Annali di Matematica pura ed applicata
(IV), Vol. CLIV (1989), pp. 259-279

On the Optimal Control and Relaxation of Nonlinear Infinite Dimensional Systems (*).

NIKOLAOS S. PAPAGEORGIOU (**)

Summary. – *In this paper we study optimal control problems for infinite dimensional systems governed by a semilinear evolution equation. First under appropriate convexity and growth conditions, we establish the existence of optimal pairs. Then we drop the convexity hypothesis and we pass to a larger system known as the « relaxed system ». We show that this system has a solution and the value of the relaxed optimization problem is equal to the value of the original one. Next we restrict our attention to linear systems and establish two « bang-bang » type theorems. Finally we present some examples from systems governed by partial differential equations.*

1. – Introduction.

In this paper we study optimal control problems for distributed parameter control systems governed by a semilinear evolution equation. We prove that under a standard growth condition relating the vector field with the integrand of the cost functional (Lagrangian) and a well known convexity hypothesis on certain orientor field, we can guarantee the existence of optimal controls. Then we examine what happens when we drop the convexity hypothesis on the orientor field made earlier. In this case in order to obtain optimal controls, we need to consider a larger system with the controls being measure valued (« relaxed controls » or « generalized curves » or « sliding regimes »). For this augmented system (« relaxed » or « convexified » system), under the same growth hypothesis as before, we can establish the existence of optimal solutions. Then we turn our attention to linear systems and we prove two « bang-bang » type theorems for them. The first can be viewed as an « approximate bang-bang » result and says that an optimal relaxed trajectory can be approximated in the supremum norm by trajectories of the original system which are generated by external controls (i.e. controls which at each time instant have their values at the

---

(*) Entrata in Redazione il 15 dicembre 1987.
Indirizzo dell'A.: University of California, 1015 Department of Mathematics, Davis, California 95616, U.S.A.
Research supported by N.S.F. Grant - 8602313.
(**) Work done while on leave at the « University of Thessaloniki, School of Technology, Mathematics Division, Thessaloniki 54006, Greece ».

extreme points of the control constraint set, « bang-bang controls »). The second, is an exact « bang-bang » result and shows that for any admissible pair, there is another with the same ends which is « bang-bang » (in the sense that the control takes values on the boundary of the control constraint multifunction). In particular then this result implies that for any optimal solution (if it exists), there is also an optimal « bang-bang » solution. So if the optimal solution is unique, then it has to be « bang-bang ». Finally, we close this paper with examples from control systems governed by partial differential equations.

Since we assume that the unbounded linear operators $A(t)$ generate a strongly continuous evolution operator $S(t, s)$ and so the trajectories admit an integral representation, our results can be viewed as an infinite dimensional extension of the work of ANGELL [1], [2]. Also several of the ideas and key hypotheses have their roots back in the fundamental works of CESARI [10], [11], [12], [13]. The idea of extending the family of admissible controls to measure valued ones goes back to the works of YOUNG [36] (« generalized curves ») FILIPPOV [21] (« sliding regimes »), WARGA [34], [35] (« relaxed controls ») and GHOUILA-HOURI [22] (« commandes limites »).

2. – Preliminaries.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)} = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)} = \{A \subseteq X\colon \text{nonempty, } (w-) \text{ compact, (convex)}\}\,.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable if for all $z \in X$, $\omega \to d(z, F(\omega)) = \inf\{\|z - x\|\colon x \in F(\omega)\}$ is measurable. When there is a σ-finite measure $\mu(\cdot)$, with respect to which Σ is complete, then the above definition of measurability is equivalent to saying that $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X (graph measurability). By S^1_F we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^1(X)$ i.e. $S^1_F = \{f \in L^1(X)\colon f(\omega) \in F(\omega)\,\mu\text{-a.e.}\}$. This set may be empty. When $F(\cdot)$ is measurable and $\omega \to |F(\omega)| = \sup\{\|x\|\colon x \in F(\omega)\}$ belongs in L^1 (such an $F(\cdot)$ is said to be integrably bounded), then $S^1_F \neq \emptyset$. Using S^1_F we can define a set valued integral fro $F(\cdot)$ by setting $\int\limits_\Omega F = \Big\{\int\limits_\Omega f\colon f \in S^1_F\Big\}$.

Let Z be a completely regular Hausdorff space and $B(Z)$ its Borel σ-field. By $M^1_+(Z)$ we will denote the sace of probability measures on Z. A transition probability is a function $\lambda\colon \Omega \times B(Z) \to [0, 1]$ s.t. for every $A \in B(Z)$, $\lambda(\cdot, A)$ is Σ-measurable and for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M^1_+(Z)$. We will denote the set of all transition probabilities from (Ω, Σ, μ) into $(Z, B(Z))$, by $R(\Omega, Z)$. Following BALDER [3], (see also Warga [35]) we can define a topology on $R(\Omega, Z)$, called the weak topology. So let $f\colon \Omega\times Z \to R$ be a Carathéodory function (i.e. $\omega \to f(\omega, x)$ is measurable, $x \to f(\omega, x)$ is continuous and $|f(\omega, x)| \leqq \alpha(\omega)\mu$-a.e, with $\alpha(\cdot)\in L^1$) and define $I_f\colon R(\Omega, Z) \to R$ by $I_f(\lambda) = \int\limits_\Omega\int\limits_Z f(\omega, z)\lambda(t)(dz)\,d\mu(\omega)$. The weakest topology on $R(\Omega, Z)$ for which those functionals $\{I_f(\cdot)\colon f = \text{Carathéodory}\}$ are continuous, is the weak topology on $R(\Omega, Z)$. Note that when Ω is a singleton $R(\Omega, Z) = M_+^1(Z)$ and the weak topology is the narrow topology (see DELLACHERIE-MEYER [16]). Finally recall that if Y, V are topological spaces $F\colon Y \to 2^V\text{-}\{\emptyset\}$ is said to be upper semicontinuous (u.s.c), if for all $U \subseteq V$ open, $F^+(U) = \{y\in Y\colon F(y)\subseteq U\}$ is open.

3. – An existence result.

Let $T = [0, b] = a$ bounded closed interval in R_+, X a separable reflexive Banach space and Z a separable Banach space which whill model the control space.

The optimal control problem that we will study is the following:

$$(*)\qquad \left|\begin{array}{l} \inf J(x, u) = \displaystyle\int_0^b h\big(t, x(t), u(t)\big)\,dt \\[2ex] \text{s.t. } \dot{x}(t) = A(t)x(t) + g\big(t, x(t), u(t)\big) \\[2ex] x(0) = x_0\,, \qquad u(t)\in U\big(t, x(t)\big) \text{ a.e.} \end{array}\right.$$

We will make the following hypotheses concerning $(*)$.

$H(A)$: $\{A(t)\colon t\in T\}$ are linear unbounded operators on $D(A(t))\subseteq X$ that generate a strongly continuous evolution operator $S(t, s)$ which is compact for $t - s > 0$.

$H(g)$: $g\colon T\times X\times Z \to X$ is a function s.t.

(1) $t \to (t, x, u)$ is measurable

(2) $(x, u) \to g(t, x, u)$ is sequentially continuous from $X\times Z_w$ into X_w (here X_w, Z_w denote the Banach spaces X, Z with the weak topology).

(3) for every $t\in T$, $g(t, \cdot, \cdot)$ is bounded.

(4) for every $\varepsilon > 0$ there exists $\psi_\varepsilon(\cdot)\in L_+^1$ s.t.

$$\|g(t, x, u)\| \leq \psi_\varepsilon(t) + \varepsilon h(t, x, u) \text{ a.e.}$$

$H(h)$: $h\colon T\times X\times Z \to \bar{R} = RU\{+\infty\}$ is a function s.t.

(1) $(t, x, u) \to h(t, x, u)$ is measurable

(2) $(x, u) \to h(t, x, u)$ is sequentially l.s.c. on $X\times Z_w$

$H(Q)$: For every $(t, x) \in T \times X$ the set

$$Q(t, x) = \{(\mu, \eta) : \mu \geqslant h(t, x, u),\ \eta = g(t, x, u),\ u \in U(t, x)\}$$

is convex.

$H(U)$: $U: T \times X \to P_{fc}(Z)$ is a multifunction s.t.

(1) $(t, x) \to U(t, x)$ is measurable and $x \in U(t, x)$ is u.s.c. from X into Z_w.

(2) $U(t, x) \subseteq W$ a.e. with $w \in P_{wkc}(Z)$.

The growth condition $H(g)(4)$ was first introduced by CESARI-LA PALM-NISHIURA [14] in order to establish certain compactness and closure properties. Since then it became a popular tool among people working on existence theorems. For an interesting comparison of this growth property with others used in the literature, we refer to CESARI [13] (section 10.4). Hypothesis $H(Q)$ is the convexity hypothesis mentioned in the introduction and is needed to establish the lower semicontinuity of the cost functional $J(x, u)$ in an appropriate topology. By a solution of the evolution equation we will understand a mild (integral) solution.

A pair of functions $x: T \to X$ and $u: T \to Z$ that satisfy all the constraints of problem $(*)$ is an « admissible pair ». In this case $x(\cdot)$ is called an « admissible trajectory » and $u(\cdot)$ an « admissible control. To avoid trivialities we will need the following hypotheses.

H: System $(*)$ has admissible pairs (x, u) s.t. $J(x, u) < +\infty$.

We can now state our existence result:

THEOREM 3.1. *If hypotheses* $H(A)$, $H(g)$, $H(h)$, $H(Q)$, $H(U)$ and H: hold, *then* problem $(*)$ admits an optimal solution (x, u)

PROOF. – Let $P: T \times X \times X \to 2^W$ be the multifunction defined by:

$$P(t, x, \eta) = \{u \in U(t, x) : \eta = g(t, x, u)\}\ .$$

Also let $r: T \times X \times X \to \bar{R}$ be the function defined by:

$$r(t, x, \eta) = \inf\{h(t, x, u) : u \in P(t, x, \eta)\} \quad (\inf \emptyset = +\infty)\ .$$

Claim 1. – $(t, x, \eta) \to r(t, x, \eta)$ *is measurable.*

Let $\lambda \in R$ and consider the level set

$$L = \{(t, x, \eta) \in T \times X \times X : r(t, x, \eta) \leq \lambda\}\ .$$

We will show that $L \in B(T) \times B(X) \times B(X)$. Let

$$\hat{L}\{(t, x, \eta, u) \in T \times X \times X \times W : h(t, x, u) \leqq \lambda,\ u \in P(t, x, \eta)\}\ .$$

Observe that $L = \text{pro}_k \hat{L}_{T\times X\times X}$. Also note that

$$\hat{L} = \text{Gr}\, P \cap \{(t, x, \eta, u) \in T\times X\times X\times W: h(t, x, u) \leqslant \lambda\}.$$

From the definition of the multifunction $P(\cdot, \cdot, \cdot)$ we have:

$$\text{Gr}\, P = \{(t, x, \eta, u) \in T\times X\times X\times W: \eta - g(t, x, u) = 0\} \cap \text{Gr}\, \hat{U}$$

where $\hat{U}: T\times X\times X \to P_{fc}(W)$ is defined by $\hat{U}(t, x, \eta) = U(t, x)$. Since by $H(U)(1)$, $U(\cdot,\cdot)$ is measurable, then so is $\hat{U}(\cdot,\cdot,\cdot)$. Thus $\text{Gr}\, \hat{U} \in B(T\times X\times X\times W) = B(T)\times B(X)\times B(X)\times B(W)$ (see Wagner [33]). Also let $\{x_n^*\}_{n\geqslant 1}$ be dense in X^* for the Mackey topology $m(X^*, X)$. Such a set exists since X is separable (see lemma III-32 of CASTAING-VALADIER [9]). Then because of hypotheses $H(g)(1)$, and (2) for every $n \geqq 1$, $t \to (x_n^*, \eta - g(t, x, u))$ is measurable and $(x, \eta, u) \to (x_n^*, \eta - g(t, x, u))$ is continuous from $X\times X\times W_w$ into R (here W_w denotes the set W with the relative weak topology). Since W_w is compact metrizable (see DUNFORD-SCHWART} [18], theorem 3, 0. 434), we can apply lemma III-14 of CASTAING-VALADIER [9] and get that $(t, x, \eta, u) \to (x_n^*, \eta - g(t, x, u))$ is measurable on

$$\begin{aligned} T\times X\times X\times W_w \Rightarrow \{(t, x, \eta, u) \in T\times X\times X\times W: \eta - g(t, x, u) = 0\} &= \\ = \bigcap_{n\geqslant 1} \{(t, x, \eta, u) \in T\times X\times X\times W: (x_n^*, \eta - g(t, x, u)) = 0\} &\in \\ \in B(T)\times B(X)\times B(X)\times B(W_w)&. \end{aligned}$$

But from corollary 2.4 of EDGAR [19] we know that $B(Z_w) = B(Z) \Rightarrow B(W_w) = B(Z_w) \cap W = B(Z) \cap W = B(W)$. So finally we conclude that $\text{Gr}\, P \in B(T)\times B(X)\times B(X)\times B(W)$. Also because of hypothesis $H(h)(1)$

$$\{(t, x, \eta, u) \in T\times X\times X\times W: h(t, x, u) \leq \lambda\} \in B(T)\times B(X)\times B(X)\times B(W).$$

Hence finally we can write that $\hat{L} \in B(T)\times B(X)\times B(X)\}B(W)$. Applying the Arsenin-Novikov theorem (see DELLACHERIE [15] or SAINT-BEUVE [32]), we get that $L = \text{pro}\, j_{T\times X\times X}\hat{L} \in B(R)\times B(X)\times B(X) \Rightarrow r(\cdot,\cdot,)$ is measurable.

Claim 2. – For every $t \in T$, $(x, \eta) \to r(t, x, \eta)$ is l.s.c. from $X\times X$ into $\bar{R}$.

We need to show that if $(x_n, \eta_n) \xrightarrow{s} (x, \eta)$ in $X\times X$, then we have $r(t, x, \varkappa) \leq \varliminf r(t, x_n, \eta_n)$. From the definition of r, for every $n \geqq 1$, we can find $u_n \in U(t, x_n)$ s.t.

$$h(t, x_n, u_n) \leqslant r(t, x_n, \eta_n) + 1/n.$$

Since $u_n \in U(t, x_n) \subseteq W \to P_{wkc}(Z)$, through the Eberliin-Smulian theorem and by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in Z. Then $\eta_n = g(t, x_n, u_n) \to \eta = g(t, x, u)$ and $u \in w - \overline{\lim}\, U(t, x_n) \subseteq U(t, x)$ since $U(t, \cdot)$ is u.s.c. from X into Z_w. Also exploiting the sequential lower semicontinuity of $h(t, \cdot, \cdot)$ (see $H(h)(2)$) we have:

$$h(t, x, u) \leqq \underline{\lim}\, h(t, x_n, u_n) \leqq \underline{\lim}\, r(t, x_n, \eta_n)\,.$$

Because $u \in P(t, x, \eta)$, from the definition of r, we conclude that:

$$r(t, x, \eta) \leqq \underline{\lim}\, r(t, x_n, \eta_n) \Rightarrow (x, \eta) \to r(t, x, \eta) \text{ is l.s.c.}$$

Claim 3. – $\eta \to r(t, x, \eta)$ *is convex.*

This follows by direct calculation, from hypothesis $H(Q)$.

Now note that if in the growth condition $H(g)(4)$, we take $\varepsilon = 1$, then we have:

$$0 \leqq \psi_1(t) + h(t, x, u) \text{ a.e.} \Rightarrow - \psi_1(t) \leqq h(t, x, u) \text{ a.e.}$$

Recalling that $\psi_1(\cdot) \in L^1$ and because of hypothesis H: we have that:

$$-\infty < m = \inf\{J(x, u) \colon (x, u) \text{ is admissible}\} < \infty\,.$$

Now let $\{(x_n, u_n)\}_{n \geqq 1}$ be a minimizing sequence of admissible pairs. We may assume without any loss of generality that for all $n \geqq 1$ $J(x_n, u_n) \leqq K$, where $K > m$. Take $t, t' \in T$, $t \leqq t'$ and $(x, u) \in \{(x_n, u_n)\}_{n \geqq 1}$. We have

$$\begin{aligned}
&\|x(t') - x(t)\| = \\
&\quad = \left\| S(t', 0)x_0 + \int_0^{t'} S(t', s)g(s, x(s), u(s))\, ds - S(t, 0)x_0 - \int_0^t S(t, s)g(s, x(s), u(s))\, ds \right\| \leqq \\
&\quad \leqq \|S(t', 0)x_0 - S(t, 0)x_0\| + \int_t^{t'} \|S(t', s)\| \cdot \|g(s, x(s), u(s))\|\, ds + \\
&\quad + \int_0^t \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\|\, ds\,.
\end{aligned}$$

We will estimate each of the three quantities of the right hand side separateliy. Because of the strong continuity of the evolution operator given $\varepsilon > 0$ there exists $\delta_1 > 0$ s.t. for $|t' - t| < \delta_1$ we have:

$$\|S(t', 0)x_0 - S(t, 0)x_0\| < \varepsilon/3\,. \tag{1}$$

Also recall that for all $(t, s) \in \{0 \leqq s \leqq t \leqq b\}$, $\|S(t, s)\| \leqslant M$ and take $\theta = \varepsilon/6M(\|\psi_1\|_1 + K)$. Then using the growth hypothesis $H(g)(4)$ we have:

$$\int_t^{t'} \|S(t', s)\| \cdot \|g(s, x(s), u(s)\| \, ds \leqq M\int_t^{t'} \Big(\psi_\theta(s) + \theta\big(\psi_1(s) + h(s, x(s), u(s))\big)\Big) \, ds \leqq$$

$$\leqq M\int_t^{t'} \psi_\theta(s)\, ds + M\theta(\|\psi_1\|_1 + K) \leqq M\int_t^{t'} \psi_\theta(s)\, ds + \varepsilon/6 \, .$$

We can find $\delta_2 > 0$ s.t. if $|t - t'| < \delta_2$, then $M\int_t^{t'} \psi_\theta(s)\, ds < \varepsilon/6$. Thus finally we have:

$$\int_t^{t'} \|S(t', s)\| \cdot \|g(s, x(s), u(s))\| \, ds < \varepsilon/3 \, . \tag{2}$$

Finally let $\varepsilon_1 > 0$ (to be chosen more precisely in the sequel) and consider:

$$\int_0^{t} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\| \, ds \leqq$$

$$\leqq \int_0^{t-\varepsilon_1} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\| \, ds + \int_{t-\varepsilon_1}^{t} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\| \, ds \, .$$

From proposition 2.1 of [28], we know that because of the compactness hypothesis on $S(t, s)$ for $t - s > 0$, we have that $t \to S(t, s)$ is continuous in the operator norm, uniformly for all s s.t. $t - s$ is bounded away from zero. Thus we can find $\delta_3 > 0$ s.t. for $|t' - t| < \delta_3$ we have $\|S(t', s) - S(t, s)\| < \varepsilon/6(\|\psi_1\|_1 + K)$ for all $s \in T$ s.t. $t - s \geqq \varepsilon_1 > 0$ and so we get that

$$\int_0^{t-\varepsilon_1} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\| \, ds \leq$$

$$\leq \big(\varepsilon/6(\|\psi_1\|_1 + K)\big) \int_0^{t-\varepsilon_1} \big(\psi_1(s) + h(s, x(s), u(s))\big) \, ds <$$

$$< \varepsilon/6\big((\|\psi_1\|_1 + K)\big) \int_0^{b} \big(\psi_1(s) + h(s, x(s), u(s))\big) \, ds < \varepsilon/6 \, .$$

Furthermore if $\theta' = \varepsilon/24M(\|\psi_1\|_1 + K)$, we have

$$\int_{t-\varepsilon_1}^{t} \|S(t', s) - S(t, s)\| \, \|g(s, x(s), u(s))\| \, ds \leqq$$

$$\leqq 2M\int_{t-\varepsilon_1}^{t} \Big(\psi_{\theta'}(s) + \theta'\big(\psi_1(s) + h(s, x(s), u(s))\big)\Big) \, ds \leqq 2M\int_{t-\varepsilon_1}^{t} \psi_{\theta'}(s)\, ds + 2M\theta(\|\psi\|_1 + K) \, .$$

So if $\varepsilon_1 > 0$ is s.t. $2M\int_{t-\varepsilon_1}^{t} \psi_{\theta'}(s)\,ds < \varepsilon/12$, we have that:

$$\int_{t-\varepsilon_1}^{t'} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\|\,ds < \varepsilon/12 + \varepsilon/12 = \varepsilon/6\,.$$

Hence finally we have that

$$\int_{0}^{t} \|S(t', s) - S(t, s)\| \cdot \|g(s, x(s), u(s))\|\,ds < \varepsilon/6 + \varepsilon/6\,. \tag{3}$$

From (1), (2) and (3) above, we conclude that for $|t' - t| < \min(\delta_k)_{k=1}^{3}$ we have for all $x \in \{x_n(\cdot)\}_{n\geqslant 1}$

$$\|x(t') - x(t)\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \Rightarrow \{x_n(\cdot)\}_{n\geqslant 1} \text{ is equicontinuous}\,.$$

Also note that for all $n \geqslant 1$ and all $t \in T$ we have:

$$\begin{aligned}
\|x_n(t)\| &= \Big\|S(t, 0)x_0 + \int_0^t S(t, s)g(s, x(s), u(s))\,ds\Big\| \leqq \\
&\leqq M\|x_0\| + M\int_0^t \big(\psi_1(s) + h(s, x(s), u(s))\big)\,ds \leqq \\
&\leqq M\|x_0\| + M\int_0^b \big(\psi_1(s) + h(s, x(s), u(s))\big)\,ds \leqq M\|x_0\| + M\|\psi_1\|_1 + MK = \bar{M}\,.
\end{aligned}$$

Thus if $B(0, \bar{M})$ is the closed ball in X of radius $\bar{M}$, centered at the origin, we ha for all $n \geqslant 1$ and all $t \in T$:

$$g(t, x_n(t), u_n(t)) \in \overline{g(t, B(0, \bar{M}), W)}^{w} = V(t)\,.$$

Because of hypothesis $H(g)(3)$, for every $t \in T$, $V(t)$ is bounded (in fact relatively w-compact since X is reflexive). From hypotheses $H(g)(1)$ and (2), we deduce that for every $n \geqslant 1$, $t \to g(t, x_n(t), u_n(t))$ is weakly measurable and since X is separable, by Pettis theorem (see DIESTEL-UHL [17], theorem 2, p. 42), is measurable. So from theorem 4.2 of WAGNER [33], we get that $t \to V_1(t) = \overline{\text{conv}}\,\{g(t, x_n(t), u_n(t))\}_{n\geqq 1}$ is measurable and $V_1(t) \subseteq \overline{\text{conv}}\,V(t) \in P_{wkc}(X)$. Thus $s \to S(t, s)V_1(s)$ is measurable and $P_{kc}(X)$-valued. Invoking Rädstrom's embedding theorem (see HIAI-UME-

GAKI [24]) we get that $\int_0^t S(t, s)\, V_1(s)\, ds \in P_{kc}(X)$. So for all $n \geq 1$ and all $t \in T$

$$x_n(t) \in S(t, 0)x_0 + \int_0^t S(t, s)\, V_1(s)\, ds \in P_{kc}(X) .$$

Now the Arzela-Ascoli theorem tells us that $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T, X)$.

Next let $g_n(t) = g(t, x_n(t), u_n(t)) \in L^1(X)$. We have already seen that for all $t \in T$, $V_1(t) \in P_{wkc}(X)$. So $t \to V_1(t)$ is a $P_{wkc}(X)$-valued integrably bounded multifunction. Hence theorem 4.2 of [29] tells us that $S^1_{V_1}$ is w-compact in $L^1(X)$ and by Eberlein-Smulian theorem sequentially w-compact. Since $g_n \in S^1_{V_1}$, by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} y \in S^1_{V_1}$ in $L^1(X)$. Consider the multifunction

$$N(t) = \{u \in W \colon u \in P(t, x(t), y(t)),\, h(t, x(t), u) \leq r(t, x(t), y(t))\} .$$

First we will show that for every $t \in T$, $N(t) \neq \emptyset$. From the definition of r we know that for every $m \geq 1$ we can find $u_m \in U(t, x(t))$ s.t.

$$h(t, x(t), u)_m \leq r(t, x(t), y(t)) + 1/m \quad \text{and } y(t) = g(t, x(t), u_m) .$$

As before, we may assume that $u_m \xrightarrow{w} u \in U(t, x(t))$. Then we get:

$$h(t, x(t), u) \leq \underline{\lim}\, h(t, x(t), u_m) \leq r(t, x(t), y(t)) \Rightarrow u \in N(t) \Rightarrow N(t) \neq \emptyset \quad \text{for all } t \in T .$$

Next observe that:

$$\operatorname{Gr} N = \{(t, u) \in \operatorname{Gr} P(\cdot, x(\cdot), y(\cdot)) \colon q(t, u) \leq p(t)\}$$

where $p(t) = r(t, x(t), y(t))$ and $q(t, u) = h(t, x(t), u)$.

Note that

$$P(t, x(t), y(t)) = \{u \in U(t, x(t)) \colon y(t) - g(t, x(t), u) = k(t, u) = 0\} .$$

Because of hypothesis $H(U)(1)$, $\operatorname{Gr} U(\cdot, x(\cdot)) \in B(T) \times B(W)$, while $(t, u) \to k(t, u)$ is clearly measurable.

Thus

$$\operatorname{Gr} P(\cdot, x(\cdot), y(\cdot)) = \{(t, u) \in \operatorname{Gr} U(\cdot, x(\cdot)) \colon k(t, u) = 0\} \in B(T) \times B(W) .$$

Since $p(\cdot)$, $q(\cdot, \cdot)$ are clearly measurable, we conclude that $\operatorname{Gr} N \in B(T) \times B(W)$. Apply Aumann's selection theorem (SAINT-BEUVE [31], theorem 3), to get $u \colon T \to W$

measurable s.t. $u(t) \in N(t)$ for all $t \in T$. Then from the definition of $N(\cdot)$ we have that

$$y(t) = g(t, x(t), u(t))\,, \qquad u(t) \in U(t, x(t))\,,$$
$$h(t, x(t), u(t)) \leqq r(t, x(t), y(t))\,.$$

Since $r(t, x, \cdot)$ is convex and X is reflexive, we can apply theorem 4 of Castaing-Clauzure [8] and get that

$$J(x, u) \leqq \int_0^b r(t, x(t), y(t))\,dt \leqq \varliminf \int_0^b r(t, x_n(t), g_n(t))\,dt \leqq \varliminf J(x_n, u_n) = m\,.$$

But we saw above that (x, u) is admissible, So $J(x, u) = m \Rightarrow (x, u)$ is the desired optimal solution. Q.E.D.

4. – Relaxation.

Scrutinizing the proof of theorem 3.1, we see that hypothesis $H(Q)$ played a key role. Namely it gave us the convexity of $r(t, x, \cdot)$ (claim 3) and then through that property, we were able to extablish the lower semicontinuity of the cost functional and so obtain an optimal pair. In this section we are going to see what happens if we drop hypothesis $H(Q)$. In this case, in order to get optimal pairs, we need to pass to a larger system with measure valued controls known as the « relaxed system ». This new augmented system has the following form:

$$(**) \qquad \text{s.t.} \begin{cases} \inf J_r(x, u) = \displaystyle\int_0^b \int_Z h(t, x(t), z)\, \lambda(t)(dz)\, dt \\ \dot{x}(t) = A(t)x(t) + \displaystyle\int_\Sigma g(t, x(t), z)\, \lambda(t)(dz) \\ x(0) = x_0\,, \qquad \lambda \in S_\Sigma \end{cases}$$

here $\Sigma(t) = \{\lambda \in M_+^1(Z)\colon \lambda(U(t)) = 1\}$ and S_Σ denotes the set of all transition probabilities which are selectros of $\Sigma(\cdot)$.

In this section, $T = [0, b]$, X is a separable Banach space and Z is a compact Polish space.

We will need the following hypotheses.

$H(A)_1$: the same as $H(A)$

$H(g)_1$: $g\colon T\times X\times Z\to X$ is a function s.t.

(1) $t\to g(t,x,z)$ is measurable;

(2) $(x,z)\to g(t,x,z)$ is continuous;

(3) for every $t\in T$, $g(t,\cdot,\cdot)$ is bounded;

(4) for every $\varepsilon>0$, there exists $\psi_\varepsilon(\cdot)\in L^1$ s.t.

$$\|g(t,x,z)\|\leq \psi_\varepsilon(t)+\varepsilon h(t,x,z) \text{ a.e.}$$

$H(h)_1$: $h\colon T\times X\times Z\to R_+$ is a function s.t.

(1) $t\to h(t,x,z)$ is measurable;

(2) $(x,z)\to h(t,x,z)$ is continuous;

(3) $h(t,x,z)\leq \alpha(t)$ a.e., with $\alpha(\cdot)\in L^1$.

$H(U)_1$: $U\colon T\to P_f(Z)$ is a measurable multifunction.

As before by a solution of the relaxed evolution equation we understand a mild (integral) solution. Again we will make a hypothesis quaranteeing that our problem has content. This hypothesis (as well as hypothesis H_α) can be viewed as controllability type hypotheses.

$H\alpha_1$: There exist admissible relaxed pairs (x,λ) for which we have $J_r(x,\lambda)<+\infty$.

We will denote the value of original problem by m and the value of the relaxed problem by m_r.

THEOREM 4.1. – *If* hypotheses $H(A)_1$, $H(g)_1$, $H(h)_1$, $H(U)_1$ and $H\alpha_1$ hold, *then* problem $(**)$ admits an optimal solution (x,λ) and furthermore $m=m_r$.

PROOF. – Let $\{(x_n,\lambda_n)\}_{n\geq1}$ be a minimizing sequence of problem $(*)$. Working as in the proof of theorem 3.1 we can show that $\{x_n\}_{n\geq1}$ is relatively compact in $C(T,X)$. Also from theorem IV-2 of CASTAING-VALADIER [9] we know that $\{\lambda_n\}_{n\geq1}$ is relatively $w\big(L^\infty(M(Z)),L^1(C(Z))\big)$-compact. Since Z is compact, $C(Z)$ is separable and then so is $L^1(C(Z))$. Thus theorem 1, p. 426, of DUNFORD-SCHWARTZ [18] tells us that $\overline{\{\lambda_n\}}^{w^*}_{n\geq1}$ is metrizable in the w^*-topology. Hence by passing to a subsequence if necessary, we may assume that $(x_n,\lambda_n)\xrightarrow{sxw^*}(x,\lambda)$ in $C(T,X)\times L^\infty(M(Z))$. By identifying the space of Carathéodory integrands with $L^1(C(Z))$, we can see that $\lambda_n\to\lambda$ in $R(T,Z)$ with the weak topology. Applying theorem 3.1 of JAWHAR [25] to the Carathéodory integrands h and $\alpha-h$, we get that

$$\lim\int_0^b\int_Z h(t,x_n(t),z)\,\lambda_n(t)(dz)\,dt=m_r=\int_0^b\int_Z h(t,x(t),z)\,\lambda(t)(dz)\,dt=J_r(x,\lambda)\,.$$

If we can show that (x, λ) is an admissible relaxed pair, we will have that (x, λ) is the desired optimal pair. By definition we have:

$$x_n(t) = S(t, 0)x_0 + \int_0^t \int_Z g(s, x_n(s), z)\, \lambda_n(s)(dz)\, ds\,.$$

Again using the result of Jawhar [25], we have that

$$\int_0^t \int_Z g(s, x_n(s), z)\, \lambda_n(s)(dz)\, ds \to \int_0^t \int_Z g(s, x(s)\, z)\, \lambda(s)(dz)\, ds\,.$$

$$\Rightarrow x(t) = S(t, 0)x_0 + \int_0^t \int_Z g(s, x(s), z)\, \lambda(s)(dz)\, ds\,.$$

$\Rightarrow x$ is a mild solution of the relaxed evolution equation

$\Rightarrow (x, \lambda)$ is admissible relaxed pair.

$\Rightarrow J_r(x, \lambda) = m_r$.

Now we will show that $m = m_r$.

Let (x, λ) be the optimal relaxed pair obtained above. Using corollary 3 of Balder [3], we know that we can find measurable selectors $u_n(\cdot)$ of $U(\cdot)$ s.t. $\delta(u_n) \to \lambda$ in $R(T, Z)$. Let $x_n(\cdot)$ be an admissible trajectory corresponding to $u_n(\cdot)$. Such trajectories exist by hypothesis H_{α_1}. Then we have:

$$x_n(t) = S(t, 0)x_0 + \int_0^t S(t, s)\, g(s, x_n(s), u_n(s))\, ds =$$
$$= S(t, 0)x_0 + \int_0^t \int_Z S(t, s)\, g(s, x_n(s), z)\, \delta(u_n(s))(dz)\, ds\,.$$

Once again we have that

$$\int_0^t \int_Z S(t, s)\, g(s, x_n(s), z)\, \delta(u_n(s))(dz)\, ds \to \int_0^t \int_Z S(t, s)\, g(s, x(s), z)\, \lambda(s)(dz)\, ds \Rightarrow x_n(t) \to x(t)\,.$$

So as above we have that $J_r(x_n, \delta(u_n)) \to J_r(x, \lambda) = m_r \Rightarrow m_r \geqq m$. On the other hand it is clear that $m_r \leq m$. So we conclude that $m = m_r$. Q.E.D.

5. – « Bang-bang » results.

In the section we turn our attention to linear systems and we obtain « bang-bang » properties of the trajectories of such systems. The first result is an « approximate bang-bang » result, while the second is an exact « bang-bang » result.

We consider the following two systems:

$$(*)' \qquad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0\,, \quad u \in S_U^1 \end{cases}$$

and

$$(**)' \qquad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0\,, \quad u \in S^1_{\overline{\mathrm{conv}}\, U}\,. \end{cases}$$

Denote the set of admissible trajectories of $(*)'$ by $P(x_0)$ and the set of admissible trajectories of $(**)'$ by $P_r(x_0)$. We want to relate those two sets.

We will need the following hypotheses.

$H(A)_2$: $\{A(t): t \in T\}$ are unbounded linear operators on $D(A(t)) \subseteq X$, that generate a strongly continuous evolution operator $0 \leqq s \leqq t \leqq b$.

$H(B)$: $B \in L^\infty(T, L(X))$.

$H(U)_2$: $U: T \to 2^Z \setminus \{\emptyset\}$ is graph measurable with $t \to |U(t)| = \sup\{\|z\|: z \in U(t)\}$ belongs in L_+^1.

In this case the state and control spaces X, Z are two separable Banach spaces.

THEOREM 5.1. – *If hypotheses* $H(A)_2$, $H(B)$, $H(U)_2$ *hold then* $\overline{P(x_0)} = \overline{P_r(x_0)}$, the closure taken in the strong topology of $C(T, X)$. Also the set is convex.

PROOF. – Let $x(\cdot) \in P_r(x_0)$ and $\varepsilon > 0$. Pick $\delta > 0$ s.t. for $A \subseteq T$ Lebesque measurable with $\lambda(A) < \delta$ ($\lambda =$ Lebesgue measurable), we have $\int_A \|B(s)\| \cdot |U(s)|\, ds < \varepsilon/4M$, where $\|S(t, s)\| \leqq M$ for all $0 \leqq s \leqq t \leqq b$. Such a set exists because of the absolute continuity of the Lebesgue integral. Let $0 = t_0 < t_1 < \ldots < t_n = b$ be an equipartition of the interval T s.t. $|t_k - t_{k-1}| < \delta$ $k \in \{1, 2, \ldots, n\}$. By definition we have:

$$x(t_1) \in S(t_1, 0)x_0 + q_r(t_1)$$

where $q_r(t_1) \in \int_0^{t_1} S(t_1, s) B(s)\, \overline{\mathrm{conv}}\, U(s)\, ds$. From theorem 3.1 of [26] (see also corollary 4.3 of HIAI-UMEGAKI [24]), we know that

$$\operatorname{cl}\int_0^{t_1} S(t_1, s)B(s)\,\overline{\operatorname{conv}}\,U(s)\,ds = \operatorname{cl}\int_0^{t_1} S(t_1, s)B(s)\,U(s)\,ds\,.$$

So we can find $u_1 \in S_U^1$ s.t.

$$\left\|q_r(t_1) - \int_0^{t_1} S(t_1, s)B(s)u_1(s)\,ds\right\| < \delta'$$

with $0 < \delta' \leqq \varepsilon'/\sum_{k=1}^{n} M^{k-1}$ and $\varepsilon' = \min(\varepsilon/2, \varepsilon/2M)$. Set

$$y_1(t) = S(t, 0)x_0 + \int_0^t S(t, s)B(s)u_1(s)\,ds \qquad t \in [0, t_1]\,.$$

Then clearly we have that $\|y_1(t_1) - x(t_1)\| < \delta'$.

Next we pass to the subinterval $[t_1, t_2]$. Again we have:

$$x(t_2) = S(t_2, t_1)x(t_1) + q_r(t_2)$$

where $q_r(t_2) \in \int_{t_1}^{t_2} S(t_2, s)B(s)\,\overline{\operatorname{ccov}}\,U(s)\,ds$. As above we can find $u_2 \in S_U^1$ s.t.

$$\left\|q_r(t_2) - \int_{t_1}^{t_2} S(t_2, t_1)B(s)u_2(s)\,ds\right\| < \delta'$$

Set $y_2(t) = S(t_2, t_1)y_1(t_1) + \int_{t_1}^{t_2} S(t_2, s)B(s)u_2(s)\,ds$ for $t \in [t_1, t_2]$. Hence

$$\|y_2(t_2) - x(t_2)\| = \left\|S(t_2, t_1)y_1(t_1) + \int_{t_1}^{t_2} S(t_2, s)B(s)u_2(s)\,ds - S(t_2, t_1)x(t_1) - q_r(t_2)\right\| \leqq$$

$$\leqq \|S(t_2, t_1)\| \cdot \|y_1(t_1) - x(t_1)\| + \left\|\int_{t_1}^{t_2} S(t_2, s)B(s)u_2(s)\,ds - q_r(t_2)\right\| \leqq M\delta' + \delta' = \delta'(M+1)\,.$$

Continuing this way we get $u_k \in S_U^1$ and define for $t \in [t_{k-1}, t_k]$

$$\text{s.t. } y_k(t) = S(t_k, t_{k-1})y_{k-1}(t_{k-1}) + \int_{t_{k-1}}^{t_k} S(t_k, s)B(s)u_k(s)\,ds\,.$$

For this we will have that $\|y_k(t_k) - x(t_k)\| < \delta' \sum_{r=1}^{k} M^{r+1}$.

If we set $\hat{u} = \sum_{k=1}^{n} \chi_{[t_{k-1},t_k]} u_k$, we set that $u \in S_U^1$ and

$$y(t) = S(t, 0)x_0 + \int_0^t S(t, s)B(s)\hat{u}(s)\,ds = \sum_{k=1}^{n} \chi_{[t_{k-1},t_k]}(t)\, y_k(t)$$

is a solution of $(*)'$ i.e. $y(\cdot) \in P_r(x_0)$. Furthermore from our construction we have that $\|y(t_k) - x(t_k)\| < \varepsilon'$ for all $k \in \{1, \ldots, n\}$.

Next let $t \in T$ arbitrary. Then $t \in (t_k, t_{k+1}]$ for some $k \in \{0, 1, \ldots, n-1\}$.

Let $u \in S^1_{\overline{\mathrm{conv}\,U}}$ be the control generating $x(\cdot)$. We have:

$$\|y(t) - x(t)\| = \left\| S(t, t_k)y(t_k) + \int_{t_k}^{t} S(t, s)B(s)\hat{u}(s)\,ds - \right.$$

$$\left. - S(t, t_k)x(t_k) - \int_{t_k}^{t_s} S(t, s)B(s)u(s)\,ds \right\| \leqq M \min(\varepsilon/2, \varepsilon/2M) + M2\varepsilon/4M = \varepsilon\,.$$

Since $t \in T$ was arbitrary we conclude that:

$$\|y - x\|_\infty \leqq \varepsilon \Rightarrow \overline{P(x_0)} = \overline{P_r(x_0)}\,.$$

Clearly $P_r(x_0)$ is convex. Q.E.D.

REMARK. – An important special case of the above theorem is when $U(t) = \operatorname{ext} V(t)$ (the extreme points of $V(t)$), where $V(\cdot)$ is a $P_{wkc}(X)$-valued, integrably bounded multifunction. From Benamara [6] we know that $U(\cdot)$ is graph measurable and so the theorem applies. Hence every trajectory generated by a control in S_V^1 can be approximated by trajectories generated from extremal controls (« bang-bang » controls). Thus theorem 5.1 can be viewed as an approximate « bang-bang » theorem. Our result extends the analogous ones by Hermes-LaSalle [23] ($X = R^n$) and Fattorini [20] (infinite dimensional but with time independent control constraint set).

Next we will present an exact « bang-bang » theorem, For this we will need to following set of hypotheses.

$H(A)_3$: The same as $H(A)$.

$H(B)_3$: The same as $H(B)_2$.

$H(U)_3$: $U\colon T \to P_{wk}(X)$ is integrably bounded.

THEOREM 5.2. – *If* hypotheses $H(A)_3$, $H(B)_3$ and $H(U)_3$ hold and if (x, u) is an admissible pair for $(*)'$ *then* there exists another admissible pair $(\hat{x}, \hat{u})$ s.t. $\hat{x}(0) = x_0$ $\hat{x}(b) = x(b)$ and $\hat{u}(t) \in b\,dU(t)$ a.e.

PROOF. – By hypothesis we have:

$$x(b) = S(b, 0)x_0 + \int_0^b S(b, s)B(s)u(s)\,ds$$

$$\in S(b, 0)x_0 + \int_0^b S(b, s)B(s)\,\overline{\text{conv}}\,U(s)\,ds$$

$$= S(b, 0)x_0 + \int_0^b S(b, s)B(s)\,\overline{\text{conv}}\,b\,dU(s)\,ds$$

$$= S(b, 0)x_0 + \int_0^b \overline{\text{conv}}\,S(b, s)B(s)b\,dU(s)\,ds \qquad \text{(from the corollary in p. 188 of [30])}$$

$$= S(b, 0)x_0 + \text{cl}\int_0^b S(b, s)B(s)b\,dU(s)\,s\,ds \qquad \text{(from theorem 3.1 of [26])}\,.$$

But from hypothesis $H(A)_3$, $S(b, s)$ is compact for $0 \leqq s < b$, while $B(s)b\,dU(s)$ is bounded. So $S(b, s)B(s)b\,dU(s)$ is compact. Then from Rädstrom's embedding theorem we know that $\int_0^b S(b, s)B(s)b\,dU(s)\,ds$ is compact. Hence finally we can write that:

$$x(b) \in S(b, 0)x_0 + \int_0^b S(b, s)B(s)b\,dU(s)\,ds\,.$$

This, from the definition of the set valued integral, menas that there exists $\hat{u} \in S^1_{bdU}$ s.t.

$$\hat{x}(b) = S(b, 0)x_0 + \int_0^b S(b, s)B(s)\hat{u}(s)\,ds\,.$$

Setting $\hat{x}(t) = S(t, 0)x_0 + \int_0^t S(t, s)B(s)\hat{u}(s)\,ds$, we see that $(\hat{x}, \hat{u})$ is the desired new admissible pair. Q.E.D.

REMARK. – In a time optimal control problem, if (x, u) is an optimal pair, then there is another time optimal pair $(\hat{x}, \hat{u})$ for which $\hat{u}(s) \in b\,dU(s)$. In particular if the time optimal pair is unique, then the control is « bang-bang » $u(t) \in b\,dU(t)$ a.e.

6. – Examples.

In this section we present some simple examples of control systems governed by partial differential equations, to which the results obtained in the previous sections apply.

EXAMPLE 1. – Let $T = [0, b]$ and W be a bounded, open domain in R^n with smooth boundary $\Gamma = \partial W$. We consider the following control system defined on $T \times W$:

$$(*)_1 \qquad \begin{cases} \dfrac{\partial x(t, z)}{\partial t} = \Delta x(t, z) + f(t, z, x(t, z))\, u(t, z) \\ x(0, z) = x_0(z) \quad \text{ on } \{0\} \times W \\ x(t, z) = 0 \quad \text{ on } T \times \Gamma \\ |u(t, z)| \leqq \varphi(z) \quad \alpha.e. \end{cases}$$

The cost functional has the following form:

$$\int_0^b \int_W h(t, z, x(t, z), u(t, z))\, dz\, dt .$$

We will make the following hypotheses.

$A)$ $f\colon T \times W \times R \to R$ is a function s.t.

(1) $(t, z) \to f(t, z, x)$ is measurable;

(2) $x \to f(t, z, x)$ is continuous;

(3) given $\varepsilon > 0$ and $M > 0$, there exists $\psi_{\varepsilon,M}(\cdot, \cdot) \in L^2(T \times W)$ s.t.

$$|f(t, z, x)u| \leqq \psi_{\varepsilon,M}(t, z) + \varepsilon h(t, z, x) \text{ a.e.}$$

for all $|u| \leqq M$

$B)$ $h\colon T \times R^n \times R \times R \to \bar{R} = RU\{+\infty\}$ is a function s.t.

(1) $(t, z, x, u) \to h(t, z, x, u)$ is measurable;

(2) $(x, u) \to h(t, z, x, u)$ is l.s.c.;

(3) $u \to h(t, z, x, u)$ is convex;

(4) $h(t\ z\ x\ u) \leqq \alpha(t, z) + b(t, z)|x|$ a.e. with $\alpha(\cdot,\cdot)$, $b(\cdot,\cdot) \in L^2(T \times W)$

$C)$ $\varphi(\cdot) \in L^2(T)$.

$D)$ $x_0(\cdot) \in L^2(W)$.

Let $A = \Delta$, with $D(A) = H_0^1(W) \cap H^2(W)$. It is well known (see for example PAVEL [27], theorem 5.2, p. 214), that A generates a compact semigroup of contractions on $X = L^2(W)$.

Let $F\colon T \times X \times X \to X$ be the Nemitsky operator associated to $f(\cdot,\cdot)(\cdot)$.

Then we have:

$$[F(t, x)u](z) = f(t, z, x(z))\, u(z)\,.$$

Our claim is that $(x, u) \to F(t, x)u$ is sequentially continuous from $X \times X_w$ into X_w. So let $(x_n, u_n) \xrightarrow{s\times w} (x, u)$ in $L^2(W) \times L^2(W)$. Then for every $\psi(\cdot) \in L^2(W)$ we have:

$$\langle \psi, F(t, x_n)u_n\rangle = \int_W \psi(z)\big(F(t, x_n)u_n\big)(z)\, dz = \int_W \psi(z) f(t, z, x_n(z))\, u_n(z)\, dz\,.$$

By passing to a subsequence if necessary, we may assume that $x_n(z) \to x(z)$ a.e. Then:

$$\int_W \psi(z) f(t, z, x_n(z))\, u_n(z)\, dz \to \int_W \psi(z) f(t, z, x(z))\, u(z)\, dz = \langle \psi, F(t, x)u\rangle \Rightarrow$$
$$\Rightarrow F(t, x_n)u_n \xrightarrow{w} F(t, x)u\,.$$

Also, from Fubini's theorem, we have that $t \to \langle \psi, F(t, x_n)u_n\rangle = \int_W \psi(z) f(t, z, x_n(r)) \cdot$ $\cdot u_n(z)\, dz$ is measurable and then so is $t \to \langle \psi, F(t, x)u\rangle$. Since $L^2(W)$ is separable, we conclude that $t \to F(t, x)u$ is measurable.

Finally note that for $M = \|\varphi\|_2$ we have

$$\|F(t, x)u\|_2 \leqq \hat{\varphi}_{\varepsilon,M}(t) + \varepsilon \hat{h}(t, x, u)$$

for all $|u| \leqq M$ and with $\hat{\varphi}_{\varepsilon,M}(t) = \|\psi_{\varepsilon,M}(t, \cdot)\|_2$ and $\hat{h}(t, x, u) = \int_W h(t, z, x(z), u(z))\, dz$.

Now rewrite system $(*)$ as the following abstract evolution equation:

$$(*)\text{i} \qquad \begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\, u(t) \\ x(0) = x_0\,, \quad \|u\| \leqq M\,. \end{cases}$$

Because of hypothesis $B(4)$ and using the growth assumption $A(3)$ with $\varepsilon = 1$, we see that $h(t, \cdot, x, u)$ is $L^2(W)$-bounded. So if $x_n \xrightarrow{s} x$ and $u_n \xrightarrow{w} u$ in X, from theorem 2.1 of Balder [4] we conclude that:

$$\underline{\lim}\, \hat{h}(t, x_n, u_n) = \underline{\lim} \int_W h(t, z, x_n(z), u_n(z))\, dz \geqq$$
$$\geqq \int_W h(t, z, x(z), u(z)) = \hat{h}(t, x, u) \Rightarrow (x, u) \to \hat{h}(t, x, u) \quad \text{is l.s.c. on } X \times X_W\,.$$

Furthermore, because of hypothesis $B(1)$, we have that $\hat{h}(\cdot,\cdot,\cdot)$ is measurable. Also because the control enters linearly in the system, it is easy to check that the orientor field $Q(t, x)$ has convex values for every $(t, x) \in T \times X$.

Finally set $B(0, M) = \{v \in L^2(W): \|v\|_2 \leqq \|\varphi\|_2 = M\}$. This is a weakly compact convex subset of X.

Therefore all hypotheses of theorem 3.1 are satisfied So according to that theorem there exists an admissible pair (x, u) of $(*)_1$ s.t. it minimizes

$$J(x, u) = \int_0^b \hat{h}(t, x(t), u(t))\, dt\,.$$

EXAMPLE 2. – In system $(*)_1$ instead of the Laplacian, consider the operator $\nabla_z(p(z)\nabla_z x(t, z))$ with $p\colon W \to R_+$ continuously differentiable. Set $Ax = \nabla_2(p\cdot\nabla_z x)$, with domain $D(A) = H_0^1(W) \cap H^2(W)$. Then from the compactness criterion of Brezis [7] (see also PAVEL [27], p. 214), we know that A is an unbounded linear operator generating a compact linear semigroup of operators defined on $X = L^2(W)$. The rest are as in example 1 analysed above.

EXAMPLE 3. – For the controlled heat equation

$$(*)_3 \qquad \begin{cases} \dfrac{\partial x(t, z)}{\partial t} = \Delta x(t, z) + u(t, z) \\ x(t, z) = 0 \quad \text{on } T \times \Gamma \\ x(0, z) = x_0(z) \quad \text{on } \{0\} \times W \\ |u(t, z)| \leqq \varphi(z) \quad \text{with } \varphi(\cdot) \in L^2(W) \end{cases}$$

the « bang-bang » result obtained in section 5 (theorem 5.2) holds. So we can find a « bang-bang » time optimal control for system $(*)_3$.

REMARK. – We can also treat boundary control problems, which, following the techniques of Barbu [5], we can transform to abstract evolution equations on a Hilbert space, that admit mild (integral) solutions. For details we refer to Barbu [5].

REFERENCES

[1] T. ANGELL, *On the optimal control of systems governed by nonlinear Volterra equations*, J. Optim. Th. Appl., **19** (1976), pp. 29-45.

[2] T. ANGELL, *Existence of optimal control without convexity and a bang-bang theorem for linear Volterra equations*, J. Optim. Th. Appl., **19** (1976), pp. 63-79.

[3] E. BALDER, *A general denseness result for relaxed control theory*, Bull. Austr. Math. Soc., **30** (1984), pp. 463-475.

[4] E. BALDER, *Lower closure for orientor fields by lower semicontinuity of outer integral functionals*, Ann. Mat. Pura Appl., **139** (1985), pp. 349-360.

[5] V. BARBU, *Boundary control problems with convex cost criterion*, SIAM J. Control Optim., **18** (1980), pp. 227-243.

[6] M. BENAMARA, *Sections mesurables extrémales d'une multiapplication*, C. R. Acad. Sci. Paris, **278** (1974), pp. 1249-1252.

[7] H. BREZIS, *New results concerning monotone operators and nonlinear semigroups*, Proc. RIMS « Analysis of Nonlinear Problems », Kyoto University (1975), pp. 2-27.

[8] C. CASTAING - P. CLAUZURE, *Semicontinuité inférieure des fonctionelles intégrales*, Seminaire Anal. Convexe, Montpellier (1981), exposé no. 15.

[9] C. CASTAING - M. VALADIER, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math., Vol. **580**, Springer, Berlin (1977).

[10] L. CESARI, *Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints, I and II*, Trans. Amer. Math. Soc., **124** (1966), pp. 369-412.

[11] L. CESARI, *Closure theorems for orientor fields and weak convergence*, Arch. Rat. Mech. Anal., **55** (1974), pp. 332-356.

[12] L. CESARI, *Convexity and property (Q) in optimal control theory*, SIAM J. Control, **12** (1974), pp. 705-720.

[13] L. CESARI, *Optimization, Theory and Applications*, Springer, New York (1983).

[14] L. CESARI - R. LAPALM - T. NISHIURA, *Remarks on some existence theorems for optimal control*, J. Optim. Th. Appl., **3** (1969), pp. 296-305.

[15] C. DELLACHERIE, *Ensembles analytiques: Théorèmes de séparation et applications*, Sém. de probabilités, IX, Univ. Strasbourg, Lecture Notes in Math., Vol. **465**, Springer, Berlin, New York (1975).

[16] C. DELLACHERIE - P. A. MEYER, *Probabilities and Potential*, North Holland, Amsterdam (1978).

[17] J. DIESTEL - J. J. UHL, *Vector Measures*, Math. Surveys, Vol. **15**, A.M.S., Providence, R. I. (1977).

[18] N. DUNFORD - J. SCHWARTZ, *Linear Operators I*, Wiley, New York (1958).

[19] G. EDGAR, *Measurability in a Banach space*, Indiana J. Math., **28** (1979), pp. 559-579.

[20] H. FATTORINI, *A remark on the Bang-Bang principle for linear control systems in infinite dimensional spaces*, SIAM J. Control, **6** (1968), pp. 109-113.

[21] A. FILIPPOV, *On certain question in the theory of optimal control*, SIAM J. Control, **1** (1962), pp. 76-84.

[22] A. GHOUILA-HOURI, *Sur la généralisation de la notion de la commande d'un système guidable*, Rev. d'Inform. et de Resch. Oper., no. 4 (1967), pp. 7-32.

[23] H. HERMES - J. LASALLE, *Functional Analysis and Time Optimal Control*, Academic Press (1969).

[24] F. HIAI - H. UMEGAKI, *Integrals, conditional expectations and martingales of multivalued mappings*, J. Multiv. Anal., **7** (1977), pp. 149-182.

[25] A. JAWHAR, *Mesures de transition et applications*, Sém. Anal. Convexe, Montpellier (1984), exposé no. 13.

[26] D. KANDILAKIS - N. S. PAPAGEORGIOU, *On the properties of the Aumann integral with applications to differential inclusions and control systems*, Czech. Math. J., to appear.

[27] N. PAVEL, *Differential Equations, Flow Invariance and Applications*, Research Notes in Math., Vol. **113**, Pitman, Boston (1984).

[28] N. S. PAPAGEORGIOU, *On multivalued evolution equations and differential inclusions in Banach spaces*, Comm. Math. Univ. Sancti Pauli, **36** (1987), pp. 21-39.

[29] N. S. PAPAGEORGIOU, *Representation of set valued operators*, Trans. Amer. Math. Soc., **292** (1985), pp. 557-572.

[30] N. S. PAPAGEORGIOU, *On the theory of Banach space valued multifunctions*. Part 1: *Integration and conditional expectation*, J. Mult. Anal., **17** (1985), pp. 185-206.

[31] M. F. SAINT-BEUVE, *On the extension of Von Neumann-Aumann's theorem*, J. Funct. Anal., **17** (1974), pp. 112-129.

[32] M. F. SAINT-BEUVE, *Une extension des théorèmes de Novikov et d'Arsenin*, Sém. Anal. Convàxe, Montpellier (1981), exposé no. 18.

[33] D. WAGNER, *Survey of measurable selection theorems*, SIAM J. Control and Optim., **15** (1977), pp. 859-903.

[34] J. WARGA, *Relaxed variational problems*, J. Math. Anal. Appl., **4** (1962), pp. 111-128.

[35] J. WARGA, *Optimal Control of Differential and Functional Equations*, Academic Press, New York (1972).

[36] L. C. YOUNG, *Lectures on the Calculus of Variations and Optimal Control Theory*, Saunders, Philadelphia (1969).