Some Properties of Condensing Maps (*) (**) (***).

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Summary. – *See Introduction.*

1. – Introduction.

In 1939, KURATOWSKI [19] introduced a measure of noncompactness of bounded sets in a metric space, called the Kuratowski measure of noncompactness, or α-measure. This along with the associated notion of an α-contraction, has proved usedful in several areas of differential equations (see, for example, [12], [14] and [25]).

DEFINITION. – Let X be a metric space. The α-measure is a map $\alpha\colon \mathcal{B} \to [0, \infty)$, where $\mathcal{B} = \{B \subset X\colon B \text{ is bounded}\}$ and $\alpha(B) = \inf\{d\colon \text{there is a finite cover of } B$ with sets in X whose diameter is less than $d\}$.

DEFINITION. – $T\colon X \to X$ is an α-contraction if there is a $k \in [0, 1)$ such that for all $B \in \mathcal{B}$ we have $\alpha(TB) \leqslant k\alpha(B)$.

To generalize this notion people began to investigate α-condensing maps.

DEFINITION. – $T\colon X \to X$ is α-condensing if for all $B \in \mathcal{B}$ we have $\alpha(TB) \leqslant \alpha(B)$ with equality if and only if $\alpha(B) = 0$.

The basic problem is to understand which properties of α-contractions also hold for α-condensing maps. Recently, Richard LEGGETT [20] showed that if X is a Banach space, T is linear and α-condensing, then there is an equivalent norm in X for which T is an α-contraction in the new norm. SADOVSKII [25] showed by transfinite induction that α-condensing maps have the fixed point property-extending a result of DARBO [9] for α-contractions. Another important contribution showing how many properties of α-contractions are true for α-condensing maps was made by COOPERMAN [8]. Cooperman developed an ingenious technique which exploited special properties of the α-measure.

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The principle results of this paper will be to generalize several of the results of Cooperman to more general measures of noncompactness as well as for certain set mappings. The proofs are more elementary than the ones in Cooperman. However, the basic lemma used by Cooperman which depended so much on properties of α-measures is not generalized. In fact, we give an example showing that it will not generally be valid for arbitrary measures of noncompactness.

Section 2 contains only notation and definitions. In section 3 we prove that a decreasing sequence of nonempty closed bounded sets with general measure of noncompactness approaching zero must have nonempty intersection. In section 4 we show condensing maps are asymptotically smooth. In section 5 we show the solution map $T_\lambda x = y$ as a function of y for $\{T_\lambda\}$ collectively condensing is upper semicontinuous with mild continuity assumptions on T_λ. In section 6 we show results proved using Sadovskii's method of transfinite sequences may be proved using ordinary sequences. We reprove a theorem by Hale and Lopes that α-condensing, compact dissipative maps have a fixed point, and show this holds for general measures of noncompactness. In section 7 we discuss the basic lemmas of Cooperman and their validity for general measures of noncompactness. In section 8 we show linear condensing maps with general measures of noncompactness are α-contractions under some equivalent norm.

2. – Definitions and notation.

Let X be a complete metric space, or a complete metrizable linear topological vector space. When we speak of distance in the latter case we may use any metric coinciding with the topology on X. Let $\mathcal{B}$ be the collection of bounded subsets of X. For any subset B of X, let $\mathrm{Cl}\,(B)$ denote the closure of B. Let $\mathcal{C}$ be the collection of subsets of X.

DEFINITION 2.1. – For any sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{B}$, the *$\omega$-limit set* of $\{B_n\}$ is defined by $\omega(\{B_n\}) = \bigcap_{k=1}^{\infty} \mathrm{Cl}\Big(\bigcup_{j\geq k} B_j\Big)$, or equivalently $\omega(\{B_n\}) = \{y \in X\colon \exists$ integers $n_k \to \infty$, $x_k \in B_{n_k}$ such that $x_k \to y\}$.

DEFINITION 2.2. – A set $A \subset X$ *attracts* a sequence of sets $\{B_n\}_{n=1}^{\infty} \subset X$ if $d(B_n, A) \to 0$ as $n \to \infty$, where $d(B, A) = \sup_{x\in B}\{d(x, A)\}$.

DEFINITION 2.3. – If $H\colon \mathcal{C} \to \mathcal{C}$ and $B \in \mathcal{C}$, the *orbit* $\gamma_H^+(B) \subset \mathcal{C}$ under H is defined by $\gamma_H^+(B) = \bigcup_{n=0}^{\infty} H^n(B)$. The *$\omega$-limit set* $\omega_H(B)$ of B under H is defined by $\omega_H(B) = \omega_H(\{H^n(B)\}) = \bigcap_{k=1}^{\infty} \mathrm{Cl}\Big(\bigcup_{j\geq k} H^j(B)\Big)$. When no confusion arises we may drop the subscript, H.

DEFINITION 2.4. – A subset $U \subset \mathcal{C}$ (i.e., a collection of subsets of X) is *invariant* under $H\colon \mathcal{C} \to \mathcal{C}$ if $HU = U$. It is *positively invariant* if $HU \subset U$.

If $T\colon X \to X$ is a map on X, then T induces a map $i_T\colon \mathcal{C} \to \mathcal{C}$ by the relation $i_T(B) = \bigcup_{x\in B} Tx$ for any $B \in \mathcal{C}$. The above definitions coincide with the usual definitions of ω-limit set and invariance.

Other mappings on a collection of subsets of X are useful. We make the following definitions.

DEFINITION 2.5. – If $H\colon \mathcal{C} \to \mathcal{C}$ is a given map, we say

H is of *type* 1 if $H(B) = \bigcup \{H(x)\colon x \in B\}$.

H is of *type* 2 if $H(B) = \bigcup \big\{H(\{x_i\})\colon \{x_i\}$ is any finite subset of $B\big\}$.

H is of *type* 3 if $H(B) = \bigcup \{H(K)\colon K \subset B$ is compact$\}$.

Similar definitions apply for $H\colon \mathcal{B} \to \mathcal{C}$ or $H\colon \mathcal{B} \to \mathcal{B}$.

A set operator of type 1 is a set operator of type 2 which is turn is a set operator of type 3. If H_1 and H_2 are set operators of the same type n, for $n = 1$ or 2, then $H_1 \circ H_2$ is a set operator of type n. This property may not hold for operators of type 3.

EXAMPLES.

(i) If $T\colon X \to X$ then $i_T\colon \mathcal{C} \to \mathcal{C}$ defined by $i_T(B) = \bigcup \{Tx \in X\colon x \in B\}$ is of type 1.

(ii) If $T\colon X \to X$, then $\gamma^+\colon \mathcal{C} \to \mathcal{C}$ defined by $\gamma^+(B) =$ positive orbit through B is of type 1.

(iii) If $T\colon X \to \mathcal{C}$ is a set valued map on X, then the map $i_T\colon \mathcal{C} \to \mathcal{C}$ defined by $i_T(B) = \bigcup \{Tx \in \mathcal{C}\colon x \in B\}$ is of type 1.

(iv) If $C \in \mathcal{C}$ is given, then $H\colon \mathcal{C} \to \mathcal{C}$ defined by either $H(B) = B \cup C$, $B \cap C$, or $B + C$, is of type 1.

(v) $H\colon \mathcal{C} \to \mathcal{C}$ given by $H(B) = \operatorname{co} B =$ the convex hull of B is of type 2.

(vi) $H\colon \mathcal{C} \to \mathcal{C}$ given by $H(B) = \operatorname{Cl}(B)$ is of type 3.

DEFINITION 2.6. – A *measure of noncompactness* on X is a map $\beta\colon \mathcal{B} \to [0, \infty)$ with the properties that (i) $\beta(B) = 0$ if and only if $\operatorname{Cl}(B)$ is compact and (ii) $\beta(B) \leqslant \beta(C)$ if $B \subset C$.

The α-measure of noncompactness of Kuratowskii defined in the introduction is a measure of noncompactness. It satisfies many more properties, some of which will be required below. They will be introduced as needed since one of our objectives is to understand which basic properties of the α-measure imply certain results.

DEFINITION 2.7. – If $H: \mathcal{B} \to \mathcal{B}$, then H is a *β-contraction* if there is a $k \in [0, 1)$ such that $\beta(H(B)) \leqslant k\beta(B)$ for any $B \in \mathcal{B}$. H is *β-condensing* if for each $B \in \mathcal{B}$, $\beta(H(B)) \leqslant \beta(B)$ with equality if and only if $\beta(B) = 0$. If $T: X \to X$ and maps bounded sets to bounded sets, we say T is a β-contraction (β-condensing) if the induced map $i_T: \mathcal{B} \to \mathcal{B}$ is a β-contraction (β-condensing).

In the applications it is sometimes convenient to not assume $H: \mathcal{B} \to \mathcal{B}$ but only that $H: \mathcal{B} \to \mathcal{C}$; that is H may not take bounded sets into bounded sets. One then calls the map a conditional β-contraction (conditionel β-condensing) if the above properties hold for each $B \in \mathcal{B}$ for which $H(B) \in \mathcal{B}$. The results below hold in this more general situation, but we do not explicitly state and prove them in this generality since it is only a minor technical detail.

DEFINITION 2.8. – Let $H: \mathcal{C} \to \mathcal{C}$ and let S be a collection of sets. A bounded set B *dissipates S-sets under H* if for any $C \in S$, there is an integer $n_0(C)$ such that $H^n(C) \subset B$ for $n \geqslant n_0(C)$. If $S = \{\{x\}: x \in X\}$ we say H is *point dissipative*, if $S = \{\{J\}: J \subset X$ is compact$\}$ we say H is *compact dissipative*; if S contains a neighborhood of any point $x \in X$, we say H is *local dissipative*; if S contains a neighborhood of any compact set, we say H is *local compact dissipative*; if S contains all bounded sets of $X (S = \mathcal{B}$ we say H is *bounded dissipative* or *ultimately bounded*.

If H is type 2 and continuous in the Hausdorff metric, then compact dissipative, local dissipative and local compact dissipative are equivalent.

DEFINITION 2.9. – A map $H: \mathcal{B} \to \mathcal{C}$ is *asymptotically smooth* if, for any $B \in \mathcal{B}$ such that $\gamma_H^+(B) \in \mathcal{B}$, there is a compact set $J \subset X$ such that J attracts B under H.

Asymptotically smooth maps play an important role in stability theory. In fact, it is known (see [8], [14]) that H asymptotically smooth and compact dissipative implies there is a maximal compact invariant set J for H which attracts neighborhoods of compact sets. In particular, J is uniformly asymptotically stable. It is important therefore to give other characterizations of asymptotically smooth maps.

LEMMA 2.1. – If $T: X \to X$ is a given map and $i_T: \mathcal{C} \to \mathcal{C}$ is the map induced by T, $i_T(B) = \bigcup \{Tx: x \in B\}$, then the following are equivalent to i_T: being asymptotically smooth:

(1) for any $B \in \mathcal{B}$ such that $\gamma^+(B) \in \mathcal{B}$, there is a compact set J that attracts B under T;

(2) for any $B \in \mathcal{B}$ such that $TB \subset B$ there is a compact set J that attracts B under T;

(3) for any $B \in \mathcal{B}$ there is a compact set J such that, for any $\varepsilon > 0$, there is an integer $n_0(B, \varepsilon)$ such that $T^n x \in B$ for $n \geqslant 0$ implies $d(T^n x, J) < \varepsilon$ for $n \geqslant n_0(B, \varepsilon)$.

Lemma 2.1 suggests other definitions of asymptotic smooth for mappings H: $\mathcal{B} \to \mathbb{C}$. More precisely one could define asymptotic smooth of type (j), $j = 1, 2$ by the relations (1) and (2) in Lemma 2.1, with T replaced by H. One finds asymptotic smooth (2) defines a smaller class than asymptotic smooth (1).

3. – A property of measures of noncompactness.

A classical result for the α-measure of noncompactness is that a decreasing sequence $\{B_n\}$ of closed bounded sets with $\alpha(B_n) \to 0$ has the property that $d(B_n, J) \to 0$ as $n \to \infty$ for some compact set J. It is the purpose of this section to show this result is true for more general measures of noncompactness. We need the following lemma.

LEMMA 3.1. – If $\{B_n\}_{n=1}^{\infty}$ is a sequence of bounded sets in X with the property that every $\{x_k\}_{k=1}^{\infty} \subset X$ is precompact if there is a sequence of integers $n_k \to \infty$ with $x_k \in B_{n_k}$, then $\omega(\{B_n\})$ is nonempty, compact, and attracts B_n.

PROOF. – $\omega(_T B_n\}) = \{y \in X$: there exists $n_k \to \infty$, $x_k \in B_{n_k}$ such that $\{x_k\}$ converges to $y\}$. $\omega(B_n)$ is nonempty since any sequence $x_k \in B_{n_k}$ with $n_k \to \infty$ has a converging subsequence which must converge to a point in $\omega(\{B_n\})$. $\omega(\{B_n\})$ is precompact since if we let $\{y_k\}$ be a sequence in $\omega(\{B_n\})$ then there is a sequence $n_k \to \infty$, $x_k \in B_{n_k}$ with $d(x_k, y_k) < 2^{-k}$. But $\{x_k\}$ has a convergent subsequence, hence so does $\{y_k\}$. All that is left is to show $\omega(\{B_n\})$ attracts B_n. Suppose it does not. Then there is an $\varepsilon > 0$, $n_k \to \infty$, $x_k \in B_{n_k}$ with $d\big(x_k, \omega(\{B_n\})\big) > \varepsilon$. But x_k has a converging subsequence which must converge to a point in $\omega(\{B_n\})$. This is a contradiction. Q.E.D.

THEOREM 3.1. – If β is a measure of noncompactness satisfying $\beta(A \cup B) = \beta(A)$ if B is a finite set then any decreasing sequence $\{B_n\} \subset \mathcal{B}$ of nonempty closed bounded sets satisfying $\beta(B_n) \to 0$ must have $\bigcap_n B_n$ nonempty, compact, and attracting B_n.

PROOF. – Let B_n be a decreasing sequence of nonempty closed bounded sets with $\beta(B_n) \to 0$. Clearly $\omega(B_n) = \bigcap_{n=0}^{\infty} B_n$. Let $n_k \to \infty$ and $x_k \in B_{n_k}$. Then $\beta(\{x_k\}) \leqslant \beta(B_n)$ for any n since $\{x_k\}$ minus a finite number of points is a subset of B_n. But then $\beta(\{x_k\}) = 0$ and $\{x_k\}$ is precompact. Lemma 3.1 implies the result.

4. – Dissipative processes.

The basic result of this section relates β-condensing maps to asymptotically smooth maps.

THEOREM 4.1. – Suppose β is a measure of noncompactness satisfying $\beta(A \cup B) = \beta(A)$ if B is a finite set. If $H: \mathcal{B} \to \mathcal{B}$ is β-condensing and of type 2 (see Definition 2.5), then H is asymptotically smooth.

PROOF. – Let B and $\gamma^+(B)$ be bounded. Let $\mathfrak{D}(B) = \{\{x_k, n_k\}: \{n_k\} \to \infty, x_k \in H^{n_k}(B)\}$. Let $P(\{x_k, n_k\}) = \{x_k\}$. Let $\eta = \sup\{\beta(Ph)/h \in \mathfrak{D}(B)\}$. Note $\eta < \infty$ since $\gamma^+(B)$ is bounded. We first show there is an $h^* = \{x_k^*, n_k^*\} \in \mathfrak{D}(B)$ such that $\beta(Ph^*) = \eta$. Let $\{h_j\} \subset \mathfrak{D}(B)$ be a sequence with $\beta(Ph_j) \to \eta$. Let $\hat{h}_j = \{(x_k, n_k) \in \in h_j: n_k > j\}$. Let $h^* = \bigcup_{k=1}^{\infty} \hat{h}_j$ reordered in any way. Then we have $h^* \in \mathfrak{D}(B)$ and so $\eta \geq \beta(h^*) \geq \beta(\hat{h}_j) = \beta(h_j) \to \eta$, as $j \to \infty$. Hence, $\beta(h^*) = \eta$.

Now for each $(x_k^*, n_k^*) \in h^*$ there is a set $\{x_k^{j*}, n_k^* - 1\}_{j=1}^{m_k} \subset H^{n_k^*-1}(B) \times Z$ such that $x_k^* \in H(\{x_k^{j*}\})$. Let $g^* = \bigcup_{k=1}^{\infty} \{x_k^{j*}, n_k^* - 1\}_{j=1}^{m_k} \in \mathfrak{D}(B)$. Hence $\eta \geq \beta(g^*) \geq \beta(Hg^*) \geq \beta(h^*) = \eta$ with equality if and only if $\beta(g^*) = 0$. Hence $\eta = 0$.

Now Lemma 3.1 implies there is a $J \subset X$ compact, which attracts $\{H^n(B)\}$, or attracts B under the map H.

COROLLARY 4.1. – If β satisfies the conditions of Theorem 4.1 and $T: X \to X$ is β-condensing, then T is asymptotically smooth.

COROLLARY 4.2. – If β satisfies the conditions of Theorem 4.1 and $T: X \to \mathfrak{C}$ is β-condensing, then T is asymptotically smooth.

COROLLARY 4.3. – If β-satisfies the conditions of Theorem 4.1 and $\beta(\text{co } B) = \beta(B)$, $T: X \to X$ is β-condensing, and $H: \mathfrak{B} \to \mathfrak{B}$ is defined $H(B) = \text{co } T(B)$ then H is asymptotically smooth. Furthermore, if T is continuous, then $\bar{H}: \mathfrak{B} \to \mathfrak{B}$ defined by $\bar{H}(B) = \text{cl } H(B)$ is asymptotically smooth.

PROOF. – H is clearly β-condensing and type 2, hence we have the first part of the corollary. For the second part we note $H(\text{cl } B) \subset \text{cl } H(B)$. Using this we get $\bar{H}^n(B) \subset \text{cl } H^n(B)$ and so $\bar{H}$ is asymptotically smooth.

COROLLARY 4.4. – If β satisfies the conditions of Corollary 4.3, $T: X \to X$ is β-condensing, $P \subset X$ is compact and $H: \mathfrak{B} \to \mathfrak{B}$ is defined by $H(B) = \text{co }(T(B) \cup P)$ then H is asymptotically smooth. If T is also continuous then $\bar{H}$ is also asymptotically smooth.

COROLLARY 4.5. – If β is a measure of noncompactness satisfying $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$ and H is β-condensing then $\gamma_H^+: \mathfrak{B} \to \mathfrak{B}$ implies γ_H^+ is β-nonexpansive.

PROOF. – Assume $\beta(\gamma_H^+(B)) > \beta(B)$. Then $\beta(\gamma_H^+(B)) = \beta(B \cup H\gamma_H^+(B)) = \max\{\beta(B), \beta(H\gamma_H^+(B))\} = \beta(H\gamma_H^+(B)) < \beta(\gamma_H^+(B))$ which is a contradiction.

The last three corollaries are useful in showing several fixed point theorems proved by Sadovskii's method of transfinite sequences may be proved with ordinary sequences. This is illustrated in Section 6.

THEOREM 4.2. – Let β be a measure of noncompactness satisfying $\beta(A \cup B) = \beta(A)$ if B is a finite set and either (1) β is continuous in the Hausdorff metric or (2) $\beta(\bar{B}) = \beta(B)$, and $\beta(A + B) = \beta(A)$ if $\beta(B) = 0$.

If $H: \mathfrak{B} \to \mathfrak{B}$ is β-condensing and type 3, then H is asymptotically smooth.

PROOF. – Let B, $\gamma^+(B) \in \mathfrak{B}$. Let $\mathfrak{D}'(B) = \{\{J_k, n_k\}: J_k \subset H^{n_k}(B),\ \beta(J_k) = 0,\ n_k \to \infty\}$. Let $\mathfrak{D}(\beta) = \{\{x_k, n_k\}: x_k \in H^{n_k}(B),\ n_k \to \infty\}$, $p'h' = p'\{J_k, n\varepsilon\} = \bigcup_k J_k$ for $h' \in \mathfrak{D}'(B)$, $ph = p\{x_k, n_k\} = \bigcup_k \{x_k\}$ for $h \in \mathfrak{D}(B)$. Let $\eta' = \sup\{\beta(p'h'): h' \in \mathfrak{D}'(B)\}$. Let $\eta = \sup\{\beta(ph): h \in \mathfrak{D}(B)\}$. We first show $\eta = \eta'$. This takes the most work. By the method of Theorem 4.1, we show there exists $h_1'^* \in \mathfrak{D}'(B)$ with $\beta(h_1'^*) = \eta'$. Let $h_1'^* = \{J_k^{1*}, n_k^*\}$ and $h_2'^* = \{J_k^{2*}, n_k^*\}$ with J_k^{2*} a finite set satisfying $d(J_k^{2*}, J_k^{1*}) \leqslant 2^{-k}$ and $J_k^{2*} \subset J_k^{1*}$. We claim $\beta(h_2'^*) = \beta(h_1'^*)$. Both are in $\mathfrak{D}'(B)$.

Case (*i*): Let $\gamma_l^* = \{R_k^{l*}, n_k^*\}$ with $R_k^{l*} = J_k^{2*}$ if $k \geqslant l$ and R_k^{l*} is a finite set with $R_k^{l*} \subset J_k^{1*}$, $d(R_k^{l*}, J_k^{1*}) < 2^{-l}$ if $k < l$. Then $\beta(h_2'^*) = \beta(\gamma_l^*) \to \beta(h_1'^*)$. Hence, $\beta(h_2'^*) = \beta(h_1'^*)$.

Case (ii): Let $h_3'^* = \{J_k^{3*}, n_k^*\} \in \mathfrak{D}'(B)$ with $J_k^{3*} \subset J_k^{1*}$ countable and dense in J_k^{1*}. Clearly, $\beta(h_3'^*) = \beta(h_1'^*)$. Now for each point $z_i^k \in J_k^{3*}$ there is an $x_i^k \in J_k^{2*}$, $y_i^k \in X$ with $z_i^k = x_i^k + y_i^k$ and $|y_i^k| = d(z_i, J_k^{2*}) \leqslant 2^{-k}$. The set $\{y_i^k / |y_i^k| \geqslant 2^{-k}\} \subset \bigcup_{k=1}^{r} J_k^{3*} - \bigcup_{k=1}^{r} J_k^{2*}$ and hence is compact.

Since r is arbitrary we know $\{y_i^k\}$ is compact. Now $h_3'^* \subset h_2'^* + \{y_i\}$ and $h_2'^* \subset h_3'^* - \{y_i\}$ implies $\beta(h_3'^*) = \beta(h_2'^*)$. This shows $\beta(h_1'^*) = \beta(h_3'^*)$.

But $h_2'^* \in \mathfrak{D}(B)$ also (when reordered). Hence, $\eta \geqslant \eta'$. It is obvious that $\eta \leqslant \eta'$. Hence $\eta = \eta'$. Now let $h^* \in \mathfrak{D}(B)$ with $\beta(h^*) = \eta$ (constructed as is Theorem 4.1). Since H is of type 3 there is an $h' \in \mathfrak{D}'(B)$ such that $h^* \subset H(h')$. We get $\eta \geqslant \beta(h') \geqslant \beta(H(h')) \geqslant \beta(h^*) = \eta$, with equality if and only if $\beta(h') = 0$. Hence, we have $\eta = \eta' = 0$. Now we may apply Lemma 3.1 to obtain the result.

5. – Continuous dependence on parameters.

Here we look at a result originally proved by ARTSTEIN [2] for α-contractions. The result was extended to α-condensing maps by Cooperman and will now be generalized to arbitrary measures of noncompactness.

DEFINITION 5.1. – We say that a *convergence structure* is given for a set V if to certain nets $\{v_n, n \in N\}$ in V (called the convergent nets) there corresponds an element v in V, denoted by $\lim v_n$, so that the following conditions are fulfilled.

(*a*) If $v_n = v$ $\forall n$ then $\lim v_n = v$.

(*b*) If $\lim v_n = v$ and $\{v_m\}$ is a subnet then $\lim v_m = v$.

(ε') If $\{v_n\}$ does not converge to v then a subnet of $\{v_n\}$ exists, no subset of which converges to v.

A set with a convergent structure on it is called a convergence space. Not every convergence space is topological. See KELLEY [18] on the « convergence of the iterated limit » property.

DEFINITION 5.2. – Let $\mathfrak{U} = \{U_\lambda\}$ be a set of operators $U_\lambda: X \to X$ with a convergence structure on it. Then $\{U_\lambda\}$ is a *collective β-contraction* if there is a $k \in [0, 1)$ such that for all $B \in \mathfrak{B}$ we have $\beta\Big(\bigcup_\lambda U_\lambda B\Big) \leqslant k\beta(B)$. $\{U_\lambda\}$ is collectively β-condensing if $\forall B \in \mathfrak{B}$, $\beta\Big(\bigcup_\lambda U_\lambda B\Big) \leqslant \beta(B)$ with equality if and only if $\beta(B) = 0$.

DEFINITION 5.3. – A multi-valued function Ω from the convergence space $\mathfrak{U}$ to the convergence space $\mathfrak{W}$ is Lu-continuous if $\lim u_n = u$, $w_n \in \Omega(u_n)$, and $\lim w_n = w$ implies $w \in \Omega(u)$.

REMARK. – If $\mathfrak{U}$ and $\mathfrak{W}$ are topological spaces this is equivalent to the graph being closed.

The following lemma is proved by ARTSTEIN in [2] and only stated here.

LEMMA 5.1. – Let $\{y_k: k \in K\}$ be a net which is contained in a bounded set of X. Denote by A_k the set $\{y_j: j \geqslant k\}$. If the numbers $\alpha(A_k)$ converge to zero then there is a convergent subnet $\{y_n, n \in K\}$ of $\{y_k\}$.

DEFINITION 5.4. – B is semi-invariant with respect to $\mathfrak{U}$ if for all $x \in B$ there is a $U \in \mathfrak{U}$ with $x = Ux$.

For the next theorem we will also use the following lemma.

LEMMA 5.2. – Let $\mathfrak{T}$ be a collectively β-condensing family of operators. Let the β-measure of noncompactness satisfy $\beta(A + B) = \beta(A)$ if $\beta(B) = 0$. Let $B_i = \{x$: there exist $(T, y) \in \mathfrak{T} \times X$ such that $Tx = y$ for some $y \in X$ with $|y - y_0| < y_i\}$. Then $\mathfrak{T}$ collectively β-condensing implies $\alpha(B_i) \to 0$ as $i \to \infty$.

PROOF. – Let $\{l_i\} \to \infty$ and $x_i \in B_{li}$. We will show $\{x_i\}$ has a converging subsequence, and then apply Lemma 3.1.

Since $x_i \in B_{li}$ there is $\{T_i\}$, $\{y_i\} \to y_0$ such that $x_i = T_i x_i + y_i$. Let $U_i: X \to X$ be defined by $U_i x = T_i x + y_i$. Since $\{y_i\} \to y_0$, it is precompact. Also $\bigcup_i U_i B \subset \bigcup_i T_i(B) + \{y_i\}$ so

$$\beta\Big(\bigcup_i U_i(B)\Big) \leqslant \beta\Big(\bigcup_i T_i(B) + \{y_i\}\Big) = \beta\Big(\bigcup_i T_i(B)\Big) \leqslant \beta(B)$$

with equality if and only if $\beta(B) = 0$. Hence $\{U_i\}$ is collectively β-condensing. Now since $\{x_i\}$ is semi-invariant with respect to $\{U_i\}$ it is precompact. This completes the proof of the Lemma.

THEOREM 5.1. – Let X be a metric space. Let $\mathcal{T}$ be a collectively β-condensing family of operators. Let the β-measure be as in Lemma 5.2. If for a certain convergence structure on $\mathcal{T}$, the map $s(T, y) = \{x: Tx = y\}$ is Lu-continuous, then for every closed and bounded B the mapping $s(T, y) \cap B: \mathcal{T} \times X \to X$ is upper semi-continuous.

PROOF. – We first make the following remark. If $\mathcal{U}$ is collectively β-condensing and B is semi-invariant with repect to $\mathcal{U}$ then $\beta(B) = 0$. This is a trivial consequence of the definitions.

Let $\{(T_k, y_k)\} \to (T_0, y_0)$ be a converging net with $T_k \in \mathcal{T}$ and $y_k \in X$. Let $x_k = T_k x_k + y_k$, and $x_k \in B$. We must show $\{x_k\}$ converges to the set $s(T, y_0) \cap B$. We notice $s(T, y_0) \cap B$ is compact since if we define $T_{y_0}: X \to X$ by $T_{y_0}(x) = Tx + y_0$ then T β-condensing implies T_{y_0} is β-condensing. Furthermore, since $s(T, y_0) \cap B$ is invariant with respect to T_{y_0}, it is precompact. Now, by the Lu-continuity of $s(T, y)$ it suffices to prove the existence of a convergent subnet. Let $A_k = \{x_n: n \geqslant k\}$. There are k_i such that $A_{ki} \subset B_i$ with B_i defined in Lemma 5.2. Lemma 5.2 limplies $\alpha(B_i) \to 0$ which implies $\alpha(A_{ki}) \to 0$, or $\alpha(A_k) \to 0$. Lemma 5.1 implies there is a convergent subnet. This completes the proof.

6. – Fixed point theorems.

We begin by stating two theorems previously proved by Sadovskii's method of transfinite sequences, and show they can be proved using ordinary sequences. Then we will reprove a result of Hale and Lopes, which used Zorn's lemma, and show this can also be proved without using Zorn's lemma.

The first is due to Sadovskii and is found in [25].

THEOREM 6.1. – Let $T: X \to X$ be β-condensing and continuous. Let the β-measure satisfy $\beta(A \cup B) = \beta(A)$ if B is finite and $\beta(\text{co } A) = \beta(A)$ for any $A \in \mathcal{B}$. Let $B \subset X$ be closed, bounded, convex, and positively invariant (i.e. $T(B) \subset B$). Then T has a fixed point.

PROOF. – Let $H: \mathcal{B} \to \mathcal{B}$ be defined by $H(B) = \text{co } T(B)$. Then $\bar{H}$ is asymptotically smooth by Corollary 4.3. Therefore, $\omega_{\bar{H}}(B) = \bigcap_{n=1}^{\infty} \bar{H}^n(B)$ is compact, convex, invariant under H, and attracts B. It is, therefore positively invariant under T. Schauder fixed point theorem implies $T: \omega_{\bar{H}}(B) \to \omega_{\bar{H}}(B)$ has a fixed point.

The second theorem is a nonrepulsive fixed point theorem by Mario MARTELLI [22].

DEFINITION 6.1. – Let Y be a nonempty subset of a topological space X and $f: Y \to Y$ be continuous. A point $x_0 \in X$ is said to be a *repulsive fixed point* for Ω

if (i) $\Omega(x_0) = x_0$, (ii) there exists a neighborhood U of x_0 such that for any neighborhood V of x_0 there exists an n_0 with the property that $\bigcup_{n \geq n_0} \Omega^n(Y \setminus V) \subset Y \setminus U$, Ω is a *nonrepulsive fixed point* if it is not a repulsive fixed point.

We will use the following theorem of BROWDER [5].

THEOREM 6.2. – Let C be a compact, convex, infinite dimensional subset of a Banach space X and let $\mathcal{F}: C \to C$ be continuous. Then $\mathcal{F}$ has a non-repulsive fixed point.

THEOREM 6.3. – Let B be a closed, bounded, convex, and infinite dimensional subset of a Banach space X and let $T: B \to B$ be a continuous β-condensing map, with β satisfying $\beta(A \cup C) = \beta(A)$ if C is a finite set, and $\beta(\text{co } A) = \beta(A)$. Then T has a nonrepulsive fixed point.

PROOF. – Let $B \in \mathfrak{B}$ be compact and infinite dimensional. Let $H_1: \mathfrak{B} \to \mathfrak{B}$ be defined by $H_1(A) = A \cup P$. Let $H_2: \mathfrak{B} \to \mathfrak{B}$ be defined by $H_2(A) = \text{co } A$. Let $H: H_2 \circ H_1 \circ T$. H is β-condensing, T, H_1, and H_2 are of type 2, hence so is H. Thus, Theorem 4.1 implies H is asymptotically smooth. Continuity of T implies $H(\bar{C}) \subset \subset \overline{H(C)}$ which implies $\bar{H}$ is asymptotically smooth. Hence $\omega_{\bar{H}}(B) = \bigcap_{n=1}^{\infty} \bar{H}^n(B)$ is compact, ∞-dimensional, invariant under $\bar{H}$, and attracts B. It is also positively invariant under T. Theorem 6.2 implies $T: \omega_{\bar{H}}(B) \to \omega_{\bar{H}}(B)$ has a non-repulsive fixed point.

The next theorem originally proved by HALE and LOPES [16] is reproved here in more detail, to end any confusion as to its validity. It is followed by a simpler proof that does not use Zorn's lemma (which Hale and Lopes use to prove result 3 below).

THEOREM 6.4. – Let T be β-condensing, continuous, and compact dissipative. With β a measure of noncompactness satisfying $\beta(A \cup B) = \max[\beta(A), \beta(B)]$ and $\beta(\text{co } A) = \beta(A)$. Then T has a fixed point.

The proof by Hale and Lopes and the results we use below are also found in HALE [14].

RESULT 1. – H compact, T β-condensing, $\gamma^+(H)$ bounded implies $\gamma^+(H)$ is precompact and $\omega(H)$ is compact, invariant and attracts H.

RESULT 2 (Horn [17]). – If $S_0 \subset S_1 \subset S_2$ are convex subsets of X, S_0, S_2 compact, and S_1 relatively open in S_2, $T: S_2 \to X$ is continuous, $\gamma^+(S_1) \subset S_2$, and S_0 dissipates S_1, the T has a fixed point.

RESULT 3 (LEMMA 4.1 in Hale [14]). – Suppose $K \subset B \subset S$ are convex subsets of X with K compact, S closed and bounded, and B open in S. If $T: S \to X$ is continuous, $\gamma^+(B) \subset S$, and K attracts points of B, then there is a closed, bounded, convex subset A of S such that $A = \overline{\text{co}}\left\{\bigcup_{j \geq 1} T^j(B \cap A)\right\}$, $A \cap K \neq \emptyset$.

RESULT 4. – Lemma 4.3 (Hale [14]). – T β-condensing with β satisfying the conditions of Theorem 6.4 then the set A in Result 3 is compact.

The proof of Result 4 is given in [14] for α-condensing but generalizes immediately to β-condensing.

PROOF 1 OF THEOREM 6.4. – Let $B_R = \{x\colon |x| < R\}$ dissipate compact sets. Since orbits of compact sets are dissipated by B_R, they are bounded. Hence, Result 1 holds for any compact set. Furthermore, for any compact set H, $\omega(H) \subset B_R$ since $\omega(H)$ is compact and invariant. Let $J = \{\omega(H)\colon H \subset X \text{ compact}\}$, $J \subset B_R$ and is invariant, hence it is precompact. It also attracts compact sets. Let $K = \overline{\text{co}}\, J$ There is a neighborhood $H_1 = K + B_\varepsilon$ of K whose orbit $\gamma_T^+(H_1)$ is bounded. This is because dissipative and local compact dissipative are equivalent when T is continuous. Let $H_0 = \overline{K + B_{\varepsilon/2}}$. Result 3 implies there is a set $A = \overline{\text{co}}\left\{\bigcup_{j\geq 1} T^j A \cap A)\right\}$. Result 4 implies A is compact. Let $S_0 = H_0 \cap A_1$, $S_1 = H_1 \cap A$, and $S_2 = A$. Clearly, $\gamma^+(S_1) \subset S_2$. Also S_1 is compact and H attracts compact sets, so H_0 dissipates S_1. This implies S_0 dissipates S_1 and Result 2, Horn's theorem, implies T has a fixed point.

PROOF 2 OF THEOREM 6.4. – $T\beta$-condensing implies T is asymptotically smooth. Hence, for any $B \in \mathcal{B}$ such that $\gamma^+(B) \in \mathcal{B}$ we have $\omega(B)$ is compact, invariant, and attracts B. Let B_R be a ball of radius R which dissipates compact sets. Then $\omega(B) \subset B_R$. Let $J = \bigcup \{\omega(B)\colon B,\ \gamma^+(B) \in \mathcal{B}\}$. We have J precompact, invariant, and attracts any $B \in \mathcal{B}$ for which $\gamma^+(B) \in \mathcal{B}$ also. In particular, it attracts neighborhoods of compact sets, since compact dissipative and local compact dissipative are equivalent. Let $K = \text{Cl co}\, J$. There is a neighborhood $H_1 = K + B_\varepsilon$ for which $\gamma_T^+(H_1) \in \mathcal{B}$ by the above reasoning. Let $H_0 = \overline{K + B_{\varepsilon/2}}$. Let $H\colon \mathcal{B} \to \mathcal{B}$ be defined by $H(B) = \text{co}\, T(\gamma^+(H_0 \cap B))$. H is of type 2 and is the composition of a β-condensing operator and the rest β-nonexpansive. Hence H is also β-condensing and Theorem 4.1 implies H is asymptotically smooth. Since $H(\bar{B}) \subset \overline{H(B)}$ we also have $\bar{H}$ asymptotically smooth. Let $S = \overline{\text{co}}\, \gamma^+(H_1) = \bar{H}(H_1)$. Then $\bar{H}(S) \subset S$ and hence $\bar{H}^n(S)$ is a decreasing sequence of sets which approaches the nonempty compact set $\omega_{\bar{H}}(S) \bigcap_{n=1}^{\infty} \bar{H}^n(S)$ which is also convex and invariant under $\bar{H}$. We also have $\omega_{\bar{H}}(S)$ positively invariant under T. Let $S_2 = \omega_{\bar{H}}(S)$, $S_1 = H_1 \cup \omega_{\bar{H}}(S)$, and $S_0 = H_0 \cap \omega_{\bar{H}}(S)$. Since K attracts H_1, H_0 dissipates H_1 and S_0 dissipates S_1. Clearly, $\gamma^+(S_1) \subset S_2$: Hence, Result 2, Horn's theorem, implies T has a fixed point.

7. – Remarks.

In this section we show how some of the proofs of Cooperman and mine are related, and also how one of Cooperman's results does not generalize to more arbitrary measures of noncompactness.

The important lemma used by COOPERMAN [8] to prove the semi-continuity of the solution map for collectively α-condensing maps, is the following lemma, which we prove for β-condensing maps.

LEMMA 7.1. – Let A_n be a decreasing sequence of bounded sets, i.e. $A_1 \supseteq A_2 \supseteq A_3 \ldots$. Let T be β-condensing with β satisfying $\beta(A \cup B) = \beta(A)$ if β is a finite set and $\beta(A + B) = \beta(A)$ if $\beta(B) = 0$. If there are two sequence $\{i_k\}$, $\{j_k\}$ such that $d(A_{i_k}, T(A_{j_k})) \to 0$, then $\alpha(A_i) \to 0$, where $d(A, B)$ is the Hausdorff metric.

REMARK. – The result is also true if the condition $\beta(A + B) = \beta(A)$ if $\beta(B) = 0$ is replaced by the condition that β is continuous in the Hausdorff metric.

PROOF. – Let $D_1 = \{\{(x_k, n_k)\}: x_k \in T(A_{n_k}),\ n_k \to \infty\}$.

If $h \in D_1$, $h = \{(x_k, n_k)\}$ let $Ph = \bigcup_k \{x_k\}$. Let $\eta_1 = \sup\{\beta(Ph): h \in D_1\}$. Let $g(h) = \max\{k: i_k \leqslant n\}$. Let $D_i = \{\{(y_k, n_k)\}: y_k \in A_{n_k},\ n_k \to \infty\}$, $\eta_2 = \sup\{\beta(Ph): h \in D_2\}$. We first show $\eta_1 = \eta_2$. Let $h \in D_1$, $h = \{(x_k, n_k)\}$, $\beta(Ph) = \beta(P\{(x_k, n_k)\}) = \beta(P\{(x_k, j_{g(n_k)})\})$. Now there is a sequence $\{(y_k, i_{g(n_k)})\} \in D_2$ with $|y_k - x_k| \to 0$ since $d(A_{i_k}, T(A_{j_k})) \to 0$. Hence, $\beta(P\{(x_k, j_{g(n_k)})\}) = \beta P\{(y_k, i_{g(n_k)})\} \leqslant \eta_2$. Hence, $\eta_1 \leqslant \eta_2$. Reversing the argument shows $\eta_2 \leqslant \eta_1$. Hence, $\eta_1 = \eta_2$. Now as in Theorem 4.1 there is an $h^* \in D_1$ with $\beta(h) = \eta_1$. But there is an $h' \in D_2$ with $TPh' = Ph^*$. So $\eta_1 \geqslant \beta(Ph') \geqslant \beta(TPh') = \beta(Ph^*) = \eta_1$ with equality if and only if $\beta(Ph') = 0$. Hence, $\eta_1 = 0$ and Lemma 3.1 implies the existence of a compact set which attracts A_n. Hence, $\alpha(A_n) \to 0$.

A result of Cooperman's [8] which does not generalize is the following.

THEOREM 7.1 (Cooperman). – Let X and Y be metric spaces, not necessarily identical, and let $T: X \to Y$ be α-condensing. Let Y be separable. Suppose $A_1 \supseteq A_2 \supseteq \ldots$, $\alpha(A_i) \to \delta$ and $\alpha(T(A_i)) \to \delta$. Then $\delta = 0$.

We give the following example to show it does not hold for more general measures of noncompactness.

EXAMPLE 7.1. – Let $T: L^2[0, 1] \times R \to L^2[0, 1] \times R$ be defined by

$$T(\mathcal{F}, a) = \begin{cases} a & \frac{3}{4} \leqslant x \\ (4x - 2)\,\mathcal{F}(2x - \frac{1}{2}) & \frac{1}{2} \leqslant x < \frac{3}{4} \\ 0 & x < \frac{1}{2}\,. \end{cases}$$

Let $B \subset L^2[0, 1] \times R$, and let $B_r = \{(\mathcal{F}_r, a)/\text{where } \mathcal{F}_r \text{ is the restriction of } \mathcal{F} \text{ to } [r, 1], (\mathcal{F}, a) \in B\}$. We define $\beta(B) = 0$ if B is compact, otherwise $\beta(B) = 1 + \inf\{r/B_r$ is compact in $L^2[r, 1], r \in [0, 1]\}$. Let $A_n = \{(\mathcal{F}, a)/\|\mathcal{F}\| \leqslant 1, |a| \leqslant 1, \mathcal{F}(x) = a$ for $x \geqslant \frac{1}{2} + 1/n\}$. Then $\beta(A_n) \to 1\frac{1}{2}$, $\beta(TA_n) \to 1\frac{1}{2}$, and T is β-condensing. Hence, the conclusion of theorem 7.1 does not hold in this case. We notice $T(A_n)$ approaches a compact set. The example could easily be modified so it does not. Also β-satisfies most nice properties for measures of noncompactness except continuity in the Hausdorff metric. If we assume β is continuous in the Hausdorff metric I do not know if theorem 7.1 is true.

8. – Linear condensing maps.

References for this section are [10], [11], [20], [28] and [29].

THEOREM 8.1. – If T is linear and β-condensing with a β-measure of noncompactness satisfying (i) $\beta(A \cup B) = \beta(A)$ if $\beta(B) = 0$ and (ii) $\beta(A + B) = \beta(A)$ if $\beta(B) = 0$ then $r_e(T) < 1$ where $r_e(T)$ is the essential spectral radius.

DEFINITION 8.1. – $r_e(T) = \inf_{A \in C} r(T + A)$ where $r(T + A)$ is the spectral radius of $T + A$ and C is the collection of compact operators.

PROOF OF THEOREM 8.1. – We will use the fact that $\partial\sigma \subset \sigma_p \cup \sigma_c$ where σ is the spectrum, σ_p is the point spectrum, and σ_c is the continuous spectrum. Clearly σ is bounded since T maps bounded sets into bounded sets. Let $B_2 = \{x: |x| < 2\}$. Let $H: \mathcal{B} \to \mathcal{B}$ be a set operator with $H(A) = T(A) \cap B_2$. H is clearly β-condensing and of type 1. Hence, Theorem 4.1 implies H is asymptotically smooth. We will show $\partial\sigma_e \subset B_k$ for some $k < 1$ where $B_k = \{z \in C: |z| < k\}$.

(*a*) If $\lambda \in \partial\sigma$, $|\lambda| \geqslant 1$, and $\lambda \in \sigma_p$ then $N(T - \lambda I)$ (the null space of $T - \lambda I$) is finite dimensional. Otherwise there is a sequence $\{x_n\}$ with $\alpha(\{x_n\}) = \eta > 0$, $\|x_n\| = 1$ such that $Tx_n = \lambda x_n$. But then $\alpha\big(H^m((1/\lambda)^m \{x_n\})\big) = \alpha\{x_n\} = \eta > 0$. Hence $\alpha(H^m(B_1)) \geqslant \eta > 0$ which contradicts the fact that H is asymptotically smooth.

(*b*) If $\lambda \in \partial\sigma$, $|\lambda| \geqslant 1$ then $R(T - \lambda I) = \operatorname{cl} R(T - \lambda I)$, where $\dim N(T - \lambda I) < \infty$. Suppose $R(T - \lambda I) \neq \operatorname{cl} R(T - \lambda I)$. Then the map $(T - \lambda I)^{-1}: R(T - \lambda I) \to X \setminus N(T - \lambda I)$ is unbounded. So there exists a sequence $\{x_n\} \in X: N(T - \lambda I)$, $\|x_n\| = 1$ and a sequence $\{y_n\} \to 0$ such that $(T - \lambda I)x_n = y_n$. There is also an $\eta > 0$ such that $\alpha\{x_n\} = \eta$. For otherwise a cluster point, $x_0 \in X \setminus N(T - \lambda I)$ would satisfy $(T - \lambda I)x_0 = 0$, which is a contradiction. It is now easily verified that $\alpha\big(H^m((1/\lambda)^m \{x_n\})\big) = \alpha(\{x_n\}) = \eta > 0$. Hence, $\alpha(H^m(B_1)) \geqslant \eta > 0$ which again contradicts the fact that H is asymptotically smooth.

(*c*) If $\lambda \in \partial\sigma$, $|\lambda| \geqslant 1$ then $\dim \mathcal{N}(T - \lambda I) < \infty$ where $\mathcal{N}(T - \lambda I) = \operatorname{cl}\Big\{\bigcup_{m \geqslant 1} N(T - \lambda I)^m\Big\}$ is the generalized null space of $T - \lambda I$. Let T' be the restriction of T to $\mathcal{N}(T - \lambda I)$, i.e. $T': \mathcal{N}(T - \lambda I) \to \mathcal{N}(T - \lambda I)$ with $T'x = T(x)$. Since T' is also β-condensing we have $\dim N(T' - \lambda I') < \infty$ (*a*) and $R(T' - \lambda I) = \operatorname{cl} R(T' - \lambda I')$ (*b*).

In this part we need to use the concept of an index. If $R(T-\lambda I) = \mathrm{cl}\, R(T-\lambda I)$ let $\alpha_{T-\lambda I} = \dim N(T-\lambda I)$ and $\beta_{T-\lambda I} = \mathrm{codim}\, R(T-\lambda I)$. The index, $K_{T-\lambda I} = \beta_{T-\lambda I} - \alpha_{T-\lambda I}$. It is clear that if $\dim \mathcal{N}(T-\lambda I) = \infty$ then $K_{T-\lambda I} \leqslant -1$. But it is known that the index is constant on an open set, and for any $\lambda \in \varrho(U)$, the resolvent set, $K_{T-\lambda I} = 0$. This contradicts $\lambda \in \partial\sigma$.

(*d*) If $\lambda \in \partial\sigma$, $|\lambda| \geqslant 1$, then $\mathrm{codim}\, \mathcal{N}(T-\lambda I) < \infty$. We know from (*c*) that $\dim \mathcal{N}(T-\lambda I) < \infty$. Hence there is a normal splitting of $X = \mathcal{N}(T-\lambda I) - Y$, for some subspace Y where $(T-\lambda I)Y \to Y$. Let $T' - \lambda I' : Y \to Y$. Since $N(T' - \lambda I') = \{0\}$ and $R(T' - \lambda I') = \mathrm{cl}\, R(T' - \lambda I')$ (*b*) we have $\lambda \notin \sigma_p(T' - \lambda I') \cup \sigma_c(T' - \lambda I')$. Hence $\lambda \notin \partial\sigma(T' - \lambda I')$. So $\lambda \in \varrho(T')$ and $T' - \lambda I'$ is $1-1$ onto. Hence, $\mathrm{codim}\, R(T-\lambda I) < \infty$.

(*e*) If $\lambda \in \partial\sigma$, $|\lambda| \geqslant 1$ then λ is a normal eigenvalue. From (*a*), (*b*), (*c*) and (*d*) we have $\alpha_{T-\lambda I} < \infty$, $\beta_{T-\lambda I} < \infty$ and $R(T-\lambda I) = \mathrm{cl}\, R(T-\lambda I)$. Hence, $K_{T-\lambda I}$ is well-defined. Since the degree is constant on an open set and $\lambda \in \partial\sigma$ we have $K_{T-\lambda I} = 0$. So, $\dim N(T-\lambda I) = \mathrm{codim}\, R(T-\lambda I)$, and λ is a normal eigenvalue.

(*f*) The set of points in $\partial\sigma \cap \{\lambda : |\lambda| \geqslant 1\}$ is finite. We have proved (*e*) that all these points are normal eigenvalues. Hence they are also isolated.

(*g*) Since we may introduce a finite dimensional operator A which subtracts off the normal eigenvalues with $|\lambda| \geqslant 1$, we get $\partial\sigma(T-A) \subset B_1$ with $B_1 = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$. Hence $r_e(T) < 1$. Q.E.D.

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