High Order Approximation of Implicitly Defined Maps (*) (**).

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Summary. – *Approximations for the function φ implicitly defined by $\varphi(u) = \Phi(u, \varphi(u))$ are obtained via the iterative scheme $\varphi_n(u) = \Phi(u, \varphi_{n-1}(u))$. In this paper the uniform convergence of high order derivatives of φ_n to the corresponding derivatives of φ is proved. This result yields a high order approximation theorem for the input-output map generated by a nonlinear control system, using linear combinations of iterated integrals of the control.*

0. – Introduction.

Consider a smooth mapping $\Phi\colon E\times F \to F$ acting on Banach spaces. A well-known consequence of the contraction mapping theorem is that, if the partial derivative of Φ with respect to the second variable satisfies,

$$\left\|\frac{\partial}{\partial x}\Phi(u, x)\right\| \leqslant \varepsilon < 1 \qquad (u, x) \in E\times F$$

then the equation $x = \Phi(u, x)$ implicitly defines a unique continuous function $x = \varphi(u)$. Moreover, the sequence of mappings

$$\varphi_0(u) \equiv 0, \; \dots, \; \varphi_n(u) = \Phi\big(u, \varphi_{n-1}(u)\big), \; \dots$$

converges to φ uniformly on bounded sets. If Φ is k-times continuously differentiable, such are φ and φ_n $(n \geqslant 0)$ as well. In this paper we show that the convergence of φ_n to φ actually takes place in the C^k norm. In theorem 1, § 2, the uniform, geometric rate of convergence of the derivatives $D^j\varphi_n$ to $D^j\varphi$ $(j = 0, \dots, k)$ is established. In § 3 we consider a second map $\Psi\colon E\times F \to F$ which approximates Φ in the C^k norm and give an estimate on the C^k norm of the difference $\varphi - \psi$, where $\psi(u)$ is implicitly defined by $\psi(u) = \Psi\big(u, \psi(u)\big)$. The proofs of the above results both rely on prolongation techniques, in the spirit of classical Lie theory [3, 4].

The primary motivation for the present study came from control theory. Indeed a control system of the form

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i\,, \qquad x(0) = \xi \in \mathbf{R}^n \tag{1}$$

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generates an input-output map $\varphi\colon \mathcal{L}^1([0, T]; \boldsymbol{R}^m) \to \mathcal{C}([0, T]; \boldsymbol{R}^n)$, $\varphi(u(\cdot)) = x(\cdot)$ implicitly defined by:

$$x(t) = \xi + \sum_{i=1}^{m} \int_0^t g_i(x(s))\, u_i(s)\, ds .$$

In general there exist no explicit formulas giving the trajectory $x(\cdot)$ directly in terms of the control. However, one can approximate φ in the following way. First replace the g_i's by vector fields q_i having polynomial components. Then compute the Picard iterates $\psi_n(u)$ for the approximate system

$$\dot{x} = \sum_{i=1}^{m} q_i(x)\, u_i , \qquad x(0) = \xi .$$

Using our abstract results, in § 4 we show that these Picard iterates do indeed approximate the input-output map φ in the $\mathcal{C}^k$-norm of functionals. The uniform approximation theorem for φ in terms of iterated integrals of the controls u_i, given in [2] for the $\mathcal{C}^0$ norm, can thus be extended to higher order norms.

1. – Preliminaries.

In this paper, differential calculus in abstract spaces is used throughout. Given two Banach spaces E and F, $k \geq 0$, we denote by $L^k(E, F)$ the space of continuous k-linear mappings Λ from $\bigotimes_k E = E \times E \times \ldots \times E$ (k times) into F with the norm

$$\|\Lambda\|_{L^k(E,F)} = \sup \{\|\Lambda(u_1, \ldots, u_k)\|_F;\ \|u_i\|_E \leq 1,\ i = 1, \ldots, k\} .$$

In the following, subscripts to the norms will be suppressed whenever this cannot generate confusion. The closed ball centered at x with radius ϱ is written $B(x, \varrho)$. If ψ is a smooth mapping from an open subset V of E into F, its k-th Fréchet derivative at a point $u \in V$ is $D^k\psi(u)$, $L^k(E, F)$. We use the conventions $D^0\psi(u) = \psi(u)$, $L^0(E, F) = F$.

It is well known that high-order derivatives are symmetric multilinear mappings. $D^k\psi(u)$ is therefore completely determined by assigning its values on elements of the form $u^{[k]} = (u, u, \ldots, u) \in \bigotimes_k E$. Partial derivatives of a function $\Psi = \Psi(u, x)$ defined on a product space $E \times F$ are denoted by ∂_u, ∂_x. High order total derivatives of a composite mapping $u \to \Psi(u, \psi(u))$ will also be used.

LEMMA 1. – *Let $\Psi\colon E \times F \to F$ and $\psi\colon E \to F$ be smooth mappings, $m \geq 1$. Then the m-th total derivative $D^m\Psi(u, \psi(u))$ is given formally by a sum of $\leq (m+1)!$ monomials having degree $\leq m$ in the terms $D^i\psi(u)$, $i = 1, \ldots, m$. Each one of these* *monomials has the form*

$$\partial_u^i \partial_x^j \Psi(u, \psi(u)) \cdot (D^1 \psi(u))^{[\alpha_1]} \dots (D^m \psi(u))^{[\alpha_m]}, \tag{1.1}$$

with

$$1 \leqslant i + j \leqslant m, \quad \sum_{l=1}^m \alpha_l = j, \quad i + \sum_{l=1}^m l \cdot \alpha_l = m. \tag{1.2}$$

Moreover, there is a unique monomial for which $\alpha_m \neq 0$, *namely* $\partial_x \Psi(u, \psi(u)) \cdot D^m \psi(u)$.

Notice that in (1.1) the expression $\partial_u^i \partial_x^j \Psi(u, \psi(u))$ denotes an $i+j$-linear map from $(\bigotimes_i E) \times (\bigotimes_j F)$ into F, and the formal power $(D^i \psi(u))^{[\alpha_i]}$ is interpreted as the vector with α_i equal components $(D^i \psi(u), \dots, D^i \psi(u)) \in \bigotimes_{\alpha_i} L^i(E, F)$.

To prove the lemma, one checks that the assertions hold when $m = 1$ and proceeds by induction. If (1.2) holds up to a certain m, differentiating (1.1) with respect to u we get the two terms

$$\partial_u^{i+1} \partial_x^j \Psi(u, \psi(u)) \cdot (D^1 \psi(u))^{[\alpha_1]} \dots (D^m \psi(u))^{[\alpha_m]}, \tag{1.3}$$

$$\partial_u^i \partial_x^{j+1} \Psi(u, \psi(u)) (D^1 \psi(u))^{[\alpha_1+1]} \dots (D^m \psi(u))^{[\alpha_m]}. \tag{1.4}$$

Moreover, for every $l = 1, \dots, m$, we get the α_l identical terms

$$\partial_u^i \partial_x^j \Psi(u, \psi(u)) (D^1 \psi(u))^{[\alpha_1]} \dots (D^l \psi(u))^{[\alpha_l - 1]} (D^{l+1} \psi(u))^{[\alpha_{l+1}+1]} \dots (D^m \psi(u))^{[\alpha_m]}. \tag{1.5}$$

Therefore each of the $\leqslant (m+1)!$ terms in the expression for $D^m \Psi(u, \psi(u))$ yields no more then $m+2$ terms in the expression for $D^{m+1} \Psi(u, \psi(u))$. An inspection of (1.3) to (1.5) shows that all of these monomials have the form (1.1) and satisfy (1.2), with m replaced by $m+1$. The last statement is clear.

A map ψ defined on an open subset V of a Banach space is $\mathcal{C}^k$ if it is k-times differentiable in the sense of Fréchet and the mappings $u \to D^j \psi(u)$ $(j = 0, \dots, k)$ are continuous on V. The $\mathcal{C}^k$ norm of ψ on a subset $U \subseteq V$ is

$$\|\psi\|_{\mathcal{C}^k(U)} = \sup \{\|D^j \psi(u)\|;\ u \in U, j = 0, \dots, k\}.$$

For the basic properties of differential calculus in Banach spaces, our general reference is DIEUDONNÉ [1].

2. – The main convergence theorem.

THEOREM 1. – *Let* E, F *be Banach spaces,* $U = B(u_0, \varrho) \subset E$, $V = B(0, \varrho_0) \subset F$. *Let* Ψ *be a* $\mathcal{C}^{k+1}$ *mapping from an open neighborhood of* $U \times V$ *into* F *such that, for* *some* $\varepsilon < 1$ *and all* $u \in U$, $x \in V$

$$\|\partial_x \Psi(u, x)\| \leqslant \varepsilon , \qquad \|\Psi(u, 0)\| \leqslant \varrho_0(1-\varepsilon) . \tag{2.1}$$

Let $\|\Psi\|_{k+1(U\times V)} \leqslant M$ *with* $1 \leqslant M < \infty$. *Then there exists a unique* $\mathcal{C}^{k+1}$ *map* $\psi: U \to V$ *satisfying*

$$\psi(u) = \Psi(u, \psi(u)) \tag{2.2}$$

for every $u \in U$. *If the sequence of mappings* $(\psi_n)_{n\geqslant 0}$ *is recursively defined by*

$$\psi_0(u) \equiv 0, \;\ldots,\; \psi_n(v) = \Psi(u, \psi_{n-1}(u)), \;\ldots \tag{2.3}$$

then, for $0 \leqslant m \leqslant k$, *the sequence of derivatives* $(D^m \psi_n(u))_{n\geqslant 0}$ *converges to* $D^m\psi(u)$ *absolutely and uniformly on* U.

PROOF. – The existence and the uniqueness of ψ are a consequence of the classica contraction mapping theorem [1, p. 260], the regularity of ψ follows from the implicit function theorem [1, p. 268]. To prove the convergence of the sequence $(D^m\psi_n)$ to $D^m\psi$, we construct a prolongation $\tilde{\Psi}$ of Ψ as follows. Let the constants ϱ_i $(i = 1, \ldots, k)$ be defined by

$$\varrho_1 = \frac{M}{1-\varepsilon}, \;\ldots,\; \varrho_i = \frac{M(i+1)!}{1-\varepsilon}(\varrho_{i-1})^i, \;\ldots . \tag{2.4}$$

Let $\tilde{F} = F \times L^1(E, F) \times L^2(E, F) \times .. \times L^k(E, F)$, $\tilde{V} = V_0 \times V_1 \times \ldots \times V_k \subset \tilde{F}$, where $V_0 = V$ and $V_i = B(0, \varrho_i) \subset L^i(E, F)$ for $i = 1, \ldots, k$.

Elements in $\tilde{F}$ are denoted by $\tilde{x} = (x_0, x_1, \ldots, x_k)$, $\|\tilde{x}\|_{\tilde{F}} = \sup\{\|x_i\|_{L^i(E,F)}, i = 0, \ldots, k\}$. Define a continuous map $\tilde{\Psi}: U \times \tilde{V} \to \tilde{F}$, $\tilde{\Psi}(u, \tilde{x}) = (\Psi_0, \Psi_1, \ldots, \Psi_k)$ by setting

$$\Psi_i(u, \tilde{x}) = D^i \Psi(u, \psi(u))|_{D^j\psi(u) = x_j} \qquad (j = 0, \ldots, i) . \tag{2.5}$$

The i-th component of $\tilde{\Psi}$ is therefore obtained by formally computing the i-th total derivative of $\Psi(u, \psi(u))$ with respect to u and by replacing the terms $D^j\psi(u)$ with the free variables x_i $(0 \leqslant j \leqslant i)$ wherever they occur. Notice that all partial derivatives of Ψ are evaluated at (u, x_0).

The system

$$\tilde{x} = \tilde{\Psi}(u, \tilde{x}) \tag{2.6}$$

is thus a set of $k+1$ implicit equations that we will solve for $\tilde{x}$ in terms of u by means of the contraction mapping theorem. Define a sequence of mappings $\psi_n\colon U \to \tilde{F}$ by

$$\tilde{\varphi}_0(u) = 0,\ \dots,\ \tilde{\varphi}_n(u) = \tilde{\Psi}(u, \tilde{\varphi}_{n-1}(u)),\ \dots. \tag{2.7}$$

From (2.7) and (2.5) it follows by induction that for every $n \geqslant 0$

$$\tilde{\varphi}_n(u) = \big(\psi_n(u), D\psi_n(u), D^2\psi_n(u), \dots, D^k\psi_n(u)\big)\,. \tag{2.8}$$

The theorem will be proved by showing the absolute and uniform convergence of the sequence $(\tilde{\varphi}_n)_{n\geqslant 0}$. This, in turn, will be a consequence of

i) $\tilde{\Psi}$ maps $U \times \tilde{V}$ continuously into $\tilde{V}$.

ii) There exists an equivalent norm $\|\ \|'$ on $\tilde{F}$ such that, for all $u \in U$, $\tilde{x}, \tilde{y} \in \tilde{V}$,

$$\|\tilde{\Psi}(u, \tilde{x}) - \tilde{\Psi}(u, \tilde{y})\|' \leqslant (2\varepsilon - \varepsilon^2)\|\tilde{x} - \tilde{y}\|'\,. \tag{2.9}$$

A preliminary extimate is needed.

LEMMA 2. – *Let* Ψ_m $(0 \leqslant m \leqslant k)$ *be defined by* (2.5). *Then for every* $(u, \tilde{x}) \in U \times \tilde{V}$ *the following bounds hold*:

$$\|\partial_{x_l}\Psi_m(u, \tilde{x})\| = 0 \qquad \textit{if } l > m\,, \tag{2.10}$$

$$\|\partial_{x_m}\Psi_m(u, \tilde{x})\| \leqslant \varepsilon\,, \tag{2.11}$$

$$\|\partial_{x_l}\Psi_m(u, \tilde{x})\| \leqslant M(m+1)!\,\varrho_m^m \qquad \textit{if } 0 \leqslant l < m\,. \tag{2.12}$$

PROOF. – By Lemma 1, $\Psi_m(u, \tilde{x})$ is the sum of no more than $(m+1)!$ terms of the form

$$A = \partial_u^i \partial_x^j \Psi(u, x_0) x_1^{[\alpha_1]} \dots x_m^{[\alpha_m]} \tag{2.13}$$

where i, j, α_l satisfy (1.2). Thus (2.10) is clear, and (2.11) holds because $\|\partial_{x_m}\Psi_m(u, \tilde{x})\| = \|\partial_x\Psi(u, x_0)\|$. Differentiating A in (2.13) with respect to x_0 and x_l $(1 \leqslant l \leqslant m)$ one gets

$$\|\partial_{x_0}A\| = \|\partial_u^i\partial_x^{j+1}\Psi(u, x_0)\, x_1^{[\alpha_1]}\cdot x \dots x_m^{[\alpha_m]}\| \leqslant M\cdot \varrho_1^{\alpha_1} \dots \varrho_m^{\alpha_m} < M\varrho_m^m\,,$$

$$\|\partial_{x_l}A\| = \alpha_l\|\partial_u^i\partial_x^j\Psi(u, x_0)x_1^{[\alpha_1]} \dots x_l^{[\alpha_l-1]} \dots x_m^{[\alpha_m]}\| \leqslant$$
$$\leqslant \alpha_l M\cdot\varrho_1^{\alpha_1} \dots \varrho_l^{\alpha_l-1} \dots \varrho_m^{\alpha_m} \leqslant mM\varrho_m^{m-1} \leqslant M\varrho_m^m\,.$$

These two inequalities yield (2.12) in the cases $l = 0$ and $l > 0$ respectively. We can now give a proof of i). By Lemma 1, $\Psi_m(u, \tilde{x})$ is the sum of less than

$(m+1)!$ monomials of the form (2.17) in the variables $x_1, \ldots, x_{m-1}$, plus the single term $\partial_x \Psi(u, x_0)\cdot x_m$. This yields the estimate

$$\|\Psi_m(u, \tilde{x})\| \leqslant M(m+1)!\varrho_{m-1}^m + \varepsilon\varrho_m \leqslant (1-\varepsilon)\varrho_m + \varepsilon\varrho_m$$

because of (2.4). Hence $\Psi_m(u, \tilde{x}) \in V_m$ for $1 \leqslant m \leqslant k$, $(u, \tilde{x}) \in U \times \tilde{V}$. The estimate for $m = 0$ is straight-forward.

To prove ii), introduce the constant

$$C = 2Mk(k+1)!\varrho_k^k\cdot(\varepsilon - \varepsilon^2)^{-1}\,. \tag{2.14}$$

Define on E the equivalent norm $\|u\|' = C\|u\|$, and denote with $\|\cdot\|'$ the induced norms on the spaces $F_i = L^i(E, F)$ and $L(F_i, F_j)$. Notice that, if $\psi\colon E \to F$ is smooth at u and if $\chi \in L(F_i, F_j)$, we have

$$\|D^i\psi(u)\|' = C^{-i}\|D^i\psi(u)\|\,, \qquad \|\chi\|' = C^{i-j}\|\chi\|\,. \tag{2.15}$$

Consider now $u \in U$, $\tilde{x}, \tilde{y} \in \tilde{V}$. Recalling the definition of the norm on the product space $\tilde{F}$ we have

$$\|\tilde{\Psi}(u, \tilde{x}) - \tilde{\Psi}(u, \tilde{y})\|' = \sup\{\|\Psi_m(u, \tilde{x}) - \Psi_m(u, \tilde{y})\|', 0 \leqslant m \leqslant k\}\,.$$

Fix some m and let $\tilde{x} = (x_0, \ldots, x_k)$, $\tilde{y} = (y_0, \ldots, y_k)$. Then mean value theorem [1, p. 155] together with Lemma 2 yields

$$\begin{aligned}\|\Psi_m(u, \tilde{x}) - \Psi_m(u, \tilde{y})\|' &\leqslant \sum_{i=0}^{m} \sup\{\|\partial_{x_i}\Psi_m(u, \tilde{z})\|';\ \tilde{z} \in \tilde{V}\}\cdot\|x_i - y_i\|' \leqslant \\ &\leqslant \sum_{i=0}^{m-1} C\cdot M(m+1)!\varrho_m^m + \varepsilon\|x_m - y_m\|' \leqslant \\ &\leqslant [k\cdot C^{-1}M(m+1)!\varrho_k^k + \varepsilon]\cdot\sup\{\|x_i - y_i\|';\ 0 \leqslant i \leqslant k\} \leqslant [(\varepsilon - \varepsilon^2) + \varepsilon]\cdot\|\tilde{x} - \tilde{y}\|'\,.\end{aligned}$$

This proves ii). The contraction mapping theorem aplied to $\tilde{\Psi}$ now implies the absolute and uniform convergence of the sequence $\tilde{\varphi}_n(u)$ to some $\tilde{\varphi}(u)k = (\psi(u), \psi^{(1)}(u), \ldots, \psi^{(k)}(u))$ in the new norm $\|\cdot\|'$, hence in the old norm as well. By (2.8), this means that for $m = 0, \ldots, k$, the sequence of derivatives $(D^m\psi_n)_{n\geqslant 0}$ tends to $\psi^{(m)}$ uniformly on U. A classical convergence theorem [1, p. 158] now implies

$$\tilde{\varphi}(u) = (\psi(u), D\psi(u), D^2\psi(u), \ldots, D^k\Psi(u))\,, \tag{2116}$$

completing the proof.

Notice that from (2.9) the geometric rate of convergence can be easily inferred.

3. – Further estimates.

Suppose we are interested in computing an approximation to the map φ implicitly defined by

$$\varphi(u) = \Phi(u, \varphi(u)) . \tag{3.1}$$

We do this by first considering a simpler mapping Ψ which is suitably close to Φ. Then we iteratively compute the mappings ψ_n defined at (2.3), which are approximate solutions of $\psi(u) = \Psi(u, \psi(u))$. The functions ψ_n can be themselves regarded as approximations of φ. Using the techniques of the previous section, an estimate on the C^k norm of the difference $\psi_n - \varphi$ is now given. To eliminate the dependence on ε of the various constants, we make the simplifying assumption $\varepsilon \leqslant \frac{1}{2}$. The general case can be treated in a similar fashion.

THEOREM 2. – *Let all of the assumptions in Theorem* 1 *hold, with* $\varepsilon = \frac{1}{2}$. *Let* Φ *be a second mapping that satisfies the exact same hypothesis made on* Ψ, *and let* $\varphi\colon U \to V$ *be the unique solution of* (3.1). *If* $\|\Phi - \Psi\|_{C^k(U\times V)} \leqslant \eta$, *then for all* $n \geqslant 0$

$$\|\psi_n - \varphi\|_{C^k(U)} \leqslant L\cdot[\eta + (\tfrac{3}{4})^n] , \tag{3.2}$$

where L *is a constant depending only on* $M = \max\left(\|\Psi\|_{C^{k+1}}, \|\Phi\|_{C^{k+1}}\right)$.

PROOF. – Define the constants $\varrho_i = \varrho_i(M)$ $(i = 1, ..., k)$ by

$$\varrho_1 = \frac{M}{2} , \; ..., \quad \varrho_i = \frac{M}{2}(i+1)!\,\varrho_{i-1}^i, \; ...$$

and define $\tilde{F}$, $\tilde{V}$ and the prolongations $\tilde{\Psi}$, $\tilde{\Phi}\colon U \times \tilde{V} \to \tilde{V}$ as in the proof of Theorem 1. By setting

$$C = 8Mk(k+1)!\,\varrho_k^k , \qquad \|\cdot\|_E' = C\|\cdot\|_E$$

and again denoting by $\|\cdot\|'$ the induced norms on the spaces $F_i = L^i(E, F)$ and on their product $\tilde{F}$, we have

$$\|\tilde{\Psi}(u, \tilde{x}) - \tilde{\Psi}(u, \tilde{y})\|' \leqslant \tfrac{3}{4}\|\tilde{x} - \tilde{y}\|' \tag{3.3}$$

for all $u \in U$ and $\tilde{x}, \tilde{y} \in \tilde{V}$, and the same holds for Φ. All of this is clearly a consequence of i), ii) in § 2.

We now seek a bound on $\|\tilde{\Phi}-\tilde{\Psi}\|'$. If $(u, \tilde{x}) \in U \times \tilde{V}$, $i+j \leqslant k$, $\sum_l \alpha_l \leqslant k$ then

$$\|\partial_u^i \partial_x^j \Phi(u, x_0) x_1^{[\alpha_1]} \dots x_k^{[\alpha_k]} - \partial_u^i \partial_x^j \Psi(u, x_0) x_1^{[\alpha_1]} \dots x_k^{[\alpha_k]}\| \leqslant \eta \cdot \|x_1\|^{\alpha_1} \dots \|x_k\|^{\alpha_k} \leqslant \eta \varrho_k^k .$$

Using Lemma 1 this yields

$$\text{(3.4)} \qquad \|\tilde{\Phi}(u, \tilde{x}) - \tilde{\Psi}(u, \tilde{x})\|' \leqslant \|\tilde{\Phi}(u, \tilde{x}) - \tilde{\Psi}(u, \tilde{x})\| = \\ = \sup\{\|\Phi_m(u, \tilde{x}) - \Psi_m(u, \tilde{x})\|;\ 0 \leqslant m \leqslant k\} \leqslant (k+1)!\,\eta \varrho_k^k$$

because the new norm $\|\cdot\|'$ is smaller than the old one on each space $L^i(E, F)$.

To complete the proof, define the sequence of maps $\tilde{\Phi}_n\colon U \to \tilde{V}$ by

$$\tilde{\varphi}_0(u) = 0,\ \dots,\ \tilde{\varphi}_n(u) = \tilde{\Phi}(u, \tilde{\varphi}_{n-1}(u)),\ \dots .$$

We claim that

$$\text{(3.5)} \qquad \|\tilde{\varphi}_n(u) - \tilde{\psi}_n(u)\|' \leqslant 4(k+1)!\,\eta \varrho_k^k$$

for all $u \in U$, $n \geqslant 0$. This is trivially true when $n = 0$. If (3.5) holds for a certain n, then (3.4) and (3.3) imply

$$\|\tilde{\varphi}_{n+1}(u) - \tilde{\psi}_{n+1}(u)\|' \leqslant \|\tilde{\Phi}(u, \tilde{\varphi}_n(u)) - \tilde{\Psi}(u, \tilde{\varphi}_n(u))\|' + \|\tilde{\Psi}(u, \tilde{\varphi}_n(u)) - \\ - \tilde{\Psi}(u, \tilde{\psi}_n(u))\|' \leqslant (k+1)!\,\eta \varrho_k^k + \tfrac{3}{4}\|\tilde{\varphi}_n(u) - \tilde{\psi}_n(u)\|' \leqslant 4(k+1)!\,\eta \varrho_k^k .$$

By induction this proves (3.5) for all n. The contraction mapping theorem applied to the map $\tilde{x} \to \tilde{\Phi}(u, \tilde{x})$ yields

$$\text{(3.6)} \qquad \|\tilde{\varphi}_n(u) - \tilde{\varphi}(u)\|' \leqslant \sum_{j=n}^{\infty} (\tfrac{3}{4})^j \|\tilde{\Phi}(u, 0)\|' \leqslant 4M(\tfrac{3}{4})^n$$

for all $u \in U$, $n \geqslant 0$. Putting together (3.5) with (3.6) and using (2.15) we get an estimate involving the old norm:

$$\|\tilde{\varphi}(u) - \tilde{\psi}_n(u)\| \leqslant C^k \|\tilde{\varphi}(u) - \tilde{\psi}_n(u)\|' \leqslant C^k[4(k+1)!\,\eta \varrho_k^k + 4M(\tfrac{3}{4})^n] .$$

By (2.8) and (2.14), this yields (3.2) with

$$\text{(3.7)} \qquad L(M) = [8Mk(k+1)!\,\varrho_k^k]^k \big(4(k+1)!\,\varrho_k^k + 4M\big) .$$

Notice that ϱ_k, and hence L, depend only on the constant M, as required.

4. – Approximation of control systems.

Consider the control system (1), which we now write in the more compact form

$$\dot{x}(t) = G\big(x(t)\big)\cdot u(t)\,, \qquad x(0) = \xi\,. \tag{4.1}$$

G is then an $n\times m$ matrix valued function defined on $\boldsymbol{R}^n$ and $u(t)\in\boldsymbol{R}^m$. Assume $G\in\mathcal{C}^{k+1}$ and call $G^{(j)}$ its j-th derivative. Taylor's formula is

$$G(x) = \sum_{j=0}^{k}\frac{1}{j!}G^{(j)}(x-x_0)^{[j]} + O\big(|x-x_0|^{k+1}\big)\,.$$

Define the mapping Φ: $\mathcal{L}^1([0,T];\boldsymbol{R}^m)\times\mathcal{C}^0([0,T];\boldsymbol{R}^n)\to\mathcal{C}^0([0,T];\boldsymbol{R}^n)$ by

$$\Phi(u,x)(t) = \xi + \int_0^t G\big(x(s)\big)u(s)\,ds\,. \tag{4.2}$$

Notice that $\Phi = \Phi''\circ\Phi'$ with $\Phi'(u,x)(t) = \big(u(t), G(x(t))\big)$ and

$$\Phi''(u,Z)(t) = \xi + \int_0^t Z(s)u(s)\,ds\,.$$

Clearly Φ' is a $k+1$ times Fréchet differentiable substitution operator and Φ'' is bilinear, hence Φ is $\mathcal{C}^{k+1}$. In particular, $\partial_x^j\Phi(u_0,x_0)(t)$ is the j-linear map

$$y^{[j]}\to\int_0^t G^{(j)}\big(x_0(s)\big)\cdot y^{[j]}(s)u_0(s)\,ds\,,$$

$\partial_u\partial_x^j\Phi(u_0,x_0)(t)$ is the multilinear map

$$(u, y^{[j]})\to\int_0^t G^{(j)}\big(x_0(s)\big)y^{[j]}(s)u(s)\,ds$$

and $\partial_u^i\partial_x^j\Phi\equiv 0$ for $i>1$, because the dependence on u is linear.

By an iterated integral of the control $u=(u_1,\dots,u_m)$ we mean a scalar map of the form

$$I(u,t) = \int_0^\tau\int_0^{\sigma_1}\dots\int_0^{\sigma_{r-1}} u_{i_r}(\sigma_r)\,d\sigma_r\dots u_{i_1}(\sigma_1)\,d\sigma_1\,,$$

where $i_1,\dots,i_r\in\{1,\dots,m\}$. Some basic approximation theorems in terms of iterated integrals are given in [2]. Using the previous abstract results, we now show that the input-output map $u(\cdot) \to x(\cdot)$ generated by (4.1) can be approximated by linear combinations of iterated integrals of u, in a high-order norm, uniformly on compact sets.

THEOREM 3. – *Let G in (4.1) be a $\mathcal{C}^{k+1}$ mapping from $\boldsymbol{R}^n$ into $\boldsymbol{R}^{n\times m}$. Then for every compact $K \subset \boldsymbol{R}^n$, T and $\varepsilon > 0$, there exists a finite family of polynomial maps $p_\alpha\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ and iterated integrals I_α such that the map*

$$y(\xi, u, t) = \sum_\alpha p_\alpha(\xi)\cdot I_\alpha(u, t) \tag{4.3}$$

satisfies

$$\|\partial_u^j y(\xi, u, t) - \partial_u^j x(\xi, u, t)\| \leqslant \varepsilon \tag{4.4}$$

for every $t \in [0, T]$, $j = 0, \ldots, k$, $\xi \in K$ and every control u with $|u_i(s)| \leqslant 1$ $(i = 1, \ldots, m)$ and such that the corresponding solution $t \to x(\xi, u, t)$ of (4.1) lies entirely inside K.

PROOF. – Fix $K \subset \boldsymbol{R}^n$, T and $\varepsilon > 0$. It is clearly not restrictive to assume that the support of G is compact. Otherwise one can replace G with a map $\bar{G}$ which coincides with G on a neighborhood of K and has compact support.

Let $M = \|G\|_{\mathcal{C}^{k+1}(\boldsymbol{R}^n)} < \infty$, let $K \subset B(0, r)$ and define the sets

$$U = \big\{u \in \mathcal{L}^1([0, T]; \boldsymbol{R}^m);\ |u_i(s)| \leqslant 1,\ i = 1, \ldots, m,\ s \in [0, T]\big\}$$
$$V = \big\{x \in \mathcal{C}^0([0, T]; \boldsymbol{R}^n);\ x(s) \in B\big(0, r + m(M + 1)\big),\ s \in [0, T]\big\}\,.$$

For every integer $\nu \geqslant 1$; a classical approximation theorem [5, p. 155] guarantees the existence of a map $F_\nu\colon \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$, having polynomial components, such that

$$\|F_\nu - G\|_{\mathcal{C}^{k+1}(B(0, r + m(M+1)))} \leqslant 1/\nu\,.$$

Consider the map Ψ_ν defined by

$$\Psi_\nu(u, x)(t) = \xi + \int_0^t F_\nu\big(x(s)\big)\, u(s)\, ds\,.$$

For each $\xi \in K$, $\nu \geqslant 1$, both Φ, defined at (4.2), and Ψ_ν map $U \times V$ into V. Moreover, by using a suitable equivalent norm on $\mathcal{C}^0[0, T]$, of the form

$$\|x(\cdot)\|^\dagger = \sup\,\{\exp\,[-\lambda t]\,|x(t)|;\ t \in [0, T]\}\,,$$

we have $\|\partial_x \Phi\|^\dagger \leqslant \frac{1}{2}$, $\|\partial_x \Psi_\nu\|^\dagger \leqslant \frac{1}{2}$ on $U \times V$, for λ large enough.

Let y_ν be the ν-th Picard iterate for the system

$$\dot{x}(t) = F_\nu(x(t))\,u(t), , \qquad x(0) = \xi .$$

Theorem 2 implies that

$$\|\partial_u^j y_\nu(\xi, u, \cdot) - \partial_u^j x(\xi, u, \cdot)\|^\dagger \to 0 \qquad \text{as } \nu \to \infty ,$$

for $j = 0, \ldots, k$, uniformly with respect to $\xi \in K$, $u \in U$. It is well known [2] that every y_ν can be written as a linear combination of iterated integrals. Hence, by setting $y = y_\nu$ with ν suitably large, the theorem is proved.

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