtain the following theorem:

THEOREM 3.1. – *For every complex manifold M the Kobayashi (resp. Hahn) pseudometric is weakly integrable and its integrated form is the Kobayashi (resp. Hahn) pseudodistance.*

REMARK. – This theorem is proved for the Kobayashi case in [17] for finite dimensional complex manifolds and in [8] for domains in normed spaces. In the cases considered in [17] and in [8] also the strongly integrability of the Kobayashi pseudometric is proved (as a consequence of the semicontinuity property of the latter), but only piecewise C^1 curves are considered as admissible. Moreover, the method used there cannot be extended in a natural way to the Hahn case, (contrary to what is asserted in [6]), since the Royden extension theorems do not hold for injective holomorphic mappings but only for holomorphic embeddings. Moreover, the upper semicontinuity of the Hahn pseudometric is an open question. For results concerning the strong integrability for the Kobayashi and Hahn metric which avoid the extension theorems of Royden and Siu see the next section.

For every complex manifold the Carathéodory pseudodistance c_M and pseudometric γ_M are respectively the pseudometrics $d^{\mathcal{F}}$ and $F^{\mathcal{F}}$, where $\mathcal{F} = \mathrm{Hol}(M, \Delta)$, the family of all holomorphic mappings from M to Δ.

Since the Poincaré metric is the derivative of the corrisponding distance by theorem 2.2 we have the following:

THEOREM 3.2. – *For every complex manifold M the Carathéodory pseudometric γ_M is the derivative of the Carathéodory pseudodistance c_M and hence the pseudometric γ_M is strongly integrable and its integrated form is the inner distance associated to c_M.*

REMARK. – Actually, it can be proved that the Carathéodory pseudodistance is continuous on TM (and Lipshitzian for Banach manifolds) and the definition (1) and (2) of section 1 of the derivative of a pseudodistance for the Carathéodory case coincide with the Carathéodory pseudometric, and theorem 3 can be proved as in [8]. We have given this less precise statement to show the symmetry of this case with the result stated in theorem 3.1.

Hence, for every complex manifold the Kobayashi pseudodistance is the integrated form of the corresponding pseudometric and the Carathéodory pseudometric is the derivative of the corresponding pseudodistance. It is known that for the domain

$$D = \{z \in \mathbf{C}^n : 1 < |z| < 2,\ n > 1\}$$

the Carathéodory distance is not inner, and hence for such a domain the Carathéodory distance is not the integrated form of the corresponding pseudometric.

We now give an example of a bounded domain for which the Kobayashi distance is not the derivative of the corresponding metric.

Let $D = \{(z, w) \in \mathbf{C}^2 : |z| < 1,\ |w| < 1,\ |zw| < a^2\}\ ,\ 0 < a < \frac{1}{2}$.

The domain D is a complete circular pseudoconvex domain. Let p be its Minkowsky functional. By [1] the Kobayashi metric $\varkappa_D$ at 0 coincides together with p and is not difficult to prove that $\varkappa_D$ is continuous on the tangent vectors at 0, Moreover, for (z_1, w_1) and (z_2, w_2) with $|z_i| < a^2$ and $|w_i| < a^2$, we have

$$k_D\big((z_1, w_1), (z_2, w_2)\big) \leqslant k_D\big((z_1, w_1), (z_2, w_1)\big) + k_D\big((z_2, w_1), (z_2, w_2)\big) = \omega(z_1, z_2) + \omega(w_1, w_2).$$

Let $f\colon \Delta \to M$ be the holomorphic mapping defined by

$$f(\zeta) = (a\zeta, a\zeta)\,,$$

let $\gamma\colon \boldsymbol{R} \to \Delta$ be the Riemannian geodesic such that $\gamma(0) = 0$ and $\gamma'(0) = 1$ and let u be the restriction of f to the real axis.

Since the mapping f preserves the Kobayashi metric at 0, and this metric is continuous at 0, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every t, with $|t| < \delta$, we have

$$\varkappa_D(u(t), u'(t)) > 1 - \varepsilon .$$

Shrinking δ if necessary, by the above inequality, we have

$$F(u(t), u'(t)) \leqslant 2a(1 - a^2 t^2) < 2a + \varepsilon .$$

Hence, having chosen $a < \frac{1}{2}$, the Kobayashi metric of the domain D is not the derivative of the Kobayashi distance.

§ 4. – In this section we investigate the strongly integrability of Kobayashi type pseudometrics on complex manifolds.

Recall that a Polish space is a topological space homeomorphic to a separable complete metric space. A Suslin subset of a Polish space X is a subset of X which is the image by a continuous mapping of a Polish space.

A subset of a Polish space X is a Polish space for the induced topology if, and only if, it is a G_δ set in X (that is, a countable intersection of open sets in X) (see e.g., [3]). Hence every open or closed subset of X is a Polish space.

A Hausdorff topological space X with countable base such that every point $x \in X$ has a Polish neighbourhood is a Polish space ([3]).

If T is a locally compact Hausdorff space with countable base and X is a Polish space then the space $C(T, X)$ of all continuous functions from T to X endowed with with the compact open topology is a Polish space ([9]).

If X is a Polish space the family of the Suslin subsets of X is closed under countable union and intersection.

If $f \in X \to Y$ is a Borel mapping between Polish spaces, then for every Suslin subset $A \subset X$ the set $f(A) \subset Y$ is a Suslin subset of Y and for every Suslin subset $B \subset Y$ the set $f^{-1}(B) \subset X$ is a Suslin subset of X.

If X and Y are Polish spaces and A and B are Suslin subsets respectively of X and Y, then $X \times Y$ is a Polish space and $A \times B$ is a Suslin subset of $X \times Y$.

A Suslin subset of $\boldsymbol{R}$ is Lebesgue measurable ([4]).

For these and other properties of Polish spaces and Suslin sets see [7].

Let E be a complex separable Fréchet space and M a connected E-manifold with countable base. Then the manifold M is a Polish space and the space $\mathrm{Hol}(\Delta, M)$, being a closed subset of $C(\Delta, M)$ is a Polish space.

THEOREM 4.1. – *Let E and M be as above. Let $\mathcal{F} \subset \mathrm{Hol}(\Delta, M)$ be a family containing all injective mappings and suppose that $\mathcal{F}$ is a Suslin subset of* $\mathrm{Hol}(\Delta, M)$.

Then, for every $a > 0$ *the set*

$$A(a) = \{\xi \in TM | F_{\mathcal{F}}(\xi) < a\}$$

is a Suslin subset of TM *and hence the pseudometric* $F_{\mathcal{F}}$ *is strongly integrable.*

PROOF. – Let $X = \mathrm{Hol}(\Delta, M) \times \boldsymbol{C}$, let $V: X \to TM$ and $W: X \to \boldsymbol{R}_+$ be the continuous functions defined on $(f, v) \in X$ respectively by $V(f, v) = f'(0)(v)$ and $W(f, v) = |v|$. By definition for every $\xi \in TM$ we have

(1) $F_{\mathcal{F}}(\xi) = \inf\{W(f, v) | V(f, v) = \xi \text{ and } f \in \mathcal{F}\}$.

Let A and B be the epigraphs of the functions $F_{\mathcal{F}}$ and W respectively, that is

$$A = \{(\xi, t) \in TM \times \boldsymbol{R}_+ | F_{\mathcal{F}}(\xi) > t\},$$
$$B = \{((f, v), t) \in X \times \boldsymbol{R}_+ | W(f, v) > t\}.$$

By (1) $A = V^*(B \cap (\mathcal{F} \times \boldsymbol{R}_+))$, where $V^*: X \times \boldsymbol{R}_+ \to TM \times \boldsymbol{R}_+$ is the function defined on $(f, v) \in X$ and $t \in R_+$ by $V^*((f, v), t) = (V(f, v), t)$. Hence the set A, as image of a Suslin set under a continuous function, is a Suslin subset of TM.

For every $a > 0$ we have

$$A(a) = \pi\big(\{A \cap (TM \times \{t \in \boldsymbol{R}_+ | t < a\}\}\big),$$

where $\pi: TM \times \boldsymbol{R}_+ \to TM$ is the natural projection. By the stability properties of Suslin sets, $A(a)$ is a Suslin subset of TM, and the first part of the theorem is proved.

Let $u: [a, b] \to M$ be an admissible curve. Then the function $t \to Du(t)$ is a Borel function (see e.g., [4]) and hence for every $a > 0$ the set

$$\{t \in [a, b] | F_{\mathcal{F}}(Du(t)) < a\} = (Du)^{-1}(A(a))$$

is a Suslin subset of $[a, b]$, thus Lebesgue measurable.

It follows that the function $t \mapsto F_{\mathcal{F}}(Du(t))$ is Lebesgue measurable, that is, by the arbitrariness of the admissible curve, the pseudometric $F_{\mathcal{F}}$ is strongly integrable.

COROLLARY 4.1. – *Let* E *and* M *be as above. Then the Kobayashi and the Hahn pseudometrics of* M *are strongly integrable pseudometrics.*

PROOF. – For the Kobayashi pseudometric the assertion follows immediatly from the above theorem. For the Hahn pseudometric it is enought to prove that the family $\mathcal{F}$ of all injective holomorphic mappings is a Suslin subset of $\mathrm{Hol}(\Delta, M)$.

We shall even prove that $\mathcal{F}$ is a G_δ set in $\mathrm{Hol}(\Delta, M)$. This fact is a consequence of the following general statement:

PROPOSITION 4.1. – *Let X and T be Hausdorff spaces and let T be locally compact with countable base. Then the family $\mathcal{G}$ of all continuous injective mappings from T to X is a G_δ set in $C(T, X)$.*

PROOF. – Let $\mathcal{U} = \{U_n\}_{n\in N}$ be a countable base of T such that $U_n \subset\subset T$ for every $n \in N$. Then

$$\mathcal{G} = \bigcap \{A(m, n) | \overline{U}_m \cap \overline{U}_n = \emptyset\}$$

where

$$A(m, n) = \{f \in C(T, X) | f(\overline{U}_m) \cap f(\overline{U}_n) = \emptyset\}$$

is open for every choice of m and n in N, and the assertion follows.

§ 5. – In this section we apply the results of section 1 to pseudodistances and pseudometrics generated by projective mappings.

Let M be a connected n-dimensional real differentiable manifold and let Γ be a torsionfree affine connection on M.

Consider on the interval I the Riemannian metric

$$ds^2 = du^2/(1 - u^2)^2$$

and let ϱ be the associated distance.

Let $\mathcal{F}$ be the family of all projective mappings from I to M (for references see [19]). The distance p_M on M constructed as in section 3 from the family $\mathcal{F}$ and the distance ϱ were introduced in [13], whereas the analogous pseudometric P_M was introduced in [19].

By the local existence theorems of geodesics it is not difficult to prove that p_M and P_M are respectively an admissible pseudodistance and pseudometric on M. Hence, theorem 2.1 yields:

THEOREM 5.1. – *The pseudometric P_M is weakly integrable and its integrated form is p_M.*

REMARK. – In [19] it is proved that the pseudometric P_M is upper-semicontinuous, hence it is actually strongly integrable. In [19] it is also proved theorem 5.1, but there only piecewise C^1 curves are considered admissible.

§ 6. – Let M be a real connected manifold, that is a manifold modelled on open domains of a real topological vector space. In section 1 we have characterized inner admissible pseudodistances as the ones which coincides with the integrated form of their derivatives.

Call inner pseudometric on M a strongly integrable pseudometric F which coincides (up to equivalence) together with the derivative of its integrated form. By theorem 1.3 the hypothesis of strongly integrability on F is not restrictive.

By the definition of derivative is not difficult to prove that for an inner pseudometric F there exists a pseudometric F' which is *convex*, that is, for every $x \in M$ and every choice of ξ_1 and ξ_2 in TM_x we have $F'(\xi_1 + \xi_2) \leqslant F'(\xi_1) + F'(\xi_2)$.

EXAMPLES. – Every Riemannian metric on a finite dimensional manifold is inner; the Carathéodory pseudometric on every complex manifold is inner; there are complex manifold whose Kobayashi metric is not inner (see section 3).

A pseudometric F on M is positive if for $\xi \in TM$, the condition $G(\xi) = 0$ implies $\xi = 0$; if that in the case, we simply call F a *metric*.

In [2] BUSEMANN and MAYER show that on a finite dimensional connected real manifold every continuous convex metric is inner.

We improve this result weakening the continuity hypothesis.

THEOREM 6.1. – *Let M be a finite dimensional real connected manifold. Every admissible lower semicontinuous convex metric is inner.*

PROOF. – Let F be a metric satisfying the hypotheses of the theorem. It is not difficult to show that there exists a non decreasing sequence of continuous convex metrics F_n such that for every $\xi \in TM$ we have $F(\xi) = \sup F_n(\xi)$. Then the assertion is a consequence of the following general proposition:

PROPOSITION 6.1. – *Let M be an arbitrary manifold and let F_n be a sequence of inner pseudometric. For every $\xi \in TM$ define*

$$F(\xi) = \sup F_n(\xi) .$$

If F is an admissible pseudometric then it is an inner pseudometric.

PROPF. – Let d_n and d be respectively the integrated forms of F_n and F, and let G be the derivative of d. We have to show that G and F are equivalent.

By theorem 1.4 $G \leqslant F$.

Conversely, for every n we have $d \geqslant d_n$. Then it follows that for every n, $= G = F(d) \geqslant F(d_n) = F_n$, and hence $G \geqslant F$.

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