Noncharacteristic Hypersurfaces for Complexes of Differential Operators (*).

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Sunto. – *Sia X una varietà differenziabile ed S una ipersuperficie orientata in X. Si consideri un complesso di operatori differenziali su X. Se S è formalmente non caratteristica, esso induce un complesso di operatori su S. Si generalizza la nozione di simbolo di un operatore differenziale al caso di multigradazioni e si dimostra che, se S è non caratteristica, modulo « trasformazioni fibra » il complesso indotto è un complesso di operatori differenziali. In particolare, se una ipersuperficie è non caratteristica rispetto alla nozione usuale di simbolo, il complesso al bordo è sempre un complesso di operatori differenziali. Nell'ultima parte del lavoro si studia il complesso al bordo indotto dal complesso di Hilbert dell'operatore $\partial\bar{\partial}$ su nna varietà complessa.*

In this paper we consider again the notion of noncharacteristic hypersurfaces for a complex of differential operators already introduced in [3]. We generalize here the notion of symbol of a differential operator to cover the case of multigradings considered in classical analysis (for instance the notion of ellipticity given by Douglis and Nirenberg). We prove that on a noncharacteristic hypersurface the boundary complex induced by a given complex of differential operators up to « fiber transformations » is a complex of differential operators (theorem 1).

In particular on a hypersurface which is noncharacteristic with respect to the usual notion of symbol as used in [3] we get that the boundary complex is always a complex of differential operators (corollary to theorem 1).

We end this paper with the investigation of the boundary complex for the Hilbert complex of the operator $\partial\bar{\partial}$ on a complex manifold (given by BIGOLIN [9]). We recover some interesting results obtained already by AUDIBERT [6], BEDFORD and FEDERBUSH ([7], [8]). For simplicity we have restricted our consideration to the C^∞ category; we believe however to have given a comprehensive set of general statements.

The Hartog type theorem for boundaries with nonvanishing Levi form is contained in papers of MARTINELLI [12] and RIZZA [13] where the first set of $(n-1)^2$ equations for the tangential operator $(\partial\bar{\partial})_S$ are first derived and interpreted geometrically.

That all the results established relating to the trace at the boundary of a pluriharmonic function (theorem 3, corollary to proposition 8, last part of corollary to

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proposition 12) should be valid under much weaker assumptions of the type used by FICHERA in [10] is very plausible.

This paper ends with a theorem asserting the nonvalidity of Poincaré's lemma in general for the boundary complex of the complex of the $\partial\bar{\partial}$-operator (theorem 4).

1. – Differential operators, multigrading and symbols, the local situation.

a) Let Ω denote an open set in the numerical space $\mathbb{R}^n$ where $x = (x_1, \ldots, x_n)$ are Cartesian coordinates. Let $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ be the symbol of differentiation and let $\mathcal{E}(\Omega)$ denote the space of C^∞ (complex valued) functions on Ω.

Let $A(x, D) = \big(a_{ij}(x, D)\big)_{1\leqslant i\leqslant q, 1\leqslant j\leqslant p}$ be a $q\times p$ matrix of differential operators with C^∞ coefficients so that $A(x, D)$ defines a linear map

$$A(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)\,.$$

Assume that we have chosen two sequences of integers

$$a_1, a_2, \ldots, a_p \qquad \text{for } \mathcal{E}^p(\Omega)\,,$$
$$b_1, b_2, \ldots, b_q \qquad \text{for } \mathcal{E}^q(\Omega)\,,$$

such that one can write, for any i, j,

$$a_{ij}(x, D) = \sum_{|\alpha|\leqslant a_j - b_i} a_{ij\alpha}(x)\, D^\alpha$$

where $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex,

$$|\alpha| = \alpha_1 + \ldots + a_n \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha_1+\ldots+\alpha_n|}}{\partial x_1^{\alpha_1}\ldots\,\partial x_n^{\alpha_n}}\,.$$

Note that if the sequences $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$ satisfy the property mentioned above, also, for any integer k, the sequences $(a_1 + k, \ldots, a_p + k)$, $(b_1 + k, \ldots, b_q + k)$ satisfy the same property.

We define

$$\hat{a}_{ij}(x, \xi) = \sum_{|\alpha| = a_j - b_i} a_{ij\alpha}(x)\, \xi^\alpha$$

for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ and where ξ^α stands for $\xi_1^{\alpha_1}\ldots\,\xi_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

We define the *symbol of the operator* $A(x, D)$ *for the multigrading* (a_j, b_i) given above, the matrix of polynomials in ξ with coefficients in $\mathcal{E}(\Omega)$:

$$\hat{A}(x, \xi) = \big(\hat{a}_{ij}(x, \xi)\big)\,.$$

b) Let

$$B(x, D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^r(\Omega)$$

be a second differential operator with C^∞ coefficients in Ω

$$B(x, D) = \big(b_{hi}(x, D)\big)_{1\leqslant h\leqslant r,\, 1\leqslant i\leqslant q}\,.$$

We fix a third sequence of integers

$$c_1, c_2, \ldots, c_r \qquad \text{for } \mathcal{E}^r(\Omega)$$

so that, for any h and i,

$$b_{hi}(x, D) = \sum_{|\alpha|\leqslant b_i - c_h} b_{hi\alpha}(x)\, D^\alpha$$

is an operator of order $b_i - c_h$.

We can then construct the symbol of the operator $B(x, D)$ for the multigrading (b_i, c_h);

$$\hat{B}(x, \xi) = \Big(\sum_{|\alpha| = b_i - c_h} b_{hi\alpha}(x)\, \xi^\alpha\Big)\,.$$

Also, one can consider the operator

$$B(x, D)\circ A(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^r(\Omega)$$

as an operator « compatible » with the multigrading (a_j, c_h). Therefore we can consider its symbol of multigrading (a_j, c_h). We have the important property

$$\widehat{B\circ A}(x, \xi) = \hat{B}(x, \xi)\hat{A}(x, \xi)$$

(*multiplicative property of the symbol*).

2. – Differential operators between vector bundles, multigrading and symbols.

a) Let X be a differentiable manifold of pure dimension n. Let

$$E \xrightarrow{\ \pi\ } X\,,$$
$$F \xrightarrow{\ \mu\ } X\,,$$

be vector bundles on X with fibres modeled respectively on $\mathbf{C}^p$, $\mathbf{C}^q$. We say that E is a vector bundle on X of rank p and F a vector bundle on X of rank q.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a system of coordinate patches on X such that on each U_i $E|_{U_i}$ and $F|_{U_i}$ are trivial. We fix trivializations on each U_i

$$E|_{U_i} \simeq U_i \times \mathbf{C}^p ,$$
$$F|_{U_i} \simeq U_i \times \mathbf{C}^q ,$$

and consequently the transition functions

$$e_{ij}\colon\ U_i \cap U_j \to GL(p, \mathbf{C}) ,$$
$$f_{ij}\colon\ U_i \cap U_j \to GL(q, \mathbf{C}) ,$$

for the bundles E and F:

$$\left.\begin{aligned} e_{ij}\,e_{jk} &= e_{ik} \\ f_{ij}\,f_{jk} &= f_{ik} \end{aligned}\right. \qquad \text{on } U_i \cap U_j \cap U_k .$$

Given a section $s\colon X \to E$, $\pi \circ s = \mathrm{id}_X$, this is represented in the local trivializations $E|_{U_i} \simeq U_i \times \mathbf{C}^p$ by $(x, s_i(x))$, $x \in U_i$, $s_i(x) \in \mathbf{C}^p$ so that

$$s_i \in \mathscr{E}^p(U_i)$$

and on $U_i \cap U_j$ we have

$$s_i(x) = e_{ij}(x)\, s_j(x) .$$

Similarly for a section of F.

b) A differential operator from the bundle E to the bundle F is a linear map

$$A(x, D)\colon\ \Gamma(X, E) \to \Gamma(X, F)$$

where $\Gamma(X, E)$ and $\Gamma(X, F)$ represent the spaces of C^∞ sections of E and F respectively such that

i) $A(x, D)$ is continuous for the Schwartz topologies of $\Gamma(X, E)$ and $\Gamma(X, F)$,

ii) $A(x, D)$ is local i.e. for any $s \in \Gamma(X, E)$

$$\operatorname{supp}\big(A(x, D)\, s\big) \subset \operatorname{supp} s .$$

From a theorem of Peetre we derive that the datum of a differential operator $A(x, D)$ is *equivalent*

(α) *to the assignment for every* U_i *of a differential operator*

$$A^{(i)}(x, D): \mathcal{E}^p(U_i) \to \mathcal{E}^q(U_i)$$

$i \in I$, *with the property that*

(β) *the diagrams*

$$(*) \qquad \begin{array}{ccc} \mathcal{E}^p(U_i) & \xrightarrow{A^{(i)}} & \mathcal{E}^q(U_i) \\ \downarrow e_{ji} & & \downarrow f_{ji} \\ \mathcal{E}^p(U_j) & \xrightarrow{A^{(j)}} & \mathcal{E}^q(U_j) \end{array}$$

commute where they are defined.

If we set

$$A^{(i)}(x, D) = \sum a_\alpha^{(i)}(x) D^\alpha$$

from the identity on $U_i \cap U_j$

$$A^{(i)} e_{ij} s_j = f_{ij} A^{(j)} s_j$$

$\forall s_j \in \mathcal{E}^p(U_j) = \Gamma(U_j, E)$, we derive that the condition $(*)$ is equivalent to the *consistency condition*

$$(\beta) \qquad \sum a_\alpha^{(i)}(x) \binom{\alpha}{\beta} D^{\alpha-\beta} e_{ij}(x) = f_{ij} a_\beta^{(j)}(x)$$

which expresses the identity of differential operators on $U_i \cap U_j$

$$(*) \qquad A^{(i)} \circ e_{ij} = f_{ij} \circ A^{(j)} .$$

c) A *grading on the bundle* E will be, by definition, an assignment for each open set U_i, $i \in I$ of a grading

$$a_1^{(i)}, \ldots, a_p^{(i)} \qquad \text{for } \mathcal{E}^p(U_i) = \Gamma(U_i, E)$$

such that, setting

$$e_{ij}(x) = \big(e_{ij,r,s}(x)\big)_{1 \leqslant r \leqslant p,\, 1 \leqslant s \leqslant p}$$

we have

$$(**) \qquad e_{ij,r,s}(x) = 0 \qquad \text{whenever } a_s^{(j)} - a_r^{(i)} \neq 0 .$$

For instance we can fix an integer $a \in \mathbb{Z}$ and set

$$a_1^{(i)} = \ldots = a_p^{(i)} = a \qquad \forall i \in I\ ;$$

then the condition $(**)$ on the transition functions becomes empty and therefore we have defined a grading on E. A grading of this sort will be called a *classical grading.*

The following proposition clarifies the structure of a graded vector bundle

PROPOSITION 0. – *Let E be a multigraded vector bundle on a connected manifold X. Then E splits into direct sum of vector bundles*

$$E = E_1 \oplus E_2 \oplus \ldots \oplus E_l$$

on each of which a classical grading is given.

PROOF. – (α) Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X such that for any $i \in I$ $E|_{U_i}$ is trivialized, $E|_{U_i} \simeq U_i \times \mathbb{C}^p$ and graded with a grading $a_1^{(i)}, \ldots, a_p^{(i)}$.

Let $i_1, i_2, \ldots, i_p$ be a permutation of $(1, 2, \ldots, p)$ such that

$$a_{i_1}^{(i)} \geqslant a_{i_2}^{(i)} \geqslant \ldots \geqslant a_{i_p}^{(i)}\ .$$

Let $\lambda^{(i)}$ denote the matrix $p \times p$ with 1 in the places $(1, i_1), (2, i_2), \ldots, (p, i_p)$ so that

$$\lambda^{(i)} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ p \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \end{pmatrix}.$$

Then $\det \lambda^{(i)} = \pm 1$. We set

$$\alpha_1^{(i)} = a_{i_1}^{(i)},\ \alpha_2^{(i)} = a_{i_2}^{(i)},\ \ldots,\ \alpha_p^{(i)} = a_{i_p}^{(i)}$$

and change the local trivializations of $E|_{U_i}$ by the isomorphisms given by the matrices $\lambda^{(i)}$.

Consider the commutative diagram (where it is defined)

$$\begin{array}{ccc} E|_{U_i} & \xrightarrow{e_{ij}} & E|_{U_i} \\ \downarrow & & \downarrow \\ E|_{U_i} & \xrightarrow{\tilde{e}_{ij}} & E|_{U_i} \end{array}$$

where $\tilde{e}_{ij} = \lambda^{(i)} e_{ij} (\lambda^{(j)})^{-1}$. With the new trivializations the $\tilde{e}_{ij}$'s will be the transition functions and these correspond to the gradings $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_p^{(i)}$ on each $U_i \in \mathscr{U}$.

We have thus proved that it is not restrictive to assume that for every $i \in I$ the chosen grading is such that

$$a_1^{(i)} \geqslant a_2^{(i)} \geqslant \ldots \geqslant a_p^{(i)} .$$

(β) Let on U_i be

$$a_1^{(i)} = \ldots = a_r^{(i)} > a_{r+1}^{(i)} = \ldots = a_{r+s}^{(i)} > \ldots$$

and let $U_j \in \mathcal{U}$ be such that $U_i \cap U_j \neq \emptyset$. Then by the prescribed conditions on the transition functions we deduce that on U_j we must have

$$\begin{aligned} a_1^{(j)} &= \ldots = a_r^{(j)} &&= k_1 = a_1^{(i)} = \ldots = a_r^{(i)} \\ a_{r+1}^{(j)} &= \ldots = a_{r+s}^{(j)} &&= k_2 = a_{r+1}^{(i)} = \ldots = a_{r+s}^{(i)} \\ &\ldots \end{aligned}$$

Since X is connected we realize that the above relations must be valid on any U_j even if $U_j \cap U_i = \emptyset$ as one can find a finite sequence of open sets $U_{j_1}, \ldots, U_{j_t}$ in $\mathcal{U}$ such that

$$U_i \cap U_{j_1} \neq \emptyset, \ U_{j_1} \cap U_{j_2} \neq \emptyset, \ \ldots, \ U_{j_{t-1}} \cap U_{j_t} \neq \emptyset, \ U_{j_t} \cap U_j \neq \emptyset .$$

We deduce then that for any i, j in I the matrices e_{ij} split into the direct sum of blocks of the form

$$e_{ij} = \begin{pmatrix} e_{ij}^1 & 0 & \cdots & 0 \\ 0 & e_{ij}^2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_{ij}^l \end{pmatrix}$$

where e_{ij}^1 is an $r \times r$ matrix, e_{ij}^2 is an $s \times s$ matrix, ... all non singular.

Set E_ν to be the bundle defined by the transition function e_{ij}^ν $1 \leqslant \nu \leqslant l$, and let us choose the classical grading on E_ν given by the integer k_ν. We have proved that up to an isomorphism

$$E \simeq E_1 \oplus E_2 \oplus \ldots \oplus E_l$$

with classical gradings k_1 on E_1, k_2 on E_2, ..., k_l on E_l.

d) Suppose now that we have given two vector bundles E and F and a differential operator

$$A(x, D) \colon \Gamma(X, E) \to \Gamma(X, F) .$$

Suppose also that we have chosen gradings

$$a_1^{(i)}, \ldots, a_p^{(i)} \qquad \text{on } E\,,$$
$$b_1^{(i)}, \ldots, b_q^{(i)} \qquad \text{on } F\,,$$

compatible with the operator $A(x, D)$. This means that setting

$$A^{(i)}(x, D) = \left(a_{rs}^{(i)}(x, D)\right)_{1\leqslant r\leqslant q,\, 1\leqslant s\leqslant p}$$

we have

$$a_{rs}^{(i)}(x, D) = \sum_{|\alpha|\leqslant a_s^{(i)}-b_r^{(i)}} a_{rs,\alpha}^{(i)}(x)\, D^\alpha\,.$$

From the consistency conditions $(*)$ we derive then the following formula

$$\hat{A}^{(i)}(x, \xi)\, e_{ij}(x) = f_{ij}(x)\hat{A}^{(j)}(x, \xi)\,. \tag{1}$$

Now note that a change of coordinates in X affects the $\xi = (\xi_1, \ldots, \xi_n)$ as if they where the components of a covariant vector. Thus (x, ξ) has to be thought of as a point in the cotangent bundle $T^*(X)$. Consider also the vector bundle

$$\operatorname{Hom}_X(E, F)\,.$$

A section $\sigma \in \Gamma(X, \operatorname{Hom}_X(E, F))$ is given by a collection $\{M_i\}_{i\in I}$ of matrices M_i, C^∞ on U_i, $i \in I$, of type $q\times p$ such that

$$M_i(x)\, e_{ij}(x) = f_{ij}(x)\, M_j(x) \qquad \forall x \in U_i \cap U_j\,.$$

Formula (1) then shows that the symbol of a differential operator

$$A(x, D)\colon\ \Gamma(X, E) \to \Gamma(X, F)$$

is a map

$$\hat{A}(x, \xi)\colon\ T^*(X) \to \operatorname{Hom}_X(E, F)$$

such that the diagram

$$\begin{array}{ccc} T^*(X) & \xrightarrow{\hat{A}(x,\xi)} & \operatorname{Hom}_X(E, F) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

is commutative, α and β being the natural projections.

e) The above representation of the symbol $\hat{A}(x, \xi)$ does not take into account the fact that the matrices $\hat{A}^{(i)}(x, \xi)$ are polynomials in ξ. For this reason we develop the following considerations.

We first consider the cotangent bundle $T^*(X)$ of X of covariant vectors on X.

We denote by $\mathscr{P}(X)$ the ring of C^∞ functions on $T^*(X)$ which are polynomials along the fibres.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a set of coordinate patches covering X and let

$$T^*(X)|_{U_i} \simeq U_i \times \mathbf{R}^n$$

be local trivializations with $x^{(i)} = (x_1^{(i)}, \dots, x_n^{(i)})$ coordinates on U_i and $\xi^{(i)} = (\xi_1^{(i)}, \dots, \xi_n^{(i)})$ coordinates along the fibres $\mathbb{R}^n$.

An element $p(x, \xi) \in \mathscr{P}(X)$ is a collection of polynomials

$$p_i(x^{(i)}, \xi^{(i)})$$

in the variables $\xi^{(i)} \in \mathbb{R}^n$ with C^∞ coefficients in $x^{(i)} \in U_i$ such that on $U_i \cap U_j$ we have

$$p_j(x^{(j)}, \xi^{(j)}) = p_i\left(x^{(i)}(x^{(j)}), \frac{\partial x^{(j)}}{\partial x^{(i)}} \xi^{(j)}\right)$$

where $\partial x^{(j)}/\partial x^{(i)}$ denotes the Jacobian matrix of the change of coordinates from U_j to U_i:

$$\begin{cases} \xi_\alpha^{(i)} = \sum_1^n \dfrac{\partial x_\beta^{(j)}}{\partial x_\alpha^{(i)}} \xi_\beta^{(j)}, \\ 1 \leqslant \alpha \leqslant n . \end{cases}$$

The space $\mathscr{P}(X)$ could be called the ring of « *codifferential symmetric forms* ».

Note that if X is parallelizable i.e. if

$$T^*(X) \simeq X \times \mathbb{R}^n$$

(as a fiber space over X) then $\mathscr{P}(X)$ is nothing but the ring $\mathscr{E}(X)[\xi_1, \dots, \xi_n]$ of polynomials in the n variables $\xi = (\xi_1, \dots, \xi_n)$ with C^∞ coefficients on X. Here $\mathscr{E}(X)$ denotes the ring of C^∞ functions on X.

Given a vector bundle E on X, trivial on the covering $\mathscr{U} = \{U_i\}_{i\in I}$ with transition functions $\{e_{ij}\}$ we can consider the space

$$\mathscr{P}(X) \otimes_{\mathscr{E}(X)} \Gamma(X, E)$$

of « *codifferential symmetric forms with values in E* ».

An element φ of this space is locally given by a collection

$$\varphi_i(x, \xi) = \begin{pmatrix} \varphi_i^1(x, \xi) \\ \vdots \\ \varphi_i^p(x, \xi) \end{pmatrix}$$

of codifferential symmetric forms $\varphi_i^j(x, \xi)$ $1 \leq j \leq p$ on each coordinate patch $U_i \in \mathcal{U}$ such that on $U_i \cap U_j$ we have

$$\varphi_i(x, \xi) = e_{ij}(x)\, \varphi_j(x, \xi) .$$

We note that the space of codifferential symmetric forms with values in E is no longer a ring but only a module over $\mathcal{P}(X)$.

Given now the vector bundles E and F over X, given a differential operator $A(x, D)\colon \Gamma(X, E) \to \Gamma(X, F)$, given a grading on E and a grading on F compatible with the differential operator $A(x, D)$, we can then consider the symbol $\hat{A}(x, \xi)$ as a $\mathcal{P}(X)$-linear map (because of formula (1))

$$\mathcal{P}(X, E) \xrightarrow{\hat{A}(x, \xi)} \mathcal{P}(X, F)$$

where by definition

$$\mathcal{P}(X, E) = \mathcal{P}(X) \otimes_{\mathcal{E}(X)} \Gamma(X, E) , \qquad \mathcal{P}(X, F) = \mathcal{P}(X) \otimes_{\mathcal{E}(X)} \Gamma(X, F)$$

are the spaces of codifferential symmetric forms with values in E and F respectively.

Finally let us consider a third vector bundle

$$G \xrightarrow{\nu} X .$$

of rank r (i.e. with fiber $\mathbf{C}^r$). Assume that we have given a second differential operator

$$B(x, D)\colon\ \Gamma(X, F) \to \Gamma(X, G)$$

and suppose that a grading

$$c_1^{(i)}, \ldots, c_r^{(i)} \qquad \text{on } G$$

is given such that it is compatible with the differential operator $B(x, D)$.

We can then consider the space $\mathcal{P}(X, G)$ of codifferential symmetric forms on X with values in G and the symbol $\hat{B}(x, \xi)$ of $B(x, D)$ as a $\mathcal{P}(X)$-linear map

$$\mathcal{P}(X, F) \xrightarrow{\hat{B}(x, \xi)} \mathcal{P}(X, G) .$$

From the multiplicative property of the symbol we derive the commutative diagram

$$\begin{array}{ccc} \mathscr{P}(X, E) & \xrightarrow{\hat{A}(x,\xi)} & \mathscr{P}(X, F) \\ & \underset{\widehat{B\circ A}(x,\xi)}{\searrow} \quad \underset{\hat{B}(x,\xi)}{\swarrow} & \\ & \mathscr{P}(X, G) & \end{array}$$

3. – Complexes of differential operators, the symbolic complex, elliptic complexes.

a) We give on X a sequence $E^0, E^1, E^2, \ldots$ of vector bundles with fibres $\mathbf{C}^{p_0}, \mathbf{C}^{p_1}, \mathbf{C}^{p_2}, \ldots$ i.e. of ranks $p_0, p_1, p_2, \ldots$, respectively.

We give a sequence of differential operators

$$\begin{aligned} A^0(x, D)&: \Gamma(X, E^0) \to \Gamma(X, E^1)\,, \\ A^1(x, D)&: \Gamma(X, E^1) \to \Gamma(X, E^2)\,, \\ A^2(x, D)&: \Gamma(X, E^2) \to \Gamma(X, E^3)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

with the property that

$$A^1 \circ A^0 = 0, \; A^2 \circ A^1 = 0, \; \ldots \text{ i.e. } A^{j+1} \circ A^j = 0 \qquad j = 0, 1, 2, \ldots .$$

We then say that we have given a complex of differential operators.

Setting for the sake of a simple notation

$$\mathscr{E}^{(j)}(X) = \Gamma(X, E^j)$$

the given complex will be denoted by:

$$\mathscr{E}^{(0)}(X) \xrightarrow{A^0(x, D)} \mathscr{E}^{(1)}(X) \xrightarrow{A^1(x, D)} \mathscr{E}^{(2)}(X) \xrightarrow{A^2(x, D)} \ldots . \tag{3}$$

b) Suppose now that we have given gradings

$$\begin{aligned} a_1^{(i)}, \ldots, a_{p_0}^{(i)} \qquad & \text{on } E^0\,, \\ b_1^{(i)}, \ldots, b_{p_1}^{(i)} \qquad & \text{on } E^1\,, \\ c_1^{(i)}, \ldots, c_{p_2}^{(i)} \qquad & \text{on } E^2\,, \\ \ldots\ldots\ldots\ldots & \end{aligned}$$

for $i \in I$, I being the index set of a covering $\mathscr{U} = \{U_i\}_{i \in I}$ of X by coordinate patches on which each bundle E^j is trivial.

We assume that these gradings are compatible with the differential operators $A^0(x, D)$, $A^1(x, D)$, ... so that for each operator $A^j(x, D)$ we can consider the corresponding symbol $\hat{A}^j(x, \xi)$. We obtain then the following sequence of $\mathscr{P}(X)$-linear maps

$$(4) \qquad \mathscr{P}(X, E^0) \xrightarrow{\hat{A}^0(x,\xi)} \mathscr{P}(X, E^1) \xrightarrow{\hat{A}^1(x,\xi)} \mathscr{P}(X, E^2) \xrightarrow{\hat{A}^2(x,\xi)} \dots .$$

This sequence is a complex by virtue of the multiplicative property of the symbols i.e. by formula (2).

The sequence (4) will be called the symbolic complex on X associated to the given complex (3).

Let us fix a point $x^0 \in X$ and let $\mathfrak{w}_{x^0}(X) \subset \mathscr{E}(X)$ denote the ideal of $\mathscr{E}(X)$ of those functions which vanish at x^0. We can tensor over $\mathscr{E}(X)$ the above sequence (4) by $\mathbf{C}_{x^0} = \mathscr{E}(X)/\mathfrak{w}_{x^0}(X)$ considered as an $\mathscr{E}(X)$-module. Then for each $j \geqslant 0$

$$\mathscr{P}(X, E^j) \otimes \mathscr{E}(X)/\mathfrak{w}_{x^0}(X) \simeq \mathscr{P}^{p_j}$$

where $\mathscr{P} = \mathbf{C}[\xi_1, \dots, \xi_n]$ is the ring of polynomials in the variables $\xi = (\xi_1, \dots, \xi_n)$. From the complex (4) we then obtain the complex

$$(4)_{x^0} \qquad \mathscr{P}^{p_0} \xrightarrow{\hat{A}^0(x^0,\xi)} \mathscr{P}^{p_1} \xrightarrow{\hat{A}^1(x^0,\xi)} \mathscr{P}^{p_2} \xrightarrow{\hat{A}^2(x^0,\xi)} \dots .$$

We call this complex *the symbolic complex associated to the given complex* (3) *at the point* $x^0 \in X$.

c) Finally we can fix $x^0 \in X$ and $\xi^0 \in \boldsymbol{R}^n - \{0\}$, on the fiber of $T^*(X)$ over x^0. From $(4)_{x_0}$ we then obtain another complex

$$(5) \qquad 0 \to \mathbf{C}^{p_0} \xrightarrow{\hat{A}^0(x^0,\xi^0)} \mathbf{C}^{p_1} \xrightarrow{\hat{A}^1(x^0,\xi^0)} \mathbf{C}^{p_2} \xrightarrow{\hat{A}^2(x^0,\xi^0)} \dots$$

where $\mathbf{C}^{p_j}$ stands for the fiber over x^0 of the bundle E^j.

We will say that the given complex (3) is an *elliptic complex at* $x^0 \in X$ if for any choice of $\xi^0 \in \mathbb{R}^n - \{0\}$ the sequence (5) is an exact sequence.

We may remark that one can consider the complexified cotangent bundle $T^*_{\mathbf{C}}(X)$ (with fibers $\mathbf{C}^n$ and the same transition functions $t_{ij}(x) = \partial x^{(j)}/\partial x^{(i)}$ of $T^*(X)$). Then the sequence (5) can be considered also for given $x^0 \in X$ and given $\xi^0 \in \mathbf{C}^n$, the fiber of $T^*_{\mathbf{C}}(X)$ over x^0.

d) Replacing the symbols ξ with the symbols of differentiation D we obtain from $(4)_{x^0}$ the complex of differential operators with constant coefficients on $\mathbb{R}^n$

$$(6) \qquad \mathscr{E}(\mathbb{R}^n)^{p_0} \xrightarrow{\hat{A}^0(x^0,D)} \mathscr{E}(\mathbb{R}^n)^{p_1} \xrightarrow{\hat{A}^1(x^0,D)} \mathscr{E}(\mathbb{R}^n)^{p_2} \xrightarrow{\hat{A}^2(x^0,D)} \dots .$$

This is what is usually called the symbolic complex for the complex (3) at the point $x^0 \in X$.

The complex (6) is a Hilbert complex (cf. [4]) if and only if

$$ {}^t(4)_{x^0} \qquad \mathscr{P}^{p_0} \xleftarrow{{}^t\hat{A}^0(x^0,\,\xi)} \mathscr{P}^{p_1} \xleftarrow{{}^t\hat{A}^1(x_0,\,\xi)} \mathscr{P}^{p_2} \xleftarrow{{}^t\hat{A}^2(x_0,\,\xi)} \dots $$

is an exact sequence.

We recall the following theorems (cf.[5])

THEOREM 0. – *Assume that at a point $x^0 \in X$ the symbolic complex* (6) *is a Hilbert complex. Then the given complex* (3) *admits the formal Poincare lemma at x^0.*

Assume that the manifold X is a real analytic manifold, that the bundles E^j are also real analytic (i.e. on a real analytic coordinate atlas $\mathscr{U} = \{U_i\}_{i\in I}$ the transition functions are real analytic functions $e^{(j)}_{ik}\colon U_i \cap U_k \to GL(p_j, \mathbb{C})$ and the differential operators $A^j(x, D)$ have real analytic coefficients). We have then ([5])

THEOREM 1. – *Assume that at a point $x^0 \in X$ the symbolic complex* (6) *is a Hilbert complex. Then* (*under the above assumptions*) *the given complex* (3) *admits the analytic Poincaré lemma at x^0.*

THEOREM 2. – *Under the same assumptions of analyticity. Assume that at a point $x^0 \in X$ the symbolic complex* (6) *is a Hilbert complex.*

Assume also that at the point x^0 the given complex is elliptic ([1]).

Then the given complex (3) *admits the C^∞ Poincaré lemma at x^0.*

It is still an open question to decide whether theorem 2 remains valid without the assumptions of analyticity on X, E^j $\forall j$, and $A^j(x, D)$.

4. – Fiber transformations and change of grading.

a) Let E be a vector bundle on X, let $\mathscr{U} = \{U_i\}_{i\in I}$ be a covering of X by coordinate patches on which the bundle E is trivial

$$ E|_{U_i} \simeq U_i \times \mathbb{C}^p $$

p being the rank of E.

Let

$$ M(x, D)\colon \Gamma(X, E) \to \Gamma(X, E) $$

be a differential operator from E to E.

([1]) In the sense that the sequence $\mathbb{C}^{p_0} \xrightarrow{\hat{A}^0(x^0,\,\xi^0)} \mathbb{C}^{p_1} \xrightarrow{\hat{A}^1(x^0,\,\xi^0)} \mathbb{C}^{p_2} \to \dots$ is exact for any $\xi^0 \in \mathbb{R}^n - \{0\}$ even if the first map is not injective.

We assume that we have given gradings

$$a_1^{(i)}, \ldots, a_p^{(i)} \qquad \text{on } E \text{ as the « source bundle »},$$
$$\alpha_1^{(i)}, \ldots, \alpha_p^{(i)} \qquad \text{on } E \text{ as the « target bundle »},$$

compatible with the differential operator $M(x, D)$. This means that locally on U_i the local representations of $M(x, D)$

$$M^{(i)}(x, D) = \left(m_{rs}^{(i)}(x, D)\right)_{1 \leqslant r \leqslant p, 1 \leqslant s \leqslant p}$$

are such that

$$\text{order of } m_{rs}^{(i)}(x, D) \leqslant a_s^{(i)} - \alpha_r^{(i)}$$

for each $i \in I$. We can then consider the symbol

$$\hat{M}(x, \xi) = \{\hat{M}^{(i)}(x, \xi)\}_{i \in I}$$

of the operator $M(x, D)$. We have the following

Proposition 1. – *We assume that*

i) *$M(x, D)$ is a differential operator of total degree zero. By this we mean that*

$$\sum_1^p a_s^{(i)} = \sum_1^p \alpha_r^{(i)} \qquad \forall i \in I$$

and therefore for each $i \in I$ det $\hat{M}^{(i)}(x, \xi)$ *is a homogeneous polynomial in ξ of degree zero, thus independent of ξ.*

ii) *For each i*

$$\det \hat{M}^{(i)}(x, \xi) = \det \hat{M}^{(i)}(x, 0) \neq 0 .$$

Then there exists a unique differential operator

$$N(x, D) \colon \Gamma(X, E) \to \Gamma(X, E)$$

compatible with the gradings

$$\alpha_1^{(i)}, \ldots, \alpha_p^{(i)} \qquad \text{on the source bundle } E,$$
$$a_1^{(i)}, \ldots, a_p^{(i)} \qquad \text{on the target bundle } E,$$

such that

$$N(x, D)\circ M(x, D) = \textit{identity on } \Gamma(X, E)\,,$$
$$M(x, D)\circ N(x, D) = \textit{identity on } \Gamma(X, E)\,.$$

This is a consequence of the local theorem proved in proposition 2 of [5]. We remark explicitly that if N_1 and N_2 are respectively left and right inverse of M we must have $N_1 = N_2$. Indeed from $N_1\circ M =$ identity, $M\circ N_2 =$ identity, as the algebra of differential operators from E to E is associative we derive

$$(N_1\circ M)\circ N_2 = N_2 \qquad \text{thus } N_1\circ(M\circ N_2) = N_2$$

hence $N_1 = N_2$ as we wanted.

If $e_{ij}\colon U_i\cap U_j \to GL(p, \mathbb{C})$ are the transition functions of E we must have

$$M^{(i)}(x, D)\circ e_{ij}(x) = e_{ij}(x)\circ M^{(j)}(x, D)$$

and

$$\hat{M}^{(i)}(x, \xi)\circ e_{ij}(x) = e_{ij}(x)\,\hat{M}^{(j)}(x, \xi)\,.$$

A differential operator $M(x, D)$ satisfying the hypothesis of proposition 1 will be called a «*fiber transformation*»; it establishes an isomorphism of $\Gamma(X, E)$ onto itself:

$$M(x, D)\colon \Gamma(X, E) \xrightarrow{\sim} \Gamma(X, E)\,.$$

REMARK. – If the grading on E as source and target bundle is a *classical grading* *i.e.* $\forall i \in I$

$$a_1^{(i)} = \ldots = a_p^{(i)} = k = \alpha_1^{(i)} = \ldots = \alpha_p^{(i)}$$

for some $k\in\mathbb{Z}$ then $M(x, D)$ is a differential operator of order zero thus locally defined by matrices $M^{(i)}(x)$ not containing derivatives and with det $M^{(i)}(x)\neq 0$, $\forall x\in U_i$.

b) Suppose now that we have a complex of differential operators

$$\mathcal{E}^{(0)}(X) \xrightarrow{A^0(x, D)} \mathcal{E}^{(1)}(X) \xrightarrow{A^1(x, D)} \mathcal{E}^{(2)}(X) \xrightarrow{A^2(x, D)} \ldots \tag{3}$$

where $\mathcal{E}^{(j)}(X) = \Gamma(X, E^j)$ for some bundle E^j of rank p_j.

Assume also that we have given gradings

$$\begin{array}{ll} a_1^{(i)}, \ldots, a_{p_0}^{(i)} & \text{on } E^0\,, \\ b_1^{(i)}, \ldots, b_{p_1}^{(i)} & \text{on } E^1\,, \\ c_1^{(i)}, \ldots, c_{p_2}^{(i)} & \text{on } E^2\,, \\ \ldots\ldots & \end{array}$$

compatible with the operators of the complex, $i \in I$ the index set of the charted covering $\mathscr{U} = \{U_i\}_{i\in I}$ on which each bundle E^j is assumed to be trivial.

Suppose that we change grading on the bundles E^j into

$$\begin{array}{lll} \alpha_1^{(i)}, \ldots, \alpha_{p_0}^{(i)} & \text{on } E^0 \text{ with } \sum_h \alpha_h^{(i)} = \sum_h a_h^{(i)} , \\ \beta_1^{(i)}, \ldots, \beta_{p_1}^{(i)} & \text{on } E^1 \text{ with } \sum_h \beta_h^{(i)} = \sum_h b_h^{(i)} , \\ \gamma_1^{(i)}, \ldots, \gamma_{p_2}^{(i)} & \text{on } E^2 \text{ with } \sum_h \gamma_h^{(i)} = \sum_h c_h^{(i)} , \\ \ldots\ldots & \ldots\ldots \end{array}$$

and that for each bundle E^j we give a fibre transformation

$$M_j(x, D)\colon \mathscr{E}^{(j)}(X) \to \mathscr{E}^{(j)}(X) ,$$

compatible with the old and new gradings on E^j.

Set, for $j = 0, 1, 2, \ldots$

$$B^j(x, D) = M_{j+1}(x, D) \circ A^j(x, D) \circ M_j^{-1}(x, D) .$$

Then

$$B^j(x, D)\colon \mathscr{E}^{(j)}(X) \to \mathscr{E}^{(j+1)}(X)$$

is a differential operator compatible with the new grading. We obtain thus a commutative diagram

$$(5) \qquad \begin{array}{ccccccc} \mathscr{E}^{(0)}(X) & \xrightarrow{A^0(x, D)} & \mathscr{E}^{(1)}(X) & \xrightarrow{A^1(x, D)} & \mathscr{E}^{(2)}(X) & \xrightarrow{A^2(x, D)} & \ldots \\ \downarrow M_0(x, D) & & \downarrow M_1(x, D) & & \downarrow M_2(x, D) & & \\ \mathscr{E}^{(0)}(X) & \xrightarrow{B^0(x, D)} & \mathscr{E}^{(1)}(X) & \xrightarrow{B^1(x, D)} & \mathscr{E}^{(2)}(X) & \xrightarrow{B^2(x, D)} & \ldots \end{array}$$

in which the horizontal rows are complexes and the vertical maps are isomorphisms.

The complex of differential operators

$$(6) \qquad \mathscr{E}^{(0)}(X) \xrightarrow{B^0(x, D)} \mathscr{E}^{(1)}(X) \xrightarrow{B^1(x, D)} \mathscr{E}^{(2)}(X) \xrightarrow{B^2(x, D)} \ldots$$

with the new gradings $\{\alpha_i\}$, $\{\beta_j\}$, $\{\gamma_h\}$, ... *will be called the transformed of the complex* (4) *by means of the fiber transformations* $M_0(x, D)$, $M_1(x, D)$, $M_2(x, D)$, ...

Let for $j \geq 0$ $H^j(X; \mathscr{E}^*(X), A^*)$ denote the j-th cohomology group of the complex (3) i.e.

$$H^j(X; \mathscr{E}^*(X), A^*) = \frac{\operatorname{Ker}\{\mathscr{E}^{(j)}(X) \xrightarrow{A^j(x, D)} \mathscr{E}^{(j+1)}(X)\}}{\operatorname{Im}\{\mathscr{E}^{(j-1)}(X) \xrightarrow{A^{j-1}(x, D)} \mathscr{E}^{(j)}(X)\}}$$

(setting $\mathcal{E}^{(-1)}(X) = 0$). Similarly, by replacing the complex (3) with the complex (6) we can define the groups $H^j(X; \mathcal{E}^*(X), B^*)$. We have the obvious

PROPOSITION 2. *If the complex* (6) *is obtained from the complex* (3) *by fiber transformations then for every* $j \geq 0$ *we have natural isomorphisms*

$$H^j(X; \mathcal{E}^*(X), A^*) \xrightarrow{\sim} H^j(X; \mathcal{E}^*(X), B^*) .$$

This isomorphism is induced by the differential operator

$$M_j(x, D): \mathcal{E}^{(j)}(X) \to \mathcal{E}^{(j)}(X) .$$

c) Let S denote a closed subset of X. We set

$$\mathcal{F}_S^{(j)}(X) = \{s(x) \in \mathcal{E}^{(j)}(X) | s(x) \text{ is flat on } S\} .$$

Let $x^0 \in S \cap U_i$ and let us represent $s(x) \in \mathcal{E}^{(j)}(X)$ locally near x^0 by a set of C^∞ functions

$$s_i(x): U_i \to \mathbf{C}^{p_j} .$$

We say that $s(x)$ is flat at x^0 if all partial derivatives of $s_i(x)$ vanish at x^0;

$$D^\alpha s_i(x_0) = 0 \qquad \forall \alpha \in \mathbf{N}^n .$$

We say that $s(x)$ is flat on S if it is flat at every point $x^0 \in S$.

The differential operator $A^j(x, D)$ sends $\mathcal{F}_S^{(j)}(X)$ into $\mathcal{F}_S^{(j+1)}(X)$. We thus obtain a subcomplex of (3)

$$\mathcal{F}_S^{(0)}(X) \xrightarrow{A^0(x, D)} \mathcal{F}_S^{(1)}(X) \xrightarrow{A^1(x, D)} \mathcal{F}_S^{(2)}(X) \xrightarrow{A^2(x, D)} \dots \tag{7}$$

whose cohomology groups will be denoted by

$$H^j(X; \mathcal{F}_S^*(X), A^*) , \qquad j = 0, 1, 2, \dots .$$

Taking the quotient complex of (3) by (7) we obtain the complex

$$\frac{\mathcal{E}^{(0)}(X)}{\mathcal{F}_S^{(0)}(X)} \xrightarrow{A^0(x, D)} \frac{\mathcal{E}^{(1)}(X)}{\mathcal{F}_S^{(1)}(X)} \xrightarrow{A^1(x, D)} \frac{\mathcal{E}^{(2)}(X)}{\mathcal{F}_S^{(2)}(X)} \xrightarrow{A^2(x, D)} \dots \tag{8}$$

where we have denoted by $A^j(x, D)$ the operators induced by the differential operators $A^j(x, D)$ on the quotient spaces. The cohomology groups of the complex (8) will be denoted by

$$H^j(X;\, \mathscr{E}^*(X)/\mathscr{F}^*_S(X),\, A^*) \qquad j = 0, 1, 2, \ldots .$$

We have the following straightforward

PROPOSITION 3. – *If the complex* (6) *is obtained from the complex* (3) *by fiber transformations then for every* $j \geq 0$ *we have also natural isomorphisms*

$$H^j(X;\, \mathscr{F}^*_S(X),\, A^*) \simeq H^j(X;\, \mathscr{F}^*_S(X),\, B^*)$$
$$H^j(X;\, \mathscr{E}^*(X)/\mathscr{F}^*_S(X),\, A^*) \simeq H^j(X;\, \mathscr{E}^*(X)/\mathscr{F}^*_S(X),\, B^*)\, .$$

5. – Noncharacteristic hypersurfaces.

a) Let

$$\varrho\colon X \to \mathbf{R}$$

be a C^∞ function on X, real valued. We consider the set

$$S = \{x \in X | \varrho(x) = 0\}\, .$$

This is a closed set. We say that S is a *hypersurface* if at each point $x^0 \in S$ we have

$$d\varrho(x^0) \neq 0\, .$$

If this is so at each point $x^0 \in S$ we can select a system of local C^∞ coordinates $x_1, \ldots, x_n$ where $x_1 = \varrho(x)$. Therefore in a small neighborhood U of x^0 we have

$$S = \{x \in U | x_1 = 0\}\, .$$

One could define a hypersurface S as a closed subset $S \subset X$ with the property that for each point $x^0 \in S$ we can find an open neighborhood U of x^0 and a C^∞ function $\varrho_U\colon U \to \mathbf{R}$ with the properties

$$d\varrho_U \neq 0 \qquad \text{on } U\, ,$$
$$S \cap U = \{x \in U | \varrho_U(x) = 0\}\, .$$

Assume we have a hypersurface S in this second sense and let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X by coordinate charts in which S is defined by the local equations $\{\varrho_i(x) = 0\}$. One verifies that on $U_i \cap U_j$ we have

$$\varrho_i(x) = h_{ij}(x)\,\varrho_j(x)$$

with $h_{ij}(x)$ C^∞ and $h_{ij}(x) \neq 0$.

We say that the hypersurface S is *orientable* if the covering $\mathcal{U} = \{U_i\}_{i\in I}$ (that we will suppose locally finite) and the local equations $\{\varrho_i(x) = 0\}_{i\in I}$ can be so chosen that $h_{ij}(x) > 0$, $\forall i, j \in I$.

We then claim that *then and only then the hypersurface S can be defined by a global equation $\{\varrho(x) = 0\}$ as in the first definition.*

PROOF. Assume S is orientable. Then if the local equations are properly chosen $h_{ij} > 0$ and thus one can consider $\log h_{ij}(x)$ for $x \in U_i \cap U_j$ as a uniquely defined real valued function. We have con $U_i \cap U_j \cap U_k$

$$\log h_{ij}(x) + \log h_{jk}(x) = \log h_{ik}(x)\,.$$

In particular for $i = k$

$$\log h_{ij}(x) = -\log h_{ij}(x)\,.$$

Let $\{\sigma_\alpha\}_{\alpha\in I}$ be a C^∞ partition of unity subordinated to the open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$. Set on U_i

$$\mu_i(x) = \sum_\alpha \sigma_\alpha(x) \log h_{\alpha i}(x)\,.$$

This has meaning as the covering $\mathcal{U}$ is locally finite. Then we have on $U_i \cap U_j$

$$\begin{aligned} \mu_j(x) - \mu_i(x) &= \sum_\alpha \sigma_\alpha(x)\{\log h_{\alpha j}(x) + \log h_{i\alpha}(x)\} \\ &= \log h_{ij}(x)\,. \end{aligned}$$

Thus $h_{ij}(x) = e^{\mu_j(x)} e^{-\mu_i(x)}$ and therefore on $U_i \cap U_j$

$$\varrho_i(x)\, e^{\mu_i(x)} = \varrho_j(x)\, e^{\mu_j(x)} = \varrho(x)$$

is a globally defined real valued C^∞ function defining S. The converse statement is obvious (take $\mathcal{U} = \{X\}$ and the unique local equation $\varrho(x) = 0$).

As we will consider only oriented hypersurfaces we will stick to the first definition. In this case X is divided into two distinct regions (closed in X)

$$X^+ = \{x \in X \,|\, \varrho(x) \geqslant 0\} \qquad \text{and} \qquad X^- = \{x \in X \,|\, \varrho(x) \leqslant 0\}$$

having only the hypersurface S in common.

b) Suppose now that we have given on X a complex of differential operators

$$\mathscr{E}^{(0)}(X) \xrightarrow{A^0(x,\,D)} \mathscr{E}^{(1)}(X) \xrightarrow{A^1(x,\,D)} \mathscr{E}^{(2)}(X) \xrightarrow{A^2(x,\,D)} \dots \tag{3}$$

with gradings on the fiber bundles E^j, $j = 0, 1, 2, \dots$, compatible with the given differential operators.

One can then consider the corresponding symbolic complex

$$\mathscr{P}(X, E^0) \xrightarrow{\hat{A}^0(x,\,\xi)} \mathscr{P}(X, E^1) \xrightarrow{\hat{A}^1(x,\,\xi)} \mathscr{P}(X, E^2) \xrightarrow{\hat{A}^2(x,\,\xi)} \dots \tag{4}$$

Let S be an oriented hypersurface in X and let $\varrho(x) = 0$ be an equation for S. At each point $x^0 \in S$ the vector

$$\operatorname{grad} \varrho(x^0) = \left(\frac{\partial \varrho}{\partial x_1}(x_0), \dots, \frac{\partial \varrho}{\partial x_n}(x^0)\right)$$

is defined. Another choice of the equation of S changes the vector grad $\varrho(x^0)$ by multiplication by a nonvanishing factor.

We will say that the hypersurface S is *noncharacteristic for the given complex* (3) *at the point* $x^0 \in S$, if the sequence

$$0 \to \mathbf{C}^{p_0} \xrightarrow{\hat{A}^0(x^0,\, \operatorname{grad} \varrho(x^0))} \mathbf{C}^{p_1} \xrightarrow{\hat{A}^1(x^0,\, \operatorname{grad} \varrho(x^0))} \mathbf{C}^{p_2} \xrightarrow{\hat{A}^2(x^0,\, \operatorname{grad} \varrho(x^0))} \dots$$

is an exact sequence.

Let $x^0 \in U_i \in \mathscr{U}$ and let on U_i

$$\begin{array}{ll} a_1^{(i)}, \dots, a_{p_0}^{(i)} & \text{be the grading for } E^0, \\ b_1^{(i)}, \dots, b_{p_1}^{(i)} & \text{be the grading for } E^1, \\ \dots\dots\dots\dots & \end{array}$$

Choose an integer $l \geqslant \alpha_\alpha^{(i)}$, $\forall \alpha$, $l \geqslant b_\alpha^{(i)}$, $\forall \alpha$, ... and let $\sigma(x)$ be another equation for S. Let

$$\operatorname{grad} \sigma(x^0) = \lambda_0 \operatorname{grad} \varrho(x^0)$$

with $\lambda_0 > 0$. We have then a commutative diagram

$$\begin{array}{ccccc} \mathbf{C}^{p_0} & \xrightarrow{\hat{A}^0(x^0,\, \operatorname{grad} \varrho(x^0))} & \mathbf{C}^{p_1} & \xrightarrow{\hat{A}^1(x^0,\, \operatorname{grad} \varrho(x^0))} & \mathbf{C}^{p_2} \to \dots \\ \downarrow N_0(\lambda_0) & & \downarrow N_1(\lambda_0) & & \downarrow N_2(\lambda_0) \\ \mathbf{C}^{p_0} & \xrightarrow{\hat{A}^0(x^0,\, \operatorname{grad} \sigma(x^0))} & \mathbf{C}^{p_1} & \xrightarrow{\hat{A}^1(x^0,\, \operatorname{grad} \sigma(x^0))} & \mathbf{C}^{p_2} \to \dots \end{array}$$

where

$$N_0(\lambda_0) = \operatorname{diag}\langle \lambda_0^{l-a_1}, \ldots, \lambda_0^{l-a_{p_0}}\rangle\,,$$
$$N_1(\lambda_0) = \operatorname{diag}\langle \lambda_0^{l-b_1}, \ldots, \lambda_0^{l-b_{p_1}}\rangle\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

This shows that the definition is independent of the choice of the equation of S.

c) Let

$$(6)\qquad \mathscr{E}^{(0)}(X) \xrightarrow{B^0(x,\,D)} \mathscr{E}^{(1)}(X) \xrightarrow{B^1(x,\,D)} \mathscr{E}^{(2)}(X) \xrightarrow{B^2(x,\,D)} \ldots$$

be another complex of differential operators between the same fiber bundles as before and with a new grading compatible with the differential operators $B^j(x, D)$.

Suppose that the complex (6) is obtained from the complex (3) by means of the fiber transformations $M_0(x, D)$, $M_1(x, D)$, $M_2(x, D)$, ...

From the commutative diagram (5) (see previous section) we derive then the commutative diagram:

$$(9)\qquad \begin{array}{ccccccc} \mathscr{P}(X, E^0) & \xrightarrow{\hat{A}^0(x,\,\xi)} & \mathscr{P}(X, E^1) & \xrightarrow{\hat{A}^1(x,\,\xi)} & \mathscr{P}(X, E^2) & \xrightarrow{\hat{A}^2(x,\,\xi)} & \ldots \\ \big\downarrow{\scriptstyle M_0(x,\,\xi)} & & \big\downarrow{\scriptstyle M_1(x,\,\xi)} & & \big\downarrow{\scriptstyle M_2(x,\,\xi)} & & \\ \mathscr{P}(X, E^0) & \xrightarrow{\hat{B}^0(x,\,\xi)} & \mathscr{P}(X, E^1) & \xrightarrow{\hat{B}^1(x,\,\xi)} & \mathscr{P}(X, E^2) & \xrightarrow{\hat{B}^2(x,\,\xi)} & \ldots \end{array}$$

Let us recall that at every point $x \in X$, $\det \hat{M}_j(x, \xi)$ is independent of ξ and different from zero.

From the commutative diagram (9) taking $x = x^0 \in S$ and $\xi = \operatorname{grad} \varrho(x^0)$ we derive the commutative diagram

$$(10)\qquad \begin{array}{ccccccc} \mathbb{C}^{p_0} & \xrightarrow{\hat{A}^0(x^0,\,\operatorname{grad}\varrho(x^0))} & \mathbb{C}^{p_1} & \xrightarrow{\hat{A}^1(x^0,\,\operatorname{grad}\varrho(x^0))} & \mathbb{C}^{p_2} & \xrightarrow{\hat{A}^2(x^0,\,\operatorname{grad}\varrho(x^0))} & \ldots \\ \big\downarrow{\scriptstyle M_0(x^0,\,\operatorname{grad}\varrho(x^0))} & & \big\downarrow{\scriptstyle M_1(x^0,\,\operatorname{grad}\varrho(x^0))} & & \big\downarrow{\scriptstyle M_2(x^0,\,\operatorname{grad}\varrho(x^0))} & & \\ \mathbb{C}^{p_0} & \xrightarrow{\hat{B}^0(x_0,\,\operatorname{grad}\varrho(x_0))} & \mathbb{C}^{p_1} & \xrightarrow{\hat{B}^1(x_0,\,\operatorname{grad}\varrho(x^0))} & \mathbb{C}^{p_2} & \xrightarrow{\hat{B}^2(x^0,\,\operatorname{grad}\varrho(x^0))} & \ldots \end{array}$$

where the vertical arrows are isomorphisms. We have therefore the following

PROPOSITION 4. – *Assume that at the point $x^0 \in S$ the hypersurface S is noncharacteristic for the complex of differential operators* (3).

If (6) *is another complex of differential operators obtained form the complex* (3) *by a (graded) fiber transformation then the hypersurface S is also noncharacteristic for this new complex at the same point $x^0 \in S$.*

A hypersurface S in X is called *noncharacteristic* if it is noncharacteristic at each one of its points (with respect to a given complex of differential operators).

If the given complex is elliptic at every point of X then any hypersurface is noncharacteristic. Conversely if any hypersurface of X is noncharacteristic the given complex must be elliptic at every point of X.

6. – Formally noncharacteristic hypersurfaces.

a) The local situation. We consider a coordinate patch on X identified by its chart with an open set $\Omega \subset \mathbb{R}^n$. On Ω we have a C^∞ function $\varrho\colon \Omega \to \mathbb{R}$ and we consider the set

$$S_\Omega = \{x \in \Omega | \varrho(x) = 0\} .$$

We assume that $d\varrho \neq 0$ on S_Ω so that S_Ω is a smooth hypersurface. Finally we replace Ω by another open set relatively compact in Ω.

LEMMA 1. – *We can find an open neighborhood U of S_Ω in Ω and a new C^∞ function $t\colon U \to \mathbb{R}$ with*

$$S_\Omega = \{x \in U | t(x) = 0\} .$$

$$dt \neq 0 \qquad \text{on } U ,$$

such that on U we have identically

$$\sum \left(\frac{\partial t}{\partial x_i}\right)^2 = 1 .$$

PROOF. – On some neighborhood U of S_Ω in Ω we have

$$\sum \left(\frac{\partial \varrho}{\partial x_i}\right)^2 > 0 .$$

Replacing Ω by U and ϱ by $\{\sum (\partial\varrho/\partial x_i)^2\}^{-\frac{1}{2}}\varrho$ we may assume that on S_Ω we have

$$\sum \left(\frac{\partial \varrho}{\partial x_i}\right)^2 \Bigg|_{S_\Omega} = 1 .$$

We consider now the following set of equations

$$(*) \qquad \begin{cases} t \dfrac{\partial \varrho}{\partial x_i}(s_1, \ldots, s_n) + s_i = x_i , & 1 \leqslant i \leqslant n , \\ \varrho(s_1, \ldots, s_n) = \varrho . \end{cases}$$

The Jacobian determinant

$$\det \frac{\partial(x_1, \ldots, x_n, \varrho)}{\partial(s_1, \ldots, s_n, t)}$$

equals, up to the sign, the quadratic form

$$\sum_{i,j=1}^{n} \left(\delta_{ij} + t \frac{\partial^2 \varrho(s)}{\partial x_i \partial x_j}\right) \frac{\partial \varrho(s)}{\partial x_i} \frac{\partial \varrho(s)}{\partial x_j} .$$

For $s \in S_\Omega$ and $t = 0$ this form is different from zero. Thus it remains different from zero in an open neighborhood U of S_Ω in Ω for $|t| < \varepsilon_U$ for $\varepsilon_U > 0$ conveniently small.

We can then solve equations $(*)$ with functions

$$t = t(x, \varrho) , \qquad s_j = s_j(x, \varrho) , \qquad 1 \leqslant j \leqslant n ,$$

defined for x in a small neighborhood U of S_Ω in Ω and for $|\varrho| < \eta_U$ with $\eta_U > 0$ conveniently small.

We consider now the functions defined on U

$$t = t(x, 0) \qquad \text{and} \qquad s_j = s_j(x, 0) .$$

We have identically

$$(**) \qquad \begin{cases} t(x, 0) \dfrac{\partial \varrho}{\partial x_i} \big(s(x, 0)\big) + s_i(x, 0) = x_i , & 1 \leqslant i \leqslant n , \\ \varrho\big(s(x, 0)\big) = 0 . \end{cases}$$

We first remark that from the nature of these equations

(α) the quantities

$$x_i - s_i(x, 0) = t(x, 0) \frac{\partial \varrho}{\partial x_i} \big(s(x, 0)\big)$$

are proportional to the quantities $(\partial \varrho / \partial x_i)\big(s(x, 0)\big)$.

Secondly from the identities on U

$$\varrho\big(s(x, 0)\big) \equiv 0 ,$$
$$\sum \left(\frac{\partial \varrho}{\partial x_i} \big(s(x, 0)\big)\right)^2 \equiv 1$$

we derive that

$$(\beta) \qquad \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i} \big(s(x, 0)\big) \frac{\partial s_i}{\partial x_j} \equiv 0 , \qquad 1 \leqslant j \leqslant n ,$$

and

$$(\gamma) \qquad \sum_{i,h=1}^{n} \frac{\partial \varrho}{\partial x_i}(s(x,0))\frac{\partial^2 \varrho}{\partial x_i\,\partial x_h}(s(x,0))\frac{\partial s_h}{\partial x_j}(x,0) \equiv 0\,, \qquad 1 \leqslant j \leqslant n\,.$$

From the equations (**) we derive also that

$$t(x,0) = \sum (x_i - s_i(x,0))\frac{\partial \varrho}{\partial x_i}(s(x,0))\,.$$

Taking partial derivatives we get

$$\frac{\partial t(x,0)}{\partial x_j} = \frac{\partial \varrho}{\partial x_j}(s(x,0))$$

because of (α), (β) and (γ). Therefore identically on U

$$\sum\left(\frac{\partial t(x,0)}{\partial x_j}\right)^2 \equiv 1\,.$$

We thus have that the function

$$t = t(x,0)$$

satisfies the desired requirements. Note that $t(x,0)$ vanishes on S_Ω and its gradient is different from zero.

REMARK. – The hypersurfaces $t =$ constant are hypersurfaces on U parallel to S_Ω.

Restricting eventually Ω we may assume without loss of generality that on Ω the function ϱ satisfies the condition

$$\sum\left(\frac{\partial \varrho}{\partial x_i}\right)^2 \equiv 1\,.$$

b) We introduce on Ω the differential operators (vector fields)

$$\frac{\partial}{\partial \varrho} = \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(x)\frac{\partial}{\partial x_i}\,,$$

$$D_{t_j} = \frac{\partial}{\partial x_j} - \frac{\partial \varrho}{\partial x_j}(x)\frac{\partial}{\partial \varrho}\,, \qquad 1 \leqslant j \leqslant n\,.$$

We have the following formulae

$$\text{(i)} \qquad D_{t_j}(\varrho h) = \varrho D_{t_j} h\,, \qquad \forall h \in \mathscr{E}(\Omega)\,,$$

(so shat D_{t_j} is a tangential operator to S in the sense that it sends the ideal $\mathscr{I}(S)$ of C^∞ functions vanishing on S into itself);

(ii) $$\left[D_{t_j}, \frac{\partial}{\partial\varrho}\right] = D_{t_j}\circ\frac{\partial}{\partial\varrho} - \frac{\partial}{\partial\varrho}\circ D_{t_j} = \sum_{i=1}^{n} \frac{\partial^2\varrho}{\partial x_i\,\partial x_j} D_{t_i}\,,$$

(iii) $$\sum_{i=1}^{n} \frac{\partial\varrho}{\partial x_j} D_{t_j} = 0\,.$$

Let dS denote the Euclidian element of hypersurface area on S. We have that

$$\frac{\partial\varrho}{\partial x_\alpha}\,dS = (-1)^{\alpha-1}\,dx_1 \ldots \widehat{dx_\alpha} \ldots dx_n|_S$$

therefore

(iv) $$dS = \sum (-1)^{\alpha-1}\frac{\partial\varrho}{\partial x_\alpha}\,dx_1 \ldots \widehat{dx_\alpha} \ldots dx_n|_S\,.$$

Given a differential operator on Ω

$$A(x, D)\colon\ \mathscr{E}^p(\Omega) \to \mathscr{E}^q(\Omega)\,,$$
$$A(x, D) = \sum_{|\alpha|\leqslant k} a_\alpha(x)\,D^\alpha$$

with $a_\alpha(x)$ matrices of type $q\times p$ with C^∞ entries one can consider the (formal) adjoint operator

$$A^*(x, D)\colon\ \mathscr{E}^q(\Omega) \to \mathscr{E}^p(\Omega)\,,$$
$$A^*(x, D) = \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}D^\alpha({}^t a_\alpha(x)\cdot)\,.$$

On has the following formula

$$\int_\Omega {}^t V A(x, D)\,U\,dx = \int_\Omega {}^t\big(A^*(x, D)\,V\big)\,U\,dx$$

for $U \in \mathscr{D}^p(\Omega)$, $V \in \mathscr{D}^q(\Omega)$ (i.e. C^∞ with compact support) and where $dx = dx_1 \ldots dx_n$.

For instance we have for the adjoint of $\partial/\partial\varrho$:

(v) $$\nabla_\varrho = -\left(\frac{\partial}{\partial\varrho}\right)^* = \frac{\partial}{\partial\varrho} + \Delta\varrho$$

where

$$\Delta\varrho = \sum\frac{\partial^2\varrho}{\partial x_i^2}\,.$$

We have the following identity

(vi) $${}^t\!\left(\frac{\partial V}{\partial\varrho}\right) U = \sum\frac{\partial}{\partial x_i}\left(\frac{\partial\varrho}{\partial x_i}\,{}^t V U\right) - {}^t V\nabla_\varrho U$$

for U, $V \in \mathscr{E}^s(\Omega)$.

Also we have for the adjoint of the « tangential » operators D_{t_j}

(vii) $$(D_{t_j})^* = -D_{t_j}\,.$$

c) Given a differential operator $A(x, D)$, using the above formulas, and the fact that $A^{**}(x, D) \equiv A(x, D)$, we can always write $A(x, D)$ in one of the forms

$$A(x, D) = A_0(x, D_t) + A_1(x, D_t)\frac{\partial}{\partial\varrho} + \dots + A_k(x, D_t)\frac{\partial^k}{\partial\varrho^k}\,,$$

$$A(x, D) = A_0(x, D_t) + \nabla_\varrho A_1(x, D_t) + \dots + \nabla_\varrho^k A_k(x, D_t)\,,$$

where the $A_j(x, D_t)$ are operators containing only the « tangential derivatives » D_{t_h}.

Let

$$\Omega^- = \{x \in \Omega | \varrho(x) \leqslant 0\}\,.$$

Let $v \in \mathcal{D}^q(\Omega)$ and $u \in \mathcal{D}^p(\Omega)$. We have the following properties

(α) Let $A(x, D_t)$: $\mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)$ be a tangential operator and let $A^*(x, D_t)$: $\mathcal{E}^q(\Omega) \to \mathcal{E}^p(\Omega)$ be its adjoint. Then

$$\int_{\Omega^-} {}^t v A(x, D_t)\, u\, dx = \int_{\Omega^-} {}^t\big(A^*(x, D_t)\, v\big)\, u\, dx\,.$$

This formula is easy to verify for an operator of the form $a(x)D_{t_i}$ and thus in general.

(β) Let the operator $A(x, D)$ be written in the form

$$\begin{aligned} A(x, D) &= A_0(x, D_t) + \nabla_\varrho A_1(x, D_t) + \dots + \nabla_\varrho^k A_k(x, D_t)\\ &= A_0(x, D_t) + \nabla_\varrho C_1(x, D)\\ &= A_0(x, D_t) + \nabla_\varrho A_1(x, D_t) + \nabla_\varrho^2 C_2(x, D_t)\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &= A_0(x, D_t) + \nabla_\varrho A_1(x, D_t) + \dots + \nabla_\varrho^k C_k(x, D) \end{aligned}$$

where

$$\begin{aligned} C_1(x, D) &= A_1(x, D_t) + \nabla_\varrho A_2(x, D_t) + \dots + \nabla_\varrho^{k-1} A_k(x, D_t)\\ C_2(x, D) &= A_2(x, D_t) + \nabla_\varrho A_3(x, D_t) + \dots + \nabla_\varrho^{k-2} A_k(x, D_t)\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ C_k(x, D) &= A_k(x, D_t)\,. \end{aligned}$$

The following formula is then valid (*Green's formula*)

$$\int_{\Omega^-} {}^t v A(x, D)\, u\, dx = \int_{\Omega^-} {}^t(A^*(x, D)v)u\, dx + \sum_{i=1}^{k}(-1)^{i-1}\int_S {}^t\left(\frac{\partial^{i-1}v}{\partial\varrho^{i-1}}\right) C_i(x, D)\, u\, dS\,.$$

The proof follows from repeated application of formula (vi):

$$\int_{\Omega^-} {}^t v A(x, D) u\, dx = \int_{\Omega^-} {}^t v\big(A_0(x, D_t) + \nabla_\varrho C_1(x, D)\big) u\, dx =$$

$$= \int_{\Omega^-} {}^t(A_0^*(x, D_t)v)u\, dx - \int_{\Omega^-} {}^t\left(\frac{\partial v}{\partial\varrho}\right) C_1(x, D) u\, dx + \int_S {}^t v C_1(x, D) u\, dS =$$

$$= \int_{\Omega^-} {}^t(A_0^*(x, D_t)v)u\, dx - \int_{\Omega^-} {}^t\left(\frac{\partial v}{\partial\varrho}\right)\big(A_1(x, D_t) + \nabla_\varrho C_2(x, D)\big)\, u\, dx + \int_S {}^t v C_1(x, D) u\, dS$$

$$= \int_{\Omega^-} {}^t\left\{\left(A_0^*(x, D_t) - A_1^*(x, D_t)\frac{\partial}{\partial\varrho}\right)v\right\} u\, dx + \int_{\Omega^-} {}^t\left(\frac{\partial^2 v}{\partial\varrho^2}\right) C_2(x, D) u\, dx +$$

$$+ \int_S {}^t v C_1(x, D)\, u\, dS - \int_S {}^t\left(\frac{\partial v}{\partial\varrho}\right) C_2(x, D) u\, dS = \dots\,.$$

d) The sheaf $\mathscr{I}_A(\boldsymbol{S})$. Let be given on X two vector bundles E and F and a differential operator

$$A(x, D)\colon \Gamma(X, E) \to \Gamma(X, F)\,.$$

Let S be an oriented hypersurface on X with the equation $\{\varrho(x) = 0\}$ where $\varrho\colon X \to \mathbb{R}$ is C^∞ and $d\varrho|S \neq 0$. We set

$$\Omega^- = \{x \in X|\varrho(x) \leqslant 0\}\,, \qquad \Omega^+ = \{x \in X|\varrho(x) \geqslant 0\}\,.$$

Let E^* and F^* denote the dual vector bundles of E and F respectively. If on the open covering $\mathscr{U} = \{U_i\}_{i\in I}$ E and F are given respectively by the transition functions e_{ij} and f_{ij} then E^* and F^* are given respectively by the transition functions ${}^te_{ij}^{-1}$ and ${}^tf_{ij}^{-1}$. Let $n = \dim X$ and let Ω^n denote the bundle of differential n-forms on X. If $T^*(X)$ is the cotangent bundle of X then with usual notations we have $\Omega^n = \Lambda^n T^*X$.

There is a uniquely defined differential operator

$$A^*(x, D)\colon \Gamma(X, F^* \otimes \Omega^n) \to \Gamma(X, E^* \otimes \Omega^n)$$

with the following property

$$\forall v\, dx \in \Gamma(X, F^* \otimes \Omega^n) \qquad \operatorname{supp}(v\, dx)\ \text{compact}\,,$$

$$\forall u \in \Gamma(X, E) \qquad \operatorname{supp}(u)\ \text{compact}\,,$$

we have

$$\int_X \langle v, A(x, D)\, u\rangle\, dx = \int_X \langle A^*(x, D)\, v, u\rangle\, dx$$

where $\langle\cdot,\cdot\rangle\, dx$ denotes the natural bilinear form of duality

$$\langle\cdot,\cdot\rangle\, dx\colon\ \Gamma(X, F)\times\Gamma(X, F^*\otimes\Omega^n)\to\Gamma(X, \Omega^n)\,,$$

$$\langle\cdot,\cdot\rangle\, dx\colon\ \Gamma(X, E)\times\Gamma(X, E^*\otimes\Omega^n)\to\Gamma(X, \Omega^n)\,.$$

The operator $A^*(x, D)$ is called the (*formal*) *adjoint* of the differential operator $A(x, D)$.

If $x^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)})$ are coordinates on U_i and if $A^{*(i)}(x, D)$ is the local expression of the operator $A^*(x, D)$ in those coordinates and if $A^{(i)}(x, D)$ is the local expression of the operator $A(x, D)$ in those same coordinates, then $A^{*(i)}(x, D)$ is the formal adjoint of $A^{(i)}(x, D)$ and on $U_i\cap U_j$ we have

$$A^{*(i)}(x, D)\circ {}^t f_{ij}^{-1} \det\frac{\partial x^{(j)}}{\partial x^{(i)}} = \det\frac{\partial x^{(j)}}{\partial x^{(i)}}\, {}^t e_{ij}^{-1}\, A^{*(j)}(x, D)\,.$$

Let now U be an open set in X and let

$$u\in\Gamma(U, E)\,;$$

we will say that u is in the domain of $A(x, D)$ along S or that u has zero Cauchy data on S for $A(x, D)$, if for every

$$\varphi\, dx\in\Gamma(U, F^*\otimes\Omega^n)\,,\qquad \operatorname{supp}\varphi\, dx \text{ compact in } U$$

we have

$$\int_{\Omega^-} \langle \varphi, A(x, D)\, u\rangle\, dx = \int_{\Omega^-} \langle A^*(x, D)\,\varphi, u\rangle\, dx\,.$$

We denote by $\mathscr{I}_A(S, U)$ the vector space

$$\mathscr{I}_A(S, U) = \{u\in\Gamma(U, E)\,|\,u \text{ has zero Cauchy data on } S \text{ for } A(x, D)\}\,.$$

If $V\subset U$ is open we have an obvious restriction map

$$\mathscr{I}_A(S, U)\to\mathscr{I}_A(S, V)\,.$$

One verifies readily that

$$U\to\mathscr{I}_A(S, U)$$

is not only a presheaf but a sheaf (denoted by $\mathscr{I}_A(\boldsymbol{S})$).

If U is covered by the chart $x = (x_1, \ldots, x_n)$.

If the equation ϱ of S in U is chosen as specified in point *a*) above.

If $A(x, D)$ has the local expression on U given in (β) of point *c*) above.

If $u = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}$ represents a local section on U of the bundle E

then $u \in \mathcal{I}_A(S, U)$ if and only if

$$C_i(x, D)\, u|_S = 0\,, \qquad 1 \leqslant i \leqslant k\,.$$

If $A(x, D)$ is a differential operator of total order k then $C_i(x, D)$ is a differential operator of order $k - i$.

In particular let

$$\mathcal{F}_S(U) = \{u \in \Gamma(U, E) | u \text{ flat } (^2) \text{ on } S\}\,.$$

From the previous remark it follows that

$$\mathcal{F}_S(U) \subset \mathcal{I}_A(S, U)\,.$$

Also $U \to \mathcal{F}_S(U)$ is a sheaf (denoted by $\mathcal{F}_S$) so that we have an exact sequence of sheaves

$$0 \to \mathcal{F}_S \to \mathcal{I}_A(\mathbf{S}) \to \mathcal{I}_A(\mathbf{S})/\mathcal{F}_S \to 0\,.$$

e) Suppose now that we have given on X a complex of differential operators

$$\mathcal{E}^{(0)}(X) \xrightarrow{A^0(x,\, D)} \mathcal{E}^{(1)}(X) \xrightarrow{A^1(x,\, D)} \mathcal{E}^{(2)}(X) \xrightarrow{A^2(x,\, D)} \ldots \tag{3}$$

where $\mathcal{E}^{(j)}(X) = \Gamma(X, E^j)$.

Let S be an oriented hypersurface in X. We can consider for any $j \geqslant 0$ the spaces $\mathcal{I}_{A^j}(S, X)$. We have with self explaining notations, with $\varphi\, dx$ compactly supported,

$$\int_{\Omega^-} \langle \varphi, A^{j+1} A^j u \rangle\, dx = 0 = \int_{\Omega^-} \langle A^{*j} A^{*j+1} \varphi, u \rangle\, dx$$

$$= \int_{\Omega^-} \langle A^{*j+1} \varphi, A^j u \rangle\, dx \qquad \text{if } u \in \mathcal{I}_{A^j}(S)\,.$$

Therefore

$$A^j(x, D)\, \mathcal{I}_{A^j}(S, X) \subset \mathcal{I}_{A^{j+1}}(S, X)\,.$$

---

(2) By this we mean that for any point $x^0 \in S$ and any chart $x = (x_1, \ldots, x_n)$ at x^0 we have $(D^\alpha u)(x^0) = 0$ $\forall \alpha \in \mathbf{N}^n$.

We have therefore the subcomplexes of (3)

$$\mathscr{I}_{A^0}(S, X) \xrightarrow{A^0(x, D)} \mathscr{I}_{A^1}(S, X) \xrightarrow{A^1(x, D)} \mathscr{I}_{A^2}(S, X) \xrightarrow{A^2(x, D)} \dots \tag{11}$$

and obviously

$$\mathscr{F}^0_S(X) \xrightarrow{A^0(x, D)} \mathscr{F}^1_S(X) \xrightarrow{A^1(x, D)} \mathscr{F}^2_S(X) \xrightarrow{A^2(x, D)} \dots \tag{12}$$

where

$$\mathscr{F}^j_S(X) = \{u \in \mathscr{E}^{(j)}(X) | u \text{ is flat on } S\} .$$

Therefore we have the quotient complex

$$\frac{\mathscr{I}_{A^0}(S, X)}{\mathscr{F}^0_S(X)} \xrightarrow{A^0(x, D)} \frac{\mathscr{I}_{A^1}(S, X)}{\mathscr{F}^1_S(X)} \xrightarrow{A^1(x, D)} \frac{\mathscr{I}_{A^2}(S, X)}{\mathscr{F}^2_S(X)} \xrightarrow{A^2(x, D)} \dots$$

We will say that the hypersurface S is *formally noncharacteristic* for the given complex (3) if the sequence

$$0 \to \frac{\mathscr{I}_{A^0}(S, X)}{\mathscr{F}^0_S(X)} \xrightarrow{A^0(x, D)} \frac{\mathscr{I}_{A^1}(S, X)}{\mathscr{F}^1_S(X)} \xrightarrow{A^1(x, D)} \frac{\mathscr{I}_{A^2}(S, X)}{\mathscr{F}^2_S(X)} \xrightarrow{A^2(x, D)} \dots \tag{13}$$

is an exact sequence.

We have the following

Proposition 5. – *Suppose that the complexes*

$$\mathscr{E}^{(0)}(X) \xrightarrow{A^0(x, D)} \mathscr{E}^{(1)}(X) \xrightarrow{A^1(x, D)} \mathscr{E}^{(2)}(X) \xrightarrow{A^2(x, D)} \dots \tag{3}$$

$$\mathscr{E}^{(0)}(X) \xrightarrow{B^0(x, D)} \mathscr{E}^{(1)}(X) \xrightarrow{B^1(x, D)} \mathscr{E}^{(2)}(X) \xrightarrow{B^2(x, D)} \dots \tag{6}$$

are graded with a classical *grading on each bundle* E^j $(j \geqslant 0)$. *Suppose that* (6) *is obtained from* (3) *by fiber transformations*

$$M_j(x)\colon\ \mathscr{E}^{(j)}(X) \to \mathscr{E}^{(j)}(X)$$

(so that $B^j = M_{j+1} \circ A^j \circ M_j^{-1}$).

Then if the hypersurface S is formally noncharacteristic for the complex (3) *it is also formally noncharacteristic for the complex* (6).

Proof. – Since the grading is classical the bundle E^j must have the same grading with respect to the complexes (3) and (6). Moreover the fiber transformations M_j

are not only of total degree zero but also in each of them each entry is an operator of degree zero; they arise therefore from an isomorphism $M_j\colon E^j \to E^j$. It follows that we have

$$\mathcal{I}_{B^j}(S, X) = M_j \mathcal{I}_{A^j}(S, X)\,.$$

Hence we have commutative diagrams

$$\begin{array}{ccc}
\dfrac{\mathcal{I}_{A^j}(S, X)}{\mathcal{F}^j_S(X)} & \xrightarrow{A^j(x,\, D)} & \dfrac{\mathcal{I}_{A^{j+1}}(S, X)}{\mathcal{F}^{j+1}_S(X)} \\
\wr\big\downarrow M_j & & \wr\big\downarrow M_{j+1} \\
\dfrac{\mathcal{I}_{B^j}(S, X)}{\mathcal{F}^j_S(X)} & \xrightarrow{B^j(x,\, D)} & \dfrac{\mathcal{I}_{B^{j+1}}(S, X)}{\mathcal{F}^{j+1}_S(X)}
\end{array}$$

with vertical isomorphisms. This establishes our proposition.

f) An example. Take $X = \mathbb{R}$ and let $\mathcal{E}(\mathbb{R})$ denote the space of C^∞ functions on $\mathbb{R}$. Let t be a Cartesian coordinate on $\mathbb{R}$ and let $S = \{0\} = \{t = 0\}$.

Consider the following commutative diagram of differential operators

$$\begin{array}{ccccccc}
\mathcal{E}(\mathbb{R}) & \xrightarrow{d/dt} & \mathcal{E}(\mathbb{R}) & \xrightarrow{0} & \mathcal{E}^2(\mathbb{R}) & \xrightarrow{\begin{pmatrix}1 & d/dt\\ 0 & t(d/dt)\end{pmatrix}} & \mathcal{E}^2(\mathbb{R}) \\
\big\uparrow 1 & & \big\uparrow 1 & & \big\uparrow \begin{pmatrix}1 & -d/dt\\ 0 & 1\end{pmatrix} & & \big\uparrow \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\
\mathcal{E}(\mathbb{R}) & \xrightarrow[d/dt]{} & \mathcal{E}(\mathbb{R}) & \xrightarrow[0]{} & \mathcal{E}^2(\mathbb{R}) & \xrightarrow[\begin{pmatrix}1 & 0\\ 0 & t(d/dt)\end{pmatrix}]{} & \mathcal{E}^2(\mathbb{R})
\end{array}$$

The two horizontal rows are complexes that can be considered obtained one from the other with fiber transformations corresponding to convenient nonclassical gradings.

We have

$$\begin{aligned}
&\mathcal{I}_{d/dt}(0, \mathbb{R}) = \{u \in \mathcal{E}(\mathbb{R}) | u(0) = 0\} = t\mathcal{E}(\mathbb{R})\,,\\
&\mathcal{I}_0(0, \mathbb{R}) = \{u \in \mathcal{E}(\mathbb{R})\} = \mathcal{E}(\mathbb{R})\,,\\
&\mathcal{I}_{\begin{pmatrix}1 & d/dt\\ 0 & t(d/dt)\end{pmatrix}}(0, \mathbb{R}) = \{(u, v) \in \mathcal{E}^2(\mathbb{R}) | v(0) = 0\} = \mathcal{E}(\mathbb{R}) \oplus t\mathcal{E}(\mathbb{R})\,,\\
&\mathcal{I}_{\begin{pmatrix}1 & 0\\ 0 & t(d/dt)\end{pmatrix}}(0, \mathbb{R}) = \{(u, v) \in \mathcal{E}^2(\mathbb{R})\} = \mathcal{E}^2(\mathbb{R})\,.
\end{aligned}$$

We denote by Φ_0 the space of formal power series in the variable t.

The sequence (13) reduces then to the following sequence, for the complex of the top horizontal line:

$$0 \to t\Phi_0 \xrightarrow{d/dt} \Phi_0 \xrightarrow{0} \Phi_0 \oplus t\Phi_0 \xrightarrow{\begin{pmatrix}1 & d/dt\\ 0 & t(d/dt)\end{pmatrix}} \Phi_0^2 .$$

This sequence is an exact sequence, hence $S = \{0\}$ is formally noncharacteristic for that complex ([3]).

For the complex of the bottom horizontal line we have instead the sequence

$$0 \to t\Phi_0 \xrightarrow{d/dt} \Phi_0 \xrightarrow{0} \Phi_0^2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & t(d/dt)\end{pmatrix}} \Phi_0^2 .$$

This sequence is not exact at the place before the last.

We conclude that the notion of formally noncharacteristic hypersurface is *not* invariant under fiber transformations of general type (arising from nonclassical gradings of the complex). The assumption in proposition 5 that the grading be classical is therefore essential.

g) We revert to the situation considered in point *e*) above.

Setting

$$Q^{(j)}(S) = \frac{\mathscr{E}^{(j)}(X)}{\mathscr{I}_{A^j}(S, X)}$$

we derive from (3) and (11) a quotient complex of the form

$$Q^{(0)}(S) \xrightarrow{A_S^0} Q^{(1)}(S) \xrightarrow{A_S^1} Q^{(2)}(S) \xrightarrow{A_S^2} \dots \tag{14}$$

where the A_S^j are induced by the differential operators $A^j(x, D)$ but are not necessarily differential operators. They are linear operators between the linear spaces $Q^{(j)}(S)$ and $Q^{(j+1)}(S)$.

The cohomology groups of the complex (14) will be denoted by

$$H^j(S; Q^*(S), A_S^*) .$$

PROPOSITION 6. – *Let* (3) *and* (6) *be graded complexes of differential operators endowed with classical gradings and obtained one from the other by fiber transformations.*

([3]) We tacitly assume that the domain along S of the « empty » operator is the whole space. Thus we have the space Φ_0^2 in the last place. (This may not be a correct view.)

Let S be an oriented hypersurface on X and let

$$Q^{(0)}(S) \xrightarrow{A_S^0} Q^{(1)}(S) \xrightarrow{A_S^1} Q^{(2)}(S) \xrightarrow{A_S^2} \dots$$
$$C^{(0)}(S) \xrightarrow{B_S^0} C^{(1)}(S) \xrightarrow{B_S^1} C^{(2)}(S) \xrightarrow{B_S^2} \dots$$

be the corresponding boundary complexes. Then for $j\geqslant 0$

$$H^j(S; Q^*(S), A_S^*) \simeq H^j(S; C^*(S), B_S^*)$$

with a natural isomorphism induced by the fiber transformations.

PROPOSITION 7. – *Let* (3) *be a given complex of differential operators on X. Let S be an oriented hypersurface on X and let* (4) *be the corresponding boundary complex.*

Assume that S is formally noncharacteristic for the complex (3). *Then we have for any $j\geqslant 0$*

$$H^j(S; Q^*(S), A_S^*) \simeq H^j(X; \mathscr{E}^*(X)/\mathscr{F}_S^*(X), A^*)\,.$$

PROOF. – We have an exact sequence of complexes

$$0 \to \frac{\mathscr{I}^*(S, X)}{\mathscr{F}_S^*(X)} \to \frac{\mathscr{E}^*(X)}{\mathscr{F}_S^*(X)} \to Q^*(S) \to 0\,.$$

By the assumption that S is formally noncharacteristic it follows that the complex $\mathscr{I}^*(S, X)/\mathscr{F}_S^*(X)$ is acyclic in all dimensions (including zero). Therefore $Q^*(S)$ and $\mathscr{E}^*(X)/\mathscr{F}_S^*(X)$ have the same cohomology.

7. – Local canonical form of a graded complex.

a) We want to prove the following local theorem:

THEOREM 1. – *Let Ω be an open set in $\mathbb{R}^n$ and let*

$$\mathscr{E}^{p_0}(\Omega) \xrightarrow{A^0(x, D)} \mathscr{E}^{p_1}(\Omega) \xrightarrow{A^1(x, D)} \mathscr{E}^{p_2}(\Omega) \xrightarrow{A^2(x, D)} \dots \tag{2}$$

be a graded finite complex of differential operators with gradings

$$a_1, \dots, a_{p_0}\,; \quad b_1, \dots, b_{p_1}\,; \quad c_1, \dots, c_{p_2}\,; \dots$$

respectively on $\mathscr{E}^{p_0}(\Omega)$, $\mathscr{E}^{p_1}(\Omega)$, $\mathscr{E}^{p_2}(\Omega)$, ...

Let S be an oriented hypersurface on Ω with equation $\{\varrho(x) = 0\}$.
We assume that at a point $x^0 \in S$, S is noncharacteristic i.e. that the sequence

$$0 \to \mathbb{C}^{p_0} \xrightarrow{\hat{A}_0(x_0,\, \operatorname{grad} \varrho(x_0))} \mathbb{C}^{p_1} \xrightarrow{\hat{A}^1(x_0,\, \operatorname{grad} \varrho(x_0))} \mathbb{C}^{p_2} \xrightarrow{\hat{A}^2(x_0,\, \operatorname{grad} \varrho(x_0))} \dots$$

is an exact sequence.

One can find an open neighborhood ω of x^0 in Ω and graded fiber transformations

$$\begin{array}{ll} M_0(x, D)\colon \mathscr{E}^{p_0}(\omega) \to \mathscr{E}^{p_0}(\omega) & \textit{of grading } (a_j, \alpha_i) \\ M_1(x, D)\colon \mathscr{E}^{p_1}(\omega) \to \mathscr{E}^{p_1}(\omega) & \textit{of grading } (b_j, \beta_i) \\ M_2(x, D)\colon \mathscr{E}^{p_2}(\omega) \to \mathscr{E}^{p_2}(\omega) & \textit{of grading } (c_j, \gamma_i) \\ \dots\dots\dots & \end{array}$$

with

$$\alpha_1, \dots, \alpha_{p_0}; \quad \beta_1, \dots, \beta_{p_1}; \quad \gamma_1, \dots, \gamma_{p_2}; \quad \dots$$

permutations respectively of

$$a_1, \dots, a_{p_0}; \quad b_1, \dots, b_{p_1}; \quad c_1, \dots, c_{p_2}; \quad \dots$$

and such that the transformed complex

$$\mathscr{E}^{p_0}(\omega) \xrightarrow{B^0(x, D)} \mathscr{E}^{p_1}(\omega) \xrightarrow{B^1(x, D)} \mathscr{E}^{p_2}(\omega) \xrightarrow{B^2(x, D)} \dots \tag{6}$$

has the following properties

i) *$S_\omega = \omega \cap S$ is formally noncharacteristic for* (6);

ii) *in the boundary complex of* (6)

$$C^{(0)}(S_\omega) \xrightarrow{B_S^0} C^{(1)}(S_\omega) \xrightarrow{B_S^1} C^{(2)}(S_\omega) \xrightarrow{B_S^2} \dots$$

for each $j \geqslant 0$ we have (denoting by $\mathscr{E}(S_\omega)$ the C^∞ functions on S_ω)

$$C^{(j)}(S_\omega) \simeq \mathscr{E}^{q_j}(S_\omega) \qquad (\textit{some } q_j \geqslant 0)$$

and

$$B_S^j\colon C^{(j)}(S_\omega) \to C^{(j+1)}(S_\omega)$$

is a differential operator;

iii) *the sheaf on ω, $U \to \mathscr{I}_{B^i}(S, U)$ and therefore also the sheaf*

$$U \to \mathscr{I}_{B^i}(S, U)/\mathscr{F}_S^i(U)$$

are soft sheaves.

We derive from this statement that if the grading of the given complex is classical then the considered fiber transformations M_j do not contain the symbols of partial derivations. One can thus apply proposition 5 to the situation considered. We obtain therefore from the above theorem the following

COROLLARY. – *Let there be given on X a graded complex*

$$\mathscr{E}^{(0)}(X) \xrightarrow{A^0(x,\,D)} \mathscr{E}^{(1)}(X) \xrightarrow{A^1(x,\,D)} \mathscr{E}^{(2)}(X) \xrightarrow{A^2(x,\,D)} \dots \tag{2}$$

of differential operators, and let us assume that the grading is classical.

Let S be an oriented hypersurface on X which we assume to be noncharacteristic for the complex (2) *at every one of its points.*

Then

i) *The boundary complex*

$$Q^{(0)}(S) \xrightarrow{A^0_S} Q^{(1)}(S) \xrightarrow{A^1_S} Q^{(2)}(S) \xrightarrow{A^2_S} \dots \tag{14}$$

is a complex of differential operators on the manifold S.

ii) *The sheaves*

$$U \to \mathscr{I}_{A^i}(S, U)$$

and

$$U \to \mathscr{I}_{A^i}(S, U)/\mathscr{F}^i_S(U)$$

are soft sheaves.

iii) *The hypersurface S is formally noncharacteristic for the given complex* (2).

The last statement iii) follows from the statement ii) since we have an exact sequence of *soft* sheaves (cf. [11], theorem 3.5.4, p. 154)

$$0 \to \mathscr{I}_{A^0}(S)/\mathscr{F}^0_S \to \mathscr{I}_{A^1}(S)/\mathscr{F}^1_S \to \mathscr{I}_{A^2}(S)/\mathscr{F}^2_S \to \dots\,.$$

b) PROOF OF THEOREM 1. – (α) Let us assume that x^0 is at the origin of the coordinates and that those are so chosen that $\varrho(x) \equiv x_n$. This can be obtained by replacing Ω, if necessary, by a smaller open neighborhood of x^0.

We set

$$x = (y_1, \dots, y_{n-1}, t)\,; \qquad \xi = (\eta_1, \dots, \eta_{n-1}, \tau)\,,$$
$$y = (y_1, \dots, y_{n-1})\,; \qquad \eta = (\eta_1, \dots, \eta_{n-1})\,,$$

so that the operator $A^i(x, D)$ and its symbol $\hat{A}^i(x, \xi)$ will be denoted also by $A^i(y, t; \partial/\partial y, \partial/\partial t)$ and $\hat{A}^i(y, t; \eta, \tau)$.

By assumption we have an exact sequence

$$0 \to \mathbf{C}^{p_0} \xrightarrow{\hat{A}^0(0, 0; 0, 1)} \mathbf{C}^{p_1} \xrightarrow{A^1(0, 0; 0, 1)} \mathbf{C}^{p_2} \xrightarrow{A^2(0, 0; 0, 1)} \dots .$$

Therefore

$$\begin{aligned} p_0 \quad &= \varrho_0 = \operatorname{rank} \hat{A}^0(0, 0; 0, 1) , \\ p_1 - \varrho_0 &= \varrho_1 = \operatorname{rank} \hat{A}^1(0, 0; 0, 1) \\ p_2 - \varrho_1 &= \varrho_2 = \operatorname{rank} \hat{A}^2(0, 0; 0, 1) \\ &\dots\dots\dots\dots\dots \end{aligned}$$

i.e.

$$p_j = \varrho_j + \varrho_{j-1} \qquad (\varrho_{-1} = 0) , \; j = 0, 1, 2, \dots .$$

Let $\mathscr{R} = \mathbf{C}(\eta, \tau)$ be the field of rotational functions in η and τ. We have for any $j \geqslant 0$

$$\varrho_j \leqslant \hat{\varrho}_j = \operatorname{rank}_{\mathscr{R}} \hat{A}^j(0, 0; \eta, \tau) .$$

Since the sequence

$$0 \to \mathscr{R}^{p_0} \xrightarrow{\hat{A}^0(0, 0; \eta, \tau)} \mathscr{R}^{p_1} \xrightarrow{\hat{A}^1(0, 0; \eta, \tau)} \mathscr{R}^{p_2} \xrightarrow{\hat{A}^2(0, 0; \eta, \tau)} \dots$$

is a complex we derive that for all $j \geqslant 0$

$$\hat{\varrho}_{j-1} + \hat{\varrho}_j \leqslant p_j \qquad (\hat{\varrho}_{-1} = 0) .$$

Therefore

$$\varrho_j = \hat{\varrho}_j \qquad \forall j \geqslant 0 .$$

Since the complex is finite we can then find a small open neighborhood ω of x^0 such that for $x = (y, t) \in \omega$ we have

$$\varrho_j = \operatorname{rank}_{\mathscr{R}} \hat{A}^j(y, t; \eta, \tau) \qquad \forall (y, t) \in \omega .$$

Indeed if $\hat{\varrho}_j(y, t)$ denotes this rank we have $\hat{\varrho}_j = \hat{\varrho}_j(y, t)$ for all j and a convenient ω. On the other hand as before we derive the inequalities

$$\hat{\varrho}_{j-1}(y, t) + \hat{\varrho}_j(y, t) \leqslant p_j \qquad (\hat{\varrho}_{-1}(y, t) = 0) \text{ (cf. [5], lemma 1)} .$$

(β) Let $\mathbb{C}_0[\tau]$ denote the graded ring of homogeneous polynomials in the variable τ. By $\mathcal{M}_{p\times p}(\mathbb{C}_0[\tau])$ we denote the ring of $p\times p$ matrices with entries in $\mathbb{C}_0[\tau]$. By a homogeneous matrix of grading $a_1, \dots, a_p$; $\alpha_1, \dots, \alpha_p$ (abbreviated (a_j, α_i)) in $\mathcal{M}_{p\times p}(\mathbb{C}_0[\tau])$ we mean a matrix $M(\tau)\in\mathcal{M}_{p\times p}(\mathbb{C}_0[\tau])$ with $M(\tau) = (m_{ij}(\tau))$ where the $m_{ij}(\tau)$ are homogeneous polynomials of degree $a_j - \alpha_i$ (the zero polynomial if $a_j - \alpha_i < 0$). The number $\sum a_j - \sum \alpha_j$ is called the total degree of $M(\tau)$. By [5], lemma 2, we can find homogeneous matrices $R(\tau)\in\mathcal{M}_{p\times p}(\mathbb{C}_0[\tau])$ and $L(\tau)\in\mathcal{M}_{p\times p}\cdot(\mathbb{C}_0[\tau])$ of total degree zero and determinant different from zero of gradings (a_j, α_i), (b_j, β_i) as specified in the statement of the theorem, such that

$$L(\tau)\hat{A}^0(0, 0; 0, \tau)R(\tau) = \begin{pmatrix} & 0 & \\ \tau^{k_1} & & \\ & \ddots & \tau^{k_{\varrho_0}} \end{pmatrix}.$$

If we apply the fiber transformations $R(\partial/\partial t)$ to $\mathcal{E}^{p_0}(\Omega)$ and $L^{-1}(\partial/\partial t)$ to $\mathcal{E}^{p_1}(\Omega)$ then

$$A^0\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) \text{ is replaced by } L\left(\frac{\partial}{\partial t}\right)A^0\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)R\left(\frac{\partial}{\partial t}\right),$$

$$A^1\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) \text{ is replaced by } A^1\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)L^{-1}\left(\frac{\partial}{\partial t}\right)$$

and the other operators are unchanged.

We can thus assume that

$$\hat{A}^0(0, 0; 0, \tau) = \begin{pmatrix} & 0 & \\ \tau^{k_1} & & \\ & \ddots & \tau^{k_{\varrho_0}} \end{pmatrix}$$

and that $b_1 \geqslant b_2 \geqslant \dots \geqslant b_{\varrho_0}$.

Then in $\hat{A}^1(0, 0; 0, \tau)$ the last ϱ_0 columns must be zero since

$$\hat{A}^1(0, 0; 0, \tau)\hat{A}^0(0, 0; 0, \tau) \equiv 0 .$$

(γ) We can find homogeneous matrices $R(\tau)\in\mathcal{M}_{\varrho_1\times\varrho_1}(\mathbb{C}_0[\tau])$ and $L(\tau)\in\mathcal{M}_{p_2\times p_2}\cdot(\mathbb{C}_0[\tau])$ of total degree zero and determinant different from zero and of gradings (b_j, β_i) and (c_j, γ_i) as specified in the statement of the theorem such that, by replacing

$$\hat{A}^0(0, 0; 0, \tau) \quad \text{by} \quad \begin{pmatrix} R(\tau)^{-1} & 0 \\ 0 & I \end{pmatrix}\hat{A}^0(0, 0; 0, \tau),$$

$$\hat{A}^1(0, 0; 0, \tau) \quad \text{by} \quad L(\tau)\hat{A}^1(0, 0; 0, \tau)\begin{pmatrix} R(\tau) & 0 \\ 0 & I \end{pmatrix},$$

we have

$$\hat{A}^0(0, 0; 0, \tau) = \begin{pmatrix} & 0 & \\ \tau^{k_1} & & \\ & \ddots & \tau^{k_{\varrho_0}} \end{pmatrix}, \qquad \hat{A}^1(0, 0; 0, \tau) = \begin{pmatrix} & 0 & & 0 \\ \tau^{h_1} & & & \\ & \ddots & \tau^{h_{\varrho_1}} & 0 \end{pmatrix}$$

and

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{\varrho_0}, \qquad c_1 \geqslant c_2 \geqslant \ldots \geqslant c_{\varrho_1}.$$

We apply the fiber transformation $\begin{pmatrix} R(\partial/\partial t) & 0 \\ 0 & I \end{pmatrix}$ to $\mathcal{E}^{p_1}(\Omega)$ and $L^{-1}(\partial/\partial t)$ to $\mathcal{E}^{p_2}(\Omega)$. Then A^0 and A^1 are transformed in the way indicated above, A^2 is replaced by

$$A^2\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) L^{-1}\left(\frac{\partial}{\partial t}\right), \qquad \text{while the other } A^j \text{ are unchanged}.$$

We can therefore assume that $\hat{A}^0$ and $\hat{A}^1$ have the forms indicated above. Moreover in $\hat{A}^2(0, 0; 0, \tau)$ the last ϱ_1 columns will be zero.

Operating with $\hat{A}^1(0, 0; 0, \tau)$ and $\hat{A}^2(0, 0; 0, \tau)$ as we did before with $\hat{A}^0(0, 0; 0, \tau)$ and $\hat{A}^1(0, 0; 0, \tau)$ and so on, we realize that we can assume without loss of generality that

$$\hat{A}^0(0, 0; 0, \tau) = \begin{pmatrix} & 0 & \\ \tau^{k_1} & & \\ & \ddots & \tau^{k_{\varrho_0}} \end{pmatrix}; \qquad \hat{A}^1(0, 0; 0, \tau) = \begin{pmatrix} & 0 & 0 \\ \tau^{h_1} & & \\ & \ddots \tau^{h_{\varrho_1}} & 0 \end{pmatrix};$$

$$\hat{A}^2(0, 0; 0, \tau) = \begin{pmatrix} & 0 & 0 \\ \tau^{l_1} & & \\ & \ddots \tau^{l_{\varrho_2}} & 0 \end{pmatrix}; \quad \ldots$$

with $b_1 \geqslant \ldots \geqslant b_{\varrho_0}$; $c_1 \geqslant \ldots \geqslant c_{\varrho_1}$; ....

(δ) According to [5], lemma 3, in an open neighbourhood ω of the origin in $\mathbb{R}^n$ we can find fiber transformations

M on $\mathcal{E}^{\varrho_0}(\omega)$		of grading « (a_j, α_i) »
N on $\mathcal{E}^{\varrho_1}(\omega)$		of grading « (b_j, β_i) »
P on $\mathcal{E}^{\varrho_2}(\omega)$		of grading « (c_j, γ_i) »
. . . .		

such that the gradings are as specified in the statement of the theorem and with the following properties: write

$$A^j(x, D) = \begin{pmatrix} A_0^{(j)} & A_1^{(j)} \\ A_2^{(j)} & A_3^{(j)} \end{pmatrix}$$

with $A_2^{(j)}$ of type $\varrho_j \times \varrho_j$ (thus

$$A^0(x, D) = \begin{pmatrix} A_0^{(0)} \\ A_2^{(0)} \end{pmatrix}\Bigg);$$

then the matrices

$$MA_2^{(0)},\ NA_2^{(1)},\ PA_2^{(2)},\ \ldots$$

are of the form

$$MA_2^{(0)} = \operatorname{diag}\left\langle\frac{\partial^{k_1}}{\partial t^{k_1}}, \dots, \frac{\partial^{k_{\varrho_0}}}{\partial t^{k_{\varrho_0}}}\right\rangle + R\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$NA_2^{(1)} = \operatorname{diag}\left\langle\frac{\partial^{h_1}}{\partial t^{h_1}}, \dots, \frac{\partial^{h_{\varrho_1}}}{\partial t^{h_{\varrho_1}}}\right\rangle + S\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$PA_2^{(2)} = \operatorname{diag}\left\langle\frac{\partial^{l_1}}{\partial t^{l_1}}, \dots, \frac{\partial^{l_{\varrho_2}}}{\partial t^{l_{\varrho_2}}}\right\rangle + T\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

with $R = (r_{ij})$, $S = (s_{ij})$, $T = (t_{ij})$, ... such that

$$\text{order of } s_{ij} \text{ in } \frac{\partial}{\partial t} < h_j\,, \qquad 1 \leqslant j \leqslant \varrho_1\,,$$

$$\text{order of } r_{ij} \text{ in } \frac{\partial}{\partial t} < k_j\,, \qquad 1 \leqslant j \leqslant \varrho_0\,,$$

$$\text{order of } t_{ij} \text{ in } \frac{\partial}{\partial t} < l_j\,, \qquad 1 \leqslant j \leqslant \varrho_2\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Then the systems $MA_2^{(0)}$, $NA_2^{(1)}$, $PA_2^{(2)}$, ... will lead to well posed Cauchy problems on $t = 0$ in the sense that those systems are in Cauchy-Kowalewska form.

We apply the fiber transformations

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} \quad \text{to} \quad \mathcal{E}^{p_0}(\omega)\,,$$

$$\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \quad \text{to} \quad \mathcal{E}^{p_1}(\omega)\,,$$

$$\begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \quad \text{to} \quad \mathcal{E}^{p_2}(\omega)\,,$$

and thus we replace

$$A^0 \quad \text{with} \quad \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} A^0,$$

$$A^1 \quad \text{with} \quad \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} A^1 \begin{pmatrix} I & 0 \\ 0 & M^{-1} \end{pmatrix},$$

$$A^2 \quad \text{with} \quad \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} A^2 \begin{pmatrix} I & 0 \\ 0 & N^{-1} \end{pmatrix},$$

$$\dots\dots\dots\dots\dots$$

This will not affect the canonical forms already obtained for the symbols $\hat{A}^j(0, 0; 0, \tau)$.

We can thus assume also that

$$A_2^{(0)} = \operatorname{diag}\left\langle \frac{\partial^{k_1}}{\partial t^{k_1}}, \ldots, \frac{\partial^{k_{\varrho_0}}}{\partial t^{k_{\varrho_0}}} \right\rangle + R\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$A_2^{(1)} = \operatorname{diag}\left\langle \frac{\partial^{h_1}}{\partial t^{h_1}}, \ldots, \frac{\partial^{h_{\varrho_1}}}{\partial t^{h_{\varrho_1}}} \right\rangle + S\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$A_2^{(2)} = \operatorname{diag}\left\langle \frac{\partial^{l_1}}{\partial t^{l_1}}, \ldots, \frac{\partial^{l_2}}{\partial t^{l_{\varrho_2}}} \right\rangle + T\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

with $R, S, T, \ldots$ of the form specified above.

(ε) We apply now lemma 5 of [5] and write

$$A_0^{(0)} = Q_0 A_2^{(0)} + R_0 ,$$
$$A_0^{(1)} = Q_1 A_2^{(1)} + R_1 ,$$
$$A_0^{(2)} = Q_2 A_2^{(2)} + R_2 ,$$
$$\cdots\cdots\cdots$$

with Q_j and R_j differential operators (of proper gradings) with

$$R_0 = (r_{ij}^{(0)}) \quad \text{and order of} \quad r_{ij}^{(0)} \text{ in } \frac{\partial}{\partial t} < k_j , \qquad 1 \leq j \leq \varrho_0 ,$$

$$R_1 = (r_{ij}^{(1)}) \quad \text{and order of} \quad r_{ij}^{(1)} \text{ in } \frac{\partial}{\partial t} < h_j , \qquad 1 \leq j \leq \varrho_1 ,$$

$$R_2 = (r_{ij}^{(2)}) \quad \text{and order of} \quad r_{ij}^{(2)} \text{ in } \frac{\partial}{\partial t} < l_j , \qquad 1 \leq j \leq \varrho_2 .$$

Performing the fiber transformations

$$\begin{pmatrix} I & -Q_0 \\ 0 & I \end{pmatrix} \quad \text{on} \quad \mathcal{E}^{p_0}(\omega) ,$$

$$\begin{pmatrix} I & -Q_1 \\ 0 & I \end{pmatrix} \quad \text{on} \quad \mathcal{E}^{p_1}(\omega) ,$$

$$\begin{pmatrix} I & -Q_1 \\ 0 & I \end{pmatrix} \quad \text{on} \quad \mathcal{E}^{p_2}(\omega) ,$$

we realize that

these fiber transformations have the gradings specified in the statement of the theorem; moreover for

$$A_j = \begin{pmatrix} A_0^{(j)} & A_1^{(j)} \\ A_2^{(j)} & A_3^{(j)} \end{pmatrix}, \qquad j = 0, 1, 2, \ldots$$

we can make the assumptions specified in points (γ), (δ), and suppose also that

$$\text{in } A_0^{(0)} = (\alpha_{ij}^{(0)}), \quad \text{order of} \quad \alpha_{ij}^{(0)} \text{ in } \frac{\partial}{\partial t} < k_j, \qquad 1 \leqslant j \leqslant \varrho_0,$$

$$\text{in } A_0^{(1)} = (\alpha_{ij}^{(1)}), \quad \text{order of} \quad \alpha_{ij}^{(1)} \text{ in } \frac{\partial}{\partial t} < h_j, \qquad 1 \leqslant j \leqslant \varrho_1,$$

$$\text{in } A_0^{(2)} = (\alpha_{ij}^{(2)}), \quad \text{order of} \quad \alpha_{ij}^{(2)} \text{ in } \frac{\partial}{\partial t} < l_j, \qquad 1 \leqslant j \leqslant \varrho_2,$$

. . . .

With all these conditions verified, we call the complex in « canonical form ». A complex of differential operators in canonical form is therefore a complex of the following type

$$0 \to \mathcal{E}^{\varrho_0}(\omega) \xrightarrow{A^0} \mathcal{E}^{\varrho_1}(\omega) \oplus \mathcal{E}^{\varrho_0}(\omega) \xrightarrow{A^1} \mathcal{E}^{\varrho_2}(\omega) \oplus \mathcal{E}^{\varrho_1}(\omega) \xrightarrow{A^2} \dots$$

with gradings on the spaces $\mathcal{E}^{\varrho_j}(\omega) \oplus \mathcal{E}^{\varrho_{j-1}}(\omega)$ $\big(\mathcal{E}^{\varrho_{-1}}(\omega) = 0\big)$ compatible with the differential operators

$$A^j\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) : \mathcal{E}^{\varrho_j}(\omega) \oplus \mathcal{E}^{\varrho_{j-1}}(\omega) \to \mathcal{E}^{\varrho_{j+1}}(\omega) \oplus \mathcal{E}^{\varrho_j}(\omega),$$

$$A^j = \begin{pmatrix} A_0^{(j)} & A_1^{(j)} \\ A_2^{(j)} & A_3^{(j)} \end{pmatrix},$$

where $A_2^{(j)}$ is of the type $\varrho_j \times \varrho_j$ with

$$A_2^{(j)}\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) = \operatorname{diag}\left\langle \frac{\partial^{k_1^{(j)}}}{\partial t^{k_1^{(j)}}}, \dots, \frac{\partial^{k_{\varrho_j}^{(j)}}}{\partial t^{k_{\varrho_j}^{(j)}}} \right\rangle + R^{(j)}\left(y, t; \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right),$$

where each entry r_{hl} in $R^{(j)}$ is an operator with

$$\text{order of } r_{hl} \text{ in } \frac{\partial}{\partial t} < k_l^{(j)}$$

and where the operator $A_0^{(j)}$ has every entry α_{hl} with

$$\text{order of } \alpha_{hl} \text{ in } \frac{\partial}{\partial t} < k_l^{(j)}.$$

We set

$$q_j = k_1^{(j)} + \dots + k_{\varrho_j}^{(j)}.$$

It is also assumed for the gradings $\{a_j\}$, $\{b_j\}$, $\{c_j\}, \dots$ that

$$b_1 \geqslant \dots \geqslant b_{\varrho_0}; \qquad c_1 \geqslant c_2 \geqslant \dots \geqslant c_{\varrho_1}; \qquad \dots$$

and that

$$\hat{A}^j(0, 0; 0, \tau) = \begin{pmatrix} 0 & 0 \\ \hat{A}_2^{(j)} & 0 \end{pmatrix}$$

with

$$\hat{A}_2^{(j)}(0, 0; 0, \tau) = \operatorname{diag}\langle \tau^{k_1^{(j)}}, \ldots, \tau^{k_{\varrho_j}^{(j)}} \rangle .$$

In all this picture $S = \{t = 0\}$ is the basic noncharacteristic hypersurface that has determined the canonical form.

(ζ) We consider for every $j \geqslant 0$ the linear map

$$\sigma\colon \mathscr{E}^{\varrho_j}(\omega) \to \mathscr{E}^{\varrho_j}(\omega) \oplus \mathscr{E}^{\varrho_{j-1}}\Omega) ,$$

$$\sigma(u) = u \oplus 0 .$$

LEMMA 2. – *Let*

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_{\varrho_j} \end{pmatrix} \in \mathscr{E}^{\varrho_j}(\omega) .$$

The following are equivalent conditions

i) $f \in \mathscr{I}_{A_2^{(j)}}(S, \omega)$,

ii) $\partial^s f_h / \partial t^s|_{t=0} = 0, \ 0 \leqslant s \leqslant k_h^{(j)} - 1, \ 1 \leqslant h \leqslant \varrho_j$,

iii) $\sigma(f) \in \mathscr{I}_{A^j}(S, \omega)$.

PROOF. – The equivalence of i) and ii) follows from Green's formula.
With the notations used in number 6 *b*), for the operator $A_2^{(j)}$ we have

$$C_1(x, D) = \operatorname{diag}\left\langle \frac{\partial^{k_1^{(j)}-1}}{\partial t^{k_1^{(j)}-1}}, \ldots, \frac{\partial^{k_{\varrho_j}^{(j)}-1}}{\partial t^{k_{\varrho_j}^{(j)}-1}} \right\rangle + \text{lower order} ,$$

$$C_2(x, D) = \operatorname{diag}\left\langle \frac{\partial^{k_1^{(j)}-2}}{\partial t^{k_1^{(j)}-2}}, \ldots, \frac{\partial^{k_{\varrho_j}^{(j)}-2}}{\partial t^{k_{\varrho_j}^{(j)}-2}} \right\rangle + \text{lower order} ,$$

. . . .

where in $C_h(x, D)$ « lower order » means that in the entry (r, s) the order in $\partial/\partial t$ is $< k_s^{(j)} - h$.

It follows then that the conditions

$$C_1(x, D) f|_S = 0 \ , \quad C_2(x, D) f|_S = 0 \ , \quad \ldots$$

are equivalent with the conditions given by ii).

Now let $\sigma(f) \in \mathscr{I}_{A^j}(S, \omega)$. This is equivalent to saying that

$$f \in \mathscr{I}_{A_0^{(j)}}(S, \omega)$$

and

$$f \in \mathscr{I}_{A_2^{(j)}}(S, \omega)\,.$$

Thus iii) $\Rightarrow$ i). But if $f \in \mathscr{I}_{A_2^{(j)}}(S, \omega)$, because of ii) and the canonical form of $A_0^{(j)}$ we deduce that automatically $f \in \mathscr{I}_{A_0^{(j)}}(S, \omega)$. Thus by the above remark $\sigma(f) \in \in \mathscr{I}_{A^j}(S, \omega)$.

Let $\mathscr{E}(S)$ be the space of C^∞ functions on $S = \{(y, t) \in \omega | t = 0\}$. Note that $\mathscr{E}(\omega)/\mathscr{F}_S(\omega) \simeq \mathscr{E}(S)\{\{t\}\}$, the space of formal power series in t with coefficients in $\mathscr{E}(S)$. From the equivalence of conditions i) and ii) in the previous lemma and from the canonical form of $A_2^{(j)}$ we deduce the following

LEMMA 3. – *For any $j \geq 0$, given $f \in (\mathscr{E}(S)\{\{t\}\})^{\varrho_j}$ we can find a unique $u \in (\mathscr{E}(s) \cdot \cdot \{\{t\}\})^{\varrho_j}$ such that*

$$\begin{cases} A_2^{(j)}\left(y, t; \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial t}\right) u = f \\ u \in \mathscr{I}_{A_2^{(j)}}(\omega, S)/\mathscr{F}_S^{\varrho_j}(\omega)\,. \end{cases}$$

LEMMA 4. – *We have the following*

i) *for any $j \geq 0$*

$$\mathscr{I}_{A^j}(S, \omega) = A^{j-1}\sigma\mathscr{I}_{A_2^{(j-1)}}(S, \omega) + \sigma\mathscr{I}_{A_2^{(j)}}(S, \omega) + \mathscr{F}_S^j(\omega)$$

with $\mathscr{F}_S^j(\omega) = (\mathscr{F}_S(\omega))^{\varrho_j + \varrho_{j-1}}$.

ii) *the map*

$$\sigma\colon \frac{\mathscr{E}^{\varrho_j}(\omega)}{\mathscr{I}_{A_2^{(j)}}(S, \omega)} \to \frac{\mathscr{E}^{\varrho_j}(\omega) \oplus \mathscr{E}^{\varrho_{j-1}}(\omega)}{\mathscr{I}_{A^j}(S, \omega)}$$

is an isomorphism.

PROOF OF i). – For $j = 0$ $u \in \mathscr{E}^{\varrho_0}(\omega)$ is such that

$$u \in \mathscr{I}_{A^0}(S, \omega)$$

if an only if

$$u \in \mathscr{I}_{A_2^{(0)}}(S, \omega)$$

(as this has for consequence that $u \in \mathscr{I}_{A_0^{(0)}}(S, \omega)$ also).

Therefore in this case

$$\mathscr{I}_{A^0}(S, \omega) = \mathscr{I}_{A_2^0}(S, \omega) .$$

Here σ is the identity map on $\mathscr{E}^{\varrho_0}(\omega)$. This shows the validity of i) for $j = 0$.

Let us assume that $j \geq 1$. Let $f = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{E}^{\varrho_j}(\omega) \otimes \mathscr{E}^{\varrho_{j-1}}(\omega)$ with $f \in \mathscr{I}_{A^j}(S, \omega)$. By lemma 3 we can find $w \in \mathscr{I}_{A_2^{(j-1)}}(S, \omega)$ such that $A_2^{(j-1)} w = v \bmod (\mathscr{F}_S(\omega))^{\varrho_{j-1}}$.

By lemma 2 $\sigma(w) = \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathscr{I}_{A^{j-1}}(S, \omega)$ and therefore $A^{j-1}\sigma(w) \in \mathscr{I}_{A^j}(S, \omega)$. Moreover

$$(*) \qquad f - A^{j-1}\sigma(w) = \begin{pmatrix} u - A_0^{(j-1)} w \\ 0 \end{pmatrix} \qquad \bmod (\mathscr{F}(\omega))^{\varrho_j + \varrho_{j-1}} .$$

By lemma 2, $u - A^{j-1} w \in \mathscr{I}_{A_2^{(j)}}(S, \omega)$. This proves our contention.

PROOF OF ii). – By lemma 2 σ is injective. To show that σ is surjective we proceed as before. If $j = 0$ there is nothing to prove. Let $f = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{E}^{\varrho_j}(\omega) \oplus \mathscr{E}^{\varrho_{j-1}}(\omega)$ with $j \geq 1$. As before we can find $w \in \mathscr{I}_{A_2^{(j-1)}}(S, \omega)$ such that $(*)$ above holds. As $A^{j-1}\sigma(w) \in \mathscr{I}_{A^j}(S, \omega)$ we obtain the desired statement.

As a corollary we obtain the following

LEMMA 5. – *The sheaf on* ω

$$U \to \mathscr{I}_{A^j}(S, U)$$

is a soft sheaf, $(j \geq 0)$.

PROOF. – To prove this fact we use the following criterion.

On a paracompact locally compact space X a sheaf $\mathscr{F}$ of abelian groups is a soft sheaf if the following holds

for any point $x \in X$ we can find an open neighborhood U of x with the property

for any compact set $F \subset U$,

for any section $s \in \Gamma(U, \mathscr{F})$,

we can find a section $s_U \in \Gamma(U, \mathscr{F})$ such that

$$\operatorname{supp} s_U \subset U ,$$

$$s_U|_F = s ,$$

(we can assume U relatively compact).

Now this property follows immediately for the sheaf considered from formula i) of the previous lemma. One has only to remark that the sheaf $U \to \mathcal{F}^j_S(U)$ is a fine sheaf as it is a sheaf of modules over the sheaf $\mathcal{E}$ of C^∞ functions on X.

From the fact that $U \to \mathcal{I}_{A^j}(S, U)$ is soft and that $U \to \mathcal{F}^j_S(U)$ is also soft (being a fine sheaf) it follows that the sheaf $U \to \mathcal{I}_{A^j}(S, U)/\mathcal{F}^j_S(U)$ is also a soft sheaf (cf. [11], theorem 3.5.3, p. 154).

Statement iii) of theorem 1 is therefore proved.

(η) We can now prove statement i) of theorem 1, i.e. that the sequence

$$0 \to \frac{\mathcal{I}_{A^0}(S,\omega)}{\mathcal{F}^0_S(\omega)} \xrightarrow{A^0} \frac{\mathcal{I}_{A^1}(S,\omega)}{\mathcal{F}^1_S(\omega)} \xrightarrow{A^1} \frac{\mathcal{I}_{A^2}(S,\omega)}{\mathcal{F}^2_S(\omega)} \xrightarrow{A^2} \cdots$$

is an exact sequence.

Let us consider first an $f \in \mathcal{I}_{A^0}(S, \omega)$, and assume that $A^0 f \in \mathcal{F}^1_S(\omega)$. We have to show then that $f \in \mathcal{F}^0_S(\omega)$. Now $A_2^{(0)} f$ is flat on S. By lemma 3 f must be flat on S i.e. $f \in \mathcal{F}^0_S(\omega)$ as we wanted, because (as we have already seen) $f \in \mathcal{I}_{A_2^{(0)}}(S, \omega)$.

Let now $j \geq 1$ and choose $w \in \mathcal{I}_{A_2^{(j-1)}}(S, \omega)$ so that $f - A^{j-1}\sigma(w) = \begin{pmatrix} u \\ 0 \end{pmatrix} \mod \mathcal{F}^j_S(\omega)$. This is possible by lemma 4 i). If $A^j f \in \mathcal{F}^{j+1}_S(\omega)$ then $A_2^{(j)} u$ is flat on S. Because of lemma 2 $u \in \mathcal{I}_{A_2^{(j)}}(S, \omega)$. By lemma 3 we deduce that u is flat on S i.e.

$$f \in A^{j-1}\sigma(w) + \mathcal{F}^j_S(\omega)\,.$$

This proves that f is in the image of A^{j-1} and thus the exactness of the sequence at the j-th place.

(θ) It remains to prove statement ii) in the formulation of the theorem.

To this purpose it will be enough to show that we have a commutative diagram

$$\begin{array}{ccccccc}
\dfrac{\mathcal{E}^{\varrho_0}(\omega)}{\mathcal{I}_{A_2^{(0)}}(S,\omega)} & \xrightarrow{A_0^{(0)}} & \dfrac{\mathcal{E}^{\varrho_1}(\omega)}{\mathcal{I}_{A_2^{(1)}}(S,\omega)} & \xrightarrow{A_0^{(1)}} & \dfrac{\mathcal{E}^{\varrho_2}(\omega)}{\mathcal{I}_{A_2^{(2)}}(S,\omega)} & \xrightarrow{A_0^{(2)}} & \cdots \\
\wr\big\downarrow\sigma & & \wr\big\downarrow\sigma & & \wr\big\downarrow\sigma & & \\
\dfrac{\mathcal{E}^{\varrho_0}(\omega)}{\mathcal{I}_{A^0}(S,\omega)} & \xrightarrow{A^0} & \dfrac{\mathcal{E}^{\varrho_1}(\omega)\oplus\mathcal{E}^{\varrho_0}(\omega)}{\mathcal{I}_{A^1}(S,\omega)} & \xrightarrow{A^1} & \dfrac{\mathcal{E}^{\varrho_2}(\omega)\oplus\mathcal{E}^{\varrho_1}(\omega)}{\mathcal{I}_{A^2}(S,\omega)} & \xrightarrow{A^2} & \cdots
\end{array}$$

where the vertical isomorphisms are those given in lemma 4.

Let $u \in \mathcal{E}^{\varrho_j}(\omega)$ be given. We can find by lemma 3 $w \in \mathcal{I}_{A_2^{(j)}}(S, \omega)$ such that

$$A_2^{(j)} w = A_2^{(j)} u \mod \mathcal{F}_S(\omega)^{\varrho_j}\,.$$

Clearly u and $u - w$ belong to the same class mod $\mathcal{I}_{A_2^{(j)}}(S, \omega)$.

Now

$$A^j\sigma(u-w)=\begin{pmatrix}A_0^{(j)}(u-w)\\ A_2^{(j)}(u-w)\end{pmatrix}=\begin{pmatrix}A_0^{(j)}(u-w)\\ 0\end{pmatrix}\quad \mathrm{mod}\ \mathscr{F}_s^j(\omega)\,.$$

This shows that in the above diagram we have commutativity (at the j-th place) since $\mathscr{F}_S^j(\omega)\subset\mathscr{I}_{A^j}(S,\omega)$.

We remark explicitly that, by lemma 2,

$$\frac{\mathscr{E}^{\varrho_j}(\omega)}{\mathscr{I}_{A_2^{(j)}}(S,\omega)}\simeq\mathscr{E}(S\cap\omega)^{q_j}\,,\qquad (q_j=k_1^{(j)}+\ldots+k_{\varrho_j}^{(j)})\,.$$

One then verifies that each operator $A_0^{(j)}$ can be written as a differential operator

$$B_j\colon\ \mathscr{E}(S\cap\omega)^{q_j}\to\mathscr{E}(S\cap\omega)^{q_{j+1}}$$

because $A_0^{(j)}$ is in canonical form.

This statement could also be deduced from Peetre's theorem and the fact that B_j is linear continuous and

$$\mathrm{supp}\,B_j u\subset\mathrm{supp}\,u\,.$$

With this the proof of theorem 1 is complete.

c) Let now ϕ be any paracompactifying family of supports ([11], p. 150) and let (2) be a complex of differential operators on the manifold X.

Let S be an oriented hypersurface on X defined by the equation $\{\varrho=0\}$.

We set

$$X^+=\{x\in X|\varrho(x)\geqslant 0\}\,,\qquad X^-=\{x\in X|\varrho(x)\leqslant 0\}\,,$$

and we define the groups

$$H_\phi^j(X,\mathscr{E}^*)\,,\quad H_\phi^j(X^+,\mathscr{E}^*)\,,\quad H_\phi^j(X^-,\mathscr{E}^*)$$

as the cohomology groups with supports in ϕ of the complexes

$$\mathscr{E}_\phi^{(0)}(X)\xrightarrow{A^0}\mathscr{E}_\phi^{(1)}(X)\xrightarrow{A^1}\mathscr{E}_\phi^{(2)}(X)\to\ldots$$

$$\mathscr{E}_\phi^{(0)}(X^\pm)\xrightarrow{A_0}\mathscr{E}_\phi^{(1)}(X^\pm)\xrightarrow{A^1}\mathscr{E}_\phi^{(2)}(X^\pm)\to\ldots$$

where

$$\mathscr{E}_\phi^{(j)}(X)=\{s\in\mathscr{E}^{(j)}(X)|\mathrm{supp}\,s\in\phi\}$$

and where $\mathscr{E}^{(j)}(X^{\pm})$ represents the C^∞ sections of E^j on $X^{\pm}$ up to the boundary S but not beyond it, while

$$\mathscr{E}^{(j)}_{\phi}(X^{\pm}) = \{s \in \mathscr{E}^{(j)}(X^{\pm}) | \operatorname{supp} s \in \phi \cap X^{\pm}\} .$$

Similarly we can define the boundary complex with supports ϕ by means of exact sequences of the form

$$0 \to \mathscr{I}^{\phi}_{A^j}(X, S) \to \mathscr{E}^{(j)}_{\phi}(X) \to Q^{(j)}_{\phi}(S) \to 0$$

$(\mathscr{I}^{\phi}_{A^j}(X, S) = \Gamma_{\phi}(X, \mathscr{I}_{A^j}))$ and the groups $H^j_{\phi}(S, Q^*)$, these being the cohomology groups of the boundary complex

$$Q^{(0)}_{\phi}(S) \xrightarrow{A^0_s} Q^{(1)}_{\phi}(S) \xrightarrow{A^1_s} Q^{(1)}_{\phi}(S) \to \dots .$$

From a standard argument we derive the following

THEOREM 2. – *Let* (2) *be a complex of differential operators on X endowed with a classical grading. Let ϕ be a paracompactifying family of supports.*

Assume that the hypersurface S is noncharacteristic. Then we have a Mayer-Vietoris exact sequence

$$0 \to H^0_{\phi}(X, \mathscr{E}^*) \to H^0_{\phi}(X^+, \mathscr{E}^*) \oplus H^0_{\phi}(X^-, \mathscr{E}^*) \to H^0_{\phi}(S, Q^*) \to$$
$$\to H^1_{\phi}(X, \mathscr{E}^*) \to H^1_{\phi}(X^+, \mathscr{E}^*) \oplus H^1_{\phi}(X^-, \mathscr{E}^*) \to H^1_{\phi}(S, Q^*) \to \dots .$$

PROOF. – By the corollary to theorem 1 we have an exact sequence of soft sheaves

$$0 \to \frac{\mathscr{I}_{A^0}}{\mathscr{F}^0_S} \to \frac{\mathscr{I}_{A^1}}{\mathscr{F}^1_S} \to \frac{\mathscr{I}_{A^2}}{\mathscr{F}^2_S} \to \dots .$$

From this, by taking sections with support in ϕ we derive an exact sequence

$$(*) \qquad 0 \to \frac{\mathscr{I}^{\phi}_{A^0}(S, X)}{\mathscr{F}^0_S(X)} \to \frac{\mathscr{I}^{\phi}_{A^1}(S, X)}{\mathscr{F}^1_S(X)} \to \dots$$

where

$$\mathscr{I}^{\phi}_{A^j}(S, X) = \{s \in \mathscr{I}_{A^j}(S, X) | \operatorname{supp} s \in \phi\} .$$

Indeed one has

$$\Gamma_{\phi}\left(X, \frac{\mathscr{I}_{A^j}}{\mathscr{F}^j_S}\right) = \frac{\mathscr{I}^{\phi}_{A^j}(S, X)}{\mathscr{F}^j_S(X)}$$

because the sheaves $\mathscr{I}_{A^j}$ and $\mathscr{F}^j_S$ are soft.

From the exactness of the sequence $(*)$ we derive the Mayer-Vietoris sequence by the usual argument (cf. [3]).

8. – An example: boundary values of pluriharmonic functions.

a) Let X be a complex manifold. For every open set $\Omega \subset X$ we set

$$A^{r,s}(\Omega) = \text{space of } C^\infty \text{ forms of type } (r, s)\,,$$
$$A^{(j)}(\Omega) = \bigoplus_{r+s=j} A^{r,s}(\Omega)\,,$$

$d =$ exterior differential, $d = \partial + \bar\partial$ where $\bar\partial$ (resp. ∂) is the exterior differentiation with respect to antiholomorphic (resp. holomorphic) local coordinates.

We consider the following complex of differential operators

$$(\alpha) \qquad A^{00}(\Omega) \xrightarrow{\partial\bar\partial} A^{11}(\Omega) \xrightarrow{d} A^{12}(\Omega) \oplus A^{21}(\Omega) \xrightarrow{d} \ldots$$
$$\ldots \xrightarrow{d} \bigoplus_{j=1}^{n-1} A^{j,n-j}(\Omega) \xrightarrow{d} A^{(n+1)}(\Omega) \xrightarrow{d} \ldots \xrightarrow{d} A^{(2n)}(\Omega) \to 0\,.$$

If Ω is open and $\mathscr{H}(\Omega)$ denotes the space of (complex valued) pluriharmonic functions on Ω we have the exact sequence (which gives an augmentation to the complex (α))

$$(\varepsilon) \qquad 0 \to \mathscr{H}(\Omega) \xrightarrow{i} A^{00}(\Omega) \xrightarrow{\partial\bar\partial} A^{11}(\Omega)\,.$$

Let $\mathscr{H}$ denote the sheaf of germs of pluriharmonic functions on X and let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on X. We have the exact sequence of sheaves

$$(\beta) \qquad 0 \to \mathbf{C} \xrightarrow{\sigma} \mathcal{O} \oplus \overline{\mathcal{O}} \xrightarrow{\tau} \mathscr{H} \to 0$$

where

$$\sigma(a) = a \oplus a \qquad (a \in \mathbf{C})\,,$$
$$\tau(f \oplus g) = f - g \qquad f \in \mathcal{O}\,,\ g \in \overline{\mathcal{O}}\,,$$

the bar over $\mathcal{O}$ denoting complex conjugation so that $\overline{\mathcal{O}}$ is the sheaf of germs of antiholomorphic functions.

This complex is a complex of differential operators with constant coefficients in any holomorphic coordinate patch.

b) For the complex (α) the bundle E^0 is the trivial bundle, the bundle E^1 is the bundle $\mathscr{T}^*(X) \otimes \overline{\mathscr{T}^*(X)}$ where $\mathscr{T}^*(X)$ is the holomorphic tangent bundle, the bundle E^2 is the bundle $\mathscr{T}^*(X) \otimes \Lambda^2 \overline{\mathscr{T}^*(X)} \oplus \Lambda^2 \mathscr{T}^*(X) \otimes \overline{\mathscr{T}^*(X)}$ etc. Gradings will be chosen classically so that there will be a jump of two units from E^0 to E^1 and of one unit from every bundle E^j to the successive E^{j+1}.

To write the symbolic complex of (α) at a point we will use the following notations:

$$\mathbb{P} = \mathbb{C}[\xi_1, \ldots, \xi_n, \bar{\xi}_1, \ldots, \bar{\xi}_n]$$

ring of polynomials in the indeterminates $\xi = (\xi_1, \ldots, \xi_n)$ and $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_n)$.

$\mathbb{P}^{r,s}$ = space of exterior forms of type r in $d\xi_1, \ldots, d\xi_n$ and of type s in $d\bar{\xi}_1, \ldots, d\bar{\xi}_n$ with coefficients in $\mathbb{P}$,

$\mathbb{P}^{(j)} = \bigoplus_{r+s=j} \mathbb{P}^{r,s}$,

$$\alpha = \sum_1^n \xi_i \, d\xi_i \, , \qquad \bar{\alpha} = \sum_1^n \bar{\xi}_i \, d\bar{\xi}_i \, .$$

A direct verification shows that the symbolic complex of (α) at any point $x_0 \in X$ (i.e. the complex denoted before as $(4)_{x_0}$) is the complex

$$(\hat{\alpha}) \qquad \mathbb{P}^{00} \xrightarrow{\wedge\alpha\wedge\bar{\alpha}} \mathbb{P}^{11} \xrightarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{12} \oplus \mathbb{P}^{21} \xrightarrow{\wedge(\alpha+\bar{\alpha})} \ldots$$

$$\ldots \xrightarrow{\wedge(\alpha+\bar{\alpha})} \bigoplus_{j=1}^{n-1} \mathbb{P}^{j,n-j} \xrightarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{(n+1)} \xrightarrow{\wedge(\alpha+\bar{\alpha})} \ldots \xrightarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{(2n)} \to 0 \, .$$

We know that (α) is an exact sequence on any countable open set of holomorphy; in particular on any open set Ω convex in a holomorphic coordinate patch [9]. This proves that the transposed complex ${}^t(\hat{\alpha})$ of $(\hat{\alpha})$ is exact. Taking into account the isomorphism

$$\mathbb{P}^{r,s} \simeq \mathbb{P}^{n-r,\, n-s}$$

the complex ${}^t(\hat{\alpha})$ can be written in the form

$${}^t(\hat{\alpha}) \qquad 0 \leftarrow N \leftarrow \mathbb{P}^{n,n} \xleftarrow{\wedge\alpha\wedge\bar{\alpha}} \mathbb{P}^{n-1,n-1} \begin{array}{c} \overset{\wedge\bar{\alpha}}{\swarrow} \mathbb{P}^{n-1,n-2} \\ \oplus \\ \underset{\wedge\alpha}{\nwarrow} \mathbb{P}^{n-2,n-1} \end{array} \xleftarrow{\wedge(\alpha+\bar{\alpha})} \ldots$$

$$\ldots \xleftarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{(n-1)} \xleftarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{(n-2)} \leftarrow \ldots \xleftarrow{\wedge(\alpha+\bar{\alpha})} \mathbb{P}^{(0)} \leftarrow 0$$

where N is the cokernel of the last map. The exactness of the sequence ${}^t(\hat{\alpha})$ can also be established directly by the results of [4] (corollaries 1 and 2, pp. 606-607).

The complex (α) is a particular Hilbert complex.

c) Let S be an oriented hypersurface given on X by an equation $\varrho = 0$. Writing $z_j = x_j + ix_{n+j}$ for the local holomorphic coordinates on X we write the gradient of ϱ in terms of holomorphic and antiholomorphic coordinates

$$\operatorname{grad} \varrho(x_0) = \left(\frac{\partial\varrho}{\partial z_1}, \dots, \frac{\partial\varrho}{\partial z_n}, \frac{\partial\varrho}{\partial \bar z_1}, \dots, \frac{\partial\varrho}{\partial \bar z_n}\right)_{x_0}.$$

We have

$$\alpha(\operatorname{grad} \varrho) = \sum \frac{\partial\varrho}{\partial z_i} d\xi_i = \partial\varrho\,,$$

$$\bar\alpha(\operatorname{grad} \varrho) = \sum \frac{\partial\varrho}{\partial \bar z_i} d\bar\xi_i = \bar\partial\varrho\,.$$

We also set

$\mathbb{C}^{r,s}$ = space of exterior forms of type r in $d\xi_1, \dots, d\xi_n$ and of type s in $d\bar\xi_1, \dots, d\bar\xi_n$ with coefficients in $\mathbb{C}$,

$\mathbb{C}^{(j)} = \bigoplus_{r+s=j} \mathbb{C}^{r,s}$,

and note that $\mathbb{C}^{r,s} \simeq \mathbb{C}^{n-r,n-s}$.

Now remark that the map

$$\mathbb{P}^{n,n} \xleftarrow{\wedge\alpha\wedge\bar\alpha} \mathbb{P}^{n-1,n-1}$$

is given by the matrix of one row

$$(\xi_1\bar\xi_1, \dots, \xi_i\bar\xi_i, \dots, \xi_n\bar\xi_n) = M_0(\xi, \bar\xi)$$

so that in $\mathbb{C}^{2n}$ where $\xi_1, \dots, \xi_n, \bar\xi_1, \dots, \bar\xi_n$ are independent variables the variety

$$V = \{(\xi, \bar\xi) \in \mathbb{C}^{2n} | M_0(\xi, \bar\xi) = 0\}$$

has no point, except the origin, where $\bar\xi$ is the conjugate of ξ [4].

In particular

$$\operatorname{grad} \varrho = \left(\frac{\partial\varrho}{\partial z_1}, \dots, \frac{\partial\varrho}{\partial z_n}, \frac{\partial\varrho}{\partial \bar z_1}, \dots, \frac{\partial\varrho}{\partial \bar z_n}\right) \notin V\,.$$

---

(4) V is the union of the two linear spaces $L = \{\xi_i = 0, 1 \leqslant i \leqslant n\}$ and $\bar L = \{\bar\xi_i = 0, 1 \leqslant i \leqslant n\}$.

From [5], proposition 1, we derive that we have an exact sequence:

$$
{}^{t}(\gamma) \qquad 0 \leftarrow \mathbf{C}^{n,n} \xleftarrow{\wedge\partial\varrho\wedge\bar\partial\varrho} \mathbf{C}^{n-1,n-1} \begin{array}{c} \mathbf{C}^{n-1,n-2} \\ \swarrow \wedge\bar\partial\varrho \\ \oplus \\ \nwarrow \wedge\partial\varrho \\ \mathbf{C}^{n-2,n-1} \end{array} \xleftarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \cdots
$$

$$
\cdots \leftarrow \mathbf{C}^{(n-1)} \xleftarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \mathbf{C}^{(n-2)} \leftarrow \cdots \xleftarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \mathbf{C}^{(0)} \leftarrow 0 .
$$

From this exact sequence or better from the exact sequence obtained by transposition:

$$
(\gamma) \qquad 0 \to \mathbf{C}^{00} \xrightarrow{\wedge\partial\varrho\wedge\bar\partial\varrho} \mathbf{C}^{11} \xrightarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \mathbf{C}^{12}\oplus\mathbf{C}^{21} \xrightarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \cdots
$$

$$
\cdots \to \mathbf{C}^{(n+1)} \xrightarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \mathbf{C}^{(n+2)} \to \cdots \xrightarrow{\wedge(\partial\varrho+\bar\partial\varrho)} \mathbf{C}^{(2n)} \to 0
$$

we deduce that the given complex is elliptic at every point and that *any hypersurface S is noncharacteristic.*

d) We have now the following

LEMMA. – *Given the complex* (α) *and the hypersurface* $S = \{\varrho = 0\}$ *in* Ω, *the successive domains of the operators* $\partial\bar\partial$, d, d, ... *of the complex* (α) *along* S *are given by*

$$
\begin{aligned}
\mathscr{I}_{\partial\bar\partial}(S, \Omega) &= \varrho^2 A^{00}(\Omega) , \\
\mathscr{I}_{d}(S, \Omega) &= \varrho A^{11}(\Omega) + \partial\varrho\wedge\bar\partial\varrho\, A^{00}(\Omega) , \\
\mathscr{I}_{d}(S, \Omega) &= \varrho\big(A^{12}(\Omega) + A^{21}(\Omega)\big) + d\varrho\, A^{11}(\Omega) , \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathscr{I}_{d}(S, \Omega) &= \varrho \sum_{j=1}^{n-1} A^{j\, n-j}(\Omega) + d\varrho \sum_{j=1}^{n-2} A^{j\, n-j-1}(\Omega) , \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathscr{I}_{d}(S, \Omega) &= \varrho A^{(2n)}(\Omega) + d\varrho\, A^{(2n-1)}(\Omega) .
\end{aligned}
$$

PROOF. – α) Let $\varphi \in A^{00}(\Omega)$ and let $u \in \mathscr{D}^{n-1\; n-1}(\Omega)$ where $\mathscr{D}$ denotes compactly supported forms. We have

$$
\int_{\Omega^-} \partial\bar\partial\varphi\wedge u = \int_S \bar\partial\varphi\wedge u + \int_S \varphi\wedge\partial u - \int_{\Omega^-} \varphi\bar\partial\partial u .
$$

Thus $\varphi \in \mathscr{I}_{\partial\bar\partial}(S, \Omega)$ if and only if $\forall u \in \mathscr{D}^{n-1\; n-1}(\Omega)$

$$
(*) \qquad \int_S \partial\varphi\wedge u + \int_S \varphi\wedge\partial u = 0 .
$$

Take $z_0 \in S$ at the origin of the coordinates. Setting $z_n = x_n + iy_n$ we may assume that $S \cap \Omega$ is given by

$$\varrho \equiv y_n - \sigma(z_1, \ldots, z_{n-1}, x_n)$$

with σ C^∞ and vanishing at the origin of second order. Taking

$$u = \big(\alpha(z_1, \ldots, z_{n-1}, x_n) + (y_n - \sigma)\beta(z_1, \ldots, z_{n-1}, x_n)\big)\, dz_1 \ldots dz_{n-1}\, d\bar{z}_1 \ldots d\bar{z}_{n-1}$$

with α and β compactly supported we realize that if

$$\varphi = \varphi^0(z_1, \ldots, z_{n-1}, x_n) + \varrho\varphi^1(z_1, \ldots, z_{n-1}, x_n) + \ldots$$

we must have $\varphi^0 = 0$, $\varphi^1 = 0$ i.e. $\varphi \in \varrho^2 A^{00}(\Omega)$. Conversely, if this holds then $(*)$ holds and $\varphi \in \mathscr{I}_{\partial\bar{\partial}}(S, \Omega)$.

β) With selfexplaining notations we have for $\varphi \in A^{11}(\Omega)$

$$\int_{\Omega^-} d\varphi \wedge (u^{n-1\, n-2} + u^{n-2\, n-1}) = \int_S \varphi \wedge (u^{n-1\, n-2} + u^{n-2\, n-1}) + \int_{\Omega^-} \varphi \wedge (\bar{\partial} u^{n-1, n-2} + \partial u^{n-2, n-1})\,.$$

Thus $\varphi \in \mathscr{I}_d(S, \Omega)$ if and only if

$$\int_S \varphi \wedge u^{n-1\, n-2} = 0 = \int_S \varphi \wedge u^{n-2\, n-1}$$
$$\forall u^{n-1\, n-2} \in \mathscr{D}^{n-1\, n-2}(\Omega) \text{ and } \forall u^{n-2\, n-1} \in \mathscr{D}^{n-2\, n-1}(\Omega).$$

Taking into account lemma 1 of ([2], part II, p. 755) we get the desired conclusion as we must have

$$\varphi \wedge \bar{\partial}\varrho|_S = 0\,,$$
$$\varphi \wedge \partial\varrho|_S = 0\,,$$

where the restriction means restriction of the coefficients of the form considered.

γ) Let $\varphi = \varphi^{12} + \varphi^{21} \in A^{12}(\Omega) + A^{21}(\Omega)$. To have $\varphi \in \mathscr{I}_d(S, \Omega)$ we must have

$$0 = \int_S \varphi^{12} \wedge u^{n-1\, n-3} = \int_S (\varphi^{12} + \varphi^{21}) \wedge u^{n-2\, n-2} = \int_S \varphi^{21} \wedge u^{n-3\, n-1}$$
$$\forall u^{n-1\, n-3} \in \mathscr{D}^{n-1\, n-3}(\Omega)\,,\ \forall u^{n-2\, n-2} \in \mathscr{D}^{n-2\, n-2}(\Omega)\,,\ \forall u^{n-3\, n-1} \in \mathscr{D}^{n-3\, n-1}(\Omega)\,.$$

From the first and last integral we derive that

$$\varphi^{12} = \varrho\alpha^{12} + \bar{\partial}\varrho \wedge \beta^{11}\,,$$
$$\varphi^{21} = \varrho\gamma^{21} + \partial\varrho \wedge \delta^{11}\,,$$

with convenient forms α^{12}, β^{11}, γ^{21}, δ^{11} in Ω. From the middle integral we get then

$$\int_S (\bar\partial\varrho\, \beta^{11} + \partial\varrho\, \delta^{11})\, u^{n-2\, n-2} = 0$$

i.e.

$$\int_S \bar\partial\varrho(\beta^{11} - \delta^{11})\, u^{n-2\, n-2} = 0$$

since $\int_S d\varrho\, \delta^{11} u^{n-2\, n-2} = 0$ because the form $d\varrho$ induces the 0-form on S. Thus $\bar\partial\varrho\, \beta^{11} = \bar\partial\varrho\, \delta^{11} + \varrho\theta^{21}$ and therefore

$$\varphi^{12} + \varphi^{21} = \varrho(\alpha^{12} + \theta^{21} + \gamma^{21}) + d\varrho\, \delta^{11} .$$

δ) The general argument is the same as in γ).

We define therefore on $S_\Omega = S \cap \Omega$

$$Q^{(0)}(S_\Omega) = \frac{A^{00}(\Omega)}{\varrho^2 A^{00}(\Omega)} \simeq A^{(0)}(S_\Omega) + \varrho A^{(0)}(S_\Omega) .$$

$$Q^{(1)}(S_\Omega) = \frac{A^{11}(\Omega)}{\varrho A^{11}(\Omega) + \partial\varrho \wedge \bar\partial\varrho\, A^{00}(\Omega)} ,$$

$$Q^{(\mu)}(S_\Omega) = \frac{\sum\limits_{\substack{r+s=\mu+1\\ r\geq 1,\, s\geq 1}} A^{r,s}(\Omega)}{\varrho \sum\limits_{\substack{r+s=\mu+1\\ r\geq 1,\, s\geq 1}} A^{r,s}(\Omega) + d\varrho \sum\limits_{\substack{r+s=\mu\\ r\geq 1,\, s\geq 1}} A^{r,s}(\Omega)} \qquad \text{for } 2 \leqslant \mu \leqslant n-1 ,$$

$$Q^{(\mu)}(S_\Omega) = A^{(\mu+1)}(S_\Omega) \qquad \text{for } n \leqslant \mu \leqslant 2n-2 .$$

where $A^{(j)}(S)$ denotes the space of forms of degree j on S. The boundary complex has therefore the form

$$Q^{(0)}(S) \xrightarrow{(\partial\bar\partial)_S} Q^{(1)}(S) \xrightarrow{d_S} Q^{(2)}(S) \xrightarrow{d_S} \dots \xrightarrow{d_S} Q^{(n)}(S) \xrightarrow{d} Q^{(n+1)}(S) \xrightarrow{d} \dots \xrightarrow{d} Q^{(2n-2)}(S) \to 0 .$$

Note that the last part from $Q^{(n)}(S)$ on coincides with the de Rham complex of exterior differentiation

$$A^{(n+1)}(S) \xrightarrow{d} A^{(n+2)}(S) \xrightarrow{d} \dots \xrightarrow{d} A^{(2n-1)}(S) \to 0 .$$

e) Explicit expression of the operator $(\partial\bar\partial)_S$. Let $z_0 \in S$ be at the origin in its coordinate patch Ω. We can assume that there

$$\varrho \equiv y_n - \sigma(z_1, \dots, z_{n-1}, x_n) , \qquad z_n = x_n + iy_n$$

with σ vanishing at the origin with its first partial derivatives. In a small neighborhood of the origin we will have expressions of the form

$$dz_n = a\,\partial\varrho + \sum_1^{n-1} \alpha_j\, dz_j\,,$$
$$d\bar{z}_n = \bar{a}\,\bar{\partial}\varrho + \sum_1^{n-1} \bar{\alpha}_j\, d\bar{z}_j\,,$$

with $\alpha(0) \neq 0$, $\alpha_j(0) = 0$, $1 \leqslant j \leqslant n-1$. Actually

$$a = -2\left(i + \frac{\partial\sigma}{\partial x_n}\right)^{-1} \quad \text{and} \quad \alpha_j = a\frac{\partial\sigma}{\partial z_j}.$$

Also

$$\partial\bar{\partial}\varrho = \sum_1^{n-1} l_{ij}\, dz_i\, d\bar{z}_j + \partial\varrho \sum_1^{n-1} \bar{\beta}_i\, d\bar{z}_i - \bar{\partial}\varrho \sum_1^{n-1} \beta_i\, dz_i + \gamma\,\partial\varrho \wedge \bar{\partial}\varrho\,.$$

Note that the analytic tangent space to S at the origin is $\{z_n = 0\}$ and that $\sum_1^{n-1} l_{ij}(0)\, dz_i\, d\bar{z}_j$ is the Levi form of ϱ at the origin restricted to the analytic tangent space.

We can take $z_1, \ldots, z_{n-1}$ and x_n as local coordinates on S near z_0.

Let

$$u_0 + \varrho u_1 \in A^0(S) + \varrho A^0(S)$$

with

$$u_0 = u_0(z_1 \ldots, z_{n-1}, x_n)\,, \qquad u_1 = u_1(z_1, \ldots, z_{n-1}, x_n)\,.$$

The equations $(\partial\bar{\partial})_S(u_0 + \varrho u_1) = 0$ can be written as

$$(\delta) \qquad \begin{cases} \partial\bar{\partial}(u_0 + \varrho u_1) \wedge \partial\varrho|_S = 0\,, \\ \partial\bar{\partial}(u_0 + \varrho u_1) \wedge \bar{\partial}\varrho|_S = 0\,, \end{cases}$$

taking into account the form of the space $Q^{(1)}(S)$. Here restriction to S means restriction of the coefficients of the form.

This allows an explicit calculation of that system of equations.

We set, for $1 \leqslant i, j \leqslant n-1$,

$$\mathscr{L}_{ij} = \frac{\partial^2}{\partial z_i\,\partial\bar{z}_j} + \frac{1}{2}\bar{\alpha}_j \frac{\partial^2}{\partial z_i\,\partial x_n} + \frac{1}{2}\alpha_i \frac{\partial^2}{\partial\bar{z}_j\,\partial x_n} + \frac{1}{4}\alpha_i\bar{\alpha}_j \frac{\partial^2}{\partial x_n^2}\,,$$
$$S_i = a\left\{\frac{\partial^2}{\partial\bar{z}_i\,\partial x_n} + \frac{1}{4}\bar{\alpha}_i \frac{\partial^2}{\partial x_n^2}\right\},$$
$$T_i = \frac{\partial}{\partial\bar{z}_i} + \frac{1}{2}\bar{\alpha}_i \frac{\partial}{\partial x_n}\,.$$

Then the system of equations $(\partial\bar{\partial})_S(u_0 + \varrho u_1) = 0$ reduces to the system of $(n-1)\cdot$ $\cdot(n+1)$ equations

$$\begin{cases} \mathscr{L}_{ij}u_0 + l_{ij}u_1 = 0\,, \\ 1 \leqslant i\,,\ j \leqslant n-1\,, \end{cases}$$

$$\begin{cases} S_i u_0 + T_i u_1 = 0\,, \\ 1 \leqslant i \leqslant n-1\,, \end{cases}$$

$$\begin{cases} \bar{S}_i u_0 + \bar{T}_i u_1 = 0\,, \\ 1 \leqslant i \leqslant n-1\,. \end{cases}$$

REMARK. - Assume that the Levi form of ϱ along the analytic tangent space to S is different from zero. If $u_0 + \varrho u_1$ and $u_0 + \varrho u_1^*$ are two solutions of the equations (δ) then $u_1 = u_1^*$. Moreover, locally, we can, from one of the first set of equations, obtain u_1 in terms of u_0 and substitute in the remaining equations. Therefore u_0 satisfies in that case a set of differential equations of second and third order (cf. [13]).

f) Hartogs type theorem. We assume now that S is compact in X and that

$X^- = \{x \in X | \varrho(x) \leqslant 0\}$ is compact,

$X^+ = \{x \in X | \varrho(x) \geqslant 0\}$ has any connected component noncompact,

$\{\varrho = 0\}$ being an equation for S.

Let $H^0(X^-, \mathscr{H})$ denote the space of C^∞ functions on X^- which are pluriharmonic in $\mathring{X}^-$.

Let $H^0(S, \mathscr{H}_S^{(1)})$ denote the space of couples of functions $u_0 + \varrho u_1 \in A^0(S) + \varrho A^0(S)$ satisfying the equations (δ):

$$(\partial\bar{\partial})_S(u_0 + \varrho u_1) = 0\,.$$

Let

$$r\colon\ H^0(X^-, \mathscr{H}) \to H^0(S, \mathscr{H}_S^{(1)})$$

be defined by

$$r(h) = u_0 + \varrho u_1$$

where

$$u_0 = h|_S\,,$$
$$u_1 = \left.\frac{dh}{d\varrho}\right|_S.$$

We have then the following

THEOREM 3. – *Under the above assumptions if the manifold X is $(n-2)$-complete $(n\geqslant 2)$ and $H^2_k(X, \mathbb{C}) = 0$ then the natural map*

$$r\colon H^0(X^-, \mathscr{H}) \to H^0(S, \mathscr{H}^{(1)}_S)$$

is an isomorphism.

PROOF. – From the Mayer-Vietoris sequence with compact supports we deduce that we have an exact sequence

$$0 \to H^0(X^-, \mathscr{H}) \xrightarrow{r} H^0(S, \mathscr{H}^{(1)}_S) \to H^1_k(X, \mathscr{H}) .$$

From the exact sequence of sheaves (β) we deduce the exact sequence

$$H^1_k(X, \mathcal{O}) \oplus H^1_k(X, \overline{\mathcal{O}}) \to H^1_k(X, \mathscr{H}) \to H^2_k(X, \mathbb{C}) .$$

By the assumption that X is $(n-2)$-complete and by the duality theorem we derive

$$H^1_k(X, \mathcal{O}) \simeq H^{n-1}(\Omega, \Omega^n) = 0$$

where Ω^n is the sheaf of holomorphic n-forms (cf. [1]).

By assumption also $H^2_k(X, \mathbb{C}) = 0$. Thus $H^1_k(X, \mathscr{H}) = 0$ and from this we deduce our conclusion.

REMARK. – The above assumptions are verified if X is a Stein manifold of dimension $n\geqslant 3$ [0] or if X is Stein of dimension 2 and $H^2_k(X, \mathbb{C}) \simeq H_2(X, \mathbb{C}) = 0$. In particular for $X = \mathbb{C}^n$ $n\geqslant 2$. We have indicated with $\mathscr{H}^{(1)}_S$ the sheaf on S defined by the exact sequence of sheaves

$$0 \to \mathscr{H}^{(1)}_S \to Q^{(0)} \xrightarrow{(\partial\bar{\partial})_S} Q^{(1)}$$

where for S_U open in S $Q^{(0)}$ and $Q^{(1)}$ denote the sheaves

$$S_U \to Q^{(0)}(S_U) , \qquad S_U \to Q^{(1)}(S_U) .$$

We denote by $\mathscr{H}_S$ the sheaf of germs of C^∞ functions u on S such that we can find a germ of C^∞ function v on S with

$$(\partial\bar{\partial})_S(u + \varrho v) = 0 .$$

If

$$\sigma\colon \frac{A^{00}(\Omega)}{\varrho^2 A^{00}(\Omega)} \to \frac{A^{00}(\Omega)}{\varrho A^{00}(\Omega)}$$

for every Ω open in X is the natural map we deduce, at the sheaf level, a natural map

$$\sigma\colon\ Q^{(0)} \to A^{(0)}$$

where $A^{(0)}$ is the sheaf of C^∞ germs on S.

From the above considerations we have a natural surjective map

$$\mathscr{H}_S^{(1)} \xrightarrow{\sigma} \mathscr{H}_S \to 0\ .$$

Taking into account the remark at the end of point e) of this section we obtain the following

PROPOSITION 8. – *Let the hypersurface S in X have the property that the Levi form of ϱ on the analytic tangent space to S is nowhere zero. Then the natural map σ is an isomorphism of sheaves:*

$$\mathscr{H}_S^{(1)} \xrightarrow{\sim} \mathscr{H}_S\ .$$

COROLLARY. – *Under the assumptions of theorem* 3 *and of proposition* 8 *the natural map*

$$H^0(X^-, \mathscr{H}) \to H^0(S, \mathscr{H}_S)$$

(given by $h \to h|_S$) is an isomorphism.

REMARK. – Let h be a C^∞ function on X^- which is pluriharmonic in $\mathring{X}^-$ (so that $\partial\bar\partial h = 0$). Then

$$\alpha = \partial h\ , \qquad \beta = \bar\partial h\ ,$$

are closed forms on X^- with C^∞ coefficients up to the boundary. Fix $z_0 \in \mathring{X}^-$ and let z be a variable point and assume that $H^1(X^-, \mathbb{C}) = 0$. Then

$$f(z) = \int_{z_0}^{z} \alpha \qquad \text{is } C^\infty \text{ on } X^- \text{ and holomorphic in } \mathring{X}^-\ ,$$

$$g(z) = \int_{z_0}^{z} \beta \qquad \text{is } C^\infty \text{ on } X^- \text{ and antiholomorphic on } \mathring{X}^-\ ,$$

and

$$h(z) - h(z_0) = f(z) + g(z)\ .$$

g) The sheaf $\mathscr{H}_S$. For Ω open in X we set $S_\Omega = S \cap \Omega$ and define

$$A^{00}(S_\Omega) = \frac{A^{00}(\Omega)}{\varrho A^{00}(\Omega)} = A^{(0)}(S_\Omega)\,,$$

$$A^{01}(S_\Omega) = \frac{A^{01}(\Omega)}{\varrho A^{01}(\Omega) + \bar\partial\varrho\, A^{00}(\Omega)}\,, \dots .$$

The operator $\bar\partial$ on X induces then a boundary operator $\bar\partial_S$ (cf. [1], [2]) and we get a complex

$$A^{00}(S_\Omega) \xrightarrow{\bar\partial_S} A^{01}(S_\Omega) \xrightarrow{\bar\partial_S} \dots .$$

Denoting by A_S^{0j} the sheaf $S_\Omega \to A^{0j}(S_\Omega)$ at the sheaf level we have a complex of sheaves

$$(\zeta) \qquad 0 \to \mathcal{O}_S \to A_S^{00} \xrightarrow{\bar\partial_S} A_S^{01} \xrightarrow{\bar\partial_S} \dots$$

where by definition $\mathcal{O}_S = \operatorname{Ker}\left\{A_S^{00} \xrightarrow{\bar\partial_S} A_S^{01}\right\}$.

Passing to the complex conjugate we define analogously the sheaves A_S^{j0}, the operator ∂_S and the complex of sheaves

$$(\bar\zeta) \qquad 0 \to \overline{\mathcal{O}}_S \to A_S^{00} \to A_S^{10} \to \dots .$$

We define $\mathscr{F}_S^{rs}(\Omega) = \{\varphi \in A^{r,s}(\Omega) | \varphi$ is « flat » on $S\}$. By « Γ » we denote the usual functor « sections ».

LEMMA 6. – *Let* $f \in \Gamma(S_\Omega, \mathcal{O}_S)$. *There exists a representative* $\tilde f \in A^{00}(\Omega)$ *of* f *such that*

$$\bar\partial \tilde f \in \mathscr{F}_S^{01}(\Omega)$$

(*and* $\tilde f|_S = f$).

This is lemma 2.2 of [2], part I, p. 240.

LEMMA 7. – *Let* $u_0 + \varrho u_1 \in \Gamma(S_\Omega, \mathscr{H}^{(1)})$, *there exists a representative* $\tilde u \in A^{00}(\Omega)$ *of* $u_0 + \varrho u_1$ *such that*

$$\partial\bar\partial \tilde u \in \mathscr{F}_S^{11}(\Omega)$$

(*and* $\tilde u|_S = u_0$, $d\tilde u/d\varrho|_S = u_1$).

PROOF. – We denote by u_0 and u_1 any fixed extensions of these functions to Ω as C^∞ functions. We have

$$\partial\bar\partial(u_0 + \varrho u_1) = \varrho\alpha_1 + \partial\varrho \wedge \bar\partial\varrho\, \beta_1$$

with $\alpha_1 \in A^{11}(\Omega)$ and $\beta_1 \in A^{00}(\Omega)$. Then with $\alpha_1' \in A^{11}(\Omega)$ conveniently chosen

$$\partial\bar{\partial}(u_0 + \varrho u_1 - \tfrac{1}{2}\varrho^2\beta_1) = \varrho\alpha_1'$$

and as ∂ and $\bar{\partial}$ applied to the left hand side give zero, we get

$$\partial\varrho\wedge\alpha_1'|_S = 0\,, \qquad \bar{\partial}\varrho\wedge\alpha_1'|_S = 0\,.$$

Hence

$$\alpha_1' = \varrho\alpha_2 + \partial\varrho\wedge\bar{\partial}\varrho\,\beta_2$$

with convenient α_2 and β_2. Thus we have

$$\partial\bar{\partial}(u_0 + \varrho u_1 - \tfrac{1}{2}\varrho^2\beta_1) = \varrho^2\alpha_2 + \varrho\,\partial\varrho\wedge\bar{\partial}\varrho\,\beta_2$$

and with a convenient α_2',

$$\partial\bar{\partial}(u_0 + \varrho u_1 - \tfrac{1}{2}\varrho^2\beta_1 - \tfrac{1}{6}\varrho^3\beta_2) = \varrho^2\alpha_2'\,.$$

Moreover one has

$$\partial\varrho\wedge\alpha_2'|_S = 0\,, \qquad \bar{\partial}\varrho\wedge\alpha_2'|_S = 0$$

so that

$$\alpha_2' = \varrho\alpha_3 + \partial\varrho\wedge\bar{\partial}\varrho\wedge\beta_3$$

with convenient α_3 and β_3.

Proceeding in this way we see that we can solve the equation

$$\partial\bar{\partial}(u_0 + \varrho u_1 + \ldots) \equiv O_S^\infty$$

with a formal power series in ϱ with coefficients C^∞ on S_Ω, where O_S^∞ denotes « vanishing of infinite order on S ». By the use of Whitney extension theorem (cf. [2], I, proposition 22, p. 337) we conclude as desired.

Let $A^{(r)}$ denote the sheaf of C^∞ exterior forms of degree r and let $\mathscr{F}_S^{(r)}$ denote the subsheaf of those forms with coefficients « flat » on S. We set

$$W_S^{(r)} = A^{(r)}/\mathscr{F}_S^{(r)}\,.$$

Exterior differentiation induces a natural map

$$d\colon\; W_S^{(r)} \to W_S^{(r+1)}\,.$$

LEMMA 8. – *The following is an exact sequence of sheaves*

$$0 \to \mathbf{C} \to W_S^{(0)} \xrightarrow{d} W_S^{(1)} \xrightarrow{d} W_S^{(2)} \xrightarrow{d} \dots .$$

PROOF. – Only the differentiable structure is concerned in this lemma. We may assume to work in $\mathbf{R}^{m+1}$ near the origin where $(x_1, \dots, x_m, t)$ are coordinates and where $S = \{t = 0\}$.

The exactness is obvious on $W_S^{(0)}$.

Let

$$\omega = \sum_0^\infty t^n(\beta_n^{(r)}(x, dx) + dt\,\sigma_n^{(r-1)}(x, dx))$$

be an element of $W_{S,0}^{(r)}$ with $r \geqslant 1$. Here $\beta_n^{(r)}$ and $\sigma_n^{(r-1)}$ denote germs of exterior forms of degree r and $r-1$ respectively.

Let d_x be the exterior differentiation on the variable x. The condition $d\omega = 0$ is equivalent to the conditions

$$d_x\beta_n^{(r)}(x, dx) = 0 \qquad \forall n ,$$

$$(n+1)\,\beta_{n+1}^{(s)}(x, dx) = d_x\sigma_n^{(r-1)}(x, dx) \qquad \forall n .$$

From the first set of equations we derive that we can find forms $\beta_n^{(r-1)}(x, dx)$ such that

$$\beta_n^{(r)}(x, dx) = d_x\beta_n^{(r-1)}(x, dx) \qquad \forall n .$$

From the second we get $d_x(\sigma_n^{(r-1)}(x, dx) - (n+1)\,\beta_{n+1}^{(r-1)}(x, dx)) = 0$. If $r \geqslant 2$ then we can find forms $\sigma_n^{(r-2)}(x, dx)$ such that

$$\sigma_n^{(r-1)}(x, dx) = d_x\sigma_n^{(r-2)}(x, dx) + (n+1)\,\beta_{n+1}^{(r-1)}(x, dx) ;$$

thus $\omega = d\left\{\sum_0^\infty t^n(\beta_n^{(r-1)}(x, dx) - dt\,\sigma_n^{(r-2)}(x, dx))\right\}$. This proves the lemma if $r \geqslant 2$. If $r = 1$ only a slight modification of the above argument is needed.

PROPOSITION 9. – *We have an exact sequence of sheaves*

$$0 \to \mathscr{L} \to \mathcal{O}_S \oplus \overline{\mathcal{O}}_S \xrightarrow{\alpha} \mathscr{H}_S \to 0$$

where

$$\alpha(f \oplus g) = f - g$$

and where $\mathscr{L} = \mathcal{O}_S \cap \overline{\mathcal{O}}_S$.

PROOF. – Given $f \in \mathcal{O}_{S,z_0}$, for $z_0 \in S$ by lemma 6 we can find an extension $\tilde{f}$ of f such that $\bar{\partial}\tilde{f} \in \mathcal{F}_S^{01}$. Then $\partial\bar{\partial}\tilde{f} \in \mathcal{F}_S^{11}$. Expanding $\tilde{f}$ in power series of ϱ we have $\tilde{f} = f + \varrho g + \dots$ and $(\partial\bar{\partial})_S(f + \varrho g) = 0$. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{H}_S$. Similarly $\overline{\mathcal{O}}_S \subset \mathcal{H}_S$ and therefore α is well defined. Clearly $\operatorname{Ker} \alpha = \mathcal{L}$.

We have to show that α is surjective.

Given $u \in \mathcal{H}_{S,z_0}$ by the definition of the sheaf $\mathcal{H}_S$ we can find a germ of C^∞ function v on S at z_0 such that $u + \varrho v \in \mathcal{H}^{(1)}_{S,z_0}$ i.e. $(\partial\bar{\partial})_S(u + \varrho v) = 0$. By lemma 7 we can find an extension $\tilde{u}$ of $u + \varrho v$ such that $\partial\bar{\partial}\tilde{u} \in \mathcal{F}_S^{11}$. By lemma 8 from $d(\bar{\partial}\tilde{u}) \in \mathcal{F}_S^{(2)}$ we deduce that we can find a germ σ of function at z_0 in the space such that

$$\bar{\partial}\tilde{u} = d\sigma \mod \mathcal{F}_S^{(1)}.$$

Hence

$$\partial\sigma \in \mathcal{F}_S^{10}$$

and

$$\bar{\partial}(\tilde{u} - \sigma) \in \mathcal{F}_S^{01}.$$

Set $\tau = \tilde{u} - \sigma$. From the above equations we derive that

$$\sigma|_S \in \overline{\mathcal{O}}_S \qquad \tau|_S \in \mathcal{O}_S$$

and that

$$u = \tau|_S + \sigma|_S.$$

This proves our contention.

PROPOSITION 10. – *Let $S_\Omega \subset S$ be the open subset of S where the Levi form of ϱ restricted to the analytic tangent plane to S is different from zero.*

We have on S_Ω

$$\mathcal{L} \simeq \mathbb{C}$$

($\mathbb{C}$ the constant sheaf) so that in the exact sequence of sheaves

$$0 \to \mathbb{C} \xrightarrow{i} \mathcal{L} \to \mathcal{N} \to 0$$

i being the natural injection, we have $\mathcal{N}|_{S_\Omega} = 0$.

PROOF. – Clearly $\mathbb{C}$ is a subsheaf of $\mathcal{L}$. Let $z_0 \in S_\Omega$ and let $f_{z_0} \in \mathcal{L}_{z_0}$. Let $\tilde{f}$ be a C^∞ extension of f to an open neighborhood U of z_0 in X. If U is sufficiently small we have

$$\bar{\partial}\tilde{f} = \varrho\alpha^{01} + \bar{\partial}\varrho\,\beta^{00},$$

$$\partial\tilde{f} = \varrho\gamma^{10} + \partial\varrho\,\sigma^{00},$$

with $\alpha, \beta, \gamma, \sigma$ convenient C^∞ forms on U. Indeed these equations translate the fact that $\bar{\partial}_S f_{z_0} = \partial_S f_{z_0} = 0$.

Therefore we have

$$d\tilde{f} = \varrho(\alpha + \gamma) + \bar{\partial}\varrho\, \beta + \partial\varrho\, \sigma\,.$$

Set $\mu = \alpha + \gamma$. As $ddf = 0$ we derive that

$$0 = d\varrho\, \mu + \varrho\, d\mu + \partial\bar{\partial}\varrho\, \beta - \bar{\partial}\varrho\, d\beta + \bar{\partial}\partial\varrho\, \sigma - \partial\varrho\, d\sigma$$

i.e. at each point of $S \cap U$

$$\partial\bar{\partial}\varrho(\beta - \sigma) + \bar{\partial}\varrho(\mu - d\beta) + \partial\varrho(\mu - d\sigma) = 0\ .$$

From this we deduce that

$$\partial\varrho \wedge \bar{\partial}\varrho \wedge \partial\bar{\partial}\varrho \wedge (\beta - \sigma)|_{S_U} = 0\ .$$

By the assumption the form $\partial\varrho\wedge\bar{\partial}\varrho\wedge\partial\bar{\partial}\varrho$ on S_U is different from zero $\Big($with the notations of point e) of this section that form equals $\big(\sum_1^{n-1} l_{ij}\, dz_i\, d\bar{z}_j\big)\,\partial\varrho\wedge\bar{\partial}\varrho\Big)$. Therefore on S_U $\beta = \sigma$.

Hence

$$d\tilde{f} = \varrho(\alpha + \gamma) + d\varrho\, \beta\,.$$

But this proves that $df_{z_0} = j^* d\tilde{f}$, j being the natural injection of S_U in U. Therefore f_{z_0} is constant in a neighborhood of z_0, i.e. $f_{z_0} \in \mathbb{C}$.

COROLLARY. – *If the Levi form of ϱ restricted to the analytic tangent space to S is everywhere different from zero and if $H^1(S, \mathbb{C}) = 0$ then we have an exact sequence*

$$0 \to \Gamma(S, \mathbb{C}) \to \Gamma(S, \mathcal{O}_S) \oplus \Gamma(S, \bar{\mathcal{O}}_S) \xrightarrow{\alpha} \Gamma(S, \mathcal{H}_S) \to 0\ .$$

We do not know if in the case S compact with $H^1(S, \mathbb{C}) = 0$ the above statement still holds without any assumption on the Levi form of ϱ.

h) The case of a Levi form or rank ≥ 2. Let us start again with the consideration of the complex (α).

We set

$$\mathcal{J}^0(S, \Omega) = \varrho A^{00}(\Omega)\,,$$

$$\mathcal{J}^1(S, \Omega) = \varrho A^{11}(\Omega) + \partial\varrho\, A^{01}(\Omega) + \bar{\partial}\varrho\, A^{10}(\Omega) + \partial\bar{\partial}\varrho\, A^{00}(\Omega)\,,$$

$$\mathcal{J}^2(S, \Omega) = \varrho\big(A^{12}(\Omega) + A^{21}(\Omega)\big) + d\varrho\, A^{11}(\Omega) = \mathcal{J}_d(S, \Omega)\,,$$

and in general for $\mu \geqslant 2$

$$\mathscr{I}^{\mu}(S, \Omega) = \mathscr{I}_d(S, \Omega) .$$

We realize that

$$(\eta) \qquad \mathscr{I}^0(S, \Omega) \xrightarrow{\partial\bar{\partial}} \mathscr{I}^1(S, \Omega) \xrightarrow{d} \mathscr{I}^2(S, \Omega) \xrightarrow{d} \mathscr{I}^3(S, \Omega) \xrightarrow{d} \dots$$

is a subcomplex of (α). Moreover the sheaves

$$\Omega \to \mathscr{I}^j(S, \Omega)$$

are fine (therefore soft) sheaves.

Note that the subcomplex of (α) given by the domains of the various operators

$$(\theta) \qquad \mathscr{I}_{\partial\bar{\partial}}(S, \Omega) \to \mathscr{I}_d(S, \Omega) \to \mathscr{I}_d(S, \Omega) \to \mathscr{I}_d(S, \Omega) \to \dots$$

is a subcomplex of (η).

We set

$$C^{(0)}(S_\Omega) = A^{00}(\Omega)/\mathscr{I}^0(S, \Omega) ,$$

$$C^{(1)}(S_\Omega) = A^{11}(\Omega)/\mathscr{I}^1(S, \Omega) ,$$

$$C^{(\mu)}(S_\Omega) = Q^{(\mu)}(S_\Omega) \qquad \text{for } \mu \geqslant 2 .$$

At the sheaf level, we have therefore a commutative diagram of sheaves and linear maps:

$$\begin{array}{ccccccccccc}
0 \to \mathscr{H}_S^{(1)} & \to & Q^{(0)} & \xrightarrow{(\partial\bar{\partial})_S} & Q^{(1)} & \xrightarrow{d_S} & Q^{(2)} & \xrightarrow{d_S} & Q^{(3)} & \xrightarrow{d_S} & \dots \\
\downarrow\lambda & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathscr{T}_S & \to & C^{(0)} & \xrightarrow{(\partial\bar{\partial})^R} & C^{(1)} & \xrightarrow{d^R} & Q^{(2)} & \xrightarrow{d_S} & Q^{(3)} & \xrightarrow{d_S} & \dots
\end{array}$$

where $(\partial\bar{\partial})^R$ and d^R are the induced linear maps of sheaves by the operators $\partial\bar{\partial}$ and d in the surrounding space, and where, by definition $\mathscr{I}_S = \operatorname{Ker}\left\{C^{(0)} \xrightarrow{(\partial\bar{\partial})^R} C^{(1)}\right\}$.

Note that $C^{(0)} \simeq A^{(0)} = \mathscr{E}_S$ the sheaf of germs of C^∞ functions on S. The sheaf $C^{(1)}$ is a sheaf of $\mathscr{E}_S$ modules.

PROPOSITION 11. – *On the set S_Ω where the Levi form of ϱ restricted to the analytic tangent space to S is different from zero the sheaf $C^{(1)}$ is a locally free sheaf of modules of rank $(n-1)^2-1$.*

PROOF. – Indeed if $z_0 \in S_\Omega$ we can choose a system of $(n-1)^2-1$ forms of type $(1, 1)$ linearly independent over the C^∞ functions $\mathscr{E}(U)$ in a small neighborhood U of z_0 in X, say

$$\omega_1, \omega_2, \ldots, \omega_{(n-1)^2-1},$$

so that

$$A^{11}(U) = \sum_{1}^{(n-1)^2-1} \mathscr{E}(U)\omega_\alpha + \partial\bar\partial\varrho\, \mathscr{E}(U) + \partial\varrho\, A^{01}(U) + \bar\partial\varrho\, A^{10}(U) .$$

COROLLARY. – *On S_Ω the linear maps $(\partial\bar\partial)^R$ and d^R are given by differential operators.*

PROOF. – Indeed $C^{(0)}, C^{(1)}, Q^{(2)}$ can be viewed as the sheaves of germs of C^∞ sections of appropriate C^∞ vector bundles on S_Ω. The operators $(\partial\bar\partial)^R$ and d^R are continuous for the usual Schwartz topology and preserve supports. One can therefore apply Peetre's theorem.

Also a direct calculation gives the same conclusion. We note that $(\partial\bar\partial)^R$ is a differential operator of the second order. It is the zero operator if $n = 2$.

From the commutativity of the above diagram we derive a natural inclusion

$$\mathscr{H}_S \subset \mathscr{T}_S .$$

Indeed $\mathscr{H}_S$ is just the image of $\mathscr{H}_S^{(1)}$ in $\mathscr{T}_S$ by the map λ.

We set

$$\mathscr{F}_S^j(\Omega) = \{s \in \mathscr{I}^j(S, \Omega) | s \text{ is flat on } S\} .$$

We know that since the complex (α) is elliptic we have an exact sequence of soft sheaves

$$(\lambda) \qquad 0 \to \frac{\mathscr{I}_{\partial\bar\partial}}{\mathscr{F}_S^0} \xrightarrow{\partial\bar\partial} \frac{\mathscr{I}_d}{\mathscr{F}_S^1} \xrightarrow{d} \frac{\mathscr{I}_d}{\mathscr{F}_S^2} \xrightarrow{d} \ldots .$$

We can also consider the complex of sheaves (all soft)

$$(\mu) \qquad 0 \to \frac{\mathscr{I}^0}{\mathscr{F}_S^0} \xrightarrow{\partial\bar\partial} \frac{\mathscr{I}^1}{\mathscr{F}_S^1} \xrightarrow{d} \frac{\mathscr{I}^2}{\mathscr{F}_S^2} \xrightarrow{d} \ldots .$$

PROPOSITION 12. – *Let $z^0 \in S$, and let $\mathscr{L}(\varrho)|_{T_{z_0}(S)}$ denote the Levi form of ϱ restricted to the analytic tangent space at z_0 to S.*

i) *If $\mathscr{L}(\varrho)|_{T_{z_0}(S)}$ is different from zero then the sequence (μ) is exact at the place $\mathscr{I}^0/\mathscr{F}_S^0$.*

ii) *If $\mathscr{L}(\varrho)|_{T_{z_0}(S)}$ is different from zero and has rank $\geqslant 2$ then the sequence (μ) is exact also at the place $\mathscr{I}^1/\mathscr{F}_S^1$ and therefore everywhere.*

PROOF. – The exactness of the sequence

$$\frac{\mathscr{I}^1}{\mathscr{F}^1_S} \xrightarrow{d} \frac{\mathscr{I}^2}{\mathscr{F}^2_S} \xrightarrow{d} \dots$$

follows from the exactness of (λ) and the fact that $\mathscr{I}_d \subset \mathscr{I}^1$.

We prove *i*). Let $w = \varrho u \in \mathscr{I}^0$. Assume that

$$\partial\bar\partial(\varrho u) \in \mathscr{F}^1_S$$

i.e.

$$\varrho\,\partial\bar\partial u + \partial\varrho\wedge\bar\partial u - \bar\partial\varrho\wedge\partial u + \partial\bar\partial\varrho\, u \in \mathscr{F}^1_S$$

therefore

$$\partial\varrho\wedge\bar\partial\varrho\wedge\partial\bar\partial\varrho\, u|_S = 0\,.$$

Because of the assumption $u|_S = 0$ i.e. $u = \varrho v$ and $w = \varrho u = \varrho^2 v \in \mathscr{I}_{\partial\bar\partial}$. Because (λ) is exact we derive that $w \in \mathscr{F}^0_S$ as we wanted.

We prove ii). Let, with obvious notations, be

$$g^{11} = \varrho\alpha^{11} + \partial\varrho\,\beta^{01} + \bar\partial\varrho\,\gamma^{10} + \partial\bar\partial\varrho\,\sigma^{00} \in \mathscr{I}^1\,.$$

Then $\varrho\sigma^{00} \in \mathscr{I}^0$ and

$$g^{11} - \partial\bar\partial(\varrho\sigma^{00}) = \varrho\theta^{11} + \partial\varrho\,\theta^{01} + \bar\partial\varrho\,\theta^{10}\,.$$

Assume that $dg^{11} \in \mathscr{F}^2_S$: We have also $d\big(g^{11} - \partial\bar\partial(\varrho\sigma^{00})\big) \in \mathscr{F}^2_S$. This gives

$$\varrho\,\partial\theta^{11} + \partial\varrho(\theta^{11} - \partial\theta^{01}) - \bar\partial\varrho\,\partial\theta^{10} + \partial\bar\partial\varrho\,\theta^{10} \in \mathscr{F}^2_S$$

and an analogous relation with $\bar\partial$ replaced by ∂. We deduce then that

$$\partial\varrho\wedge\bar\partial\varrho\wedge\partial\bar\partial\varrho\wedge\theta^{10}|_S = 0\,.$$

Because of the assumption we must have

$$\theta^{10} = \varrho\lambda^{10} + \partial\varrho\mu^{00}\,.$$

(we can assume ϱ as in the form of point *e*) of this section with $\sum_1^{n-1} l_{ij}\,dz_i\,d\bar z_j = \sum_1^{n-1} \varepsilon_j\,dz_j\,d\bar z_j$ in diagonal form at the origin and $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$. Setting $\theta^{10} = \sum_1^{n-1} a_j\,dz_j + \mu^{00}\,\partial\varrho + \varrho\lambda^{10}$ we deduce that $a_j = 0$ for $1 \leqslant j \leqslant n-1$ at the origin. From this we get our conclusion.)

Similarly

$$\theta^{01} = \varrho\lambda^{01} + \nu^{00}\bar{\partial}\varrho .$$

Hence

$$g^{11} - \partial\bar{\partial}(\varrho\sigma^{00}) = \varrho(\theta^{11} + \partial\varrho\,\lambda^{01} + \bar{\partial}\varrho\,\lambda^{10}) + \partial\varrho\wedge\bar{\partial}\varrho(\nu^{00} - \mu^{00}) .$$

Therefore $g^{11} - \partial\bar{\partial}(\varrho\sigma^{00}) \in \mathscr{I}_a$ and consequently by the exactness of (λ) we get $g^{11} - \partial\bar{\partial}(\varrho\sigma^{00}) \in \mathscr{F}^1_S$.

Let Σ be an open portion of S. We can consider on Σ the following complexes for any family of supports ϕ:

$$\text{(i)}\ \ \Gamma_\phi(\Sigma, A^{00}/\mathscr{F}^{00}_S) \xrightarrow{\partial\bar{\partial}} \Gamma_\phi(\Sigma, A^{11}/\mathscr{F}^{11}_S) \xrightarrow{d} \Gamma_\phi(\Sigma, (A^{12}\oplus A^{21})/(\mathscr{F}^{12}_S \oplus \mathscr{F}^{21}_S)) \xrightarrow{d} \dots ,$$

$$\text{(ii)}\ \ \Gamma_\phi(\Sigma, Q^{(0)}) \xrightarrow{(\partial\bar{\partial})_S} \Gamma_\phi(\Sigma, Q^{(1)}) \xrightarrow{d_S} \Gamma_\phi(\Sigma, Q^{(2)}) \xrightarrow{d_S} \dots ,$$

$$\text{(iii)}\ \ \Gamma_\phi(\Sigma, C^{(0)}) \xrightarrow{(\partial\bar{\partial})^R} \Gamma_\phi(\Sigma, C^{(1)}) \xrightarrow{d^R} \Gamma_\phi(\Sigma, Q^{(2)}) \xrightarrow{d_S} \dots .$$

We set

$$\hat{\mathscr{H}} = \operatorname{Ker}\{A^{00}/\mathscr{F}^{00}_S \xrightarrow{\partial\bar{\partial}} A^{11}/\mathscr{F}^{11}_S\} , \qquad \mathscr{H}^{(1)}_S = \operatorname{Ker}\{Q^{(0)} \xrightarrow{(\partial\bar{\partial})_S} Q^{(1)}\} ,$$

$$\mathscr{T}_S = \operatorname{Ker}\{C^{(0)} \xrightarrow{(\partial\bar{\partial})^R} C^{(1)}\}$$

and we denote the cohomology groups of the above complexes with the notations, for any $j \geqslant 0$

$$H^j_\phi(\Sigma, [\hat{\mathscr{H}}]) , \quad H^j_\phi(\Sigma, [\mathscr{H}^{(1)}_S]) , \quad H^j_\phi(\Sigma, [\mathscr{T}_S]) .$$

From the previous proposition we deduce then the following

COROLLARY. - *Set*

$$S^{(2)}_\Omega = \{x \in S | \operatorname{rank} \mathscr{L}(\varrho)|T_x(S) \geqslant 2\} .$$

Let Σ be any open subset of $S^{(2)}_\Omega$. Then for any family of supports ϕ (paracompactifying) we have

$$H^j_\phi(\Sigma, [\mathscr{H}^{(1)}_S]) = H^j_\phi(\Sigma, [\hat{\mathscr{H}}]) = H^j_\phi(\Sigma, [\mathscr{T}_S])$$

for any $j \geqslant 0$.

In particular for $j = 0$ and ϕ the family of closed sets, germifying Σ at any point of $S^{(2)}_\Omega$ we get that

$$\lambda\colon \mathscr{H}^{(1)}_S \to \mathscr{T}_S$$

is an isomorphism of sheaves over $S^{(2)}_\Omega$. *Thus on* $S^{(2)}_\Omega$

$$\mathscr{H}^{(1)}_S \xrightarrow{\sim} \mathscr{H}_S \xrightarrow{\sim} \mathscr{T}_S .$$

COROLLARY. – *If* $S^{(2)}_\Omega = S$ *we have a Mayer-Vietoris sequence*

$$0 \to H^0_\phi(X, \mathscr{H}) \to H^0_\phi(X^+, \mathscr{H}) \oplus H^0_\phi(X^-, \mathscr{H}) \to H^0_\phi(S, [\mathscr{T}_S]) \to$$
$$\to H^1_\phi(X, \mathscr{H}) \to H^1_\phi(X^+, \mathscr{H}) \oplus H^1_\phi(X^-, \mathscr{H}) \to H^1_\phi(S, [\mathscr{T}_S]) \to \dots .$$

REMARK. – We have denoted with the peculiar notation $H^i_\phi(\Sigma, [\hat{\mathscr{H}}])$ etc. the cohomology groups of the complexes (i), (ii), and (iii) above as they may *not* be isomorphic to the cohomology groups with values in the corresponding sheaves. Indeed the complexes of sheaves

$$\begin{array}{lllll} 0 \to \hat{\mathscr{H}} \to \dfrac{A_{00}}{\mathscr{F}^{00}_S} \to \dfrac{A^{11}}{\mathscr{F}^{11}_S} \to \dfrac{A^{12} \oplus A^{21}}{\mathscr{F}^{12}_S \oplus \mathscr{F}^{21}_S} \to \dots \\ 0 \to \mathscr{H}^{(1)}_S \to Q^{(0)} \to Q^{(1)} \to Q^{(2)} \to \dots \\ 0 \to \mathscr{T}_S \to C^{(0)} \to C^{(1)} \to C^{(2)} \to \dots \end{array}$$

may not be exact. This situation will be discussed in the next point.

i) We consider the locally closed region $X^- = \{x \in X | \varrho(x) \leqslant 0\}$. We have defined (section 7 *c*)) the cohomology groups $H^j(X^-, \mathcal{O})$, $H^j(X^-, \overline{\mathcal{O}})$ and $H^j(X^-, \mathscr{H})$ by means of the complexes of Dolbeault, of its « conjugate » and of the complex (α).

We can also consider the usual cohomology groups $H^j(X^-, \mathbb{C})$.

We first claim that

LEMMA 9. – $H^j(X^-, \mathbb{C})$ *is the j-th cohomology group of the complex*

$$A^0(X^-) \xrightarrow{d} A^{(1)}(X^-) \xrightarrow{d} A^{(2)}(X^-) \xrightarrow{d} \dots$$

where $A^{(j)}(X^-)$ *is the space of* C^∞ *forms of degree j defined on* X^- *up to the boundary but not beyond it, and where d is exterior differentiation.*

PROOF. – This lemma deals only with the C^∞ structure of X. We are reduced to prove the following. Let $X = \mathbb{R}^m$ and $X^- = \{(x_1, \dots, x_m) \in \mathbb{R}^m | x_m \leqslant 0\}$. Similarly for X^+. Let $A^{(j)}$ be the sheaf of C^∞ forms on $\mathbb{R}^m$ of degree j and let $\mathscr{F}^{(j)} = \{s \in A^{(j)} | s$ is « flat » on $X^+\}$ so that setting $A^{(j)}_- = A^{(j)}/\mathscr{F}^{(j)}$ we have $A^{(j)}(X^-) = \Gamma(X, A^{(j)}_-)$. One has to show that at a point $x^0 = (x^0_1, \dots, x^0_{m-1}, 0) \in \partial X^-$ we have the Poincaré lemma, i.e. that the sequence

$$0 \to \mathbb{C} \to A^{(0)}_{-x^0} \xrightarrow{d} A^{(1)}_{-x^0} \xrightarrow{d} A^{(2)}_{-x^0} \xrightarrow{d} \dots$$

is an exact sequence. This follows with the usual proof of Poincaré lemma for the operator d, or by the use of the Mayer-Vietoris sequence on a small ball centered at x^0.

Let us now consider the map for $j \geqslant 1$

$$\sigma\colon H^j(X^-, \mathcal{O}) \oplus H^j(X^-, \overline{\mathcal{O}}) \to H^j(X^-, \mathcal{H})$$

given as follows: Let $\{\varphi^{0j}\}$, $\bar\partial\varphi^{0j} = 0$ and $\{\varphi^{j0}\}$, $\partial\varphi^{j0} = 0$ be cohomology classes in $H^j(X^-, \mathcal{O})$ amd $H^j(X^-, \overline{\mathcal{O}})$ so that φ^{0j} (φ^{j0}) is a C^∞ form of type $(0, j)$ $((j, 0))$ defined on X^- but not beyond. We define

$$\sigma(\{\varphi^{0j}\} \oplus \{\varphi^{j0}\}) = \{\partial\varphi^{0j} + \bar\partial\varphi^{j0}\}\,.$$

This map is linear and well defined.

LEMMA 10. - *If* $H^j(X^-, \mathbf{C}) = 0 = H^{j+1}(X^-, \mathbf{C})$ *and* $j \geqslant 1$ *we have that* σ *is an isomorphism*:

$$H^j(X^-, \mathcal{O}) \oplus H^j(X^-, \overline{\mathcal{O}}) \underset{\sigma}{\xrightarrow{\sim}} H^j(X^-, \mathcal{H})\,.$$

PROOF. - The map σ is injective. Assume that $j \geqslant 2$ and that with obvious notations,

$$\partial\varphi^{0j} + \bar\partial\varphi^{j0} = d(\eta^{1\,j-1} + \ldots + \eta^{j-1\,1})\,.$$

Then since $\bar\partial\varphi^{0j} = 0 = \partial\varphi^{j0}$ we get

$$d(\varphi^{0j} + \varphi^{j0} - \eta^{1\,j-1} - \ldots - \eta^{j-1\,1}) = 0\,.$$

By lemma 9 we deduce then, since $H^j(X^-, \mathbf{C}) = 0$,

$$\varphi^{0j} + \varphi^{j0} - \eta^{1\,j-1} - \ldots - \eta^{j-1\,1} = d(\theta^{0\,j-1} + \ldots + \theta^{j-1\,0})\,.$$

Hence $\varphi^{0j} = \bar\partial\theta^{0\,j-1}$ and $\varphi^{j0} = \partial\theta^{j-1\,0}$. This proves our contention.

The map σ is surjective. We shall assume $j \geqslant 2$. Let

$$\varphi^{1j} + \varphi^{2\,j-1} + \ldots + \varphi^{j1}$$

with

$$d(\varphi^{1j} + \ldots + \varphi^{j1}) = 0$$

represent a class of $H^j(X^-, \mathcal{H})$. Since $H^{j+1}(X^-, \mathbf{C}) = 0$ we have, by lemma 9,

$$\varphi^{1j} + \varphi^{2\,j-1} + \ldots + \varphi^{j1} = d(\theta^{0j} + \theta^{1\,j-1} + \ldots + \theta^{j0})$$

i.e.

$$\begin{cases} \bar\partial\theta^{0j} = 0\,, \\ \varphi^{1j} = \partial\theta^{0j} + \bar\partial\theta^{1\,j-1}\,, \\ \varphi^{2\,j-1} = \partial\theta^{1\,j-1} + \bar\partial\theta^{2\,j-2}\,, \\ \dots\dots\dots\dots \\ \varphi^{j1} = \partial\theta^{j-1\,1} + \bar\partial\theta^{j0}\,, \\ \partial\theta^{j0} = 0\,. \end{cases}$$

Hence

$$\varphi^{1j} + \dots + \varphi^{j1} = \partial\theta^{0j} + \bar\partial\theta^{j0} + d(\theta^{1\,j-1} + \dots + \theta^{j-1\,1})\,.$$

This proves the surjectivity of σ.

It remains to treat the case $j = 1$. Let φ^{01} and φ^{10} be such that $\bar\partial\varphi^{01} = 0$, $\partial\varphi^{10} = 0$ and assume that

$$\partial\varphi^{01} + \bar\partial\varphi^{10} = \partial\bar\partial\theta^{00}\,.$$

Then

$$d(\varphi^{01} + \varphi^{10} - \bar\partial\theta^{00}) = 0\,.$$

Since $H^1(X^-, \mathbb{C}) = 0$ we must have

$$\varphi^{01} + \varphi^{10} - \bar\partial\theta^{00} = d\eta^{00}$$

i.e.

$$\varphi^{01} = \bar\partial\theta^{00} + \bar\partial\eta^{00}\,, \qquad \varphi^{10} = \partial\eta^{00}\,.$$

This shows the injectivity of σ also for $j = 1$.

Consider now φ^{11} with $d\varphi^{11} = 0$. As $H^2(X, \mathbb{C}) = 0$ we have

$$\varphi^{11} = d(\eta^{10} + \eta^{01})\,.$$

Therefore

$$\varphi^{11} = \bar\partial\eta^{10} + \partial\eta^{01}$$

with

$$\partial\eta^{10} = 0 = \bar\partial\eta^{01}\,.$$

Thus σ is surjective also in the case $j = 1$.

THEOREM 4. – *Let $z_0 \in S$ be a point where the Levi form of ϱ restricted to the analytic tangent space to S is nondegenerate with p positive and $n-1-p=q$ negative eigenvalues. Then in the boundary complex of sheaves*

$$Q^{(0)} \xrightarrow{(\partial\bar{\partial})_S} Q^{(1)} \xrightarrow{d_S} Q^{(2)} \xrightarrow{d_S} \dots .$$

The Poincaré lemma fails to be true at $Q^{(p)}_{z_0}$ and at $Q^{(q)}_{z_0}$ but holds at any other place.

PROOF. – The theorem being of local nature we can assume that $X = \mathbb{C}^n$, that z_0 is at the origin of the coordinates and that ϱ is in the form used at point e) of this section.

Set

$$B_n = \left\{ z \in \mathbb{C}^n \,\middle|\, \Sigma |z_j|^2 < \frac{1}{n^2} \right\}, \qquad B_n^+ = \{ z \in B_n | \varrho \geqslant 0 \}.$$
$$B_n^- = \{ z \in B_n | \varrho \leqslant 0 \}, \qquad \Sigma_n = S \cap B_n .$$

For $j \geqslant 1$ and n large we have $H^j(B_n, \mathbb{C}) = H^j(B_n^+, \mathbb{C}) = H^j(B_n^-, \mathbb{C}) = 0$. Therefore for n large

$$H^j(B_n, \mathscr{H}) \simeq H^j(B_n, \mathcal{O}) \oplus H^j(B_n, \overline{\mathcal{O}}) = 0 .$$

Hence from the Mayer-Vietoris sequence we derive that, for $j \geqslant 1$,

$$H^j(\Sigma_n, [\mathscr{H}^{(1)}]) \simeq H^j(B_n^+, \mathscr{H}) \oplus H^j(B_n^-, \mathscr{H}) .$$

Also by lemma 10

$$(*) \qquad \begin{cases} H^j(B_n^+, \mathscr{H}) \simeq H^j(B_n^+, \mathcal{O}) \oplus H^j(B_n^+, \overline{\mathcal{O}}) , \\ H^j(B_n^-, \mathscr{H}) \simeq H^j(B_n^-, \mathcal{O}) \oplus H^j(B_n^-, \overline{\mathcal{O}}) . \end{cases}$$

Taking direct limits we get

$$\varinjlim_n H^j(\Sigma_n, [\mathscr{H}^{(1)}]) \simeq \varinjlim_n H^j(B_n^+, \mathscr{H}) \oplus \varinjlim_n H^j(B_n^-, \mathscr{H}) .$$

Taking into account the isomorphisms $(*)$ and theorem 3 of ([2], II, p. 795) we get

$$\varinjlim_n H^j(B_n^\pm, \mathscr{H}) = 0 \qquad \text{if } j \neq 0, p, q .$$

Thus the Poincaré lemma holds for $j \neq 0, p, q$.

For $j = p, q$ and for a proper sign of ϱ we have

$$\varinjlim_n H^p(B_n^+, \mathscr{H}) \qquad \text{is infinite dimensional},$$
$$\varinjlim_n H^q(B_n^-, \mathscr{H}) \qquad \text{is infinite dimensional},$$

(by theorem 4 of [2], II, p. 798; see also [1], theorem 9.6.1, p. 165). This shows that

$$\lim_{\overrightarrow{n}} H^j(\Sigma_n, [\mathscr{H}^{(1)}]) \neq 0 \qquad \text{for } j = p\,,\ j = q$$

(indeed these spaces are infinite dimensional). Hence the statement of the theorem.

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