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On the Homogeneous Ideal of Projectively Normal Curves (*).

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Summary. – *Fix integers* k, d, g *with*

$$g \geqslant 0\,, \quad d \geqslant g+3\,, \quad k>0\,, \quad 2k<(d-g)\,, \quad d \geqslant (g(k+1)/k)+k+1\,.$$

Here we prove that for a general curve X *of genus* g *and a general* $L \in \mathrm{Pic}^d(X)$, L *is normally presented.*

A very ample line bundle L on a smooth, complete curve X is called normally generated (cf. [15]) if $h_L(X)$ is projectively normal, where $h_L\colon X \to \boldsymbol{P}(H^0(X, L)^*)$ is the embedding associated to X. L is called normally presented if it is normally generated and the homogeneous ideal of $h_L(X)$ in $\boldsymbol{P}(H^0(X, L)^*)$ is generated by quadrics. Here we prove the following result.

THEOREM 1. – *Fix integers* k, d, g *with* $g \geqslant 0$, $d \geqslant g+3$, $k>0$; *assume either*

(*a*) $2k+1 \leqslant d-g$, $d \geqslant (g(k+1)/k)+k+1$, *or*

(*b*) $2k+2 \leqslant d-g$, $d \geqslant (g(k+2)/(k+1))+k+2$.

Then for a general curve X *of genus* g *and a general* $L \in \mathrm{Pic}^d(X)$, L *is normally presented* (*over any algebraically closed field*).

B. SAINT-DONAT [16] and T. FUJITA [4] proved that for every $d \geqslant 2g+2$, for every smooth curve X of genus g, every $L \in \mathrm{Pic}^d(X)$ is normally presented. M. GREEN ([5], [6]) generalized this statement to higher syzygies and to higher dimensional varieties, developing a very useful general framework for this kind of problems; the notion of finite presentation for a very ample line bundle is now only the first case, N_1, of the problem N_p about the p-th step of a minimal free resolution. For related work, see F. O. SCHREYER ([17], [18]). In [12] LANGE and MARTENS proved the statement of theorem 1 under the assumption that $d \geqslant (3g+4+(8g+1)^{\frac{1}{2}})/2$. Theorem 1 is stronger. If for example $(2g)^{\frac{1}{2}}$ is an integer k, theorem 1 works if $d \geqslant g+2+2(2g)^{\frac{1}{2}}$ which is not too far from the conjectural bound « $d \geqslant g+1+(3g+1)^{\frac{1}{2}}$ », which, if true, would be sharp.

Note that the thesis in theorem 1 is just a statement about linearly normal curves of degree d, genus g in $T := \boldsymbol{P}^r$, $r = d-g$, which are arithmetically Cohen-

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Macaulay, with non special hyperplane section and with homogeneous ideal generated by quadrics. In § 4 it will be constructed a reducible curve $C \subset T$ satisfying these conditions and apply a result of Sernesi [19] or Hartshorne-Hirschowitz [9] to show that C is smoothable. Then the thesis will follow by semicontinuity (see § 2 for more details).

To prove that C has the properties we want, it is sufficient to look at a hyperplane section $C \cap V$ of C, V hyperplane of T, and see that the homogeneous ideal of $C \cap V$ in V is generated by the right number of quadrics. In § 3 we will find a configuration of points $S \subset V$ whose homogeneous ideal in V is generated by the right number of quadrics. In § 4 we will find by elementary means a suitable curve $C \subset T$ with $C \cap V = S$. The proofs in § 3 use the results of § 1 on the elementary transformations of a vector bundle on V along a divisor of V (cf. [13]). The idea of using «simple points» (see § 1) for proving results about syzygies is due to Hirschowitz ([10]) who refined the general method introduced by HARTSHORNE and HIRSCHOWITZ in [8]. For much more on this method, see the thesis of M. Idà [11].

1. – Elementary transformations and simple points.

Set $V := \boldsymbol{P}^n$. Fix homogeneous coordinates $x_0, \ldots, x_n$ on V. Consider the dual of the Euler's sequence ([7] Ch. II, § 8):

$$0 \to \Omega_V(t) \to (n+1)\mathcal{O}_V(t-1) \xrightarrow{f} \mathcal{O}_V(t) \to 0 \tag{1}$$

where f is defined by $f((s_0, \ldots, s_n)) = x_0 s_0 + \ldots + x_n s_n$.

Let H be a hyperplane of V. We have $\Omega_V|H \cong \Omega_H \oplus \mathcal{O}_H(-1)$ and this isomorphism defines a surjective morphism $h\colon \Omega_V \to \Omega_H$. Set $F_{V,H} = \operatorname{Ker}(h)$. By definition we have an exact sequence

$$0 \to F_{V,H} \to \Omega_V \to \Omega_H \to 0 . \tag{2}$$

By (13) $F_{V,H}$ is a rank n vector bundle on V. The following two lemmas are known to the specialists (by definition of specialists), but, as far as we know, not published. Since we need them, we will give complete proofs of them.

LEMMA 1.1. – *If* $\dim(V) = 2$, *then* $F_{V,H}(2) \cong 2\mathcal{O}_V$.

PROOF. – By (1) and a twist of (2), we obtain

$$h_0(V, F_{V,H}(2)) = 2 , \quad h_0(V, F_{V,H}(1)) = 0 .$$

Hence $F_{V,H}(2)$ has a section s vanishing at most in codimension 2. Since $c_1(F_{V,H}(2)) = c_2(F_{V,H}(2)) = 0$ by (1), (2), s cannot vanish at all and $F_{V,H}(2)$ is a direct sum of two trivial line bundles. ∎

LEMMA 1.2. – *We have* $F_{V,H}(2) \cong n\mathcal{O}_V$.

PROOF. – By induction on n. First we assume that for one hyperplane R in V, $R \neq H$, $F_{V,H}|R \cong F_{R,H\cap R} \oplus \mathcal{O}_R(-2)$. Then by induction $F_{V,H}(2)|R$ is trivial. By [1] Cor. 1.7 (which holds over any algebraically closed base field), $F_{V,L}(2)$ is trivial. Hence we assume that for all hyperplanes R with $R \neq H$, $F_{V,H}(2) \not\cong F_{R,H\cap R} \oplus \mathcal{O}_R(-2)$. Take a hyperplane R, $R \neq H$. Set $L = H \cap R$. From (2) and $\Omega_V|R \cong \Omega_R \oplus \mathcal{O}_R(-1)$, we obtain a diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & F_{V,H}|R & \to & \Omega_R \oplus \mathcal{O}_R(-1) & \xrightarrow{a} & \Omega_L \oplus \mathcal{O}_L(-1) & \to & 0 \\ & & & & \| & & \uparrow{\scriptstyle h} & & \\ 0 & \to & F_{R,L} \oplus \mathcal{O}_R(-2) & \to & \Omega_R \oplus \mathcal{O}_R(-1) & \xrightarrow{b} & \Omega_L \oplus \mathcal{O}_L(-1) & \to & 0\,. \end{array} \tag{3}$$

Note that

$$\operatorname{Hom}(\Omega_R, \mathcal{O}_L(-1)) \cong \operatorname{Hom}(\Omega_R|L, \mathcal{O}_L(-1)) \cong \operatorname{Hom}(\Omega_L \oplus \mathcal{O}_L(-1), \mathcal{O}_L(-1))\,.$$

We have

$$\dim\big(\operatorname{Hom}(\mathcal{O}_L(-1), \mathcal{O}_L(-1))\big) = 1\,,$$

$$\dim\big(\operatorname{Hom}(\Omega_L, \mathcal{O}_L(-1))\big) = h_0(L, T_L(-1)) = \dim(L) + 1 \text{ (Euler's sequence)}\,,$$

$$\operatorname{Hom}(\mathcal{O}_R(-1), \Omega_L(-1)) = \operatorname{Hom}(\mathcal{O}_L(-1), \Omega_L(-1)) = 0 \text{ by (1);}$$

$\dim(\operatorname{Hom}(\Omega_L, \Omega_L)) = 1$ because Ω_L is stable. Hence the map a in (3) is given by $n+1$ constants $(A; a_1, \ldots, a_{n-1}; c)$; $A \neq 0$ since a is surjective. By definition the map b in (3) is given by the $n+1$ constants $(1; 0, \ldots, 0; 1)$. If $c \neq 0$, we may find in (3) an isomorphism h which makes commutative the right square; hence $F_{V,H}|R \cong F_{R,L} \oplus \mathcal{O}_R(-2)$, contradiction. Hence for all $R \neq H$, the corresponding constant c vanishes. We may construct a diagram (3)′ similar to (3) with $\Omega_R(-1) \oplus \mathcal{O}_R(-1)$ instead of $F_{R,L} \oplus \mathcal{O}_R(-2)$ and with a very different b: $b = (1; 1, \ldots, 1; 0)$ i.e. $b|\Omega_R$ is the restriction map, $b|\mathcal{O}_R(-1) = 0$. Since $c = 0$, we may find h making commutative (3)′ and obtain $F_{V,H}|R \cong \Omega_R(-1) \oplus \mathcal{O}_R(-1)$. Since $n > 2$, any line in V is contained in a hyperplane R with $R \neq H$. Hence $F_{V,H}$ would be a uniform vector bundle of splitting type $(-3, -2, \ldots, -2, -1)$. By [2] (and Ein [3] in positive characteristic), $F_{V,H}$ is a direct sum of line bundles, contracting $F_{V,H}|R \cong \Omega_R(-1) \oplus \mathcal{O}_R(-1)$. ■

$F_{V,H}(t)$ is called the elementary transformation of $\Omega_V(t)$ by the surjection $\Omega_V(t) \to \Omega_H(t)$.

Let F be a vector bundle on a variety S. By definition a simple point t for F is a point $t \in \boldsymbol{P}(F)$. The support of the simple point t is the image of t under the pro$_k$ection map $\boldsymbol{P}(F) \to S$. On $\boldsymbol{P}(F)$ there is the tautological line bundle $\mathcal{O}_{\boldsymbol{P}(F)}(1)$ with the property that $H^0(\boldsymbol{P}(F), \mathcal{O}_{\boldsymbol{P}(F)}(1)) \cong H^0(S, F)$.

Fix a projective space V and a hyperplane H of V. In this paper we will consider only a particular type of simple points for the vector bundle $F_{V,H}(t)$. We write $\mathcal{T}(t)$ or $\mathcal{T}$ instead of $\boldsymbol{P}(F_{V,H}(t))$. The splitting $\Omega_V(t)|H \simeq \Omega_H(t) \oplus \mathcal{O}_H(t-1)$, the definition of $F_{V,H}(t)$ and the commutativity of elementary transformations ([13], prop. 2.2) give the following exact sequence:

$$0 \to \Omega_V(t-1) \to F_{V,H}(t) \xrightarrow{u_t} \mathcal{O}_H(t-1) \to 0 \,. \tag{4}$$

Fix a point $P \in H$. The surjection u_t in (4) induces a surjection $u_t(P)\colon F_{V,H}(t)|P \to \mathcal{O}_H(t-1)|P$, hence a simple point t_P of $F_{V,H}(t)$; t_P will be called « a simple point for $F_{V,H}(t)$ with respect to $\mathcal{O}_H(t-1)$ » or a « simple » point for $F_{V,H}(t)$, for short. Fix a hyperplane R of V, $R \neq H$, with $P \notin R$. By the commutativity of elementary transformations ([13], prop. 2.2) it is an harmless abuse of notations to say that t_P is a « simple » point for $F_{V,H}(t) \otimes \mathcal{I}_{R,V} \simeq F_{V,H}(t-1)$. A subset J of e « simple » points for $F_{V,H}(t)$ imposes independent conditions to $H^0(F_{V,H}(t))$ if $h^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1) \oplus \mathcal{I}_{J,\boldsymbol{P}}) = h^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1)) - e$. Take $s \in H^0(V, \Omega_V(t))$ and assume that s induces the zero section on $\Omega_H(t)$. Then s induces $s' \in H^0(V, F_{V,H}(t))$. For any $P \in H$, $s(P) = 0$ if and only if the section of $\mathcal{O}_{\boldsymbol{P}}(1)$ induced by s' vanishes on t_P; in this case we will say often that s' vanishes on t_P.

2. – Sketch of the proof and preliminaries.

Fix integers n, m with $2 \leqslant m \leqslant n$. In § 3 we will construct for every integer d with $0 < d \leqslant m(n-m+2)$ a subset $S := S(d, m) \subset V := \boldsymbol{P}^n$ card $(S) = d$, such that the restriction map $r_{S,V}(k)\colon H^0(V, \mathcal{O}_V(k)) \to H^0(S, \mathcal{O}_S(k))$ is surjective if $k = 2$ (hence if $k > 2$ by Castelnuovo-Mumford's lemma [14], p. 99) and the homogeneous ideal of S in V is generated by quadrics.

Set $r = n + 1$, $k = m - 2$ and think V as a hyperplane in $T := \boldsymbol{P}^r$. In § 4 we will find that if either (*a*) $2k + 1 \leqslant r \leqslant d \leqslant r + k(r - 2k - 1)$, or (*b*) $2k + 2 \leqslant r \leqslant d \leqslant r + (k+1)(r - 2k - 2)$, there is a curve $C \subset T$ with $H^1(C, \mathcal{O}_C(1)) = 0$, C reduced and connected, C spanning T and linearly normal (i.e. $H^1(T, \mathcal{I}_C(1)) = 0$), C intersecting transversally V and with $C \cap V = S(d, k+2)$. Note that such a curve has arithmetic genus $d - r$; hence setting $g := d - r$ the conditions (*a*), (*b*) are exactly the conditions (*a*), (*b*) in the statement of theorem 1. The exact sequence on T

$$0 \to \mathcal{I}_C(t-1) \xrightarrow{h} \mathcal{I}_C(t) \to \mathcal{I}_{S(d,k+2),V}(t) \to 0$$

(where h is the multiplication by the equation of V in T), and the linear normality of C show that C is arithmethically Cohen-Macaulay and that the homogeneous ideal of C is generated by quadrics. C is not smooth. Indeed C is built in the following way. Start from a suitable curve C' of degree r and arithmetic genus 0, C' spanning T; C' is connected, with only ordinary double points, and every irreducible component of C' is a line. C is the union of C' and $d-r$ lines D_i, $i=1,\ldots,d-r$, each D_i intersecting C' exactly at two smooth points. By the results of Sernesi [19] or Hartshorne-Hirschowitz [9], C is smoothable. By semicontinuity we may find a projectively normal curve $X\subset T$, X of degree d and genus g, whose homogeneous ideal is generated by quadrics; the last assertion will be obvious after we will recall (see below) a cohomological interpretation of the fact that the degree 3 part of the homogeneous ideal of X is generated by the degree 2 part; it was very well-known ([15], [14], p. 99) that the homogeneous ideal of a projectively normal curve with non-special hyperplane section is generated by its degree 2 and degree 3 part. By the dual of the Euler's sequence (i.e. by (1)) the degree 3 part of the homogeneous ideal of a scheme Z in a projective space T is generated by its degree 2 part if $H^1(T,\Omega_T(3)\otimes \mathfrak{I}_{Z,T})=0$. The last condition is also necessary if $H^1(T,\mathfrak{I}_{Z,T}(2))$ (hence if $Z=C$). Thus we can use semicontinuity for the condition about the degree of the generators of the homogeneous ideal of C and X (for general X).

3. – The homogeneous ideal for a suitable configuration of points.

Fix integers n,m with $2\leqslant m\leqslant n$. Here we construct for every integer d with $0<d\leqslant m(n-m+2)$ a subset $S:=S(d,m)\subset V:=\boldsymbol{P}^n$, such that the restriction map $r_{S,V}(2)\colon H^0(V,\mathcal{O}_V(2))\to H^0(S,\mathcal{O}_S(2))$ is surjective, card $(S)=d$, and the homogeneous ideal of S in V is generated by quadrics.

LEMMA 3.1. – *Let J be the union of n « simple » points on H for $F_{V,H}(2)$ (with respect to $\mathcal{O}_H(1)$) with support S spanning H. Then no non zero section of $F_{V,H}(2)$ vanishes on J.*

PROOF. – Note that $h^0(V,F_{V,H}(2))=n$ by 1.1, 1.2. Hence the lemma means that J imposes independent conditions for $H^0(V,F_{V,H}(2))$. Assume that this is not true. Then we find $s\in H^0(V,F_{V,H}(2))$, $s\neq 0$, s inducing 0 on $\mathcal{O}_H(1)$. Hence by (4) s induces $s''\in H^0(V,\Omega_V(1))$, $s''\neq 0$, contradicting (1). ■

LEMMA 3.2. – *Fix integers m,n with $2\leqslant m\leqslant n$. Take m hyperplanes $V_1,\ldots,V_m$ of V and let V^i be the intersection of every hyperplane V_j with $j\neq i$. Assume* dim $(V_1\cap V^1)=n-m$. *Take subsets $E_i\subset V^i\setminus(V^i\cap V_1)$, $1\leqslant i\leqslant m$, with the points in each E_i linearly independent for every i. Let S be the union of all the E_i's. Then*

(i) $H^1(V,\mathfrak{I}_{S,V}(2))=0$;

(ii) $H^1(V,\Omega_V(3)\otimes\mathfrak{I}_{S,V})=0$ *(hence the homogeneous ideal of S in V is generated by the right number of quadrics).*

PROOF. – We use induction on n, the case $n=2$ being obvious.
Set $W:=V_1$ and $E:=E_1$.

(i) Consider the exact sequence

$$0 \to \mathfrak{I}_{E,V}(1) \to \mathfrak{I}_{S,V}(2) \to \mathfrak{I}_{(S\setminus E),W}(2) \to 0 \,. \tag{5}$$

Since the points in E are linearly independent, we have $H^1(V, \mathfrak{I}_{E,V}(1)) = 0$. By induction we have $H^1(W, \mathfrak{I}_{(S\setminus E)},(2)) = 0$, hence the thesis follows from (5).

(ii) By induction we have $H^1(W, \Omega_W(3) \otimes \mathfrak{I}_{(S\setminus E),W}) = 0$. Let J be the union of the «simple» points for $F_{V,W}(3)$ with support $(S\setminus E)$. It is sufficient to prove that $E \cup J$ imposes $n(\operatorname{card}(E)) + \operatorname{card}(J)$ independent conditions for $H^0(V, F_{V,W}(3))$. Set $\boldsymbol{P} := \boldsymbol{P}(F_{V,W}(3))$; for any subset A of V, let A' be its counterimage in $\boldsymbol{P}$ under the projection from $\boldsymbol{P}$ to V. We have to check that $H^1(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1) \otimes \mathfrak{I}_{E' \cup J}) = 0$. Let f be the union of the «simple» points with support in E_2 and R a hyperplane containing E with $E_2 \cap R = \emptyset$; for instance take $R' = \boldsymbol{P}(F_{V,W}(3)|R)$. By 3.1 the composition of the restriction maps $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1)) \to H^0(R, \mathcal{O}_{R'}(1)) \to$ $\to H_0(E', \mathcal{O}_{E'}(1))$ is surjective. Hence to prove that $H^1(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}_0}(1) \otimes \mathfrak{I}_{E' \cup f}) = 0$, it is sufficient to note that $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1) \otimes \mathfrak{I}_{R'}) \cong H^0(V, F_{V,W}(2))$ and apply 3.1. Let e_i be the union of the «simple» points for $F_{V,W}(3)$ with support in E_i: Note that $E_3 \cap V_3 = \emptyset$, but $E \cup E_2 \subset V_3$. Using V_3 and the same trick, we obtain that $E' \cup e_2 \cup e_3$ imposes $n(\operatorname{card}(E)) + \operatorname{card}(E^i) + \operatorname{card}(E_3)$ independent conditions to $H^0(V, F_{V,W}(3))$. After $m - 3$ steps, we obtain the thesis. ■

4. – End of proof of theorem 1.

Recall that it is sufficient to find a curve $C \subset T := \boldsymbol{P}^r$, $r = n + 1$, C as described in § 2, with $C \cap V = S(d, k + 2)$, V hyperplane of T, $S(d, k + 2)$ described in the statement of 3.2. V will be a hyperplane of $T := \boldsymbol{P}^r$.

LEMMA 4.1. – *Fix two hyperplanes H, R of a projective space V, $H \neq R$. Set $L = H \cap R$. Let U be a hyperplane of H, $U \neq L$. Take points $P \in U\setminus L$, $Q \in R\setminus L$ and lines D, D' in T with $D' \cap V = \{P\}$, $D \cap V = \{Q\}$, $D \cap D' = \emptyset$. Fix a finite number of hyperplanes U_i of U, L_j of L. Then there are points $r \in U$, $y \in L$, and lines F, F' in T such that:*

1) *$x \notin U_i$ for every i; $y \notin L_j$ for every j; $F \cap V = \{x\}$, $F' \cap V = \{y\}$;*

2) *$F \cap D \neq \emptyset$, $F' \cap D' \neq \emptyset$, $F \cap F' \neq \emptyset$.*

PROOF. – For a general point $x \in U$, the linear space $I := \langle x, D, D' \rangle$ spanned by x, D and D', has dimension 4. Take a general $y \in I \cap L$, $y \in U$. In I the planes $\langle x, D \rangle$ and $\langle y, D' \rangle$ must intersect, say at a point z. Set $F := \langle x, z \rangle$, $F' := \langle z, y \rangle$. To show that we may satisfy the condition on y, we reverse the construction. Start with a general $y \in L$. Set $J := \langle y, D, D' \rangle$ and take a general $x \in J \cap U$. Take $z \in \langle x, D \rangle \langle \cap \langle y, D' \rangle$ and set $F = \langle x, z \rangle$, $F' = \langle z, y \rangle$. ■

The same proof gives the following lemma.

LEMMA 4.2. – *Fix two hyperplanes H, R of V, $H \neq R$, a hyperplane U of H, a hyperplane U' of R (both different from $L := H \cap R$), a point $P \in U \setminus (U \cap L)$, a point $Q \in U' \setminus (U' \cap L)$. Take two lines D, D' in T with $D \cap D' = \emptyset$, $D' \cap V = \{P\}$, $D \cap V = \{Q\}$, and a finite number of hyperplanes U_i of U, L_j of L. Then there are points $x \in U$, $y \in U'$, $x \notin U_i$, $y \notin U'_i$ for every i, j, and lines F, F' in T with $\{x\} = F \cap V$, $\{y\} = F' \cap V$, $D \cap F \neq \emptyset$, $D' \cap F' \neq \emptyset$, $F \cap F' \neq \emptyset$.*

PROOF OF THEOREM 1. – (*a*) Fix integers r, k with $k > 0$, $r > 2k$. In $T := \boldsymbol{P}^r$ take $k+1$ hyperplanes $V, W_1, \ldots, W_k$ and two hyperplanes H, R of V; set $L := R \cap H$. Let W^i be the intersection of all W_j with $j \neq i$; set $V_i = W_i \cap V$ $V^i = W^i \cap V$. Assume $\dim(L \cap W_k \cap W^k) = r - 3 - k$. Set $U := H \cap W_k \cap W^k$, $U' := R \cap W_k \cap W^k$; (hence in V we have a configuration of $k+2$ hyperplanes as in 3.2). Fix a general point $P \in U$ and a line $D \subset W_k \cap W^k$ with $D \cap V = \{P\}$. For each j, $j = 1, \ldots, k$, fix a general point $P_j \in L \cap W^j$. Assume $P_j \notin W_j$ for every j and in particular $P_j \notin U \cup U'$. Let D_j, $j = 1, \ldots, k$, be disjoint lines with $D_j \subset W^j$, D_j intersecting D and $P_j \in D_j$ (hence $D_j \not\subset W_j$). Assume $D_j \cap V = \{P_j\}$ for every j. Let $R_1, \ldots, R_k$ be disjoint lines with $R_j \cap D_j \neq \emptyset$, R_j intersecting V at a point $Q_j \in (U' \setminus L)$. In particular $R_j \subset W^j$. We may assume that the points $Q_1, \ldots, Q_k$ are linearly independent (here we use that $r > 2k$). We apply 4.1 to D and R_j in W^j for $j = 1, \ldots, k$ (with respect to $V \cap W^j$, $H \cap W^j$, $R \cap W^j$ and $U := H \cap W_k \cap W^k$ as hyperplane of $H \cap W^j$); we may apply 4.1 k times simultaneously because the conditions of linear independence in the thesis of 4.1 are « open » conditions. Hence by 4.1 we find a line A_1, a point $B_1 := (A_1 \cap V) \in U$, and lines L_{1j}, $j = 1, \ldots, k$, with L_{1j} intersecting A_1 and R_j, $L_{1j} \not\subset V$, with $B_{1j} := L_{1j} \cap V$ in $L \cap W^j$ and with $B_1 \neq P$, $B_{1j} \neq P_j$; hence $L_{1j} \not\subset W^j$. Since $B_1 \in W_k \cap W^k$ and $D \subset W_k \cap W^k$, we have $A_1 \subset (W_k \cap W^k)$. Then we apply again this construction $r - 2k - 2$ times. Again by 4.1 we find lines $A_1, \ldots, A_{r-2k-1}$ in $W_k \cap W^k$ with, for $i > 1$, A_i intersecting A_{i-1} and containing a point $B_i \in U$, lines L_{ij}, $1 \leqslant i \leqslant r - 2k - 1$, $1 \leqslant j \leqslant k$, with L_{ij} intersecting both A_i and R_j, and intersecting V at a point $B_{ij} \in L \cap W^j$, $B_{ij} \notin V_j$. By 4.1 we may assume that $P, B_1, \ldots, B_{r-2k-1}$ are linearly independent and that for every j with $j = 1, \ldots, k$, the points P_j, B_{ij}, $1 \leqslant i \leqslant r - 2k - 1$, are linearly independent in $L \cap W^j$. Let C' be the union of D, A_i for $1 \leqslant i \leqslant r - 2k - 1$, D_j, R_j, $1 \leqslant j \leqslant k$. Let C be the union of C' and $d - r$ of the lines L_{ij}. We may take the points P, B_i, $1 \leqslant i \leqslant r - 2k - 1$, P_j, Q_j, $1 \leqslant j \leqslant k$, spanning V. Hence by 3.2, § 2 and [19] or [9], C is the curve we were looking for.

(*b*) Now assume $r \geqslant 2k + 2$. Take another line R_{k+1} intersecting D and intersecting V at a point $Q_{k+1} \in (U' \setminus L)$. Assume that the points $Q_1, \ldots, Q_{k+1}$ are linearly independent. By 4.1 and 4.2 we may find lines $A_1, \ldots, A_{r-2k-2}$, L_{ij}, $1 \leqslant i \leqslant r - 2k - 2$ $1 \leqslant j \leqslant k$, satisfying the previous conditions, and lines $F_1, \ldots, F_{r-2k-2}$, such that for every i, $1 \leqslant i \leqslant r - 2k - 2$, F_i intersects A_i and R_{k+1}, F_i intersects V at a point $Z_i \in U' \setminus L$; by 4.2 we may assume that the points $Z_1, \ldots, Z_{r-2k-2}, Q_1, \ldots, Q_{k+1}$ are linearly independent. We take as C the union of D, A_i for $1 \leqslant i \leqslant r-2k-2$, E_j for $1 \leqslant j \leqslant k+1$, L- for $1 \leqslant u \leqslant k$, and $d-r$ of the lines L_{ij}, F_i, $1 \leqslant i \leqslant r-2k-2$ $1 \leqslant j \leqslant k$. ∎

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