Periodic Solutions of a Class of Nonlinear Evolution Equations (*).

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Summary. – *In this paper we prove the existence of periodic solutions of abstract evolution equations which are modelled after parabolic problems. More precisely we prove that existence results follow from degree type hypotheses on the «projection» of the problem onto a suitable finite dimensional space.*

1. – Let Ω be a bounded open set in $\boldsymbol{R}^n$ having sufficiently smooth boundary $\partial\Omega$ and let $f\colon \boldsymbol{R}\times\bar{\Omega}\times\boldsymbol{R}$ be a continuous function such that

$$f(t+1, x, u) = f(t, x, u)\,, \qquad (t, x, u)\in \boldsymbol{R}\times\bar{\Omega}\times\boldsymbol{R}\,.$$

In this paper we shall be interested in the existence of 1-periodic solutions of evolution equations, which are modelled after the problem

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + \varepsilon f(t, x, u)\,, \qquad t\in\boldsymbol{R},\ x\in\Omega \\ &\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0\,. \end{aligned}\right. \tag{1}$$

From our abstract results we may, for example, deduce that (1) will have a t-periodic «mild» solution $u(t, x)$ of period one for all ε small whenever there exist constants a and b, $a<b$ such that

$$g(a)g(b) < 0\,, \tag{2}$$

where g is the mapping $g\colon \boldsymbol{R}\to\boldsymbol{R}$ defined by

$$s \mapsto \int_\Omega\int_0^1 f(t, x, s)\,dt\,dx\,, \tag{3}$$

in which case u in addition satisfies

$$a < u(t, x) < b\,.$$

(*) Entrata in Redazione il 28 maggio 1983.

In case f is independent of x, this, of course, reduces to a well-known result for ordinary differential equations (see e.g. [3] and [6]).

REMARK 1. – A « mild » solution is a solution in a weak sense (to be specified later). If f satisfies enough smoothness requirements (continuity suffices in case f is independent of x) such mild solutions are in fact classical.

2. – Let E be a real Banach space and let $f\colon \boldsymbol{R}\times E \to E$ be a continuous mapping, such that $f(t+1, u) = f(t, u)$, $(t, u) \in \boldsymbol{R}\times E$. Let A with domain $D(A) \subset E$ be a linear operator which is the infinitesimal generator of an analytic semigroup (see [3]) $\{T(t)\}_{t\geqslant 0}$. We shall be interested in the existence of solutions $u\colon \boldsymbol{R} \to E$, which are 1-periodic, of the nonlinear evolution equation

$$\frac{du}{dt} = Au(t) + \varepsilon f(t, u(t)) . \tag{4}$$

The additional hypotheses imposed on A (see below) will easily be seen to be satisfied by a large class of linear elliptic second (or higher) order operators on bounded domains provided the coefficients of A and the boundary $\partial\Omega$ are sufficiently smooth.

The assumptions on A are the following:

(H1) The semigroup $\{T(t)\}$ generated by A is compact, i.e. for each $t > 0$, $T(t)$ is a compact operator, $T(t)\colon E \to E$.

(H2) Let $N = \ker A$ and assume that $N = \ker\big(\mathrm{id} - T(1)\big)$.

REMARK 2. – It follows from (H1) that $N \subset \ker\big(\mathrm{id} - T(t)\big)$, $t > 0$ and hence N is a finite dimensional subspace of E and (H2) asserts that A has no eigenvalues of the form $2k\pi i$, $k \neq 0$, an integer.

(H3) $E = N \oplus M$ where M is left invariant under $T(t)$, i.e. the algebraic and geometric multiplicity of 0 as an eigenvalue of A coincide.

(H4) There exists a Banach space V, $D(A) \subset V$, such that $V \subset E$ in both the algebraic and topological sense, and $V = N \oplus (V \cup M)$, moreover V is assumed compactly embedded in E.

(H5) The evolution operator

$$f(t) \mapsto \int_0^t T(t-s) f(s)\, ds$$

maps $C([0, 1], E)$ compactly into $C([0, 1], V)$.

REMARK 3. – It has been shown in [1] that the convolution operator considered in (H5) does not map $C([0, 1], E)$ into $C\big([0, 1], D(A)\big)$, even if A is a bounded above self-adjoint operator in a Hilbert space. Because of this we shall consider the existence of « mild » solutions of (4), i.e. continuous maps $u\colon [0,1] \to E$ such that

$$u(t) = T(t)u(1) + \int_0^t T(t-s)f\big(s, u(s)\big)\, ds\,, \qquad 0 \leqslant t \leqslant 1\,. \tag{5}$$

Such solutions, since $T(0) = \mathrm{id}$ will satisfy $u(0) = u(1)$ and will hence be 1-periodic solutions of (4).

REMARK 4. – Any mild solution of (4) (a solution of (5)) will in fact be Hölder-continuous with respect to t and will take values in suitable interpolation spaces between $D(A)$ and E provided f satisfies suitable Hölder conditions, and u will then in fact be a classical solution (see [1]).

REMARK 5. – The compactness condition (H5) will be satisfied if, for instance, the norm of $T(t)$ as an operator from E to V is integrable in a neighborhood of 0 (see [7]).

3. – Let P and Q denote canonical projections (with respect to A) of E onto N and M, respectively, and consider the continuous mapping

$$a \mapsto g(a) = \int_0^t Pf(t, a)\, dt\,. \tag{6}$$

Concerning g we shall assume:

(H6) There exists a bounded open subset $\mathcal{O}_N \subset N$ such that

$$g^{-1}(0) \cap \partial\mathcal{O}_N = \emptyset\,, \qquad d_B(t, \mathcal{O}_N, 0) \neq 0\,,$$

where $d_B(\cdot,\cdot,\cdot)$ denotes Brouwer degree.

We shall establish the following existence theorem.

THEOREM 1. – Let (H1)-(H6) be satisfied, then for all small ε equation (4) has a 1-periodic mild solution, i.e. there exists $u \in C([0,1], E)$ solving (5).

PROOF. – Our proof is patterned after some ideas early used in [2]. We consider the Banach Space

$$\mathcal{E} = C\big([0,1], E\big) \oplus (V \cap M)\,,$$

a bounded open neighborhood $\mathcal{O}_M$ of $0 \in M$ (with respect to the M norm), a bounded open neighborhood $\mathcal{O}_V$ of $0 \in V \cap M$ (with respect to the V-norm) such that

$$T(1)(\overline{\mathcal{O}}_M) \subset \mathcal{O}_V\,.$$

In $\mathcal{E}$ we define the open set $\mathcal{O}$ by

$$\mathcal{O} = \{(u, b) \in \mathcal{E}: u(t) \in \mathcal{O}_E = \mathcal{O}_N \oplus \mathcal{O}_M,\ 0 \leqslant t \leqslant 1,\ b \in \mathcal{O}_V\} .$$

and the family of completely continuous vector fields $k_{\lambda,\varepsilon}(u, b) = (v, \beta)$ given by

$$(7) \qquad \begin{cases} Pv(t) = Pu(1) + \varepsilon \displaystyle\int_0^{a(t,\lambda)} Pf(s, P\tilde{u}(s, \lambda) + \lambda Qu(s))\, ds \\ Qv(t) \ = T(t)Qu(1) + \lambda\varepsilon \displaystyle\int_0^t T(t-s)Qf(s, u(s))\, ds \\ \beta \qquad = Qv(1)\,, \end{cases}$$

where

$$a(t, \lambda) = \lambda t + (1-\lambda)\,, \qquad 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant t \leqslant 1$$
$$\tilde{u}(t, \lambda) = \lambda Pu(t) + (1-\lambda)Pu(1)\,, \qquad 0 \leqslant \lambda \leqslant 1\,.$$

We next show that there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, $\varepsilon \neq 0$ and $\lambda \in \in [0, 1]$, $k_{\lambda,\varepsilon}$ has no fixed point in $\partial\mathcal{O}$. For if this were not the case we may find sequences $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $\{\lambda_n\} \subseteq [0, 1]$, $\{(u_n, b_n)\} \subseteq \partial\mathcal{O}$ such that

$$k_{\lambda_n,\varepsilon_n}(u_n, b_n) = (u_n, b_n)\,.$$

Since $k_{\lambda,\varepsilon}$ is completely continuous we may assume that $\lambda_n \to \lambda$, $u_n \to u$, $b_n \to b$ as $n \to \infty$ and we obtain

$$(8) \qquad \begin{cases} Pu(t) = Pu(1) \\ Qu(1) = T(1)Qu(1) \\ b \qquad = Qu(1)\,. \end{cases}$$

Thus $b \in M$ is a fixed point of $T(1)$ implying that $b = 0$ (see H2). Moreover, $a = Pu(1) \in \partial\mathcal{O}_N$ implying that $g(a) \neq 0$. Putting $t = 1$ in the first equation of (7) we obtain

$$0 = \int_0^1 Pf(s, Pu_n(s, \lambda_n) + \lambda_n Qu_n(s))\, ds$$

and as $n \to \infty$

$$g(a) = \int_0^1 Pf(s, a)\, ds = 0\,,$$

a contradiction.

We hence obtain that the Leray-Schauder degree $d_{LS}(\mathrm{id} - k_{\lambda,\varepsilon}, \mathcal{O}, 0)$ is constant for small $|\varepsilon| > 0$, $0 \leqslant \lambda \leqslant 1$, and we thus may compute this degree for $\lambda = 0$, i.e. we may consider the maps

$$
(9) \qquad \begin{cases} Pv(t) = Pu(1) + \varepsilon \int_0^t Pf(s, Pu(s))\, ds \\ Qv(t) \;= T(t)Qu(1) \\ \beta \qquad = T(1)Q(1)\,. \end{cases}
$$

The last two equations define a nonsingular linear operator whose degree in $C([0, 1]; E) \oplus (M \cap V)$ is different from 0, hence the degree of $k_{0,\varepsilon}$ is given by the first equation, i.e.

$$
d_{Ls}(\mathrm{id} - k_{1,\varepsilon}, \mathcal{O}, 0) = d_{Ls}(\mathrm{id} - k_{0,\varepsilon}, \mathcal{O}, 0) = \\ = d_B(g, \mathcal{O}_N, 0) \neq 0\,,
$$

thus showing that $k_{1,\varepsilon}$ has a fixed point in $\mathcal{O}$ for $\varepsilon \neq 0$, sufficiently small.

REMARK 6. - If for $\bar{\varepsilon} \leqslant \varepsilon \leqslant 1$, $k_{1,\varepsilon}$ has no fixed points in $\partial\mathcal{O}$, then in fact (5) will have also a solution for $\varepsilon = 1$, as follows from a homotopy argument using ε as a homotopy parameter.

4. - In this section we shall consider some applications of Theorem 1; the example given at the beginning is easily seen to be contained in the first of these.

Consider the system of parabolic partial differential equations of the form

$$
(10) \qquad \frac{\partial u_k}{\partial t} = L_k u + \varepsilon f_k(t, x, u_1, \ldots, u_n)\,, \qquad t \in \boldsymbol{R},\; x \in \Omega,\; 1 \leqslant k \leqslant m\,,
$$

where for $1 \leqslant k \leqslant n$, L_k is a given uniformly elliptic second order operator together with linear homogeneous boundary conditions

$$
(11) \qquad B_k u_k = 0\,, \quad 1 \leqslant k \leqslant m\,.
$$

We define

$$
D(L_k) = \{u \in C^1(\bar{\Omega}) \colon B_k u = 0 \text{ and } L_k u \in C^0(\bar{\Omega})\}
$$

and let E_k the closure in $C^0(\bar{\Omega})$ of $D(L_k)$. We suppose that L_k is the generator of an analytic semigroup.

REMARK 7. - Both the usual Neumann and Dirichlet, as well as more general boundary conditions give such examples (see [9]).

We put

$$
E \quad = E_1 \quad \oplus \ldots \oplus E_m
$$

$$
D(L) = D(L_1) \oplus \ldots \oplus D(L_m)
$$

$$\begin{aligned} u &= u_1 \oplus \ldots \oplus u_m \\ Lu &= L_1 u_1 \oplus \ldots \oplus L_m u_m \\ Bu &= B_1 u_1 \oplus \ldots \oplus B_m u_m \end{aligned}$$

and we find that (H1)-(H5) are satisfied. Also

$$N = N_1 \oplus \ldots \oplus N_m\,, \qquad N_k = \ker L_k\,.$$

(11)-(12) may then be written as

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} = Lu + \varepsilon f(t, x, u)\,, && t \in \boldsymbol{R},\ x \in \Omega \\ &Bu = 0\,, && x \in \partial\Omega\,. \end{aligned}\right. \tag{13}$$

Associated with (13) we have the linear homogeneous problem

$$\left\{\begin{aligned} &Lu = 0\,, && x \in \Omega \\ &Bu = 0\,, && x \in \partial\Omega \end{aligned}\right. \tag{14}$$

and we obtain as a Corollary to Theorem 1:

THEOREM 2. – Let N be the finite dimensional space of all solutions of (14) and suppose $\dim N > 0$, further assume that no eigenvalue of (14) has the form $2k\pi i$, $k \neq 0$ an integer. Let $g(a)$ be defined by

$$g(a) = \int_0^1 Pf(\cdot, t, a)\, dt$$

and let there exist a bounded open neighborhood $\mathcal{O}_N$ of $0 \in N$ with $g^{-1}(0) \cap \partial\mathcal{O}_N = \emptyset$ and $d_B(g, \mathcal{O}_N, 0) \neq 0$, then (13) has a 1-periodic mild solution for all $\varepsilon \neq 0$, small.

To illustrate this result, let us consider the special case:

$$\left.\left\{\begin{aligned} &\frac{\partial u}{\partial t} = d_1 \Delta u + \varepsilon f_1(t, x, u, v) \\ &\frac{\partial v}{\partial t} = d_2 \Delta v + \varepsilon f_2(t, x, u, v)\,, \end{aligned}\right.\right\} t \in \boldsymbol{R},\ x \in \Omega,\ d_1 > 0,\ d_2 > 0\,. \tag{15}$$

Subject to the boundary conditions

$$\left.\left\{\begin{aligned} &u = 0 \\ &\frac{\partial v}{\partial \nu} = 0\,, \end{aligned}\right.\right\} t \in \boldsymbol{R},\ x \in \partial\Omega\,. \tag{16}$$

Here $N = 0 \oplus \boldsymbol{R}$ and $g\colon \boldsymbol{R} \to \boldsymbol{R}$ is given by

$$g(a) = \int_\Omega \int_0^1 f_2(t, x, 0, a)\, dt\, dx\,,$$

thus we need to require the existence of constants $a_1 < a_2$ such that

$$g(a_1) g(a_2) < 0$$

and we conclude the existence of a mild periodic solution of (15)-(16) for $\varepsilon \neq 0$ small and if f_1 and f_2 are Hölder continuous this mild solution will be classical.

We conclude by returning to the example at the beginning and we impose the stronger requirement:

There exist constants $a < b$ such that

$$f(t, x, a) > 0 > f(t, x, b)(t, x) \in \boldsymbol{R} \times \bar{\Omega}\,, \tag{17}$$

then in fact (1) will have a periodic solution for $\varepsilon = 1$. To see this, we observe that (17) clearly implies (2). In this example $\mathcal{O} = \{u \in C^0\colon a < u(t, x) < b\}$ and we see by the maximum principle that for $\varepsilon > 0$, (1) cannot have periodic solutions $u \in \partial\Omega$, thus the remark following Theorem 1 applies.

Similar results, using the ideas of invariant regions, may be obtained for systems of parabolic equations (see e.g. [8]). To illustrate this we consider the following two dimensional system

$$\begin{cases} \left.\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + \varepsilon a_1 u(1 - u - b_1 v) \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + \varepsilon a_2 v(1 - v - b_2 u) \end{aligned}\right\} (t, x) \in \boldsymbol{R} \times \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0\,, \qquad (t, x) \in \boldsymbol{R} \times \partial\Omega\,, \end{cases} \tag{18}$$

where $a_i(t, x)$, $b_i(t, x)$ are continuous positive functions of period 1 in t and $d_1 > 0$, $d_2 > 0$.

One may then show that if

$$\max_{(t,x) \in \boldsymbol{R} \times \bar{\Omega}} b_i(t, x) < 1\,,$$

then the square

$$\Sigma = \{(u, v)\colon \delta < u < 1,\ \delta < v < 1\}\,,$$

$0 < \delta \ll 1$ will be such that every solution of (18) whose range lies in $\bar{\Sigma}$ must already have its range contained in Σ, for all $\varepsilon > 0$. We hence may apply the remark following Theorem 1 once we show that the Brouwer degree of the mapping

$$(u, v) \mapsto \Big(\int_\Omega\int_0^1 a_1(x, t)\,u(1 - u - b_1(x, t)\,v)\,dx\,dt\,,\ \int_\Omega\int_0^1 a_2(x, t)\,v(1 - v - b_2(x, t)\,u)\,dx\,dt\Big) =$$

$$= \big(A_1u(1 - u) - \tilde{A}_1uv, \tilde{A}_2v(1 - v) - A_2uv\big)$$

where A_i, $\tilde{A}_i$ are positive numbers with $\tilde{A}_i < A_i$, $i = 1, 2$, relative to Σ is unequal to zero. An easy computation shows that this degree is 1.

This example gives an extension of a result in [5].

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