Finite Additivity and Upper and Lower Distribution Functions (*).

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Summary. — Suppose U is a set and F is a field of subsets of U . Suppose μ is a real-nonnegative-valued finitely additive function defined on F and α is a function from F into a collection of subsets of $\mathbb{R}$. An extension of the «usual» notion of distribution function is introduced by defining the notions of upper and lower distribution functions of α with respect to μ . Two fundamental inequalities, for the case of α with bounded range union, are shown. These are shown to imply characterization theorems that imply previous integrability and summability theorems (*Ann. di Mat. pur e applic.*, vol. 96, 1973, pp. 265-287) of the author. A measurability (*Boll. U.M.I.*, (4), vol. 7, 1973, pp. 42-49) characterization theorem is shown; this theorem is a consequence of the above mentioned characterization for μ -integrability, and in turn implies the μ -summability characterization. Set function analogues of the «standard» distribution function integral representation theorems are shown.

1. — Introduction.

In a previous paper [4] the author introduced the notion of a distribution function, g_α , for a function, α , from a field F of sets into $\exp(\mathbb{R})$, with respect to a real-nonnegative-valued finitely additive function, μ , defined on F . The principal results of [4] concerned a characterization of the integrability (see section 2) of α with respect to μ , for α with bounded range union, and a characterization of the summability (see [1], [4] and section 2) of α with respect to μ , together with the «standard» equivalences of certain set function integrals and summability operator values to certain integrals of functions with respect to g_α .

The proofs of the characterizations mentioned above rested heavily on the Bochner-Radon-Nikodym Theorem for the Lipschitz case. In this paper we consider a «non-integrable» generalization (see [6] and section 2) of this theorem, introduce the notions of upper and lower distribution function, and prove inequalities and equations that imply upper and lower distribution function versions of the aforementioned characterization theorems.

We begin as follows: Suppose U is a set, F is a field of subsets of U , $\mathfrak{p}$ is the set of all functions from F into $\exp(\mathbb{R})$, and $\mathfrak{p}_A^+$ is the set of all real-nonnegative-valued finitely additive functions defined on F . Suppose μ is in $\mathfrak{p}_A^+$. For each α in $\mathfrak{p}$ and x in $\mathbb{R}$ we define $\beta(\alpha, x)$ as the element of $\mathfrak{p}$ such that for each I in F , $\beta(\alpha, x)(I) \subseteq \{0, 1\}$

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and contains 1 iff there is y in $\alpha(I)$ such that $y < x$, and contains 0 iff there is y in $\alpha(I)$ such that $x \geq y$. We let $u(\alpha)$ and $v(\alpha)$ denote functions defined on $\mathbb{R}$, respectively by

$$u(\alpha)(x) = \int_{\bar{v}} L(\beta(\alpha, x)\mu)(I) \quad \text{and} \quad v(\alpha)(x) = \int_{\bar{v}} G(\beta(\alpha, x)\mu)(I),$$

where L and G denote, respectively, the sum supremum and sum infimum functionals (see section 2). We note that each of $u(\alpha)$, $v(\alpha)$ and $u(\alpha) - v(\alpha)$ is nonnegative-valued and each of $u(\alpha)$ and $v(\alpha)$ is nondecreasing. We also note, and leave the details to the reader, that the distribution function g_α defined in [4] is given by $g_\alpha(x) = u(\alpha)(x +)$. We let $\mathfrak{p}_B$ denote the set of all elements of $\mathfrak{p}$ with bounded range union. The inequalities mentioned above are the following two theorems:

THEOREM 3.1. (section 3). – If α is in $\mathfrak{p}_B$ and $t < r$, then

$$u(\alpha)(t) - v(\alpha)(r) \leq (r - t)^{-1} \int_{\bar{v}} L(\alpha\mu)(I) - G(\alpha\mu)(I).$$

THEOREM 4.1 (section 4). – If α is in $\mathfrak{p}_B$ and h is a real-valued nondecreasing continuous function defined on $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} h \, du(\alpha) \leq \int_{\bar{v}} G(h(\alpha)\mu)(I) \leq \int_{\bar{v}} L(h(\alpha)\mu)(I) \leq \int_{-\infty}^{\infty} h \, dv(\alpha).$$

These inequalities first enable us to prove the following two integrability characterization and representation theorems:

THEOREM 5.1 (section 5). – If α is in $\mathfrak{p}_B$, then the following three statements are equivalent:

- 1) $\int_{\bar{v}} \alpha(I) \mu(I)$ exists,
- 2) $u(\alpha)(x +) = v(\alpha)(x +)$ on $\mathbb{R}$, and
- 3) $\int_{-\infty}^{\infty} x \, du(\alpha)(x) = \int_{-\infty}^{\infty} x \, dv(\alpha)(x).$

THEOREM 5.2 (section 5). – If α is in $\mathfrak{p}_B$, k is a real-valued continuous function defined on $\mathbb{R}$ and $\int_{\bar{v}} \alpha(I) \mu(I)$ exists, then we have the following existence and equivalence:

$$\int_{\bar{v}} k(\alpha(I)) \mu(I) = \int_{-\infty}^{\infty} k(x) \, du(\alpha)(x) = \int_{-\infty}^{\infty} k(x) \, dv(\alpha)(x).$$

We next prove the following measurability (see [5] and section 6) characterization theorem:

THEOREM 6.1 (section 6). – If α is in $\mathfrak{p}$, then α is μ -measurable iff

- 1) $u(\alpha)(x+) = v(\alpha)(x+)$ on $\mathbb{R}$, and
- 2) $v(\alpha)(x) \rightarrow \mu(U)$ as $x \rightarrow \infty$, and $u(\alpha)(x) \rightarrow 0$ as $x \rightarrow -\infty$.

We end the paper with the following two paraphrases of two summability (see [1], [2] and section 7) characterization and representation theorems of [4]:

THEOREM 7.1 (section 7). – If α is in $\mathfrak{p}$, then α is μ -summable iff:

- i) $u(\alpha)(x+) = v(\alpha)(x+)$ on $\mathbb{R}$,
- ii) $v(\alpha)(x) \rightarrow \mu(U)$ as $x \rightarrow \infty$ and $u(\alpha)(x) \rightarrow 0$ as $x \rightarrow -\infty$, and
- iii) $\int_{-\infty}^{\infty} x du(\alpha)(x)$ (and hence $\int_{-\infty}^{\infty} x dv(\alpha)(x)$) exists.

THEOREM 7.2 (section 7). – Suppose h is a continuous function from $\mathbb{R}$ into $\mathbb{R}$ such that $\{|h(x)|/|x|: 1 \leq |x|\}$ is bounded. If α is μ -summable, then $\int_{-\infty}^{\infty} h(x) du(\alpha)(x)$ (and hence $\int_{-\infty}^{\infty} h(x) dv(\alpha)(x)$) exists and is $s_{\mu}(h(\alpha))(U)$, where s_{μ} is the « μ -summability operator » (section 7).

2. – Preliminary theorems and definitions.

We refer the reader to [1] for the definitions, notation and basic facts concerning the notions of subdivision, refinement, integral, Σ -boundedness, sum supremum functional, L , and sum infimum functional, G . We let « $\mathfrak{P} \ll \mathfrak{Q}$ » mean « $\mathfrak{P}$ is a refinement of $\mathfrak{Q}$ ». We also refer the reader to [1] for a statement of Kolmogoroff's [7] differential equivalence theorem, as well as certain of its consequences. In this paper there are statements about integrals, following easily from differential equivalence, that we shall simply make, leaving the arguments to the reader.

We state a previous theorem of the author [5].

THEOREM 2.A.1. – If α is in $\mathfrak{p}$, $\mathfrak{D} \ll \{U\}$, α is Σ -bounded on U with respect to $\mathfrak{D}$, and c is a number, then, for each V in $\mathbf{F}$,

$$cL(\alpha)(V) = L(c\alpha)(V) \text{ and } cG(\alpha)(V) = G(c\alpha)(V) \quad \text{if } 0 \leq c,$$

and

$$cL(\alpha)(V) = G(c\alpha)(V) \quad \text{and} \quad cG(\alpha)(V) = L(c\alpha)(V) \quad \text{if } c < 0.$$

We state an easy extension of a theorem from the above mentioned paper.

THEOREM 2.A.2. — Suppose each of α and β is in $\mathfrak{p}$, $\mathfrak{D} \ll \{U\}$, α is Σ -bounded on U with respect to $\mathfrak{D}$, $0 < c$, and if $\mathfrak{E} \ll \mathfrak{D}$ and for each I in $\mathfrak{E}$, $b(I)$ is in $\beta(I)$, then for each I in $\mathfrak{E}$, there is $a(I)$ in $\alpha(I)$ such that $\left| \sum_{\mathfrak{E}} a(I) - b(I) \right| < c$. Then β is Σ -bounded on U with respect to $\mathfrak{D}$, and for each subdivision $\mathfrak{S}$ of U ,

$$-c + \sum_{\mathfrak{S}} G(\alpha)(V) \leq \sum_{\mathfrak{S}} G(\beta)(V) \leq \sum_{\mathfrak{S}} L(\beta)(V) \leq \sum_{\mathfrak{S}} L(\alpha)(V) + c.$$

We now state the generalization of the Lipschitz portion of the Bochner-Radon-Nikodym Theorem mentioned in the introduction.

THEOREM 2.A.3 [6]. — If α is in $\mathfrak{p}_B$ and μ is in $\mathfrak{p}_A^+$, then $\inf \left\{ \sup \left\{ \sum_{\mathfrak{E}} \sum_{\mathfrak{E}(V)} |a(V) - a(I)| \mu(I) : \mathfrak{E} \ll \mathfrak{D}, \text{ for each } V \text{ in } \mathfrak{E}, \mathfrak{E}(V) \ll \{V\} \text{ and } a(V) \text{ is in } \alpha(V), \text{ for each } I \text{ in } \mathfrak{E}(V), a(I) \text{ is in } \alpha(I) \right\} : \mathfrak{D} \ll \{U\} \right\} = \int_U L(\alpha\mu)(J) - G(\alpha\mu)(J).$

We end this section with a theorem about «truncations».

THEOREM 2.1. — Suppose α is in $\mathfrak{p}$, μ is in $\mathfrak{p}_A^+$, x is in $\mathbb{R}$ and $a < b$. The following statements hold:

i) If $a < x \leq b$, then $\beta(\max \{\min \{\alpha, b\}, a\}, x) = \beta(\alpha, x)$, which implies that $u(\max \{\min \{\alpha, b\}, a\})(x) = u(\alpha)(x)$, and $v(\max \{\min \{\alpha, b\}, a\})(x) = v(\alpha)(x)$.

ii) If $x \leq a$, then $\beta(\max \{\min \{\alpha, b\}, a\}, x)(I) = \{0\}$ for all I in $\mathbf{F}$, so that $0 = u(\max \{\min \{\alpha, b\}, a\})(x) = v(\max \{\min \{\alpha, b\}, a\})(x)$. If $b \leq x$, then for each $y > x$, $\beta(\max \{\min \{\alpha, b\}, a\}, y) = \{1\}$ for all I in $\mathbf{F}$, so that

$$\mu(U) = u(\max \{\min \{\alpha, b\}, a\})(y) = v(\max \{\min \{\alpha, b\}, a\})(y);$$

this implies that

$$\mu(U) = u(\max \{\min \{\alpha, b\}, a\})(x+) = v(\max \{\min \{\alpha, b\}, a\})(x+).$$

INDICATION OF PROOF. — In proving i), we make the salient observations and leave the details to the reader. We simply note that if y is in $\mathbb{R}$, then

$$\max \{\min \{y, b\}, a\} < x \quad \text{iff } y < x;$$

this implies that $\max \{\min \{y, b\}, a\} \geq x$ iff $y \geq x$.

Statement ii) is self explanatory and we accordingly dispense with further argument.

3. – An upper and lower distribution function inequality.

In this section we prove Theorem 3.1, as stated in the introduction.

PROOF OF THEOREM 3.1. – Suppose $0 < c$. By Theorem 2.A.3 there is $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$ and for each V in $\mathfrak{E}$, $a(V)$ is in $\alpha(V)$, $\mathfrak{E}(V) \ll \{V\}$ and for each I in $\mathfrak{E}(V)$, $a(I)$ is in $\alpha(I)$, then

$$\sum_{\mathfrak{E}} \sum_{\mathfrak{E}(V)} |a(V) - a(I)| \mu(I) < \int_U L(\alpha\mu)(J) - G(\alpha\mu)(J) + (r-t)c/32.$$

For each V in $\mathfrak{D}$ there is $\mathfrak{D}_1^*(V) \ll \{V\}$ and for each I in $\mathfrak{D}_1^*(V)$, $b^*(\alpha, t)(I)$ in $\beta(\alpha, t)(I)$ such that

$$L(\beta(\alpha, t)\mu)(V) - \sum_{\mathfrak{D}_1^*(V)} b^*(\alpha, t)(I) \mu(I) < c/(8N),$$

where N is the number of elements of $\mathfrak{D}$, so that

$$\sum_{\mathfrak{D}} L(\beta(\alpha, t)\mu)(V) - \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_1^*(V)} b^*(\alpha, t)(I) \mu(I) < c/8.$$

Let $\mathfrak{D}_1 = \bigcup_{\mathfrak{D}} \mathfrak{D}_1^*(V)$. For each I in $\mathfrak{D}_1$ there is $\mathfrak{D}_2(I) \ll \{I\}$ and for each J in $\mathfrak{D}_2(I)$, $b^{**}(\alpha, r)(J)$ in $\beta(\alpha, r)(J)$ such that

$$\left[\sum_{\mathfrak{D}_2(I)} b^{**}(\alpha, r)(J) \mu(J) \right] - G(\beta(\alpha, r)\mu)(I) < c/(8M),$$

where M is the number of elements of $\mathfrak{D}_1$, so that

$$\sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^{**}(\alpha, r)(J) \mu(J) - \sum_{\mathfrak{D}_1} G(\beta(\alpha, r)\mu)(I) < c/8.$$

For each I in $\mathfrak{D}_1$, there is $a^*(I)$ in $\alpha(I)$ such that if $b^*(\alpha, t)(I) = 0$, then $t \leq a^*(I)$, and if $b^*(\alpha, t)(I) = 1$, then $a^*(I) < t$. For each J in $\bigcup_{\mathfrak{D}_1} \mathfrak{D}_2(I)$ there is $a^{**}(J)$ in $\alpha(J)$ such that if $b^{**}(\alpha, r)(J) = 0$, then $r \leq a^{**}(J)$, and if $b^{**}(\alpha, r)(J) = 1$, then $a^{**}(J) < r$. Now,

$$\begin{aligned} (r-t)c/32 + \int_U [L(\alpha\mu)(J') - G(\alpha\mu)(J')] &> \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_1^*(V)} \sum_{\mathfrak{D}_2(I)} |a^*(I) - a^{**}(J)| \mu(J) \geq \\ &\geq \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_1^*(V)} \sum_{\mathfrak{D}_2(I)} |a^*(I) - a^{**}(J)| b^*(\alpha, t)(I) (1 - b^{**}(\alpha, r)(J)) \mu(J) \geq \\ &\geq \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_1^*(V)} \sum_{\mathfrak{D}_2(I)} (r-t) b^*(\alpha, t)(I) (1 - b^{**}(\alpha, r)(J)) \mu(J), \end{aligned}$$

so that

$$c/32 + (r-t)^{-1} \int_U L(\alpha\mu)(J') - G(\alpha\mu)(J') > \sum_{\mathfrak{D}} \sum_{\mathfrak{D}_1^*(V)} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I) (1 - b^{**}(\alpha, r)(J)) \mu(J).$$

Now

$$\begin{aligned}
\int_{\bar{U}} G(\beta(\alpha, r)\mu)(J') &\geq \sum_{\mathfrak{D}_1} G(\beta(\alpha, r)\mu)(I) > -c/8 + \sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^{**}(\alpha, r)(J)\mu(J) \geq \\
&-c/8 + \sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I) b^{**}(\alpha, r)(J)\mu(J) = \\
&-c/8 + \sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I) (1 - (1 - b^{**}(\alpha, r)(J)))\mu(J) = \\
&\sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I)\mu(J) - \sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I) (1 - b^{**}(\alpha, r)(J))\mu(J) > \\
&\sum_{\mathfrak{D}_1} \sum_{\mathfrak{D}_2(I)} b^*(\alpha, t)(I)\mu(J) - c/32 - (r-t)^{-1} \int_{\bar{U}} L(\alpha\mu)(J') - G(\alpha\mu)(J') > \\
&-c/8 + \sum_{\mathfrak{D}} L(\beta(\alpha, t)\mu)(V) - c/32 - (r-t)^{-1} \int_{\bar{U}} L(\alpha\mu)(J') - G(\alpha\mu)(J') \geq \\
&-c/8 - c/32 + \int_{\bar{U}} L(\beta(\alpha, t)\mu)(J') - (r-t)^{-1} \int_{\bar{U}} L(\alpha\mu)(J') - G(\alpha\mu)(J') ,
\end{aligned}$$

which implies that

$$(r-t)^{-1} \int_{\bar{U}} L(\alpha\mu)(J') - G(\alpha\mu)(J') + c/8 + c/32 \geq \int_{\bar{U}} L(\beta(\alpha, t)\mu)(J') - \int_{\bar{U}} G(\beta(\alpha, r)\mu)(J') .$$

so that

$$\begin{aligned}
(r-t)^{-1} \int_{\bar{U}} L(\alpha\mu)(J') - G(\alpha\mu)(J') &\geq \int_{\bar{U}} L(\beta(\alpha, t)\mu)(J') - \int_{\bar{U}} G(\beta(\alpha, r)\mu)(J') = \\
&= u(\alpha)(t) - v(\alpha)(r) .
\end{aligned}$$

4. - An upper and lower distribution function inequality involving a continuous nondecreasing function.

In this section we prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM 4.1. - Let $u = u(\alpha)$ and $v = v(\alpha)$. There is r and s such that $r < x < s$ for all x in the range union of α . Note that for $x \leq r$, $0 = u(x) = v(x)$ and for $s \leq x$, $u(x) = v(x) = \mu(U)$. There is an interval subdivision S' of $[r; s]$ such that if T is an interval refinement of S' and for each $[p; q]$ in T , z is in $[p; q]$, then

$$\sum_T u(z) [h(q) - h(p)] < \int_r^s u dh + c/16 .$$

There is $\mathfrak{D} \ll \{U\}$ such that if $[p; q]$ is in S' and $\mathfrak{E} \ll \mathfrak{D}$ and for each I in $\mathfrak{E}$, $b(I)$ is in $\beta(\alpha, q)(I)$, then

$$\sum_{\mathfrak{E}} b(I)\mu(I) < \int_{\bar{U}} L(\beta(\alpha, q)\mu)(J) + c/(16(1 + h|_r^s)) .$$

By observations similar to those in the proof of Theorem 3.1, for each V in $\mathfrak{D}$ there is $\mathfrak{E}(V) \ll \{V\}$ and for each I in $\mathfrak{E}(V)$, $a^*(I)$ in $\alpha(I)$ such that

$$\sum_{\mathfrak{D}} \left[\sum_{\mathfrak{E}(V)} h(a^*(I)) \mu(I) - G(h(\alpha)\mu)(V) \right] < c/16,$$

so that

$$\int_U G(h(\alpha)\mu)(I) \geq \sum_{\mathfrak{D}} G(h(\alpha)\mu)(V) > -c/16 + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}(V)} h(a^*(I)) \mu(I).$$

Let $\mathfrak{E} = \bigcup_{\mathfrak{D}} \mathfrak{E}(V)$. If I is in $\mathfrak{E}$, $[p; q]$ is in S' and x is p or q , then there is $b^*(\alpha, x)(I)$ in $\beta(\alpha, x)(I)$ such that if $a^*(I) < x$, then $b^*(\alpha, x)(I) = 1$, and if $x \leq a^*(I)$, then $b^*(\alpha, x)(I) = 0$. Now

$$\begin{aligned} \sum_{\mathfrak{E}} h(a^*(I)) \mu(I) &= \sum_{\mathfrak{E}} h(a^*(I)) \left[\sum_{S'} [b^*(\alpha, q)(I) - b^*(\alpha, p)(I)] \right] \mu(I) = \\ &= \sum_{\mathfrak{E}} \sum_{S'} h(a^*(I)) (b^*(\alpha, q)(I) - b^*(\alpha, p)(I)) \mu(I) \geq \\ &= \sum_{\mathfrak{E}} \sum_{S'} h(p) (b^*(\alpha, q)(I) - b^*(\alpha, p)(I)) \mu(I) = \\ &= \sum_{S'} h(p) \left(\sum_{\mathfrak{E}} b^*(\alpha, q)(I) \mu(I) - \sum_{\mathfrak{E}} b^*(\alpha, p)(I) \mu(I) \right) = \\ &= h(s) \sum_{\mathfrak{E}} b^*(\alpha, s)(I) \mu(I) - h(r) \sum_{\mathfrak{E}} b^*(\alpha, r)(I) \mu(I) - \\ &= \sum_{S'} \left[\sum_{\mathfrak{E}} b^*(\alpha, q)(I) \mu(I) \right] [h(q) - h(p)] \geq \\ &= h(s) u(s) - h(r) u(r) - \sum_{S'} \left(u(q) + c/(16(1 + h_r^s)) \right) (h(q) - h(p)) = \\ &= hu_r^s - \sum_{S'} u(q) (h(q) - h(p)) - (ch_r^s)/(16(1 + h_r^s)) > \\ &= hu_r^s - \int_r^s u dh - c/16 - c/16 = \int_r^s h du - c/8. \end{aligned}$$

Therefore

$$c/16 + \int_U G(h(\alpha)\mu)(J) \geq \int_r^s h du - c/8,$$

so that

$$\int_U G(h(\alpha)\mu)(J) \geq \int_r^s h du.$$

In a similar fashion it follows that

$$\int_U L(h(\alpha)\mu)(J) \leq \int_r^s h dv.$$

Therefore

$$\int_{-\infty}^{\infty} h du \leq \int_{\bar{U}} G(h(\alpha)\mu)(J) \leq \int_{\bar{U}} L(h(\alpha)\mu)(J) \leq \int_{-\infty}^{\infty} h dv.$$

5. - An integrability characterization and an integral representation.

We let $u = u(\alpha)$ and $v = v(\alpha)$.

We first prove Theorem 5.1, as stated in the introduction.

PROOF OF THEOREM 5.1. - First, suppose that 1) is true. Then, if $r < t$, then

$$u(r) - v(t) \leq (t - r)^{-1} \int_{\bar{U}} L(\alpha\mu)(I) - G(\alpha\mu)(I) = (t - r)^{-1} 0 = 0,$$

so that $u(r) \leq v(t)$. Thus, if x is in $\mathbb{R}$ and $x < r < t$, then $u(x+) \leq u(r) \leq v(t)$, which clearly implies that $u(x+) \leq v(x+)$. Since $v(y) \leq u(y)$ for all y in $\mathbb{R}$, it follows that for each x in $\mathbb{R}$, $v(x+) \leq u(x+) \leq v(x+)$, so that $v(x+) = u(x+)$. Therefore 1) implies 2).

Before proceeding we make the observation that since α in is $\mathfrak{p}_B$, there is a number interval $[a; b]$ such that $v(a) = u(a) = 0 = m(U) - v(b) = m(U) - u(b)$.

Now suppose 2) is true. From well-known theorems it follows that if g is a continuous function from $[a; b]$ into $\mathbb{R}$, then

$$\int_a^b g du = \int_a^b g dv.$$

From this and the preceding paragraph it follows that

$$\int_{-\infty}^{\infty} x du(x) = \int_{-\infty}^{\infty} x dv(x).$$

Therefore 2) implies 3).

Now suppose 3) is true. The function h defined on $\mathbb{R}$ by $h(x) = x$ clearly satisfies the hypothesis of Theorem 4.1. It is therefore an immediate consequence of Theorem 4.1 that $\int_{\bar{U}} G(\alpha\mu)(I) = \int_{\bar{U}} L(\alpha\mu)(I)$, so that $\int_{\bar{U}} \alpha(I)\mu(I)$ exists. Therefore 3) implies 1).

Therefore 1), 2) and 3) are equivalent.

We now prove theorem 5.2, as stated in the introduction.

PROOF OF THEOREM 5.2. - Again, there is a number interval $[a; b]$ such that $v(a) = u(a) = 0 = \mu(U) - v(b) = \mu(U) - u(b)$. It further follows from Theorem 5.1 that $u(x+) = v(x+)$ for all x in $\mathbb{R}$, so that if h is a real-valued continuous function defi-

ed on $\mathbb{R}$, then we have the following existence and equality:

$$\int_{-\infty}^{\infty} h \, du = \int_a^b h \, du = \int_a^b h \, dv = \int_{-\infty}^{\infty} h \, dv.$$

Now, suppose k is a real-valued continuous function defined on $\mathbb{R}$ and $0 < c$. It is well-known that there is an interval subdivision H of $[a; b]$ such that if $a \leq x \leq b$, then

$$\left| k(x) - \left[k(a) + \sum_H [k|_p^q / (q - p)] \max \{ \min \{ x - p, q - p \}, 0 \} \right] \right| < c/2(1 + \mu(U)).$$

From Theorem 4.1 it follows that if $[p, q]$ is in H and h is given by

$$h(x) = \max \{ \min \{ x - p, q - p \}, 0 \},$$

then

$$\int_{-\infty}^{\infty} h \, du \leq \int_U G(h(x)\mu)(I) \leq \int_U L(h(x)\mu)(I) \leq \int_{-\infty}^{\infty} h \, dv,$$

which implies the following existence and equality:

$$\int_{-\infty}^{\infty} h \, du = \int_U h(\alpha(I))\mu(I) = \int_{-\infty}^{\infty} h \, dv.$$

From this we have

$$\begin{aligned} \int_{-\infty}^{\infty} k \, du &= \int_a^b k \, du \leq \int_a^b \left[k(a) + \sum_H [k|_p^q / (q - p)] \max \{ \min \{ x - p, q - p \}, 0 \} + \right. \\ &\quad \left. c/2(1 + \mu(U)) \right] du(x) = k(a)\mu(U) + \left[\sum_H [k|_p^q / (q - p)] \int_a^b \max \{ \min \{ x - p, \right. \\ &\quad \left. q - p \}, 0 \} du(x) \right] + [c/2(1 + \mu(U))]\mu(U) = k(a)\mu(U) + \\ &\quad \left[\sum_H [k|_p^q / (q - p)] \int_U \max \{ \min \{ \alpha(I) - p, q - p \}, 0 \} \mu(I) \right] + c\mu(U)/2(1 + \mu(U)) = \\ &\quad k(a)\mu(U) + \left[\sum_H [k|_p^q / (q - p)] \int_a^b \max \{ \min \{ x - p, q - p \}, 0 \} du(x) \right] + \\ &\quad c\mu(U)/2(1 + \mu(U)) = \int_a^b \left[k(a) + \sum_H [k|_p^q / (q - p)] \max \{ \min \{ x - p, q - p \}, 0 \} + \right. \\ &\quad \left. c/2(1 + \mu(U)) \right] du(x) \leq \int_a^b [k + c/(1 + \mu(U))] du = \int_a^b k \, du + c\mu(U)/(1 + \mu(U)) < \\ &\quad \int_{-\infty}^{\infty} k \, du + c. \end{aligned}$$

We let t denote the function given by

$$t(x) = k(a) + \sum_H [k|_p^q(q-p)] \max \{ \min \{x-p, q-p\}, 0 \}.$$

The above inequality implies the following existence and inequality:

$$\int_{-\infty}^{\infty} k \, du \leq \int_U t(\alpha(I)) \mu(I) + c\mu(U)/2(1 + \mu(U)) < \int_{-\infty}^{\infty} k \, du + c.$$

From Theorem 2.A,2 we have that

$$\begin{aligned} -c\mu(U)/2(1 + \mu(U)) + \int_U t(\alpha(I)) \mu(I) &= -c\mu(U)/2(1 + \mu(U)) + \\ &\int_U G(t(\alpha)\mu)(I) \leq \int_U G(k(\alpha)\mu)(I) \leq \int_U L(k(\alpha)\mu)(I) \leq \\ &c\mu(U)/2(1 + \mu(U)) + \int_U L(t(\alpha)\mu)(I) = c\mu(U)/2(1 + \mu(U)) + \int_U t(\alpha(I)) \mu(I), \end{aligned}$$

so that

$$-c + \int_{-\infty}^{\infty} k \, du \leq \int_U G(k(\alpha)\mu)(I) \leq \int_U L(k(\alpha)\mu)(I) < \int_{-\infty}^{\infty} k \, du + c.$$

It therefore follows that $\int_U k(\alpha(I)) \mu(I)$ exists and is each of $\int_{-\infty}^{\infty} k \, du$ and $\int_{-\infty}^{\infty} k \, dv$.

6. - Upper and lower distribution functions and measurability.

We begin by stating the definition of set function measurability (see also [3]): If α is in $\mathfrak{p}$ and μ is in $\mathfrak{p}_A^+$, then the statement that α is μ -measurable means:

- 1) If $p \leq 0 \leq q$, then $\int_U \max \{ \min \{ \alpha(I), q \}, p \} \mu(I)$ exists, and
- 2) If $0 \leq c$, then there is $T > 0$ and $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$, for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$ and $\mathfrak{E}^* = \{I : I \text{ in } \mathfrak{E}, |a(I)| > T\}$, then $\sum_{\mathfrak{E}^*} \mu(I) < c$.

We now prove Theorem 6.1, as stated in the introduction.

PROOF OF THEOREM 6.1. - First, suppose that α is μ -measurable. We first show that 1) holds. Suppose that x is in $\mathbb{R}$. There is a and b in $\mathbb{R}$ such that $a < \min \{x, 0\} \leq \max \{x, 0\} < b$. Since $\int_U \max \{ \min \{ \alpha(I), b \}, a \} \mu(I)$ exists, it follows from Theorem 5.1 and Theorem 2.1 that

$$u(\alpha)(x+) = u(\max \{ \min \{ \alpha, b \}, a \})(x+) = v(\max \{ \min \{ \alpha, b \}, a \})(x+) = v(\alpha)(x+),$$

so that 1) holds.

We now show that 2) holds. Suppose $0 < c$.

There is $K > 0$ and $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$, for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$, and $\mathfrak{E}^* = \{I: I \text{ in } \mathfrak{E}, |a(I)| > K\}$, then $\sum_{\mathfrak{E}^*} \mu(I) < c/4$. Now suppose $K < x$. There is $\mathfrak{F} \ll \mathfrak{D}$ such that

$$\sum_{\mathfrak{F}} G(\beta(\alpha, x)\mu)(I) < \int_{\bar{U}} G(\beta(\alpha, x)\mu)(J) + c/4.$$

For each I in $\mathfrak{F}$, there is $\mathfrak{H}(I) \ll \{I\}$ and for each J in $\mathfrak{H}(I)$, $b(J)$ in $\beta(\alpha, x)(J)$ such that

$$\sum_{\mathfrak{H}(I)} b(J)\mu(J) < G(\beta(\alpha, x)\mu)(I) + c/4N,$$

where N is the number of elements in $\mathfrak{H}$, so that

$$\sum_{\mathfrak{F}} \sum_{\mathfrak{H}(I)} b(J)\mu(J) < \sum_{\mathfrak{F}} G(\beta(\alpha, x)\mu)(I) + c/4,$$

so that

$$\sum_{\mathfrak{F}} \sum_{\mathfrak{H}(I)} b(J)\mu(J) < \int_{\bar{U}} G(\beta(\alpha, x)\mu)(J) + c/2.$$

Let $\mathfrak{E} = \bigcup_{\mathfrak{F}} \mathfrak{H}(I)$ and $\mathfrak{E}' = \{J: J \text{ in } \mathfrak{E}, b(J) = 1\}$. Suppose there is J in $\mathfrak{E}$ such that $b(J) = 0$. For each J in $\mathfrak{E} - \mathfrak{E}'$ there is $a(J)$ in $\alpha(J)$ such that $K < x \leq a(J)$. Thus $\sum_{\mathfrak{E} - \mathfrak{E}'} \mu(J) < c/4$, so that

$$0 \leq \mu(U) - v(\alpha)(x) = \sum_{\mathfrak{E} - \mathfrak{E}'} \mu(J) + \sum_{\mathfrak{E}'} b(J)\mu(J) - \int_{\bar{U}} G(\beta(\alpha, x)\mu)(J) \leq \sum_{\mathfrak{E} - \mathfrak{E}'} \mu(J) + c/2 < c/4 + c/2 < c.$$

In fashions similar to the above, the remainder of 2) follows; we leave the conventional arguments to the reader.

Now suppose that each of 1) and 2) hold. First, suppose $a \leq 0 \leq b$. There is a' and b' such that $a' < a \leq 0 \leq b < b'$. From Theorem 2.1 it follows first that if $a' < y < b'$, then $u(\max\{\min\{\alpha, b'\}, a'\})(y) = u(\alpha)(y)$ and $v(\max\{\min\{\alpha, b'\}, a'\})(y) = v(\alpha)(y)$, so that if $a' \leq x < b'$, then, since 1) holds,

$$u(\max\{\min\{\alpha, b'\}, a'\})(x+) = u(\alpha)(x+) = v(\alpha)(x+) = v(\max\{\min\{\alpha, b'\}, a'\})(x+)$$

If $y < a'$, then, by Theorem 2.1,

$$0 = u(\max\{\min\{\alpha, b'\}, a'\})(y) = v(\max\{\min\{\alpha, b'\}, a'\})(y),$$

so that if $x < a'$, then

$$0 = u(\max\{\min\{\alpha, b'\}, a'\})(x+) = v(\max\{\min\{\alpha, b'\}, a'\})(x+).$$

If $b' \leq x$, then, again by Theorem 2.1,

$$\mu(U) = u(\max \{\min \{\alpha, b'\}, a'\})(x+) = v(\max \{\min \{\alpha, b'\}, a'\})(x+).$$

This implies, by Theorem 5.1, that $\int_U \max \{\min \{\alpha(I), b'\}, a'\} \mu(I)$ exists, which implies the following existence and equality:

$$\begin{aligned} \int_U \max \left\{ \min \left\{ \int_I \max \left\{ \min \{\alpha(J), b'\}, a'\} \mu(J) \right\}, b\mu(I) \right\}, a\mu(I) \right\} = \\ \int_U \max \left\{ \min \left\{ \max \left\{ \min \{\alpha(I), b'\}, a'\} \mu(I), b\mu(I) \right\}, a\mu(I) \right\} = \\ \int_U \max \{\min \{\alpha(I), b\}, a\} \mu(I). \end{aligned}$$

We now show that the latter portion of the definition of measurability holds. Suppose $0 < c$. Since 2) holds, there is $K > 0$ such that $u(\alpha)(-K) < c/4$ and $\mu(U) - v(\alpha)(K) < c/4$. There is $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$, then

$$\sum_{\mathfrak{E}} L(\beta(\alpha, -K)\mu)(I) < \int_U L(\beta(\alpha, -K)\mu)(J) + c/4 = u(\alpha)(-K) + c/4 < c/2.$$

Now, suppose $\mathfrak{E} \ll \mathfrak{D}$, and for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$, and $\mathfrak{E}^* = \{I: I \text{ in } \mathfrak{E}, a(I) < -K\}$. For each I in $\mathfrak{E}$ there is $b(I)$ in $\beta(\alpha, -K)(I)$ such that $b(I) = 1$ if $a(I) < -K$. Thus

$$\sum_{\mathfrak{E}^*} \mu(I) \leq \sum_{\mathfrak{E}} b(I) \mu(I) \leq \sum_{\mathfrak{E}} L(\beta(\alpha, -K)\mu)(I) < c/2.$$

There is $\mathfrak{D}' \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}'$, then

$$v(\alpha)(K) = \int_U G(\beta(\alpha, K)\mu)(J) < \sum_{\mathfrak{E}} G(\beta(\alpha, K)\mu)(I) + c/4.$$

Suppose $\mathfrak{E} \ll \mathfrak{D}'$, for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$ and $\mathfrak{E}^* = \{I: I \text{ in } \mathfrak{E}, a(I) > K\}$. For each I in $\mathfrak{E}$ there is $b(I)$ in $\beta(\alpha, K)(I)$ such that $b(I) = 0$ if $a(I) > K$. Thus

$$\begin{aligned} \sum_{\mathfrak{E}^*} \mu(I) &\leq \sum_{\mathfrak{E}} (1 - b(I)) \mu(I) = \mu(U) - \sum_{\mathfrak{E}} b(I) \mu(I) \leq \\ &\mu(U) - \sum_{\mathfrak{E}} G(\beta(\alpha, K)\mu)(I) < \mu(U) - v(\alpha)(K) + c/4 < c/2. \end{aligned}$$

There is $\mathfrak{D}''$ such that $\mathfrak{D}'' \ll \mathfrak{D}$ and $\mathfrak{D}'' \ll \mathfrak{D}'$. Suppose $\mathfrak{E} \ll \mathfrak{D}''$, for each I in $\mathfrak{E}$ $a(I)$ is in $\alpha(I)$ and $\mathfrak{E}^{**} = \{I: I \text{ in } \mathfrak{E}, |a(I)| > K\} = \{I: I \text{ in } \mathfrak{E}, a(I) < -K \text{ or } K <$

$< a(I)\}$. It follows from the preceding paragraph that

$$\sum_{\mathfrak{E}^{**}} \mu(I) < c/2 + c/2 = c.$$

Therefore α is μ -measurable.

7. – A summability characterization and representation theorem.

We refer the reader to [1] for the basic definitions, initial theorems, conventions and notations concerning the notion of set function summability. For the purpose of reading this section, one may dispense with a detailed consideration of the matters just mentioned, and instead let the following characterization theorem have the force of a definition:

THEOREM 7.A.1 [2]. – If α is in $\mathfrak{p}$, then the following two statements are equivalent:

- 1) α in μ -summable, and
- 2) if $p < 0 < q$, then $\int_U \max \{ \min \{ \alpha(I), q \}, p \} \mu(I)$ exists and there is an element of $\mathfrak{p}_{AB}$ such that

$$\int_U |\xi(I) - \max \{ \min \{ \alpha(I), q \}, p \} \mu(I)| \rightarrow 0, \quad \min \{ q, -p \} \rightarrow \infty.$$

THEOREM 7.A.2 [2]. – Under the hypothesis of Theorem 7.A.1, if α is μ -summable and $0 < c$, then there is $T > 0$ and $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$, for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$, and $\mathfrak{E}' = \{I: I \text{ in } \mathfrak{E}, |a(I)| > T\}$, then

$$\sum_{\mathfrak{E}'} \mu(I) < c.$$

Clearly Theorems 7.A.1 and 7.A.2 imply that if α is in $\mathfrak{p}$ and is μ -summable, then α is μ -measurable.

We now prove Theorem 7.1, as stated in the introduction.

PROOF OF THEOREM 7.1. – First, suppose α is μ -summable. Since α is μ -measurable, it follows from Theorem 6.1 that i) and ii) of Theorem 7.1 hold. We now show that iii) holds. First, i) implies that if $p < q$ and $u(p) = v(p)$ and $u(q) = v(q)$, then

$$\int_p^q x du(x) = \int_p^q x dv(x);$$

consequently we can confine our attention to u . By Theorem 7.A.1 there is $M \geq 0$ such that if $r \leq 0 \leq s$, then

$$\left| \int_U \max \{ \min \{ \alpha(I), s \}, r \} \mu(I) \right| \leq M.$$

Now, suppose $0 < q$. By Theorem 2.1, if $0 < x \leq q + 1$, then $u(x) = u(\max \{ \min \{ \alpha, q + 1 \}, 0 \})(x)$. This and Theorem 5.2 imply, letting $w = u(\max \{ \min \{ \alpha, q + 1 \}, 0 \})$, that

$$\int_0^q x du(x) = \int_0^q x dw(x) \leq \int_0^\infty x dw(x) = \int_{-\infty}^\infty x dw(x) = \int_U \max \{ \min \{ \alpha(I), q + 1 \}, 0 \} \mu(I) \leq M.$$

Therefore $\int_{-\infty}^\infty x du(x)$ exists. Now, suppose $p < 0$. Again, by Theorem 2.1, if $p - 1 < x \leq 0$, then $u(x) = u(\max \{ \min \{ \alpha, 0 \}, p - 1 \})(x)$. This and Theorem 5.2 imply, letting $z = u(\max \{ \min \{ \alpha, 0 \}, p - 1 \})$, that

$$\int_p^0 x du(x) = \int_p^0 x dz(x) \geq \int_{-\infty}^0 x dz(x) = \int_{-\infty}^\infty x dz(x) = \int_U \max \{ \min \{ \alpha(I), 0 \}, p - 1 \} \mu(I) \geq -M.$$

Therefore $\int_{-\infty}^0 x du(x)$ exists. Therefore $\int_{-\infty}^\infty x du(x)$ exists, and consequently $\int_{-\infty}^\infty x dv(x)$ exists. Therefore iii) holds.

Now suppose i), ii) and iii) hold. First, i) and ii) imply that α is μ -measurable, so that, if $r \leq 0 \leq s$, then $\int_U \max \{ \min \{ \alpha(I), s \}, r \} \mu(I)$ exists. Suppose that $0 < q$. Let $w = u(\max \{ \min \{ \alpha, q \}, 0 \})$. By Theorem 2.1, if $0 < x \leq q$, then $u(x) = w(x)$. By Theorem 5.2,

$$\begin{aligned} \int_U \max \{ \min \{ \alpha(I), q \}, 0 \} \mu(I) &= \int_{-\infty}^\infty x dw(x) = \int_0^\infty x dw(x) = \\ &= \int_0^q x dw(x) + \int_q^\infty x dw(x) = \int_0^q x du(x) + q[\mu(U) - w(q)] = \\ &= \int_0^q x du(x) + q[\mu(U) - u(y) + u(y) - u(q)] \leq \\ &\leq \int_0^q x du(x) + q[\mu(U) - u(y)] + \int_y^q x du(x) \quad \text{for each } y > q. \end{aligned}$$

Again, ii) implies that $u(y) \rightarrow \mu(U)$ as $y \rightarrow \infty$, so that we see that if $q < y$, then

$$\begin{aligned} \int_U \max \{ \min \{ \alpha(I), q \}, 0 \} \mu(I) &\leq \int_0^q x du(x) + \int_q^y x du(x) + q[\mu(U) - u(y)] \leq \\ &\int_0^\infty x du(x) + q[\mu(U) - u(y)] \rightarrow \int_0^\infty x du(x) \quad \text{as } y \rightarrow \infty. \end{aligned}$$

Since $\max \{ \min \{ \alpha, q \}, 0 \} = \min \{ \max \{ \alpha, 0 \}, q \}$ for each $q > 0$, it follows that $\max \{ \alpha, 0 \}$ is μ -summable.

Now suppose that $p < 0$. Let $z = u(\max \{ \min \{ \alpha, 0 \}, p \})$. Then

$$\int_U \max \{ \min \{ \alpha(I), 0 \}, p \} \mu(I) = \int_{-\infty}^\infty x dz(x) = \int_{-\infty}^0 x dz(x) = \int_{-\infty}^p x dz(x) + \int_p^0 x dz(x) = \int_p^0 x dz(x).$$

Now, if $p < x \leq 0$, then, by Theorem 2.1, $u(x) = z(x)$. Thus, if $p < r < 0$, then

$$\begin{aligned} \int_p^0 x dz(x) &= \int_p^r x dz(x) + \int_r^0 x du(x) \geq p[u(r) - 0] + \int_r^0 x du(x) = \\ &p[u(r) - u(p)] + \int_r^0 x du(x) + pu(p) \rightarrow \int_p^0 x du(x) + pu(p) \quad \text{as } r \rightarrow p, \end{aligned}$$

this last limit assertion a consequence of differential equivalence. Thus, if $v < p$, then

$$\begin{aligned} \int_U \max \{ \min \{ \alpha(I), 0 \}, p \} \mu(I) &\geq \int_p^0 x du(x) + pu(p) = \\ \int_p^0 x du(x) + p[u(p) - u(v)] + pu(v) &\geq \int_p^0 x du(x) + pu(v) \rightarrow \int_p^0 x du(x) + p0 \quad \text{as } v \rightarrow \infty. \end{aligned}$$

So $\int_U \max \{ \min \{ \alpha(I), 0 \}, p \} \mu(I) \geq \int_{-\infty}^0 x du(x)$. Since

$$\max \{ \min \{ \alpha, 0 \}, p \} = - \min \{ - \min \{ \alpha, 0 \}, -p \} \quad \text{for all } p < 0,$$

it follows that $- \min \{ \alpha, 0 \}$ is μ -summable. Since $\alpha = \max \{ \alpha, 0 \} + \min \{ \alpha, 0 \}$, it follows from the theorems of [1] that α is μ -summable.

As for Theorem 7.2, the preceding theorems of this paper make the argument for $u(\alpha)$ and $v(\alpha)$ nearly a complete repetition of the argument for the function, g_α , of [4]; consequently we omit it.

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