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Generating Functions for a Class of q-Polynomials (*).

H. M. SRIVASTAVA - A. K. AGARWAL

Summary. – *Some simple ideas are used here to prove a theorem on generating functions for a certain class of q-polynomials. This general theorem is then applied to derive a fairly large number of known as well as new generating functions for the familiar q-analogues of various polynomial systems including, for example, the classical orthogonal polynomials of Hermite, Jacobi, and Laguerre. A number of other interesting consequences of the theorem are also discussed.*

1. – Introduction, Notations, and the Main Result.

A great surge of activities in the theory of q-series and q-polynomials has been witnessed in recent years. Various q-extensions of well-known hypergeometric identities and quadratic transformations have recently been obtained by several workers. These q-extensions are known to have important applications in many areas of pure as well as applied mathematics, physics, and engineering. Workers in the field of q-series and q-polynomials are realizing the need of extending all the important results involving special functions to hold true for their q-analogues. With this objective in mind, we prove a general theorem on generating functions for an important class of q-polynomials, and then apply this theorem not only to derive q-extensions of several familiar generating functions, but also to deduce (for example) Jackson's q-Pfaff transformation [8] which ANDREWS [3, p. 527] used to prove q-analogues of Kummer's summation theorem and (the so-called) Gauss's second theorem, Hahn's q-analogue [7] of Kummer's first formula, and Jackson's q-analogue [9] of the celebrated Pfaff-Saalschütz theorem.

For real or complex q, $|q| < 1$, let

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \tag{1.1}$$

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Indirizzo degli AA.: H. M. SRIVASTAVA: Department of Mathematics, University of Victoria, Victoria, British Columbia V8W 2Y2, Canada; A. K. AGARWAL: Department of Mathematics, Pennsylvania State University (Mont Alto Campus), Mont Alto, Pennsylvania 17237, U.S.A.

for arbitrary λ and μ, so that

$$(1.2)\qquad \begin{cases} (\lambda; q)_0 = 1; \\ (\lambda; q)_n = (1-\lambda)(1-\lambda q)\dots(1-\lambda q^{n-1}), \quad \forall n \in \{1, 2, 3, \dots\}, \quad \text{and} \\ (\lambda; q)_\infty = \prod_{j=0}^{\infty}(1-\lambda q^j). \end{cases}$$

Define, as usual, a generalized basic (or q—) hypergeometric function by (*cf.* [11, Chapter 3]; see also [13, p. 347, Equation (272)])

$$(1.3)\qquad {}_{p+1}\Phi_{p+r}\left[\begin{matrix}\alpha_1, \dots, \alpha_{p+1}; \\ \\ \beta_1, \dots, \beta_{p+r};\end{matrix}\quad q, z\right] = \sum_{n=0}^{\infty}(-1)^{rn}q^{\frac{1}{2}rn(n-1)}\frac{(\alpha_1; q)_n \dots (\alpha_{p+1}; q)_n}{(\beta_1; q)_n \dots (\beta_{p+r}; q)_n}\frac{z^n}{(q; q)_n},$$

where, for convergence, $|q| < 1$, and $|z| < \infty$ when r is a positive integer, or $|z| < 1$ when $r = 0$, provided that no zeros appear in the denominator.

We shall also need the Gaussian polynomial (or q-binomial coefficient) defined, for all non-negative integers n and k, by (see, *e.g.*, [4, p. 35])

$$(1.4)\qquad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{j=1}^{k}\left(\dfrac{1-q^{n-j+1}}{1-q^j}\right), & \text{if } 1 \leqq k \leqq n, \\ 0, & \text{if } k > n. \end{cases}$$

For a non-negative integer m, the familiar q-binomial theorem (*cf.* [4, p. 17, Theorem 2.1])

$$(1.5)\qquad {}_1\Phi_0\left[\begin{matrix}\lambda\,; \\ \\ -;\end{matrix}\quad q, t\right] \equiv \sum_{n=0}^{\infty}\frac{(\lambda; q)_n}{(q; q)_n}t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty}, \qquad |t| < 1, \qquad |q| < 1$$

can be rewritten at once as

$$(1.6)\qquad \sum_{n=0}^{\infty}\frac{(\lambda; q)_{m+n}}{(q; q)_n}t^n = \frac{(\lambda; q)_m}{(\lambda t; q)_m}\frac{(\lambda t; q)_\infty}{(t; q)_\infty}, \qquad |t| < 1, \quad |q| < 1,$$

which, in view of (1.2), yields (1.5) when $m = 0$ (or when λ is replaced by λq^{-m}). Making use of (1.6), we shall prove the following

THEOREM. – *In terms of a bounded complex sequence* $\{S_{n,q}\}_{n=0}^{\infty}$ *generated by*

$$(1.7)\qquad F_\omega(\lambda, \mu, q, t) = \sum_{n=0}^{\infty}\frac{(\lambda; q)_{\omega n}}{(\lambda\mu; q)_{\omega n}(q; q)_{\omega n}}S_{n,q}t^n,$$

define a family of basic (or $q-$) polynomials $\{f_{n,N}(x;q)\}_{n=0}^{\infty}$ by

$$f_{n,N}(x;q)=\sum_{k=0}^{[n/N]}\begin{bmatrix} n \\ Nk \end{bmatrix} S_{k,q}x^k \quad (n=0,1,2,\ldots)\,, \tag{1.8}$$

where N is a positive integer.

Then

$$\sum_{n=0}^{\infty}\frac{(\lambda;q)_n}{(q;q)_n}f_{n,N}(x;q)t^n=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}F_N(\lambda,t,q,xt^N)\,, \tag{1.9}$$

provided that each side exists, $|t|<1$, and $|q|<1$.

2. – Proof of the Theorem.

Denote, for convenience, the left-hand side of our assertion (1.9) by $\Omega(t)$. Substituting for $f_{n,N}(x;q)$ from the definition (1.8) into $\Omega(t)$, and inverting the order of summation, we have

$$\Omega(t)=\sum_{k=0}^{\infty}S_{k,q}\frac{(xt^N)^k}{(q;q)_{Nk}}\sum_{n=0}^{\infty}\frac{(\lambda;q)_{n+Nk}}{(q;q)_n}t^n\,,$$

provided that the series involved converge absolutely.

Now we sum the inner series by appealing to (1.6) with $m=Nk$, and we find for $|t|<1$ and $|q|<1$ that

$$\Omega(t)=\frac{(\lambda t;q)_\infty}{(\lambda;q)_\infty}\sum_{k=0}^{\infty}\frac{(\lambda;q)_{Nk}}{(\lambda t;q)_{Nk}(q;q)_{Nk}}S_{k,q}(xt^N)^k\,.$$

Interpreting this last expression by means of the generating relation (1.7), we are led immediately to the theorem.

REMARK. – For substantially more general classes of q-generating functions, and for their multivariable extensions, the reader should refer to Section 3 of a recent paper by SRIVASTAVA [12].

3. – Applications.

We begin by applying our theorem to derive generating functions for the q-analogues of many of the classical orthogonal polynomials. Setting

$$S_{n,q}=(-1)^n\frac{q^{n(n-1)}}{(\alpha q;q)_n}$$

in our theorem, we find from (1.8) that

$$f_{n,1}(x;q) = {}_1\Phi_1\begin{bmatrix} q^{-n}; & \\ & q, -xq^n \\ \alpha q; & \end{bmatrix} = \frac{(q;q)_n}{(\alpha q;q)_n} L_n^{(\alpha)}(x;q),$$

where $L_n^{(\alpha)}(x;q)$ denotes the q-Laguerre polynomial defined by (*cf.* [6])

$$L_n^{(\alpha)}(x;q) = \frac{(\alpha q;q)_n}{(q;q)_n}\,{}_1\Phi_1\begin{bmatrix} q^{-n}; & \\ & q, -xq^n \\ \alpha q; & \end{bmatrix}. \tag{3.1}$$

Thus our theorem yields the following generating function for the q-Laguerre polynomials:

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\alpha q;q)_n} L_n^{(\alpha)}(x;q)\,t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\,{}_1\Phi_2\begin{bmatrix} \lambda; & \\ & q, -xt \\ \alpha q, \lambda t; & \end{bmatrix}, \tag{3.2}$$

which provides a q-extension of a well-known generating function for Laguerre polynomials [14, p. 132, Equation (5)].

Next we consider the little q-Jacobi polynomials defined by (*cf.* [6])

$$p_n^{(\alpha,\beta)}(x;q) = \frac{(\alpha q;q)_n}{(q;q)_n}\,{}_2\Phi_1\begin{bmatrix} q^{-n}, \alpha\beta q^{n+1}; & \\ & q, qx \\ \alpha q; & \end{bmatrix}, \tag{3.3}$$

and our theorem with $N = 1$, and

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha\beta q;q)_n}{(\alpha q;q)_n},$$

gives us the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\alpha q;q)_n} p_n^{(\alpha,\beta q^{-n})}(xq^n;q)\,t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\,{}_2\Phi_2\begin{bmatrix} \lambda, \alpha\beta q; & \\ & q, xqt \\ \alpha q, \lambda t; & \end{bmatrix}. \tag{3.4}$$

For $\lambda = 0$, (3.4) reduces immediately to

$$\sum_{n=0}^{\infty} p_n^{(\alpha,\beta q^{-n})}(xq^n;q)\frac{t^n}{(\alpha q;q)_n} = \frac{1}{(t;q)_\infty}\,{}_1\Phi_1\begin{bmatrix} \alpha\beta q; & \\ & q, xqt \\ \alpha q; & \end{bmatrix}, \tag{3.5}$$

which is a q-extension of a known generating function for Jacobi polynomials ([1, p. 159, Equation (3.5)]; see also [14, p. 170, Problem 19 (i)]).

Setting

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha\beta q;\, q)_n (\nu q/x;\, q)_n}{(\beta q;\, q)_n (\nu q;\, q)_n},$$

we observe from (1.8) that

$$f_{n,1}(x;\, q) = \frac{(\alpha^{-1};\, q)_n}{(\beta q;\, q)_n} Q_n(\alpha x;\, \alpha q^{-n}, \beta, \nu|q)$$

in terms of the q-Hahn polynomials defined by

$$Q_n(x;\, \alpha, \beta, \nu|q) = {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, x; & \\ & q, q \\ \alpha q, \nu q \quad ; & \end{matrix}\right] \tag{3.6}$$

or, equivalently, by

$$Q_n(x;\, \alpha, \beta, \nu|q) = \frac{(\beta q;\, q)_n}{(1/\alpha q^n;\, q)_n} {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, \nu q/x; & \\ & q, \dfrac{x}{\alpha} \\ \beta q, \nu q \quad ; & \end{matrix}\right]. \tag{3.7}$$

Our theorem when applied to the q-Hahn polynomials yields the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda;\, q)_n (\alpha^{-1};\, q)_n}{(\beta q;\, q)_n (q;\, q)_n} Q_n(x;\, \alpha q^{-n}, \beta, \nu|q)\, t^n = \frac{(\lambda t;\, q)_\infty}{(t;\, q)_\infty} {}_3\Phi_3 \left[\begin{matrix} \lambda, \alpha\beta q, \nu q/x; & \\ & q, \dfrac{xt}{\alpha} \\ \beta q, \nu q, \lambda t \; ; & \end{matrix}\right]. \tag{3.8}$$

Similarly, for the q-Meixner polynomials defined by

$$M_n(x;\, \beta, \gamma|q) = (\beta;\, q)_n\, {}_2\Phi_1 \left[\begin{matrix} q^{-n}, x; & \\ & q, \dfrac{q^{n+1}}{\gamma} \\ \beta \quad ; & \end{matrix}\right], \tag{3.9}$$

we obtain the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda;\, q)_n}{(\beta;\, q)_n (q;\, q)_n} M_n(x;\, \beta, \gamma|q)\, t^n = \frac{(\lambda t;\, q)_\infty}{(t;\, q)_\infty} {}_2\Phi_2 \left[\begin{matrix} \lambda, x\, ; & \\ & q, \dfrac{qt}{\gamma} \\ \beta, \lambda t; & \end{matrix}\right]. \tag{3.10}$$

In particular, (3.10) with $\lambda = \beta$ yields

$$\sum_{n=0}^{\infty} M_n(x;\, \beta, \gamma|q) \frac{t^n}{(q;\, q)_n} = \frac{(\beta t;\, q)_\infty}{(t;\, q)_\infty} {}_1\Phi_1 \left[\begin{matrix} x\, ; & \\ & q, \dfrac{qt}{\gamma} \\ \beta t; & \end{matrix}\right], \tag{3.11}$$

which provides a q-extension of a known generating function for the Meixner polynomials [5, p. 225, Equation 10.24 (13)].

The definitions (3.3) and (3.9) imply the following relationship between q-Meixner polynomials and the little q-Jacobi polynomials:

$$M_n(x;\beta,\gamma|q) = (q;q)_n\, p_n^{(\beta/q,\, x/\beta q^n)}\left(\frac{q^n}{\gamma}\right), \tag{3.12}$$

which can be used to show that the generating functions (3.4) and (3.10), and indeed also (3.5) and (3.11), are essentially the same.

Now we turn to the q-Charlier polynomials defined by

$$c_n(x;\alpha|q) = {}_2\Phi_1\left[\begin{matrix} q^{-n}, x; \\ \\ 0\;; \end{matrix}\quad q, -\frac{q^{n+1}}{\alpha}\right] \tag{3.13}$$

for which our theorem with $N = 1$, and

$$S_{n,q} = q^{\frac{1}{2}n(n+1)}(x;q)_n,$$

readily yields the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} c_n(x;\alpha|q)\, t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_2\Phi_2\left[\begin{matrix} \lambda, x; \\ \\ \lambda t, 0; \end{matrix}\quad q, -\frac{qt}{\alpha}\right]. \tag{3.14}$$

In its special case when $\lambda = 0$, (3.14) reduces immediately to

$$\sum_{n=0}^{\infty} c_n(x;\alpha|q) \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix} x; \\ \\ 0; \end{matrix}\quad q, -\frac{qt}{\alpha}\right], \tag{3.15}$$

which is a q-extension of a known generating function for Charlier polynomials [5, p. 226, Equation 10.25 (6)].

Setting

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)},$$

the definition (1.8) assumes the form:

$$f_{n,1}(x;q) = (x;q)_n,$$

and our theorem immediately yields the identity:

$$(3.16)\qquad {}_2\Phi_1\left[\begin{matrix}\lambda, x; \\ \\ 0\ ;\end{matrix}\quad q, t\right] = \frac{(\lambda t; q)_\infty}{(t; q)_\infty}\, {}_1\Phi_1\left[\begin{matrix}\lambda\ ; \\ \\ \lambda t;\end{matrix}\quad q, xt\right],$$

which is, in fact, contained in Jackson's formula (3.27) below.

On the other hand, in view of Heine's transformation (*cf.* [4, p. 19, Corollary 2.3]; see also [13, p. 348, Equation (275)])

$$(3.17)\qquad {}_2\Phi_1\left[\begin{matrix}a, b; \\ \\ c\ ;\end{matrix}\quad q, z\right] = \frac{(b; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}z, c/b; \\ \\ az\ ;\end{matrix}\quad q, b\right],$$

the first member of (3.16) can also be expressed as

$$(3.18)\qquad {}_2\Phi_1\left[\begin{matrix}\lambda, x; \\ \\ 0\ ;\end{matrix}\quad q, t\right] = \frac{(x; q)_\infty (\lambda t; q)_\infty}{(t; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}t, 0; \\ \\ \lambda t\ ;\end{matrix}\quad q, x\right].$$

Comparing (3.16) and (3.18), we readily obtain [7, p. 374, Equation (10.2)]

$$(3.19)\qquad {}_2\Phi_1\left[\begin{matrix}a, 0; \\ \\ b\ ;\end{matrix}\quad q, z\right] = \frac{1}{(z; q)_\infty}\, {}_1\Phi_1\left[\begin{matrix}b/a; \\ \\ b\ ;\end{matrix}\quad q, az\right],$$

which is a q-extension of Kummer's first formula for the confluent hypergeometric function [10, p. 125, Theorem 42].

The orthogonal q-polynomials $\Phi_n^{(\alpha)}(x; q)$ studied by AL-SALAM and CARLITZ [2, p. 48, Equation (1.11)] are precisely the polynomials defined by (1.8) with $N = 1$, and

$$S_{n,q} = (\alpha; q)_n\,.$$

Thus our theorem yields the following generating function for $\Phi_n^{(\alpha)}(x; q)$:

$$(3.20)\qquad \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n}\, \Phi_n^{(\alpha)}(x; q)\, t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}\lambda, \alpha; \\ \\ \lambda t\ ;\end{matrix}\quad q, xt\right],$$

which, for $\lambda = 0$, reduces to the following result due to Al-Salam and Carlitz [2, p. 48, Equation (1.13)]:

$$(3.21)\qquad \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x; q)\, \frac{t^n}{(q; q)_n} = \frac{(\alpha x t; q)_\infty}{(t; q)_\infty (xt; q)_\infty}\,.$$

Setting $\alpha = 0$ in (3.20) and then applying (3.19), we have

$$\sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} H_n(x; q)\, t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty (xt; q)_\infty}\, {}_1\Phi_1 \left[\begin{matrix} t\,; \\ \lambda t\,; \end{matrix} \; q, \lambda x t \right], \tag{3.22}$$

where $H_n(x; q)$ denotes the q-Hermite polynomial defined by (*cf.* [15]; see also [4, p. 49])

$$H_n(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k . \tag{3.23}$$

Formula (3.22) may be compared with a *divergent* generating function for the classical Hermite polynomials (see, *e.g.*, [14, p. 138, Equation (7)]). On the other hand, a further special case of (3.21) when $\alpha = 0$ [that is, (3.22) with $\lambda = 0$] is a well-known result [4, p. 49, Example 3].

Yet another interesting application of our theorem with $x = \beta/\alpha$, $N = 1$, and

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha; q)_n}{(\beta; q)_n},$$

leads us to the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n}\, {}_2\Phi_1 \left[\begin{matrix} q^{-n}, \alpha\,; \\ \beta\,; \end{matrix} \; q, \frac{\beta}{\alpha} q^n \right] t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty}\, {}_2\Phi_2 \left[\begin{matrix} \lambda, \alpha\,; \\ \beta, \lambda t\,; \end{matrix} \; q, \frac{\beta t}{\alpha} \right]. \tag{3.24}$$

In view of the q-summation formula [11, p. 97, Equation (3.3.2.6)]:

$${}_2\Phi_1 \left[\begin{matrix} q^{-n}, b\,; \\ c\,; \end{matrix} \; q, \frac{c}{b} q^n \right] = \frac{(c/b; q)_n}{(c; q)_n}, \tag{3.25}$$

the generating function (3.24) can be rewritten fairly easily as

$$\sum_{n=0}^{\infty} \frac{(\lambda; q)_n (\beta/\alpha; q)_n}{(q; q)_n (\beta; q)_n} t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty}\, {}_2\Phi_2 \left[\begin{matrix} \lambda, \alpha\,; \\ \beta, \lambda t\,; \end{matrix} \; q, \frac{\beta t}{\alpha} \right] \tag{3.26}$$

or, equivalently, as Jackson's q-Pfaff transformation [8, p. 145, Equation (4)]

$${}_2\Phi_1 \left[\begin{matrix} a, b\,; \\ c\,; \end{matrix} \; q, z \right] = \frac{(az; q)_\infty}{(z; q)_\infty}\, {}_2\Phi_2 \left[\begin{matrix} a, c/b\,; \\ c, az\,; \end{matrix} \; q, bz \right]. \tag{3.27}$$

Formula (3.27) is the main lemma of Andrews [3] which he used to derive q-analogues of Kummer's summation theorem and (the so-called) Gauss's second theorem.

Finally, we set $x = \gamma\delta/\alpha\beta$, $N = 1$, and

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha; q)_n(\beta; q)_n}{(\gamma; q)_n(\delta; q)_n},$$

and our theorem yields the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha, \beta; \\ \\ \gamma, \delta\ ; \end{matrix}\quad q, \frac{\gamma\delta}{\alpha\beta} q^n\right] t^n = \frac{(\lambda t; q)_\infty}{(t; q)_\infty} {}_3\Phi_3 \left[\begin{matrix} \lambda, \alpha, \beta\ ; \\ \\ \gamma, \delta, \lambda t; \end{matrix}\quad q, \frac{\gamma\delta t}{\alpha\beta}\right]. \tag{3.28}$$

The ${}_3\Phi_2$ occurring in (3.28) can be transformed by appealing to the familiar identity:

$${}_3\Phi_2 \left[\begin{matrix} q^{-n}, a, b\ ; \\ \\ c, d\ \ ; \end{matrix}\quad q, \frac{cd}{ab} q^n\right] = \frac{(c/a; q)_n}{(c; q)_n} {}_3\Phi_2 \left[\begin{matrix} q^{-n}, a, d/b\ ; \\ \\ aq^{1-n}/c, d\ ; \end{matrix}\quad q, q\right], \tag{3.29}$$

which incidentally is involved in the equivalence of (3.6) and (3.7), and we thus find from (3.28) that

$$\sum_{n=0}^{\infty} \frac{(\lambda; q)_n(\gamma/\alpha; q)_n}{(q; q)_n(\gamma; q)_n} {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha, \delta/\beta; \\ \\ \alpha q^{1-n}/\gamma, \delta\ ; \end{matrix}\quad q, q\right] t^n = \tag{3.30}$$

$$= \frac{(\lambda t; q)_\infty}{(t; q)_\infty} {}_3\Phi_3 \left[\begin{matrix} \lambda, \alpha, \beta\ ; \\ \\ \gamma, \delta, \lambda t; \end{matrix}\quad q, \frac{\gamma\delta t}{\alpha\beta}\right].$$

In its special case when $\gamma = \beta$, the right-hand side of (3.30) becomes identical with the right-hand side of (3.26) with, of course, β replaced by δ. Equating the coefficients of t^n in the first members of (3.26) and (3.30), in this special case, we obtain the q-summation formula:

$${}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha, \delta/\beta; \\ \\ \alpha q^{1-n}/\beta, \delta\ ; \end{matrix}\quad q, q\right] = \frac{(\beta; q)_n(\delta/\alpha; q)_n}{(\delta; q)_n(\beta/\alpha; q)_n} \tag{3.31}$$

or, equivalently,

$${}_3\Phi_2 \left[\begin{matrix} a, b, q^{-n}\ ; \\ \\ c, abq^{1-n}/c; \end{matrix}\quad q, q\right] = \frac{(c/a; q)_n(c/b; q)_n}{(c; q)_n(c/ab; q)_n}, \tag{3.32}$$

which is Jackson's q-analogue of the celebrated Pfaff-Saalschütz theorem (*cf.* [9, p. 111, Equation (B)]; see also [11, p. 97, Equation (3.3.2.2)]). Conversely, setting $\gamma=\beta$ in (3.30) and summing the resulting ${}_3\Phi_2$ series by appealing to Jackson's result (3.32), we shall arrive at (3.26) or (3.27). Thus our formula (3.30) may also be looked upon as a generalization of the principal result employed by Andrews [3, p. 527].

We conclude by remarking that many of the q-generating functions considered in this section can alternatively be deduced from the following consequence of our theorem (see also [12, Section 3]):

$$\sum_{n=0}^{\infty}\frac{(\lambda;q)_n}{(q;q)_n}\,{}_{p+1}\Phi_p\left[\begin{matrix}q^{-n},\alpha_1,\ldots,\alpha_p;\\ \\ \beta_1,\ldots,\beta_p;\end{matrix}\quad q,xq^n\right]t^n=\tag{3.33}$$

$$=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}\,{}_{p+1}\Phi_{p+1}\left[\begin{matrix}\lambda,\alpha_1,\ldots,\alpha_p;\\ \\ \lambda t,\beta_1,\ldots,\beta_p;\end{matrix}\quad q,xt\right],\qquad |t|<1,\quad |q|<1,$$

which provides a q-analogue of a well-known hypergeometric generating function (*cf.*, *e.g.*, [14, p. 138, Equation (8)]). Formula (3.33) can indeed be specialized also to derive generating functions for a number of q-hypergeometric polynomials in addition to those that are considered here.

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