On the Nonlinear Parabolic Systems in Divergence Form. Hölder Continuity and Partial Hölder Continuity of the Solutions (*) (**).

SERGIO CAMPANATO (Pisa)

Sunto. – *Ω è un aperto limitato di R^n, $n \geqslant 1$. Nel cilindro $Q = \Omega \times (-T, 0)$, di punto $X = (x, t)$, si considera il sistema non lineare, in forma di divergenza,*

$$(1) \qquad -\sum_{1=i}^{n} D_i A^i(X, u, Du) + \frac{\partial u}{\partial t} = -\sum_{i=1}^{n} D_i B^i(X, u) + B^0(X, u, Du)$$

dove u, A^i, B^i, B^0 sono vettori di R^N, $N > 1$. Si suppone che il sistema (1) sia fortemente parabolico e che i vettori A^i, B^i, B^0 abbiano andamenti strettamente controllati. In queste ipotesi, si studia la regolarità, o la parziale regolarità, hölderiana delle soluzioni

$$u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega)) .$$

Preliminare è lo studio dei sistemi non lineari del tipo

$$(2) \qquad -\sum_i D_i A^i(Du) + \frac{\partial u}{\partial t} = 0 \quad in\ Q$$

che hanno lo stesso ruolo che, nella teoria lineare, hanno i sistemi a coefficienti costanti e ridotti alla parte principale. Questo studio, che ha interesse in sè, viene fatto nei paragrafi 3, 4, 5 e 6. Per le soluzioni del sistema (2), si dimostrano la locale differenziabilità, le maggiorazioni tipo Poincaré e le cosidette maggiorazioni fondamentali dalle quali si deduce, in particolare, che le soluzioni del sistema (2) sono hölderiane in Q se $n \leqslant 2$. Per maggiori dettagli si veda l'introduzione.

1. – Introduction.

For the sake of simplicity, throughout the present work we will be concerned with second order differential systems, even if what we will prove could be extended to systems of even order $2m$.

Let Ω be a bounded open subset of R^n, with $n \geqslant 1$, whose boundary $\partial\Omega$ is as smooth as necessary; x is a point of R^n; $t \in R$ and $X = (x, t)$ is a point of $R^n \times R$.

N is an integer > 1 (1), $(|)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in R^k, respectively. We will drop the subscript k when there is no fear of confusion.

(*) Entrata in Redazione il 27 luglio 1983.

(**) Lavoro eseguito nell'ambito di un progetto nazionale di ricerca finanziato dal Ministero della Pubblica Istruzione (40%-1982).

(1) For the case $N = 1$ (equations) see for instance [10].

Set $Q = \Omega\times(-T, 0)$ with $T > 0$.

If $X_0 = (x^0, t_0)$ we define

$$B(x^0, \sigma) = \{x \in R^n : \|x - x^0\| < \sigma\}$$

$$Q(X_0, \sigma) = B(x^0, \sigma)\times(t_0 - \sigma^2, t_0) .$$

Moreover, we say that $Q(X_0, \sigma) \subset\subset Q$ if

$$B(x^0, \sigma) \subset\subset \Omega \quad \text{and} \quad \sigma^2 < t_0 + T \leqslant T .$$

If $u: Q \to R^N$, we set $Du = (D_1 u, \ldots, D_n u)$ where, as usual, $D_i = \partial/\partial x_i$. Clearly $Du \in R^{nN}$ and we denote by $p = (p^1, \ldots, p^n)$, $p^i \in R^N$, a typical vector of R^{nN}.

Let $a^i(X, u, p)$, $i = 1, \ldots, n$, and $B^0(X, u, p)$ be vectors of R^N, defined in $\Lambda = Q\times R^N \times R^{nN}$, measurable in X and continuous in (u, p).

Let us consider the nonlinear differential operator of second order

$$Eu = -\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} - B^0(X, u, Du) . \tag{1.1}$$

Having set

$$\begin{cases} A^i(X, u, p) = a^i(X, u, p) - a^i(X, u, 0) \\ B^i(X, u) = -a^i(X, u, 0) \end{cases} \tag{1.2}$$

it can be written in the form

$$Eu = E_0 u + \sum_i D_i B^i(X, u) - B^0(X, u, Du) \tag{1.3}$$

where

$$E_0 u = -\sum_i D_i A^i(X, u, Du) + \frac{\partial u}{\partial t} \tag{1.4}$$

is the principal part of the operator E.

Let us suppose that the vector mappings $p \to a^i(X, u, p)$ are differentiable with derivatives $\partial a^i/\partial p^j_k$ measurable in X, continuous in (u, p) and bounded in Λ:

$$\left\{\sum_{ij}\sum_{hk}\left|\frac{\partial a^i_h}{\partial p^j_k}\right|^2\right\}^{\frac12} \leqslant M , \quad \forall (X, u, p) \in \Lambda . \tag{1.5}$$

Set

$$A_{ij} = \{A^{hk}_{ij}\} \quad \text{with} \quad A^{hk}_{ij}(X, u, p) = \int_0^1 \frac{\partial a^i_h(X, u, \tau p)}{\partial p^j_k}\, d\tau .$$

The A_{ij} are $N\times N$ matrices, measurable in X, continuous in (u, p) and

$$A^i(X, u, p) = \sum_j A_{ij}(X, u, p)p^j \tag{1.6}$$

$$E_0 u = -\sum_{ij} D_i(A_{ij}(X, u, Du)\, D_j u) + \frac{\partial u}{\partial t}\,. \tag{1.7}$$

We will say that E is quasi-linear if

$$A_{ij} = A_{ij}(X, u)\,. \tag{1.8}$$

We will say that the operator E has linear principal part if

$$A_{ij} = A_{ij}(X)\,. \tag{1.9}$$

Let us suppose that the operator E is strongly parabolic in the following sense: there exists $\nu > 0$ such that

$$\sum_{ij}\sum_{hk} \frac{\partial a_h^i(X, u, p)}{\partial p_k^j}\xi_h^i \xi_k^j \geqslant \nu\|\xi\|^2 \tag{1.10}$$

for every $(X, u, p) \in \Lambda$ and for any $\xi = (\xi^1, \dots, \xi^n) \in R^{nN}$.

Denote by $H^k = H^{k,2}$ and $H_0^k = H_0^{k,2}$ the usual Sobolev spaces and set

$$a(u, \varphi) = \int_Q \sum_i (a^i(X, u, Du)|D_i\varphi)_N - \left(u \left| \frac{\partial\varphi}{\partial t}\right.\right)_N dX \tag{1.11}$$

$$W(Q) = L^2(-T, 0, H_0^1(\Omega)) \cap H^1(-T, 0, L^2(\Omega))\,. \tag{1.12}$$

Throughout this paper, by a solution of the system

$$Eu = 0 \quad \text{in } Q$$

we will mean a vector

$$u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega)) \tag{1.13}$$

such that

$$a(u, \varphi) = \int_Q (B^0(X, u, Du)|\varphi)\, dX\,, \tag{1.14}$$

$$\forall \varphi \in W(Q)\colon \varphi(x, -T) = \varphi(x, 0) = 0 \quad \text{in } \Omega \text{ (}^2\text{)}\,.$$

[2]) Remark that $W(Q) \subset H^1(Q)$, so that there exist the traces $\varphi(x, -T)$ and $\varphi(x, 0)$ in $H^{\frac{1}{2}}(\Omega)$.

We define q_0 this way:

$$(1.15)\qquad \begin{cases} q_0 = \dfrac{2(n+2)}{n}, & \text{if } n > 2 \\ q_0 \text{ is any number} \in [1, 4), & \text{if } n = 2 \\ q_0 = 4, & \text{if } n = 1 \end{cases}$$

then, it is known (see lemma 2.I) that

$$W(Q) \subset L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega)) \subset L^{q_0}(Q).$$

Therefore, to guarantee the existence of the integrals which appear in (1.14), it is sufficient to assume that the vectors a^i and B^0 have the following growths that we will say *controlled*

$$(1.16)\qquad \|a^i(X, u, p)\| \leqslant c(1 + \|u\|^\alpha + \|p\|)$$

$$(1.17)\qquad \|B^0(X, u, p)\| \leqslant c(1 + \|u\|^\beta + \|p\|^\gamma)\ \text{(3)}$$

with

$$(1.18)\qquad \begin{cases} 1 \leqslant \alpha \leqslant \dfrac{n+2}{n} & \text{if } n > 2 \\ 1 \leqslant \alpha < 2 & \text{if } n = 2 \\ 1 \leqslant \alpha \leqslant 2 & \text{if } n = 1 \end{cases}$$

$$(1.19)\qquad \begin{cases} 1 \leqslant \beta \leqslant \dfrac{n+4}{n} & \text{if } n > 2 \\ 1 \leqslant \beta < 3 & \text{if } n = 2 \\ 1 \leqslant \beta \leqslant 3 & \text{if } n = 1 \end{cases}$$

$$(1.20)\qquad \begin{cases} 1 \leqslant \gamma \leqslant \dfrac{n+4}{n+2}, & \text{if } n > 2 \\ 1 \leqslant \gamma < \frac{3}{2}, & \text{if } n = 2 \\ 1 \leqslant \gamma \leqslant \frac{3}{2}, & \text{if } n = 1. \end{cases}$$

These growths assure that, if u verifies assumption (1.13), then

$$(1.21)\qquad a^i(X, u, Du) \in L^2(Q) \quad \text{and} \quad B^0(X, u, Du) \in L^{q_0'}(Q), \qquad \left[\frac{1}{q_0} + \frac{1}{q_0'} = 1\right].$$

If $\alpha = \beta = \gamma = 1$, we will say that the growths (1.16), (1.17) are linear.

---

(3) More generally, in the right-hand side, the constant 1 may be replaced with appropriate integrable functions $f_i(X)$ and $f_0(X)$

$$f_i(X) \in L^2(Q) \quad \text{and} \quad f_0 \in L^{q_0'}(Q), \qquad \frac{1}{q_0} + \frac{1}{q_0'} = 1.$$

We observe that, from (1.6), (1.5), (1.2) it follows that

$$\|A^i(X, u, p)\| \leqslant M\|p\| \tag{1.22}$$

$$\|B^i(X, u)\| \leqslant c(1 + \|u\|^\alpha) . \tag{1.23}$$

In this paper, like in [5], we will suppose that the growths of the vectors a^i and B^0 are *strictly controlled*, that is we will suppose that

$$\begin{cases} 1 \leqslant \alpha < \dfrac{n+2}{n}, & \text{if } n \geqslant 2 \\ 1 \leqslant \alpha < 2, & \text{if } n = 1 \end{cases} \tag{1.24}$$

$$\begin{cases} 1 \leqslant \beta < \dfrac{n+4}{n}, & \text{if } n \geqslant 2 \\ 1 \leqslant \beta < 3, & \text{if } n = 1 \end{cases} \tag{1.25}$$

$$\begin{cases} 1 \leqslant \gamma < \dfrac{n+4}{n+2}, & \text{if } n \geqslant 2 \\ 1 \leqslant \gamma < \frac{3}{2}, & \text{if } n = 1 \end{cases} \tag{1.26}$$

this aims to avoid some technical difficulties. Notwithstanding this I believe that all the results of the present paper are true also in the case of controlled growths, as it is proved in [2] for non-linear elliptic systems.

In this paper we will study the Hölder continuity, or the partial Hölder continuity, of the solutions of system $Eu = 0$ (as meant in (1.13), (1.14)). Clearly, the Hölder continuity is related to the parabolic metric

$$d(X, Y) = \max\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\}, \quad \text{if } X = (x, t) \text{ and } Y = (y, \tau) . \tag{1.27}$$

We recall that a vector $v: Q \to R^N$ is said to be partially μ-Hölder continuous in Q, if there exists a subset $Q_0 \subset Q$ (Q_0 is the singular set of v), such that

$$Q_0 \text{ is closed in } Q$$

$$\text{meas } Q_0 = 0$$

$$v \in C^{0,\mu}(Q \setminus Q_0, d) .$$

The partial Hölder continuity of the solutions is bein already studied for quasi-linear systems with linear growth in [4], [14] and for quasi-linear systems with strictly controlled growth in [5], [13]. Here we want to prove, for the non-linear parabolic systems of second order, results which are analogous to those proved in [1] and [2] for non-linear elliptic systems ([4]).

The Hölder continuity and the partial Hölder continuity will be obtained, as it is usual by now, as a particular case of regularity, or partial regularity, in the $\mathfrak{L}^{2,\lambda}(Q, d)$ spaces (see [8], [6]).

For this purpose we will first consider the nonlinear systems of the following type:

$$-\sum_i D_i a^i(Du) + \frac{\partial u}{\partial t} = 0 \quad \text{in } Q \tag{1.28}$$

which satisfy the conditions (1.5), (1.10) and (1.16). Without any loss of generality, we can suppose that $a^i(0) = 0$; then

$$\|a^i(p)\| = \|A^i(p)\| \leqslant M\|p\| , \quad \forall p \in R^{nN} . \tag{1.29}$$

In the theory of the $\mathfrak{L}^{2,\lambda}$-regularity for non-linear parabolic systems, those of type (1.28) play an analogous role to that played by linear systems, with constant coefficients and reduced to the principal part, in the theory of linear or quasi-linear systems (see [6], [4], [5]).

The solutions $u \in L^2(-T, 0, H^1(\Omega))$ of system (1.28) are locally differentiable (see section 3), i.e.

$$D_{ij} u \in L^2_{\text{loc}}(Q) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^2_{\text{loc}}(Q) \tag{1.30}$$

and for every $Q(2\sigma) = Q(X_0, 2\sigma) \subset\subset Q$

$$\int_{Q(\sigma)} \sum_{ij} \|D_{ij} u\|^2 \, dX \leqslant \frac{c}{\sigma^2} \int_{Q(2\sigma)} \|Du\|^2 \, dX \tag{1.31}$$

where c does not depend on σ.

From this we get that the solutions of system (1.28) verify the fundamental estimates which follow (see section 5):

There exists $\varepsilon \in (0, 1)$ such that $\forall Q(\sigma) = Q(X_0, \sigma) \subset\subset Q$ and $\forall \lambda \in (0, 1)$

$$\int_{Q(\lambda\sigma)} \sum_{ij} \|D_{ij} u\|^2 \, dX \leqslant c\lambda^{\varepsilon(n+2)} \int_{Q(\sigma)} \sum_{ij} \|D_{ij} u\|^2 \, dX . \tag{1.32}$$

There exists $\varepsilon \in (0, n/(n+2))$ such that $\forall Q(\sigma) \subset\subset Q$ and $\forall \lambda \in (0, 1)$

$$\int_{Q(\lambda\sigma)} \|Du\|^2 \, dX \leqslant c\lambda^{2+\varepsilon(n+2)} \int_{Q(\sigma)} \|Du\|^2 \, dX . \tag{1.33}$$

([4]) In the elliptic case, quoted in the text, we have supposed a q-nonlinearity, with $q > 1$, on the contrary here we confine ourselves to consider the case $q = 2$.

There exists $\varepsilon \in (0, n/(n+2))$ such that $\forall Q(\sigma) \subset\subset Q$ and $\forall \lambda \in (0,1)$

$$\int_{Q(\lambda\sigma)} \|u - u_{Q(\lambda\sigma)}\|^2 \, dX \leq c\lambda^{4+\varepsilon(n+2)} \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2 \, dX \tag{1.34}$$

where the constants c which appear in (1.32), (1.33), (1.34) do not depend on σ and λ, and

$$u_A = \fint_A u(X) \, dX = \frac{1}{\text{meas}\, A} \int_A u \, dX \, . \tag{1.35}$$

In particular (see section 6) from inequality (1.34) it follows that, if $n \leq 2$, then

$$u \in C^{0,\mu}(Q, d) \qquad \text{with} \qquad \mu = 2 - \frac{1-\varepsilon}{2}(n+2) \, . \tag{1.36}$$

Furthermore, if the derivatives $\partial a^i / \partial p_k^j$ are uniformly continuous in R^{nN}, then the vector Du is partially μ-Hölder continuous in Q, $\forall \mu \in (0,1)$, and, Q_0 being the singular set of Du,

$$H_{n+2-q}(Q_0) = 0 \qquad \text{for some } q > 2 \, . \tag{1.37}$$

Here H_β is the β-dimensional Hausdorff measure with respect to the parabolic metric d (see for instance (3.10) in [4]).

In section 7 we study the solutions of the strongly parabolic systems

$$-\sum_i D_i A^i(X, Du) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du) \tag{1.38}$$

still with strictly controlled growth.

We prove (theorems 7.I, 7.II) that, if $n \leq 2$ and the vectors $A^i(X, p)$ satisfy an uniform continuity condition with respect to X (see (7.5)), then the result (1.36) holds again:

For a suitable $\varepsilon \in (0, n/(n+2))$

$$u \in C^{0,\mu}(Q, d) \qquad \text{with} \qquad \mu = 2 - \frac{1-\varepsilon}{2}(n+2) \, . \tag{1.39}$$

If the derivatives $\partial A^i / \partial p_k^j$ are uniformly continuous in $\bar{Q} \times R^{nN}$, then, whatever n may be, the solutions of system (1.38) are partially μ-Hölder continuous in Q, $\forall \mu \in (0,1)$. If Q_0 is their singular set, one can merely say that

$$\text{meas}\, Q_0 = 0 \tag{1.40}$$

(see theorem 7.IV).

In section 8 we study the solutions of the strongly parabolic systems of general type

$$-\sum_i D_i a^i(X, u, Du) = B^0(X, u, Du) \tag{1.41}$$

which have strictly controlled growth.

One proves (theorem 8.II) that, if $n \leqslant 2$ and the vectors $A^i(X, u, p)$ satisfy an uniform continuity condition with respect to (X, u) (see (8.6)), then u is partially μ-Hölder continuous in Q with $\mu = 2 - (1-\varepsilon)(n+2)/2$ and, Q_0 being its singular set,

$$H_n(Q_0) = 0\,. \tag{1.42}$$

Furthermore, if the derivatives $\partial a^i/\partial p_k^j$ are uniformly continuous in $\bar{\Lambda}$, then, for any n, the solutions of system (1.41) are partially μ-Hölder continuous in Q, $\forall \mu \in (0, 1)$. About their singular set Q_0, one can merely say that

$$\operatorname{meas} Q_0 = 0 \tag{1.43}$$

(see theorem 8.IV).

2. – Some notations and preliminary results.

Where there is no fear of confusion we will write simply $B(\sigma)$ and $Q(\sigma)$ instead of $B(x^0, \sigma)$ and $Q(X_0, \sigma)$, respectively. We define $u_{Q(\sigma)}$ as in (1.35), and we set

$$|u|_{0,q,A} = \Big(\int_A \|u\|^q\, dX\Big)^{1/q}, \qquad |u|_{0,A} \text{ if } q = 2\,. \tag{2.1}$$

$$\tilde{\Phi}(u, X_0, \sigma) = \int_{Q(X_0,\sigma)} \|Du\|^2 + \sigma^{-2}\|u - u_{Q(\sigma)}\|^2\, dX \tag{2.2}$$

$$\Phi(u, X_0, \sigma) = \int_{Q(X_0,\sigma)} 1 + \|u\|^{q_0} + \|Du\|^2 + \sigma^{-2}\|u - u_{Q(\sigma)}\|^2\, dX \tag{2.3}$$

where q_0 is defined in (1.15).

LEMMA 2.I. – *If* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$, *then* $u \in L^{q_0}(Q)$ *and* $\forall q \in [2, q_0]$

$$|u|^q_{0,q,Q} \leqslant c(n, q)\, d_\Omega^\delta \cdot \sup_{(-T,0]} |u|^{q-2}_{0,\Omega} \cdot \int_Q \|Du\|^2 + d_\Omega^{-2}\|u\|^2\, dX \tag{2.4}$$

where d_Ω *is the diameter of* Ω *and*

$$\delta = n + 2 - \frac{nq}{2}. \tag{2.5}$$

PROOF. - If $n \geqslant 2$

$$\int_\Omega \|u\|^q \, dx = \int_\Omega \|u\|^2 \cdot \|u\|^{q-2} \, dx \leqslant |u|_{0,\Omega}^{q-2} \cdot \left(\int_\Omega \|u\|^{4/(4-q)} \, dx \right)^{(4-q)/2}. \tag{2.6}$$

And, by Sobolev's theorem, we have

$$\left(\int_\Omega \|u\|^{4/(4-q)} \, dx \right)^{(4-q)/2} \leqslant c(n, q) \left\{ \int_\Omega \{\|Du\| + d_\Omega^{-1}\|u\|\}^{4n/(n(4-q)+4)} \, dx \right\}^{(n(4-q)+4)/2n} \leqslant$$
$$\leqslant c(n, q) \, d_\Omega^{(n(2-q)+4)/2} \cdot \int_\Omega \|Du\|^2 + d_\Omega^{-2}\|u\|^2 \, dx. \tag{2.7}$$

Inequality (2.4) easily follows from (2.6) and (2.7).

On the contrary, if $n = 1$

$$\int_\Omega \|u\|^q \, dx \leqslant \sup_\Omega \|u\|^2 \cdot \int_\Omega \|u\|^{q-2} \, dx \leqslant \sup_\Omega \|u\|^2 \cdot |u|_{0,\Omega}^{q-2} \cdot d_\Omega^{2-q/2}. \tag{2.8}$$

Moreover, by Sobolev's theorem,

$$\sup_\Omega \|u\|^2 \leqslant c d_\Omega \int_\Omega \|Du\|^2 + d_\Omega^{-2}\|u\|^2 \, dx. \tag{2.9}$$

Inequality (2.4) follows again from (2.8) and (2.9).

LEMMA 2.II. - *If* $u \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big) \cap H^{\frac{1}{2}}\big(t_0 - \sigma^2, t_0, L^2(B(\sigma))\big)$, *then*

$$\tilde{\Phi}(u, X_0, \sigma) \leqslant c(n) \left\{ \int_{Q(\sigma)} \|Du\|^2 \, dX + \int_{t_0-\sigma^2}^{t_0} d\tau \int_{Q(\sigma)} \frac{\|u(x,\tau) - u(x,t)\|^2}{|t-\tau|^2} \, dx \, dt \right\}. \tag{2.10}$$

This lemma is well known (see for instance [4] lemma 2.I).

LEMMA 2.III. - *If* $u \in L^2\big(t_0 - \sigma^2, t_0, H^2(B(\sigma))\big) \cap H^1\big(t_0 - \sigma^2, t_0, L^2(B(\sigma))\big)$, *then*

$$\tilde{\Phi}(Du, X_0, \sigma) \leqslant c(n) \int_{Q(\sigma)} \sum_{ij} \|D_{ij}u\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 dX. \tag{2.11}$$

Inequality (2.11) is well known too (see for instance [7] lemma 2.II).

LEMMA 2.IV. – *Let us suppose that* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *and* α *satisfies the condition* (1.24). *Having set*

$$\Psi(u, X_0, \sigma) = \int_{Q(X_0,\sigma)} (1 + \|u\|^{2\alpha})\, dX \tag{2.12}$$

then, $\forall Q(\sigma) \subset Q$ *and* $\forall \lambda \in (0, 1)$, *we have the inequality*

$$\Psi(u, X_0, \lambda\sigma) \leqslant c(n, \alpha)\, \lambda^{n+2} \Psi(u, X_0, \sigma) + C(u)\, \sigma^{n(1-\alpha)+2}\, \tilde{\Phi}(u, X_0, \sigma) \tag{2.13}$$

where

$$C(u) = c(n, \alpha) \sup_{(-T,0)} |u|_{0,\Omega}^{2(\alpha-1)} . \tag{2.14}$$

PROOF. – For every $\lambda \in (0, 1)$

$$\int_{Q(\lambda\sigma)} \|u\|^{2\alpha}\, dX \leqslant c(n, \alpha)\, \lambda^{n+2} \int_{Q(\sigma)} \|u\|^{2\alpha}\, dX + c(n, \alpha) \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^{2\alpha}\, dX .$$

Inequality (2.13) follows from this estimate and from (2.4), where we assume $q = 2\alpha$.

LEMMA 2.V. – *If* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *and* $B^0(X, u, p)$ *verifies the growth condition* (1.17), *then,* $\forall Q(X_0, \sigma) \subset Q$ *and for every* r, *with* $2(n+2)/(n+4) \leqslant r \leqslant q_0/\beta \wedge 2/\gamma$

$$|B^0(X, u, Du)|_{0,r,Q(\sigma)}^2 \leqslant C(u)\, \sigma^\delta\, \Phi(u, X_0, \sigma) \tag{2.15}$$

where

$$\delta = 2(n+2)\left(\frac{1}{r} - \frac{\beta}{q_0} \vee \frac{\gamma}{2}\right) \tag{2.16}$$

and

$$C(u) = c(n) \left\{ \int_Q 1 + \|u\|^{q_0} + \|Du\|^2\, dX \right\}^{(2\beta/q_0)\vee\gamma - 1} . \tag{2.17}$$

PROOF. – It is easy to obtain the following inequalities:

$$\left(\int_{Q(\sigma)} \|B^0\|^r\, dX\right)^{2/r} \leqslant c(n)\, \sigma^{2(n+2)(1/r - \beta/q_0 \vee \gamma/2)} \left(\int_{Q(\sigma)} \|B^0\|^{q_0/\beta \wedge 2/\gamma}\, dX\right)^{2\beta/q_0 \vee \gamma} \leqslant$$

$$\leqslant c(n)\, \sigma^\delta \left(\int_{Q(\sigma)} 1 + \|u\|^{q_0} + \|Du\|^2\, dX\right) \leqslant$$

$$\leqslant c(n)\, \sigma^\delta\, \Phi(u, X_0, \sigma) \left(\int_Q 1 + \|u\|^{q_0} + \|Du\|^2\, dX\right)^{(2\beta/q_0)\vee\gamma - 1} .$$

Let $A_{ij}(X)$, $ij = 1, \dots, n$, be $N \times N$ matrices defined in Q, and suppose that

$$A_{ij} \in L^\infty(Q) \quad \text{and} \quad \sup_Q \left\{ \sum_{ij} \|A_{ij}\|^2 \right\}^{\frac{1}{2}} = M \tag{2.18}$$

$$\sum_{ij} (A_{ij}\xi^j|\xi^i) \geqslant \nu \|\xi\|^2, \quad \nu > 0, \quad \forall X \in Q \text{ and } \forall \xi \in R^{nN}. \tag{2.19}$$

Let $B^i(X)$, $i = 1, \dots, n$, and $B^0(X)$ be vectors of R^N, such that

$$B^i \in L^2(Q) \quad \text{and} \quad B^0 \in L^{2(n+2)/(n+4)}(Q). \tag{2.20}$$

The following result is well known (see [11], [10], [12]):

LEMMA 2.VI. – *There exists a unique* $u \in L^2(-T, 0, H_0^1(\Omega))$, *which is the solution of the Cauchy-Dirichlet* [C.D.] *problem*:

$$\int_Q \sum_{ij} (A_{ij} D_j u|D_i\varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i|D_i\varphi) + (B^0|\varphi)\, dX, \tag{2.21}$$
$$\forall \varphi \in W(Q): \varphi(x, 0) = 0 \quad \text{in } \Omega$$

and the following inequality holds

$$\int_Q \|Du\|^2 dX + \int_{-T}^0 d\tau \int_Q \frac{\|u(x,t) - u(x,\tau)\|^2}{|t-\tau|^2} dx\, dt \leqslant \tag{2.22}$$
$$\leqslant c(\nu, M) \left\{ \sum_i |B^i|_{0,Q}^2 + |B^0|_{0,2(n+2)/(n+4),Q} \right\}.$$

In particular, $\forall Q(X_0, \sigma) \subset Q$

$$\tilde{\Phi}(u, X_0, \sigma) \leqslant c(\nu, M) \left\{ \sum_i |B^i|_{0,Q}^2 + |B^0|_{0,2(n+2)/(n+4),Q}^2 \right\}. \tag{2.23}$$

More generally, in section 4 of [5] the following L^q_{loc}-regularity result for the vector Du is proved [5]:

LEMMA 2.VII. – *There exist* $\bar{q} > 2$ *and a continuous and increasing function* $r(q)$, *defined on* $[2, \bar{q}]$, *with these properties*:

$$\frac{2(n+2)}{n+4} \leqslant r(q) \leqslant 2, \qquad \lim_{q \leftarrow 2} r(q) = \frac{2(n+2)}{n+4} \tag{2.24}$$

---

(5) In [5] I considered the case $n > 2$, but in case $n = 1$ and $n = 2$ the proof remains the same.

such that, if $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *is a solution of system*

$$(2.25) \qquad \int_Q \sum_{ij} (A_{ij} D_j u | D_i \varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i | D_i \varphi) + (B^0 | \varphi)\, dX, \qquad \forall \varphi \in C_0^\infty(Q)$$

under the hypotheses (2.18), (2.19) *and*

$$(2.26) \qquad B^i \in L^q(Q), \quad B^0 \in L^{r(q)}(Q), \quad 2 \leq q \leq \bar{q}$$

then, for every $Q(X_0 2\sigma) \subset Q$ *and* $\forall \eta \in R^N$

$$(2.27) \qquad \left(\fint_{Q(\sigma)} \|Du\|^q \, dX\right)^{1/q} \leq c \left\{\fint_{Q(2\sigma)} \sum_i \|B^i\|^q \, dX\right\}^{1/q} + \\ + c\sigma \left\{\fint_{Q(2\sigma)} \|B^0\|^{r(q)} \, dX\right\}^{1/r(q)} + c\left\{\fint_{Q(2\sigma)} \|Du\|^2 + \sigma^{-2}\|u - \eta\|^2 \, dX\right\}^{\frac{1}{2}}$$

where the constants c *do not depend on* σ.

Denote by A^*_{ij} the adjoint of the matrix A_{ij}; set

$$(2.28) \qquad A^+_{ij} = \tfrac{1}{2}(A_{ij} + A^*_{ji}), \quad A^-_{ij} = \tfrac{1}{2}(A_{ij} - A^*_{ji})$$

and define

$$(2.29) \qquad M_- = \sup_Q \left\{\sum_{ij} \|A^-_{ij}\|^2\right\}^{\frac{1}{2}}.$$

LEMMA 2.VIII. – *For every* $\mu \geq 0$ *and* $\forall \xi \in R^{nN}$

$$(2.30) \qquad \sup_Q \left\{\sum_i \|(M + \mu)\xi^i - \sum_j A_{ij}\xi^j\|^2\right\}^{\frac{1}{2}} \leq \{M - \nu + \sqrt{\mu^2 + M_-^2}\}\, \|\xi\|.$$

Moreover, if $\mu > (M_-^2 - \nu^2)/2\nu$, *then*

$$(2.31) \qquad K(\mu) = \frac{M - \nu + \sqrt{\mu^2 + M_-^2}}{M + \mu} < 1.$$

As it concerns inequality (2.30) see [3] section 1 and [Q] lemma 8.III, p. 88. To verify (2.31), an elementary calculation is enough.

LEMMA 2.IX. – *If* A_{ij}, $ij = 1, \ldots, n$, *are* $N \times N$ *matrices which satisfy the conditions* (2.18), (2.19), *and if* $u \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ *is a solution in* $Q(X_0, \sigma)$ *of system*

$$(2.32) \qquad \int_{Q(\sigma)} \sum_{ij} (A_{ij} D_j u | D_i \varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = 0, \quad \forall \varphi \in C_0^\infty(Q(\sigma))$$

then there exists $\varepsilon(\nu, M) \in (0,1)$, *such that* $\forall\lambda \in (0,1)$

$$\int_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c(\nu, M)\, \lambda^{\varepsilon(n+2)} \int_{Q(\sigma)} \|Du\|^2\, dX\,. \tag{2.33}$$

PROOF. – Fix $\mu = (M^2 - \nu^2)/\nu$. In $Q(\sigma)$ we decompose u as $u = v + w$, where $w \in L^2\big(t_0 - \sigma^2, t_0, H^1_0(B(\sigma))\big)$ is the solution of the Cauchy-Dirichlet problem

$$\int_{Q(\sigma)} (M+\mu) \sum_i (D_i w | D_i\varphi) - \left(w \left| \frac{\partial\varphi}{\partial t}\right.\right) dX = \int_{Q(\sigma)} \sum_i \Big((M+\mu)\, D_i u - \sum_i A_{ij}\, D_j u \Big| D_i\varphi\Big)\, dX\,, \tag{2.34}$$

$$\forall\varphi \in W(Q(\sigma)) : \varphi(x, t_0) = 0 \quad \text{in } B(\sigma)$$

whereas $v \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system

$$\int_{Q(\sigma)} (M+\mu) \sum_i (D_i v | D_i\varphi) - \left(v \left| \frac{\partial\varphi}{\partial t}\right.\right) dX = 0\,, \qquad \forall\varphi \in C^\infty_0(Q(\sigma))\,. \tag{2.35}$$

From the linear theory, it is known that w verifies the inequality

$$\int_{Q(\sigma)} \|Dw\|^2\, dX \leqslant \frac{1}{(M+\mu)^2} \int_{Q(\sigma)} \sum_i \Big\|(M+\mu)\, D_i u - \sum_j A_{ij}\, D_j u\Big\|^2\, dX$$

therefore, by lemma 2.VIII,

$$\int_{Q(\sigma)} \|Dw\|^2\, dX \leqslant K^2(\mu) \int_{Q(\sigma)} \|Du\|^2\, dX \tag{2.36}$$

v satisfies the fundamental inequality which follows (see [6] and [4], lemma 2.II)

$$\int_{Q(\lambda\sigma)} \|Dv\|^2\, dX \leqslant c(\nu, M)\, \lambda^{n+2} \int_{Q(\sigma)} \|Dv\|^2\, dX\,, \qquad \forall\lambda \in (0,1)\,. \tag{2.37}$$

From (2.36) and (2.37), it easily follows that $\forall\lambda \in (0,1)$

$$|Du|_{0,Q(\lambda\sigma)} \leqslant \{c(1+K)\,\lambda^{(n+2)/2} + K\}\, |Du|_{0,Q(\sigma)}\,. \tag{2.38}$$

As $K \in (0,1)$, from (2.38), we get the estimate (2.33) by means of lemma 1.V, p. 12 of [Q].

One can prove the following Caccioppoli's type inequality:

LEMMA 2.X. – *If* A_{ij}, $ij = 1, \dots, n$, *are* $N\times N$ *matrices, which satisfy the conditions* (2.18), (2.19), *and if* $u \in L^2(-T, 0, H^1(\Omega))$ *is a solution in* Q *of system*

$$\int_Q \sum_{ij} (A_{ij} D_j u | D_i \varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = 0\,, \quad \forall \varphi \in C_0^\infty(Q) \tag{2.39}$$

then, $\forall B(x^0, 2\sigma) \subset\subset \Omega$, $\forall 2a \in (0, T)$, *and* $\forall \eta \in R^N$

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|Du\|^2\, dx \leqslant c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right) \int_{-2a}^{0} dt \int_{B(2\sigma)} \|u - \eta\|^2\, dx\,. \tag{2.40}$$

PROOF. – Let $\theta(x) \in C_0^\infty(R^n)$ be a function with these properties

$$0 \leqslant \theta \leqslant 1\,, \quad \theta = 1 \text{ in } B(\sigma)\,, \quad \theta = 0 \text{ in } R^n \setminus B(\tfrac{3}{2}\sigma)\,, \quad \|D\theta\| \leqslant c\sigma^{-1}\,. \tag{2.41}$$

Let $\varrho_m(t)$, with m integer $> 2/a$, be a function defined on R this way

$$\begin{cases} \varrho_m(t) = 1\,, & \text{if } -a \leqslant t \leqslant -\dfrac{2}{m} \\ \varrho_m(t) = 0\,, & \text{if } t > -\dfrac{1}{m} \text{ or } t < -2a \\ \varrho_m(t) = \dfrac{t}{a} + 2\,, & \text{if } -2a \leqslant t \leqslant -a \\ \varrho_m(t) = -(mt+1)\,, & \text{if } -\dfrac{2}{m} \leqslant t \leqslant -\dfrac{1}{m}\,. \end{cases} \tag{2.42}$$

Finally, let $\{g_s(t)\}$ be a sequence of symmetric mollifying functions

$$\begin{cases} g_s(t) \in C_0^\infty(R)\,, \quad g_s(t) \geqslant 0\,, \quad g_s(t) = g_s(-t) \\ \operatorname{supp} g_s \subset \left[-\dfrac{1}{s}, \dfrac{1}{s}\right] \\ \displaystyle\int_R g_s(t)\, dt = 1\,. \end{cases} \tag{2.43}$$

As (2.39) is true for any $\varphi \in W(Q)$: $\varphi(x, -T) = \varphi(x, 0) = 0$ in Ω, then, if $s > m \vee 1/(T - 2a)$, we can assume in (2.39)

$$\varphi(X) = \theta^2 \varrho_m[(\varrho_m(u - \eta)) * g_s] \tag{2.44}$$

and we get that

$$\int_Q \theta^2 \varrho_m \sum_{ij} (A_{ij} D_j u|(\varrho_m D_i u) * g_s)\, dX + \tag{2.45}$$
$$+ 2\int_Q \theta\varrho_m \sum_{ij} \left(A_{ij} D_j u | D_i\theta\cdot[(\varrho_m(u-\eta)) * g_s]\right) dX -$$
$$-\int_Q \left(u-\eta|\theta^2\varrho'_m[(\varrho_m(u-\eta)) * g_s]\right) dX = \int_Q \left(u-\eta|\theta^2\varrho_m[(\varrho_m(u-\eta)) * g_s]'\right) dX\,.$$

By symmetry of the $g_s(t)$, the integral in the right-hand side equals zero; furthermore, when $s\to+\infty$, then

$$[\varrho_m(u-\eta)] * g_s \to \varrho_m(u-\eta) \quad \text{in } L^2(-T, 0, H^1(\Omega))\,.$$

And so, from (2.45), taking the limit for $s\to+\infty$, we obtain that

$$\int_Q \theta^2 \varrho_m^2 \sum_{ij} (A_{ij} D_j u|D_i u)\, dX = \tag{2.46}$$
$$= 2\int_Q \theta\varrho_m^2 \sum_{ij} (A_{ij} D_j u|D_i\theta\cdot(u-\eta))\, dX + \int_Q \theta^2\varrho_m\varrho'_m\|u-\eta\|^2\, dX\,.$$

We may estimate the integral in the left-hand side by the ellipticity condition (2.19), and we easily estimate the terms in the right-hand side by the Hölder's inequality. Therefore we obtain that $\forall\varepsilon>0$

$$\nu\int_Q \theta^2\varrho_m^2\|Du\|^2\, dX \leqslant \varepsilon\int_Q \theta^2\varrho_m^2\|Du\|^2\, dX +$$
$$+ c(\varepsilon, M)\int_Q \varrho_m^2\|D\theta\|^2\|u-\eta\|^2\, dX + \int_Q \theta^2\varrho_m\varrho'_m\|u-\eta\|^2\, dX\,.$$

Choosing ε sufficiently small and taking into account that $\varrho_m\varrho'_m\leqslant 0$, if $t>-2/m$, we get that

$$\int_{-a}^{-2/m} dt \int_{B(\sigma)} \|Du\|^2\, dx \leqslant c(\nu, M)\left(\frac{1}{\sigma^2}+\frac{1}{a}\right)\int_{-2a}^{-2/m} dt \int_{B(2\sigma)} \|u-\eta\|^2\, dx\,.$$

Now the thesis, i.e. the (2.40), follows by taking the limit for $m\to+\infty$.

Let $a^i(p)$, $i=1,\ldots,n$, be vectors of R^N of class $C^1(R^{nN})$, which satisfy the conditions (1.5), (1.10) and (1.29). Let us suppose that $B^i(X)$, $i=1,\ldots,n$, and $B^0(X)$ are vectors of R^N, which satisfy condition (2.20). Then, we can prove the following existence lemma.

LEMMA 2.XI. – *For any $w \in L^2(-T, 0, H^1(\Omega))$ there exists a unique vector $u \in L^2(-T, 0, H^1_0(\Omega))$, which is the solution of the* C.D. *problem*

$$(2.47)\quad \int_Q \sum_i (a^i(Du + Dw)|D_i\varphi) - \left(u\left|\frac{\partial\varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i|D_i\varphi) + (B^0|\varphi)\, dX\,,$$

$$\forall\varphi \in W(Q)\colon \varphi(x, 0) = 0 \quad \text{in } \Omega\,.$$

Moreover, we have the inequality

$$(2.48)\quad \int_Q \|Du\|^2\, dX + \int_{-T}^{0} d\tau \int_Q \frac{\|u(x,t) - u(x,\tau)\|^2}{|t-\tau|^2}\, dx\, dt \leqslant$$

$$\leqslant c(\nu, M)\left\{\sum_i |B^i - a^i(Dw)|^2_{0,Q} + |B^0|^2_{0,2(n+2)/(n+4),Q}\right\}.$$

We give a proof for the reader's convenience.

PROOF. – Fix $\mu = (M^2 - \nu^2)/\nu$. For any $u \in L^2(-T, 0, H^1_0(\Omega))$ the condition

$$(M + \mu)D_i u - a^i(Du + Dw) \in L^2(Q)$$

holds, and then (see lemma 2.VI) there is a unique solution $U = \mathcal{C}(u) \in L^2(-T, 0, H^1_0(\Omega))$ of C.D. problem

$$(2.49)\quad \int_Q (M + \mu) \sum_i (D_i U|D_i\varphi) - \left(U\left|\frac{\partial\varphi}{\partial t}\right.\right) dX =$$

$$= \int_Q \sum_i ([M + \mu]\, D_i u - a^i(Du + Dw) + B^i|D_i\varphi) + (B^0|\varphi)\, dX\,,$$

$$\forall\varphi \in W(Q)\colon \varphi(x, 0) = 0 \quad \text{in } \Omega\,.$$

$\mathcal{C}$ is a contraction mapping sending the Banach space $L^2(-T, 0, H^1_0(\Omega))$ into itself. In fact, if $U = \mathcal{C}(u)$, $V = \mathcal{C}(v)$, $\tilde{U} = U - V$ and $\tilde{u} = u - v$, then from (2.49) it follows that $\tilde{U} \in L^2(-T, 0, H^1_0(\Omega))$ is the solution of the C.D. problem

$$(2.50)\quad \int_Q (M + \mu) \sum_i (D_i\tilde{U}|D_i\varphi) - \left(\tilde{U}\left|\frac{\partial\varphi}{\partial t}\right.\right) dX =$$

$$= \int_Q \sum_i ([M + \mu]\, D_i\tilde{u} - a^i(Du + Dw) + a^i(Dv + Dw)|D_i\varphi)\, dX\,,$$

$$\forall\varphi \in W(Q)\colon \varphi(x, 0) = 0 \quad \text{in } \Omega\,.$$

After the $N\times N$ matrices

$$A_{ij} = \{A^{hk}_{ij}\} \qquad \text{with} \qquad A^{kh}_{ij} = \int_0^1 \frac{\partial a^i_h(\tau\, Du + (1-\tau)\, Dv + Dw)}{\partial p^j_k}\, d\tau$$

have been introduced, problem (2.50) becomes

$$(2.51)\quad \int_Q (M+\mu)\sum_i (D_i\tilde{U}|D_i\varphi) - \left(\tilde{U}\left|\frac{\partial\varphi}{\partial t}\right.\right) dX = \\ = \int_Q \sum_i \left([M+\mu]\, D_i\tilde{u} - \sum_j A_{ij}\, D_j\tilde{u}\,|D_i\varphi\right) dX\,, \\ \forall\varphi\in W(Q)\colon \varphi(x,0)=0 \quad \text{in } \Omega\,.$$

As $A_{ij}\in L^\infty(Q)$ satisfy the conditions (2.18), (2.19), by lemma 2.VIII we get this inequality (see (2.36)):

$$(2.52)\qquad \int_Q \|D\tilde{U}\|^2\, dX \leqslant K^2(\mu)\int_Q \|D\tilde{u}\|^2\, dX$$

where $K(\mu)<1$. We conclude that $\mathcal{T}$ has a unique fixed point u which is the solution of problem (2.47).

As far as inequality (2.48) is concerned, we argue as follows:

Introduce the $N\times N$ matrices

$$A_{ij}=\{A_{ij}^{hk}\}\,,\qquad \text{with}\qquad A_{ij}^{hk}=\int_0^1 \frac{\partial a_h^i(\tau\, Du + Dw)}{\partial p_k^j}\, d\tau$$

then problem (2.47) may be written this way

$$(2.53)\quad \int_Q \sum_{ij} (A_{ij}\, D_j u|D_i\varphi) - \left(u\left|\frac{\partial\varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i - a^i(Dw)|D_i\varphi) + (B^0|\varphi)\, dX\,, \\ \forall\varphi\in W(Q)\colon \varphi(x,0)=0 \quad \text{in } \Omega$$

so that $u\in L^2(-T,0,H_0^1(\Omega))$ is the solution of a C.D. linear problem with coefficients A_{ij} which satisfy the conditions (2.18) and (2.19). Then, from (2.22) inequality (2.48) follows.

THE CASE $A^i = A^i(p)$

3. – Local differentiability of the solutions.

Let $u\in L^2(-T,0,H^1(\Omega))$ be a solution in Q of system

$$(3.1)\qquad -\sum_i D_i A^i(Du) + \frac{\partial u}{\partial t} = 0$$

in the sense that

$$(3.2)\qquad \int_Q \sum_i (a^i(Du)|D_i\varphi) - \left(u\left|\frac{\partial\varphi}{\partial t}\right.\right) dX = 0\,,\quad \forall\varphi\in C_0^\infty(Q)$$

$A^i(p)$, $i = 1, \ldots, n$, are vectors of R^N, of class $C^1(R^{nN})$, which satisfy the conditions (1.5), (1.10) and (1.29).

We prove the following

THEOREM 3.I. – *The vector u is locally differentiable in Q, i.e. there exist*

$$D_{ij}u \in L^2_{\text{loc}}(Q)\,, \qquad \frac{\partial u}{\partial t} \in L^2_{\text{loc}}(Q) \tag{3.3}$$

and $\forall B(x^0, 2\sigma) \subset\subset \Omega$, $\forall 2a \in (0, T)$

$$\int_{-a}^{0} dt \int_{B(\sigma)} \sum_{ij} \|D_{ij}u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 dx \leqslant c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right) \int_{-2a}^{0} dt \int_{B(2\sigma)} \|Du\|^2\, dx\,. \tag{3.4}$$

PROOF. – Let $\theta(x)$, $\varrho_m(t)$ and $\{g_s(t)\}$ be defined as in (2.41), (2.42), (2.43); $\{g_s\}$ being a sequence of symmetric mollifying functions. Define

$$\tau_{r,h}u = u(x + he^r, t) - u(X) \tag{3.5}$$

where $\{e^r\}_{r=1,\ldots,n}$ is the standard base of R^n, and suppose that

$$|h| < \frac{\sigma}{2}\,.$$

Since (3.2) is valid $\forall \varphi \in W(Q)$: $\varphi(x, -T) = \varphi(x, 0) = 0$ in Ω, for each fixed m and $\forall s > m \vee 1/(T - 2a)$, we may assume in (3.2)

$$\varphi = \tau_{r,-h}\{\theta^2 \varrho_m[(\varrho_m \tau_{r,h}u) * g_s]\}$$

and hence we obtain that

$$\int_Q \sum_i (\tau_{r,h}a^i(Du)|D_i\{\theta^2 \varrho_m[(\varrho_m \tau_{r,h}u) * g_s]\})\, dX = \\ = \int_Q (\tau_{r,h}u|\theta^2\{\varrho_m[(\varrho_m \tau_{r,h}u) * g_s]\}')\, dX\,. \tag{3.6}$$

Account taken of symmetry of $g_s(t)$, it turns out that

$$\int_Q (\tau_{r,h}u|\theta^2 \varrho_m[(\varrho_m \tau_{r,h}u) * g_s]')\, dX = 0 \tag{3.7}$$

If, moreover, we set

$$A_{ij} = \{A_{ij}\}\,, \qquad \text{with} \qquad A_{ij}^{k} = \int_0^1 \frac{\partial a_h^i}{\partial p_k^j}(Du + \eta\tau_{r,h}\, Du)\, d\eta$$

we have that

$$\tau_{r,h}a^i(Du) = \sum_{j=1}^{n} A_{ij}(\tau_{r,h}\, D_j u)\,. \tag{3.8}$$

By keeping in mind (3.7) and (3.8), from (3.6) we obtain

$$\int_Q \theta^2 \varrho_m \sum_{ij} (A_{ij}\tau_{r,h}\, D_j u|(\varrho_m \tau_{r,h}\, D_i u) * g_s)\, dX = \tag{3.9}$$
$$= -2\int_Q \theta\varrho_m \sum_{ij} (A_{ij}\tau_{r,h}\, D_j u|D_i\theta\cdot[(\varrho_m\tau_{r,h}u) * g_s])\, dX + \int_Q \theta^2\varrho'_m(\tau_{r,h}u|(\varrho_m\tau_{r,h}u) * g_s)\, dX\,.$$

When $s \to +\infty$, then

$$(\varrho_m \tau_{r,h}u) * g_s \to \varrho_m\tau_{r,h}u \quad \text{in } L^2(-T, 0, H^1(\Omega))\,.$$

Therefore, from (3.9), taking the limit for $s \to +\infty$, we get that

$$A = \int_Q \theta^2\varrho_m^2 \sum_{ij} (A_{ij}\tau_{r,h}\, D_j u|\tau_{r,h}\, D_i u)\, dX = \tag{3.10}$$
$$= -2\int_Q \theta\varrho_m^2 \sum_{ij} (A_{ij}\tau_{r,h}\, D_j u|D_i\theta\cdot\tau_{r,h}u)\, dX + \int_Q \theta^2\varrho_m\varrho'_m\|\tau_{r,h}u\|^2\, dX = B + C\,.$$

By hypothesis (1.10)

$$A \geqslant \nu\int_Q \theta^2\varrho_m^2\|\tau_{r,h}\, Du\|^2\, dX \tag{3.11}$$

and moreover, by (1.5) together with the Hölder's inequality, we have for every $\varepsilon > 0$

$$|B| \leqslant \varepsilon\int_Q \theta^2\varrho_m^2\|\tau_{r,h}\, Du\|^2\, dX + c(\varepsilon, M)\int_Q \|D\theta\|^2\varrho_m^2\|\tau_{r,h}u\|^2\, dX\,. \tag{3.12}$$

Finally, as $\varrho_m\varrho'_m \leqslant 0$ if $t > -2/m$,

$$C \leqslant \frac{c}{a}\int_{-2a}^{-2/m} dt \int_{B(\frac{3}{2}\sigma)} \|\tau_{r,h}u\|^2\, dx\,. \tag{3.13}$$

From (3.10) ... (3.13), choosing ε small enough, it follows that

$$\int_{-a}^{-2/m} dt \int_{B(\sigma)} \|\tau_{r,h}\, Du\|^2\, dx \leqslant c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right)\int_{-2a}^{0} dt \int_{B(\frac{3}{2}\sigma)} \|\tau_{r,h}u\|^2\, dx \leqslant$$
$$\leqslant c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right)|h|^2 \int_{-2a}^{0} dt \int_{B(2\sigma)} \|Du\|^2\, dx$$

and taking the limit for $m \to +\infty$

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|\tau_{r,h}\, Du\|^2\, dx \leq c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right)|h|^2 \int_{-2a}^{0} dt \int_{B(2\sigma)} \|Du\|^2\, dx\,. \tag{3.14}$$

From this, because of a well known Nirenberg's lemma (see for instance [Q], p. 26), we conclude that there exist $D_r\, Du \in L^2(B(\sigma)\times(-a, 0))$, $r = 1, \dots, n$, and the following inequality

$$\int_{-a}^{0} dt \int_{B(\sigma)} \sum_{ij} \|D_{ij}u\|^2\, dx \leq c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right) \int_{-2a}^{0} dt \int_{B(2\sigma)} \|Du\|^2\, dx \tag{3.15}$$

holds. From this it easily follows that

$$\exists\, \frac{\partial u}{\partial t} \in L^2(B(\sigma)\times(-a, 0))\,. \tag{3.16}$$

In fact, from (3.2) we get that, $\forall \varphi \in C_0^\infty(B(\sigma)\times(-a, 0))$

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dx = -\int_{-a}^{0} dt \int_{B(\sigma)} \left(\sum_i D_i a^i(Du) | \varphi\right) dx \tag{3.17}$$

and, because of (1.5) and (3.15)

$$\sum_i D_i a^i(Du) = \sum_{ij} \sum_k D_{ij} u_k \frac{\partial a^i(Du)}{\partial p_k^j} \in L^2(B(\sigma)\times(-a, 0))\,.$$

From (3.17) and (3.15) we can furthermore get

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dx = \int_{-a}^{0} dt \int_{B(\sigma)} \left\|\sum_i D_i a^i(Du)\right\|^2 dx \leq c(\nu, M)\left(\frac{1}{\sigma^2} + \frac{1}{a}\right) \int_{-2a}^{0} dt \int_{B(2\sigma)} \|Du\|^2\, dx\,. \tag{3.18}$$

This completes the proof of the estimate (3.4).

4. – Poincaré's type estimates for the solutions in Q of system (3.1).

Let $u \in L^2(-T, 0, H^1(\Omega))$ be a solution in Q of system (3.1), under the hypotheses (1.5), (1.10) and (1.29).

Let $Q(X_0, 2\sigma) \subset\subset Q$. Because of theorem 3.I, the vector u belongs to $L^2\big(t_0 - 4\sigma^2, t_0, H^2(B(2\sigma))\big) \cap H^1\big(t_0 - 4\sigma^2, t_0, L^2(B(2\sigma))\big)$.

THEOREM 4.I. – *The following Poincaré's type inequalities*

$$\tilde{\Phi}(u, X_0, \sigma) \leqslant c(\nu, M) \int_{Q(2\sigma)} \|Du\|^2 \, dX \tag{4.1}$$

$$\tilde{\Phi}(Du, X_0, \sigma) \leqslant c(\nu, M) \int_{Q(\sigma)} \sum_{ij} \|D_{ij} u\|^2 \, dX \tag{4.2}$$

hold ($\tilde{\Phi}$ *is defined in* (2.2)).

PROOF. – From lemma 2.II and estimate (3.4), where one assumes $a = \sigma^2$, inequality (4.1) follows by taking into account that

$$\int_{t_0-\sigma^2}^{t_0} d\tau \int_{Q(\sigma)} \frac{\|u(x,t) - u(x,\tau)\|^2}{|t-\tau|^2} \, dx \, dt \leqslant c\sigma^2 \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dX \, .$$

Inequality (4.2) follows from lemma 2.III together with the fact that, in $Q(\sigma)$ we have

$$\left\| \frac{\partial u}{\partial t} \right\|^2 = \left\| \sum_i D_i a^i(Du) \right\|^2 \leqslant c(M) \sum_{ij} \|D_{ij} u\|^2 \, .$$

5. – Some fundamental estimates for the solutions in Q of system (3.1).

Let $u \in L^2(-T, 0, H^1(\Omega))$ be a solution in Q of system (3.1), under the hypotheses (1.5), (1.10) and (1.29). Let $Q(X_0, \sigma) \subset\subset Q$. Because of theorem 3.I it turns out that

$$u \in L^2\big(t_0 - \sigma^2, t_0, H^2(B(\sigma))\big) \cap H^1\big(t_0 - \sigma^2, t_0, L^2(B(\sigma))\big) \, .$$

From (3.2), assuming $\varphi = D_s \psi$, $s = 1, \ldots, n$, with $\psi \in C_0^\infty(Q(\sigma))$, we obtain

$$\int_{Q(\sigma)} \sum_i (D_s a^i(Du) | D_i \psi) - \left(D_s u \left| \frac{\partial \psi}{\partial t} \right. \right) dX = 0$$

and setting

$$A_{ij} = \{A_{ij}^{hk}\} \, , \quad \text{with} \quad A_{ij}^{hk}(p) = \frac{\partial a_k^i(p)}{\partial p_h^j} \, .$$

We have

$$\int_{Q(\sigma)} \sum_{ij} (A_{ij}(Du) D_j D_s u | D_i \psi) - \left(D_s u \left| \frac{\partial \psi}{\partial t} \right. \right) dX = 0 \, , \tag{5.1}$$

$\forall \psi \in C_0^\infty(Q(\sigma))$ and $s = 1, \ldots, u$.

Furthermore, we introduce

$$U = Du$$

$$(5.2)\qquad \mathcal{A}_{ij}(p) = \left|\begin{array}{c:c:c} A_{ij} & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & A_{ij} \end{array}\right| \; n^2 \text{ blocks}$$

then, from (5.1) it follows that $U \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system

$$(5.3)\qquad \int\limits_{Q(\sigma)} \sum_{ij} (\mathcal{A}_{ij}(U)\, D_j U | D_i \Psi) - \left(U \left| \frac{\partial \Psi}{\partial t}\right.\right) dX = 0\,, \quad \forall \Psi \in C_0^\infty(Q(\sigma))\,.$$

We observe that $\mathcal{A}_{ij}(p)$ are $nN \times nN$ matrices, bounded, continuous and elliptic, i.e.

$$(5.4)\qquad \sum_{ij} (\mathcal{A}_{ij}(p)\xi^i | \xi^j) \geqslant \nu \|\xi\|^2\,, \quad \forall p \in R^{nN} \text{ and } \forall \xi \in R^{Nn^2}.$$

Then, we can prove some fundamental estimates for the vectors u, Du and $D_{ij}u$.

THEOREM 5.I. – *If* $u \in L^2(-T, 0, H^1(\Omega))$ *is a solution in* Q *of system* (3.2), *then there exists an* $\varepsilon \in (0, 1)$, *such that,* $\forall Q(X_0, \sigma) \subset\subset Q$ *and* $\forall \lambda \in (0, 1)$,

$$(5.5)\qquad \int\limits_{Q(\lambda\sigma)} \sum_{ij} \|D_{ij}u\|^2\, dX \leqslant c(\nu, M)\, \lambda^{\varepsilon(n+2)} \int\limits_{Q(\sigma)} \sum_{ij} \|D_{ij}u\|^2\, dX\,.$$

PROOF. – System (5.3) can be regarded as a linear, strongly parabolic, system of second order, with coefficients $\mathcal{A}_{ij}(U(X)) \in L^\infty(Q(\sigma))$. Therefore, estimate (5.5) is a consequence of inequality (2.33) of lemma 2.IX.

THEOREM 5.II. – *If* $u \in L^2(-T, 0, H^1(\Omega))$ *is a solution in* Q *of system* (3.2), *then there exists an* ε, $0 < \varepsilon < n/(n+2)$, *such that,* $\forall Q(X_0, \sigma) \subset\subset Q$ *and* $\forall \lambda \in (0, 1)$,

$$(5.6)\qquad \int\limits_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c(\nu, M)\, \lambda^{2+\varepsilon(n+2)} \int\limits_{Q(\sigma)} \|Du\|^2\, dX\,.$$

PROOF. – Let us suppose $0 < \lambda < \tau < \frac{1}{2}$. Then,

$$(5.7)\qquad \int\limits_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c(n) \left(\frac{\lambda}{\tau}\right)^{n+2} \int\limits_{Q(\tau\sigma)} \|Du\|^2\, dX + c \int\limits_{Q(\tau\sigma)} \|Du - (Du)_{Q(\tau\sigma)}\|^2\, dX\,.$$

On the other hand, account taken of inequalities (4.2) and (5.5)

$$\int_{Q(\tau\sigma)} \|Du - (Du)_{Q(\tau\sigma)}\|^2\, dX \leqslant c(\tau\sigma)^2 \int_{Q(\tau\sigma)} \sum_{ij} \|D_{ij}u\|^2\, dX \leqslant c(\nu, M)\sigma^2\tau^{2+\varepsilon(n+2)} \int_{Q(\sigma/2)} \sum_{ij} \|D_{ij}u\|^2\, dX$$

where $\varepsilon(\nu, M) \in (0, 1)$. In conclusion

$$\int_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c\left(\frac{\lambda}{\tau}\right)^{n+2} \int_{Q(\tau\sigma)} \|Du\|^2\, dX + c\sigma^2\tau^{2+\varepsilon(n+2)} \int_{Q(\sigma/2)} \sum_{ij} \|D_{ij}u\|^2\, dX\,. \tag{5.8}$$

Now chose $\varepsilon < n/(n+2)$; then by lemma 1.I, p. 7 of [Q]

$$\int_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c\left(\frac{\lambda}{\tau}\right)^{2+\varepsilon(n+2)} \int_{Q(\tau\sigma)} \|Du\|^2\, dX + c\sigma^2\lambda^{2+\varepsilon(n+2)} \int_{Q(\sigma/2)} \sum_{ij} \|D_{ij}u\|^2\, dX\,.$$

Taking the limit for $\tau \to \frac{1}{2}$, we obtain that, $\forall 0 < \lambda < \frac{1}{2}$,

$$\int_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c\lambda^{2+\varepsilon(n+2)} \left\{ \int_{Q(\sigma)} \|Du\|^2\, dX + \sigma^2 \int_{Q(\sigma/2)} \sum_{ij} \|D_{ij}u\|^2\, dX \right\}$$

and, because of inequality (3.4)

$$\int_{Q(\lambda\sigma)} \|Du\|^2\, dX \leqslant c\lambda^{2+\varepsilon(n+2)} \int_{Q(\sigma)} \|Du\|^2\, dX\,. \tag{5.9}$$

This shows the thesis when $0 < \lambda < \frac{1}{2}$, however (5.6) is clearly true also for $\frac{1}{2} \leqslant \lambda < 1$ too.

THEOREM 5.III. – *If* $u \in L^2(-T, 0, H^1(\Omega))$ *is a solution in* Q *of system* (3.2), *then there exists an* ε, $0 < \varepsilon < n/(n+2)$, *such that,* $\forall Q(X_0, \sigma) \subset\subset Q$ *and* $\forall \lambda \in (0, 1)$,

$$\int_{Q(\lambda\sigma)} \|u - u_{Q(\lambda\sigma)}\|^2\, dX \leqslant c(\nu, M)\, \lambda^{4+\varepsilon(n+2)} \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2\, dX\,. \tag{5.10}$$

PROOF. – Let us suppose $0 < \lambda < \frac{1}{4}$. Inequality (5.6) is valid

$$\int_{Q(2\lambda\sigma)} \|Du\|^2\, dX \leqslant c(\nu, M)\, \lambda^{2+\varepsilon(n+2)} \int_{Q(\sigma/2)} \|Du\|^2\, dX\,. \tag{5.11}$$

Also the Poincaré's type inequality (4.1) is true

$$\int_{Q(\lambda\sigma)} \|u - u_{Q(\lambda\sigma)}\|^2\, dX \leqslant c(\nu, M)\, \lambda^2\sigma^2 \int_{Q(2\lambda\sigma)} \|Du\|^2\, dX\,. \tag{5.12}$$

Finally, system (3.1) may be written in the form (see (1.6))

$$-\sum_{ij} D_i(A_{ij}(Du)\, D_j u) + \frac{\partial u}{\partial t} = 0 \quad \text{in } Q \tag{5.13}$$

where

$$A_{ij} = \{A_{ij}^{hk}\}\,, \qquad \text{with} \qquad A_{ij}^{hk}(p) = \int_0^1 \frac{\partial a_h^i(\tau p)}{\partial p_k^j}\, d\tau\,.$$

Therefore, the system has the form of a linear, strongly parabolic, system with coefficients $A_{ij}(Du(X)) \in L^\infty(Q)$. Then, estimate (2.40) of lemma 2.X

$$\int_{Q(\sigma/2)} \|Du\|^2\, dX \leqslant \frac{c(\nu, M)}{\sigma^2} \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2\, dX \tag{5.14}$$

holds. From (5.12), (5.11), (5.14), inequality (5.10) follows when $0 < \lambda < \frac{1}{4}$; however inequality (5.10) trivially holds for $\frac{1}{4} \leqslant \lambda < 1$ too.

COROLLARY 5.I. – *If $u \in L^2(-T, 0, H^1(\Omega))$ is a solution in Q of system* (3.2), *there exists an ε, $0 < \varepsilon < n/(n+2)$, such that, $\forall Q(X_0, \sigma) \subset\subset Q$ and $\forall \lambda \in (0, 1)$,*

$$\tilde{\Phi}(u, X_0, \lambda\sigma) \leqslant c(\nu, M)\, \lambda^{2+\varepsilon(n+2)}\, \tilde{\Phi}(u, X_0, \sigma)\,. \tag{5.15}$$

In fact, (5.15) is a consequence of theorems 5.II and 5.III.

Consider now system

$$-\sum_i D_i A^i(Du) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X)\,, \quad \text{in } Q \tag{5.16}$$

under the conditions (1.5), (1.10), (1.29) and the hypothesis

$$B^i(X) \in L^2(Q)\,. \tag{5.17}$$

THEOREM 5.IV. – *Let $u \in L^2(-T, 0, H^1(\Omega))$ be a solution of system* (5.16), *that is*

$$\int_Q \sum_i (a^i(Du)|D_i\varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i|D_i\varphi)\, dX\,, \qquad \forall \varphi \in C_0^\infty(Q) \tag{5.18}$$

Then, $\forall Q(X_0, \sigma) \subset\subset Q$ and $\forall \lambda \in (0, 1)$,

$$\tilde{\Phi}(u, X, \lambda\sigma) \leqslant c(\nu, M) \left\{\lambda^{2+\varepsilon(n+2)}\, \tilde{\Phi}(u, X_0, \sigma) + \sum_i |B^i|^2_{0,Q(\sigma)}\right\} \tag{5.19}$$

where, as usual, $\varepsilon \in (0, n/(n+2))$.

PROOF. – In $Q(\sigma)$ we decompose u as $u = v - w$, where $w \in L^2\big(t_0 - \sigma^2, t_0, H_0^1(B(\sigma))\big)$ is the solution of the C.D. problem

$$(5.20)\qquad \int\limits_{Q(\sigma)} \sum_i \big(a^i(Dw + Du)|D_i\varphi\big) - \left(w \left| \frac{\partial\varphi}{\partial t}\right.\right) dX = \int\limits_{Q(\sigma)} \sum_i \big(a^i(Du) - B^i|D_i\varphi\big)\, dX\,,$$
$$\forall\varphi \in W\big(Q(\sigma)\big): \varphi(x, t_0) = 0 \quad \text{in } B(\sigma)\,.$$

As $a^i - B^i \in L^2(Q)$, by lemma 2.XI such a w exists and is unique. Moreover, w verifies inequality (2.48). Hence $\forall\lambda \in (0, 1)$

$$(5.21)\qquad \tilde{\Phi}(w, X_0, \lambda\sigma) \leqslant c(\nu, M) \sum_i |B^i|^2_{0,Q(\sigma)}\,.$$

Clearly $v \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system

$$(5.22)\qquad \int\limits_{Q(\sigma)} \sum_i \big(a^i(Dv)|D_i\varphi\big) - \left(v \left| \frac{\partial\varphi}{\partial t}\right.\right) dX = 0\,, \qquad \forall\varphi \in C_0^\infty\big(Q(\sigma)\big)\,.$$

Then, by corollary 5.I, there exists $\varepsilon \in \big(0, n/(n+2)\big)$, such that $\forall\lambda \in (0, 1)$

$$(5.23)\qquad \tilde{\Phi}(v, X_0, \lambda\sigma) \leqslant c(\nu, M)\, \lambda^{2+\varepsilon(n+2)} \tilde{\Phi}(v, X_0, \sigma)\,.$$

As $u = v - w$, from (5.21) and (5.23) estimate (5.19) easily follows.

HÖLDER CONTINUITY AND PARTIAL HÖLDER CONTINUITY

6. – The case $A^i = A^i(p)$.

Consider now system

$$(6.1)\qquad -\sum_i D_i a^i(Du) + \frac{\partial u}{\partial t} = 0 \quad \text{in } Q$$

under the hypotheses (1.5), (1.10) and (1.29). From theorem 5.III it follows that

THEOREM 6.I. – *If* $u \in L^2(-T, 0, H^1(\Omega))$ *is a solution in* Q *of system* (6.1), *and* $n \leqslant 2$, *then*

$$(6.2)\qquad u \in C^{0,\mu}(Q, d)\,, \quad \text{with} \quad \mu = 2 - \frac{1-\varepsilon}{2}(n+2)\,, \qquad \left[\varepsilon \in \left(0, \frac{n}{n+2}\right)\right]$$

and for every cylinder $A \subset\subset Q$

$$[u]_{\mu,\bar{A}} \leqslant c|u|_{0,Q} \tag{6.3}$$ (6)

where c depends on M, ν *and on the distance between A and the parabolic boundary of Q* (7).

In fact, from the fundamental estimate (5.10) it follows that for every cylinder $A \subset\subset Q$

$$[u]_{\mathcal{L}^{2,4+\varepsilon(n+2)}(A,d)} \leqslant c|u|_{0,Q} .$$

If $n \leqslant 2$, then $4 + \varepsilon(n+2) > n+2$, and thus (see [8], theorem 3.I)

$$[u]_{\mu,\bar{A}} \leqslant c[u]_{\mathcal{L}^{2,4+\varepsilon(n+2)}(A,d)} \leqslant c|u|_{0,\mu} .$$

If the derivatives $\partial^i a/\partial p_k^j$ are uniformly continuous in R^{nN}, then also the vector Du is partially μ-Hölder continuous in Q, $\forall \mu < 1$, and this fact holds for any n. Indeed (see section 5) the vector $U = Du$, at least locally, is a solution of the quasi-linear and strongly parabolic system

$$\int_Q \sum_{ij} (\mathcal{A}_{ij}(U)\, D_j U | D_i \varphi) - \left(U \middle| \frac{\partial \varphi}{\partial t}\right) dX = 0 , \qquad \forall \varphi \in C_0^\infty(Q) \tag{6.4}$$

where coefficients $\mathcal{A}_{ij}(p)$ are uniformly continuous in R^{nN}. Therefore, the following theorem holds (see [9], [4])

THEOREM 6.II. – *There exists a set* $Q_0 \subset Q$, *closed in Q and with measure zero, such that*

$$U = Du \in C^{0,\mu}(Q \setminus Q_0, d) , \qquad \forall \mu < 1 . \tag{6.5}$$

Furthermore,

$$H_{n+2-q}(Q_0) = 0 \quad \text{for a } q > 2 . \tag{6.6}$$

By (3.18) of [4] and inequality (4.2), the singular set of the vector Du may be defined this way

$$Q_0 = \left\{ X \in Q : \lim_{\sigma \to 0}{}' \sigma^{-n} \int_{Q(X,\sigma)} \|Du\|^2 + \sum_{ij} \|D_{ij} u\|^2 \, dY > 0 \right\} . \tag{6.7}$$

---

(6) $[u]_{\mu,\bar{A}} = \sup_{X,Y \in \bar{A}} \dfrac{\|u(X) - u(Y)\|}{d^\mu(X,Y)}$.

(7) I.e. $[\Omega \times \{-T\}] \cup [\partial\Omega \times (-T, 0)]$.

However, u is a solution of system (5.13), strongly parabolic in Q and with coefficients $A_{ij}(Du(X)) \in L^\infty(Q)$, whereas $U = Du$ is a solution of system (6.4), strongly parabolic and with coefficients $\mathcal{A}_{ij}(U(X)) \in L^\infty(Q)$. Therefore, because of lemma 2.VII and inequalities (4.1) and (4.2), there exists a $q > 2$, such that

$$(6.8) \qquad \left\{ \fint_{Q(X,\sigma)} \|Du\|^q + \sum_{ij} \|D_{ij}u\|^q \, dY \right\}^{1/q} \leq c\{\tilde{\Phi}(u, X, 2\sigma) + \tilde{\Phi}(Du, X, 2\sigma)\}^{\frac{1}{2}} \leq \leq \left\{ \fint_{Q(X,2\sigma)} \|Du\|^2 + \sum_{ij} \|D_{ij}u\|^2 \, dY \right\}^{\frac{1}{2}} .$$

Equality (6.6) follows from (6.8) (see for instance [Q] theorem 0.I, p. 142).

Account taken of theorem 6.I, the following conjecture seems to be reasonable:

If $n > 2$ and Q_0^* is the singular set of the vector u, then

$$(6.9) \qquad H_{n-2}(Q_0^*) = 0 .$$

7. – The case $A^i = A^i(X, p)$.

Consider the system

$$(7.1) \qquad -\sum_i D_i A^i(X, Du) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du) , \qquad \text{in } Q$$

where $A^i(X, p)$ are vectors of R^N, which satisfy conditions (1.5), (1.10) and (1.29), whereas B^i and B^0 are vectors of R^N, measurable in X and continuous in u and (u, p) respectively, each having strictly controlled growth

$$(7.2) \qquad \|B^i(X, u)\| \leq c(1 + \|u\|^\alpha)$$

$$(7.3) \qquad \|B^0(X, u, Du)\| \leq c(1 + \|u\|^\beta + \|p\|^\gamma)$$

where α, β, γ are subject to the conditions (1.24), (1.25), (1.26).

Let $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ be a solution of system (7.1)

$$(7.4) \qquad \int_Q \sum_i (A^i|D_i\varphi) - \left(u \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i|D_i\varphi) + (B^0|\varphi) \, dX , \qquad \forall \varphi \in C_0^\infty(Q) .$$

In order to study the Hölder regularity of the u we consider two cases: $n \leq 2$ and $n > 2$.

The case $n \leqslant 2$.

Suppose that vectors A^i satisfy the following uniform continuity condition in X:

There exists a bounded non-negative function $\omega(\sigma)$, defined for $\sigma > 0$, which is non-decreasing and goes to zero as $\sigma \to 0$, such that $\forall X, Y \in Q$ and $\forall p \in R^{nN}$

$$(7.5) \qquad \sum_i \|A^i(X, p) - A^i(Y, p)\|^2 \leqslant \omega\big(d(X, Y)\big)\big(1 + \|p\|^2\big) .$$

Fix $Q(X_0, 2\sigma) \subset\subset Q$ and let $w \in L^2\big(t_0 - \sigma^2, t_0, H^1_0(B(\sigma))\big)$ be the solution in $Q(\sigma)$ of the C.D. problem

$$(7.13) \quad \int_{Q(\sigma)} \sum_i \left(A^i(X_0, Dw + Du)|D_i\varphi\right) - \left(w \left|\frac{\partial\varphi}{\partial t}\right.\right) dX =$$
$$= \int_{Q(\sigma)} \sum_i \left(A^i(X, Du)|D_i\varphi\right) - \left(B^0(X, u, Du)|\varphi\right) dX ,$$
$$\forall\varphi \in W\big(Q(\sigma)\big) : \varphi(x, t_0) = 0 \quad \text{in } B(\sigma) .$$

By lemma 2.XI, such a w exists and is unique, in fact (see (1.21))

$$A^i(X, Du) \in L^2(Q) \qquad \text{and} \qquad B^0(X, u, Du) \in L^{2(n+2)/(n+4)}(Q) .$$

Set $v = u + w$. Obviously, $v \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system

$$(7.14) \qquad \int_{Q(\sigma)} \sum_i \left(A^i(X_0, Dv)|D_i\varphi\right) - \left(v \left|\frac{\partial\varphi}{\partial t}\right.\right) dX = \int_{Q(\sigma)} \sum_j \left(B^i(X, u)|D_i\varphi\right) dX ,$$
$$\forall\varphi \in C^\infty_0\big(Q(\sigma)\big) .$$

Estimate on w.

Using lemmas 2.XI and 2.II, we get the following estimate on the vector w: $\forall\lambda \in (0, 1]$

$$(7.15) \quad \tilde{\Phi}(w, X_0, \lambda\sigma) \leqslant c(\nu, M) \int_{Q(\sigma)} \sum_i \|A^i(X, Du) - A^i(X_0, D_k)\|^2 \, dX +$$
$$+ c(\nu, M) |B^0(X, u, Du)|^2_{0,2(n+2)/(n+4),Q(\sigma)} .$$

On the other hand, because of the hypothesis (7.5),

$$\int_{Q(\sigma)} \sum_i \|A^i(X, Du) - A^i(X_0, Du)\|^2 \, dX \leqslant \omega(\sigma)\, \Phi(u, X_0, \sigma)$$

and taking into account lemma 2.V

$$|B^0(X, u, Du)|^2_{0,2(n+2)/(n+4),Q(\sigma)} \leqslant C(u)\, \sigma^{n+4-2(n+2)(\beta/a_0)\vee(\gamma/2)}\, \Phi(u, X_0, \sigma) .$$

We conclude that, $\forall\lambda\in(0,1)$,

$$\tilde{\Phi}(w, X_0, \lambda\sigma)\leqslant o(\sigma)\,\Phi(u, X_0, \sigma) \tag{7.16}$$

where $o(\sigma)$ goes to zero in respect of σ.

Estimate on v. – By theorem 5.IV, we get the following estimate on v: There exists an $\varepsilon\in\big(0, n/(n+2)\big)$ such that $\forall\lambda\in(0,1)$

$$\tilde{\Phi}(v, X_0, \lambda\sigma)\leqslant c(\nu, M)\left\{\lambda^{2+\varepsilon(n+2)}\tilde{\Phi}(v, X_0, \sigma) + \sum_i |B^i(X, u)|^2_{0,Q(\sigma)}\right\}. \tag{7.17}$$

On the other hand $\big($see (2.12)$\big)$

$$\sum_i |B^i(X, u)|^2_{0,Q(\sigma)}\leqslant c\int\limits_{Q(X_0,\sigma)}(1+\|u\|^{2\alpha})\,dX = c\Psi(u, X_0, \sigma)$$

moreover, because of lemma 2.IV, $\forall\lambda\in(0,1)$

$$\Psi(u, X_0, \lambda\sigma)\leqslant c\lambda^{n+2}\Psi(u, X_0, \sigma) + C(u)\sigma^{n(1-\alpha)+2}\,\tilde{\Phi}(u, X_0, \sigma). \tag{7.18}$$

Account taken of lemma 1.II, p. 8 of [Q], from (7.17) and (7.18) it follows that $\forall\lambda\in(0,1)$ and $\forall\varepsilon'\in(0,\varepsilon)$

$$\begin{aligned}\tilde{\Phi}(v, X_0, \lambda\sigma)\leqslant c(\nu, M)\,\lambda^{2+\varepsilon'(n+2)}\,\tilde{\Phi}(v, X_0, \sigma) + \\ + c\Phi(u, X_0, \sigma)\{\lambda^{2+\varepsilon'(n+2)} + \sigma^{n(1-\alpha)+2}\}.\end{aligned} \tag{7.19}$$

As $u=v-w$ in $Q(\sigma)$, from (7.16) and (7.19) we conclude with the theorem below

THEOREM 7.1. – *If* $u\in L^2\big(-T, 0, H^1(\Omega)\big)\cap L^\infty\big(-T, 0, L^2(\Omega)\big)$ *is a solution of system* (7.1), *under the assumptions* (1.5), (1.10), (1.29) *and* (7.2), (7.3), (7.5), *then* $\forall Q(X_0,\sigma)\subset\subset Q$ *and* $\forall\lambda\in(0,1)$ *the following estimate holds*

$$\Phi(u, X_0, \lambda\sigma)\leqslant c\Phi(u, X_0, \sigma)\{\lambda^{2+\varepsilon'(n+2)} + 0(\sigma)\} \tag{7.20}$$

where $0(\sigma)$ *goes to zero with* σ.

PROOF. – As $u=v-w$ in $Q(\sigma)$, from (7.16) and (7.19) we obtain that $\forall\lambda\in(0,1)$

$$\tilde{\Phi}(u, X_0, \lambda\sigma)\leqslant c\Phi(u, X_0, \sigma)\{\lambda^{2+\varepsilon'(n+2)} + 0(\sigma)\} \tag{7.21}$$

where $0(\sigma)\to 0$ when $\sigma\to 0$. To the left-hand side of (7.21) we can add the integral

$$\int\limits_{Q(\lambda\sigma)} 1 + \|u\|^{\alpha_0}\,dX$$

because

$$\int_{Q(\lambda\sigma)} 1 + \|u\|^{q_0}\, dX \leqslant c(n)\,\lambda^{n+2}\int_{Q(\sigma)} 1 + \|u\|^{q_0}\, dX + c(n)\int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^{q_0}\, dX$$

and from (6.37) of [5], if $n > 2$, or from (2.4), if $n \leqslant 2$, we have

$$\int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^{q_0}\, dX \leqslant C(u)\, 0(\sigma)\, \Phi(u, X_0, \sigma)$$

where $0(\sigma)$ goes to zero in respect of σ.

Inequality (7.20) allows us to achieve the Hölder continuity in Q of the vector u when $n \leqslant 2$.

In fact, from (7.20) and from lemma 1.III, p. 9 of [Q], it follows that $\forall \varepsilon' < \varepsilon$ there exists a $\sigma(\varepsilon')$ such that, $\forall \lambda \in (0, 1)$ and $0 < \sigma \leqslant \sigma(\varepsilon')$

$$\Phi(u, X_0, \lambda\sigma) \leqslant c\lambda^{2+\varepsilon'(n+2)}\Phi(u, X_0, \sigma)\,. \tag{7.22}$$

This inequality is quite analogous to the fundamental estimate (5.15) which holds for the solutions of system (6.1). In particular, from (7.22) we obtain

$$\int_{Q(\lambda\sigma)} \|u - u_{Q(\lambda\sigma)}\|^2\, dX \leqslant c\lambda^{4+\varepsilon'(n+2)}\sigma^2\Phi(u, X_0, \sigma)\,. \tag{7.23}$$

Therefore,

$$u \in \mathfrak{L}^{2,4+\varepsilon'(n+2)}_{\text{loc}}(Q, d) \tag{7.24}$$

and certainly it results $4 + \varepsilon'(n + 2) > n + 2$, if $n \leqslant 2$. In general, the validity of the previous inequality depends on the value of ε, which in its turn depends on the constants ν and M of the system.

THEOREM 7.II. – *If* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *is a solution in* Q *of system* (7.1), *under the assumptions* (1.5), (1.10), (1.29) *and* (7.2), (7.3), (7.5), *then*

$$u \in C^{0,\mu}(Q, d)\,, \qquad \textit{with} \qquad \mu = 2 - \frac{1-\varepsilon}{2}(n + 2)\,. \tag{7.25}$$

As it is known from (7.24), by theorem 3.I of [8], (7.25) follows.

The case $n > 2$.

As (see (1.6))

$$A^i(X, p) = \sum_j A_{ij}(X, p)p^j$$

system (7.1) may be written in the form

$$(7.26) \qquad -\sum_{ij} D_i A_{ij}(X, Du)\, D_j u + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du) .$$

Let us suppose that the derivatives $\partial a^i j \partial p^j_k$, as well as the matrices $A_{ij}(X, p)$, are uniformly continuous in $\bar{Q}\times R^{nN}$. As the $\partial a^i/\partial p^j_k$ are also bounded (see (1.5)), it follows that there exists a non-negative function $\omega(\sigma)$, defined for $\sigma \geq 0$ with $\omega(0) = 0$, non-decreasing, continuous, bounded and concave, such that $\forall X, Y \in Q$ and $\forall p$, $\bar{p} \in R^{nN}$

$$(7.27) \qquad \left\{\sum_{ij} \|A_{ij}(X, p) - A_{ij}(Y, \bar{p})\|^2\right\}^{\frac{1}{2}} \leq \omega\big(d^2(X, Y) + \|p - \bar{p}\|^2\big) .$$

We premise a result of L^q_{loc}-regularity for the vector Du. This result can be proved for systems of the general type.

LEMMA 7.I. – *If* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *is a solution in* Q *of the system*

$$(7.28) \qquad -\sum_i D_i A^i(X, u, Du) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du)$$

which is strongly parabolic, with strictly controlled growth, and satisfies hypothesis (1.5), *then* $\exists \bar{q}$

$$(7. \qquad 2 < \bar{q} \leq \frac{q_0}{\alpha}$$

such that $\forall q \in (2, \bar{q}]$ *and* $\forall Q(X_0, 2\sigma) \subset\subset Q$, *with* $\sigma \leq 1$,

$$(7.30) \qquad \left(\int_{Q(\sigma)} \|Du\|^q\, dX\right)^{2/q} \leq c(u)\, \sigma^{(n+2)(2/q-1)}\, \Phi(u, X_0, 2\sigma)$$

where $C(u)$ *is defined in* (2.17) (8).

PROOF. – As (see (1.6))

$$A^i(X, u, p) = \sum_j A_{ij}(X, u, p) p^j$$

system (7.28) can be regarded as a linear strongly parabolic system with coefficients $A_{ij}(X, u(X), Du(X)) \in L^\infty(Q)$.

(8) An analogous result for quasi-linear parabolic systems is proved in [5], section 5.

Because of lemma 2.I

$$B^i(X, u) \in L^{q_0/\alpha}(Q) \quad \text{and} \quad B^0(X, u, Du) \in L^{(q_0/\beta)\wedge(2/\gamma)}(Q)\,.$$

Therefore, by lemma 2.VII, there exists $\bar{q}$

$$2 < \bar{q} \leqslant \frac{q_0}{\alpha} \tag{7.31}$$

such that $\forall q \in (2, \bar{q}]$ and $\forall Q(X_0, 2\sigma) \subset\subset Q$

$$\frac{2(n+2)}{n+4} < r(q) < \frac{q_0}{\beta} \wedge \frac{2}{\gamma} \tag{7.32}$$

and

$$\left(\fint_{Q(\sigma)} \|Du\|^q\, dX\right)^{2/q} \leqslant c\left(\fint_{Q(2\sigma)} \sum_i \|B^i\|^q\, dX\right)^{2/q} + \\ + c\sigma^2\left(\fint_{Q(2\sigma)} \|B^0\|^{r(q)}\, dX\right)^{2/r(q)} + c\sigma^{-(n+2)}\tilde{\Phi}(u, X_0, 2\sigma)\,. \tag{7.33}$$

On the other hand, account taken of (7.31),

$$\left(\fint_{Q(2\sigma)} \sum_i \|B^i\|^q\, dX\right)^{2/q} \leqslant c\fint_{Q(2\sigma)} 1 + \|u\|^{q_0}\, dX \leqslant c\sigma^{-(n+2)}\,\Phi(u, X_0, 2\sigma) \tag{7.34}$$

and by lemma 2.V and (7.32),

$$\left(\fint_{Q(2\sigma)} \|B^0\|^{r(q)}\, dX\right)^{2/r(q)} \leqslant C(u)\,\sigma^{-2(n+2)((\beta/q_0)\vee(\gamma/2))}\,\Phi(u, X_0, 2\sigma)\,. \tag{7.35}$$

Since $\sigma \leqslant 1$, estimate (7.30) easily follows from (7.33), (7.34), (7.35).

That being stated, we prove the following theorem which, in case $n > 2$, replaces theorem 7.I.

THEOREM 7.III. – *If u is a solution of system* (7.1), *under the hypotheses* (1.5), (1.10), (1.29), (7.2), (7.3) *and if the derivatives $\partial a^i/\partial p^j_k$ are uniformly continuous in $\bar{Q}\times R^{nN}$, then $\forall Q(X_0, \sigma) \subset\subset Q$, with $\sigma \leqslant 2$, $\forall \lambda \in (0, 1)$ and $\forall \varepsilon \in (0, n\alpha)$*

$$\Phi(u, X_0, \lambda\sigma) \leqslant \\ \leqslant A\Phi(u, X_0, \sigma)\left\{\lambda^{n+2-\varepsilon} + o(\sigma) + \left[\omega\left(\sigma^2 + \fint_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dX\right)\right]^{1-2/q}\right\} \tag{7.36}$$

where $o(\sigma)$ goes to zero in respect of σ.

PROOF. – We argue as in theorem 7.I. System (7.1) can be written in the form (7.26). Fix $Q(X_0, 2\sigma) \subset\subset Q$, with $\sigma \leqslant 1$, and, for the sake of simplicity, set

$$\bar{A}_{ij} = A_{ij}\big(X_0, (Du)_{Q(\sigma)}\big)\,. \tag{7.37}$$

Let $w \in L^2\big(t_0 - \sigma^2, t_0, H_0^1(B(\sigma))\big)$ be the solution of the C.D. problem

$$\int\limits_{Q(\sigma)} \sum_{ij} (\bar{A}_{ij} D_j w | D_i \varphi) - \left(w \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = \int\limits_{Q(\sigma)} \sum_{ij} ([\bar{A}_{ij} - A_{ij}(X, Du)]\, D_j u | D_i \varphi)\, dX + \\ + \int\limits_{Q(\sigma)} (B^0(X, u, Du)|\varphi)\, dX\,, \quad \forall \varphi \in W(Q(\sigma)) : \varphi(x, t_0) = 0 \quad \text{in } B(\sigma)\,. \tag{7.38}$$

Clearly, $v = u - w \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of the system

$$-\sum_i D_i(\bar{A}_{ij} D_j v) + \frac{\partial v}{\partial t} = -\sum_i D_i B^i(X, u) \quad \text{in } Q(\sigma)\,. \tag{7.39}$$

Estimate on v. – (7.39) is a linear system with constant coefficients; therefore, by lemma 2.II of [4], we have that $\forall \lambda \in (0, 1)$

$$\tilde{\Phi}(v, X_0, \lambda\sigma) \leqslant c(\nu, M)\Big\{\lambda^{n+2} \tilde{\Phi}(v, X_0, \sigma) + \sum_i |B^i(X, u)|^2_{0,Q(\sigma)}\Big\}$$

Since

$$\sum_i |B^i(X, u)|^2_{0,Q(\sigma)} \leqslant c \int\limits_{Q(\sigma)} 1 + \|u\|^{2\alpha}\, dX = c\Psi(u, X_0, \sigma)$$

using lemma 2.IV and lemma 1.II, p. 8 of [Q], we conclude that $\forall \lambda \in (0, 1)$ and $\forall \varepsilon \in (0, n\alpha)$

$$\tilde{\Phi}(v, X_0, \lambda\sigma) \leqslant c(\nu, M)\, \lambda^{n+2-\varepsilon} \tilde{\Phi}(v, X_0, \sigma) + c\Phi(u, X_0, \sigma)\{\lambda^{n+2-\varepsilon} + \sigma^{n(1-\alpha)+2}\}\,. \tag{7.40}$$

Estimate on w. – Because of lemma 2.VI, we have thet $\forall \lambda \in (0, 1]$

$$\tilde{\Phi}(w, X_0, \lambda\sigma) \leqslant c(\nu, M) \int\limits_{Q(\sigma)} \sum_{ij} \|A_{ij}(X, Du) - \bar{A}_{ij}\|^2 \cdot \|Du\|^2\, dX + \\ + c(\nu, M) |B^0(X, u, Du)|^2_{0,2(n+2)/(n+4),Q(\sigma)}\,. \tag{7.41}$$

On the other hand, taking into account (7.27), lemma 7.I, the boundedness and concavity of ω, we obtain

$$\int\limits_{Q(\sigma)} \sum_{ij} \|A_{ij}(X, Du) - \bar{A}_{ij}\|^2 \cdot \|Du\|^2\, dX \leqslant \int\limits_{Q(\sigma)} \omega(\ldots) \|Du\|^2\, dX \leqslant \\ \leqslant c\sigma^{n+2} \Big(\fint_{Q(\sigma)} \|Du\|^q\, dX \Big)^{2/q} \Big(\fint_{Q(\sigma)} \omega(\ldots)\, dX \Big)^{1-2/q} \leqslant \\ \leqslant c(u)\, \Phi(u, X_0, 2\sigma) \left[\omega\Big(\sigma^2 + \fint_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dX\Big)\right]^{1-2/q}\,. \tag{7.42}$$

Finally, $|B^0|^2_{0,2(n+2)/(n+4),Q(\sigma)}$ can be estimated as in lemma 2.V. Then, we conclude that $\forall\lambda \in (0,1]$

$$\tilde{\Phi}(w, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, 2\sigma)\left\{o(\sigma) + \left[\omega\left(\sigma^2 + \fint_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dX\right)\right]^{1-2/q}\right\}. \tag{7.43}$$

As $u = v + w$ in $Q(\sigma)$, from (7.40) and (7.42) we conclude that $\forall\lambda \in (0,1)$ and $\forall\varepsilon > 0$

$$\tilde{\Phi}(u, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, 2\sigma)\{\lambda^{n+2-\varepsilon} + 0(\sigma) + [\omega(\ldots)]^{1-2/q}\}$$

and, from this, (7.36) follows as the integral

$$\int_{Q(\lambda\sigma)} 1 + \|u\|^{q_0}\, dX$$

can be added to the left-hand side for the same motivations we pleaded in the proof of theorem 7.I.

The following theorem on partial Hölder continuity of u is a consequence of theorem 7.III.

Set

$$Q_1 = \left\{X \in Q : \lim_{\sigma\to 0}{}'' \fint_{Q(X,\sigma)} \|Du - (Du)_{Q(X,\sigma)}\|^2\, dY > 0\right\}. \tag{7.44}$$

The properties of Lebesgue integral imply that

$$\text{meas}\, Q_1 = 0\,. \tag{7.45}$$

THEOREM 7.IV. – *If u is a solution of system* (7.1), *under the hypotheses* (1.5), (1.10), (1.29), (7.2), (7.3), *and if the derivatives $\partial a^i/\partial p^j_k$ are uniformly continuous in $\bar{Q}\times R^{nN}$, then there exists a set Q_0, closed in Q, with*

$$Q_0 \subset Q_1\,, \quad \textit{and therefore} \quad \text{meas}\, Q_0 = 0 \tag{7.46}$$

such that

$$u \in C^{0,\mu}(Q \setminus Q_0, d)\,, \quad \forall\mu < 1\,. \tag{7.47}$$

This theorem may be proved by reasoning exactly as in the proof of theorem 5.I of [2].

8. – The case $A^i = A^i(X, u, p)$.

Consider now a system of the general type

$$-\sum_i D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du)\,, \quad \text{in } Q$$

which may be written in the form (see section 1)

$$(8.1) \qquad -\sum_i D_i A^i(X, u, Du) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du)$$

where $A^i(X, u, p)$, $B^i(X, u)$, $B^0(X, u, p)$ are vectors of R^N, which satisfy conditions (1.5), (1.10) and have strictly controlled growths

$$(8.2) \qquad \|A^i(X, u, p)\| \leqslant c(M)\|p\|$$

$$(8.3) \qquad \|B^i(X, u)\| \leqslant c(1 + \|u\|^\alpha)$$

$$(8.4) \qquad \|B1(X, u, p)\| \leqslant c(1 + \|u\|^\beta + \|p\|^\gamma)$$

where α, β, γ are subject to the conditions (1.24), (1.25), (1.26).

Let $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ be a solution of system (8.1) i.e.

$$(8.5) \qquad \int_Q \sum_i (A^i|D_i\varphi) - \left(u\left|\frac{\partial\varphi}{\partial t}\right.\right) dX = \int_Q \sum_i (B^i|D_i\varphi) + (B^0|\varphi)\, dX\,, \qquad \forall\varphi \in C_0^\infty(Q)\,.$$

Also for these systems, in order to study the Hölder regularity of the vector u, we consider the cases $n \leqslant 2$ and $n > 2$ separately.

The case $n \leqslant 2$.

Our proof is quite analogous to that of the case $n \leqslant 2$ in section 7.

Suppose that the vectors A^i verify this uniform continuity condition with respect to (X, u) (see (7.5)):

There exists a non-decreasing, bounded, continuous, concave function $\omega(\sigma)$, defined for $\sigma \geqslant 0$ with $\omega(0) = 0$, such that $\forall X, Y \in Q$, $\forall u, v \in R^N$ and $\forall p \in R^{nN}$

$$(8.6) \qquad \sum_i \|A^i(X, u, p) - A^i(Y, v, p)\|^2 \leqslant \omega\big(d^2(X, Y) + \|u - v\|^2\big)\|p\|^2\,.$$

This condition is easily fulfilled if, for instance, $A^i \in C^1(\bar{\Lambda})$ and, in agreement with (8.2),

$$(8.7) \qquad \left\|\frac{\partial A^i}{\partial t}\right\| + \sum_s \left\|\frac{\partial A^i}{\partial x_s}\right\| + \sum_k \left\|\frac{\partial A^i}{\partial u_k}\right\| \leqslant c\|p\|\,.$$

In fact, from (8.2)

$$\|A^i(X, u, p) - A^i(Y, v, p)\| \leq c_1(M)\|p\| \tag{8.8}$$

and by (8.7)

(8.9) $\|A^i(X, u, p) - A^i(Y, v, p)\| =$

$$= \left\| \int_0^1 \frac{d}{d\eta} A^i(\eta(X - Y) + Y, \eta(u - v) + v, p)\, d\eta \right\| \leq c_2 \sqrt{T}\, \|p\| \cdot \{d^2(X, Y) + \|u - v\|^2\}^{\frac{1}{2}} .$$

Therefore, it is enough to assume

$$\omega(\sigma) = n c_1(M) \min \{c_1(M),\, c_2\sqrt{T}\sqrt{\sigma}\} . \tag{8.10}$$

Having fixed $Q(X_0, 2\sigma) \subset\subset Q$, with $g \leq 1$, we set, for simplicity,

$$\bar{A}^i(p) = A^i(X_0, u_{Q(\sigma)}, p) . \tag{8.11}$$

Let $w \in L^2\big(t_0 - \sigma^2, t_0, H_0^1(B(\sigma))\big)$ be the solution in $Q(\sigma)$ of the C.D. problem

(8.12) $$\int_{Q(\sigma)} \sum_i (\bar{A}^i(Dw + Du)|D_i\varphi) - \left(w \left| \frac{\partial \varphi}{\partial t}\right.\right) dX =$$

$$= \int_{Q(\sigma)} \sum_i (A^i(X, u, Du)|D_i\varphi) - (B^0(X, u, Du)|\varphi)\, dX ,$$

$\forall \varphi \in W(Q(\sigma)) : \varphi(x, t_0) = 0 \quad$ in $B(\sigma)$.

Because of lemma 2.XI and of (1.21), w exists and is unique.

Set $v = u + w$. Clearly $v \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system

(8.13) $$\int_{Q(\sigma)} \sum_i (\bar{A}^i(Dv)|D_i\varphi) - \left(v \left| \frac{\partial \varphi}{\partial t}\right.\right) dX = -\int \sum_i (B^i(X, u)|D_i\varphi)\, dX ,$$

$\forall \varphi \in C_0^\infty(Q(\sigma))$.

Inequality (7.19) holds on v i.e.: $\exists \varepsilon \in \big(0, n/(n + 2)\big)$ such that $\forall \lambda \in (0, 1)$

$$\tilde{\Phi}(v, X_0, \lambda\sigma) \leq c\lambda^{2+\varepsilon(n+2)} \tilde{\Phi}(v, X_0, \sigma) + c\Phi(u, X_0, \sigma)\{\lambda^{2+\varepsilon(n+2)} + o(\sigma)\} . \tag{8.14}$$

Inequality (7.15) holds on w, therefore, $\forall \lambda \in (0, 1]$,

$$\tilde{\Phi}(w, X_0, \lambda\sigma) \leq o(\sigma)\, \Phi(u, X_0, \sigma) + c(v, M) \int_{Q(\sigma)} \sum_i \|A^i(X, u, Du) - \bar{A}^i(Du)\|^2\, dX . \tag{8.15}$$

On the other hand, taking into account hypothesis (8.6), lemma 7.I, the concavity and boundedness of ω, we get

$$(8.16)\quad \int_{Q(\sigma)} \sum_i \|A^i(X, u, Du) - \bar{A}^i(Du)\|^2\, dX \leqslant \int_{Q(\sigma)} \omega(\sigma^2 + \|u - u_{Q(\sigma)}\|^2)\|Du\|^2\, dX \leqslant$$

$$\leqslant c\sigma^{n+2}\Big(\fint_{Q(\sigma)} \|Du\|^q\, dX\Big)^{2/q}\Big(\fint_{Q(\sigma)} \omega(\ldots)\, dX\Big)^{1-2/q} \leqslant$$

$$\leqslant C(u)\,\Phi(u, X_0, 2\sigma)[\omega(c\sigma^{-n}\,\Phi(u, X_0, \sigma))]^{1-2/q}.$$

We conclude that $\forall\lambda \in (0, 1]$

$$(8.17)\qquad \tilde{\Phi}(w, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, 2\sigma)\{o(\sigma) + [\omega(c\sigma^{-n}\,\Phi(u, X_0, \sigma))]^{1-2/q}\}.$$

Therefore, the following theorem holds

THEOREM 8.I. – *If* $u \in L^2(-T, 0, H^1(\Omega)) \cap L^\infty(-T, 0, L^2(\Omega))$ *is a solution of system* (8.1), *under the hypotheses* (1.5), (1.10), (8.2), (8.3), (8.4), (8.6), *then there exists* $\varepsilon \in (0, n/(n+2))$ *such that* $\forall Q(X_0, \sigma) \subset\subset Q$, *with* $\sigma < 2$, *and* $\forall\lambda \in (0, 1)$

$$(8.18)\qquad \Phi(u, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, \sigma)\{\lambda^{2+\varepsilon(n+2)} + o(\sigma) + [\omega(c\sigma^{-n}\,\Phi(u, X_0, \sigma))]^{1-2/q}\}$$

where $o(\sigma)$ *goes to zero with* σ.

In fact, as $u = v - w$ in $Q(\sigma)$, from (8.14), (8.17) it follows that $\forall\lambda \in (0, 1)$

$$\tilde{\Phi}(u, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, 2\sigma)\{\lambda^{2+\varepsilon(n+2)} + o(\sigma) + [\omega(\ldots)]^{1-2/q}\}.$$

The previous inequality is trivial for $1 \leqslant \lambda < 2$. Finally, to the left-hand side we can add the integral

$$\int_{Q(\lambda\sigma)} 1 + \|u\|^{q_0}\, dX$$

for the same reasons we pleaded in the proof of theorem 7.I.

From the previous theorem we draw forth the partial Hölder continuity of the vector u, by reasoning exactly as in [4] section 3.

Set

$$(8.19)\qquad Q_0 = \Big\{X \in Q : \lim_{\sigma\to 0}{}' \sigma^{-n}\,\Phi(u, X, \sigma) > 0\Big\}$$

we have that

$$(8.20)\qquad H_n(Q_0) = 0$$

(it is sufficient to argue as in theorem 2 of [9]).

THEOREM 8.II. – *If u is a solution of system* (8.1), *under the hypotheses* (1.5), (1.10), (8.2), (8.3), (8.4), (8.6), *then Q_0 is closed in Q and*

$$u \in C^{0,\mu}(Q \setminus Q_0, d)\,, \quad \forall \mu < 2 - \frac{1-\varepsilon}{2}(m+2)\,. \tag{8.21}$$

The case $n > 2$.

Our proof is quite analogous to that of the case $n > 2$ in section 7.

System (8.1) can be written as follows (see (1.7))

$$-\sum_{ij} D_i(A_{ij}(X, u, Du)\, D_j u) + \frac{\partial u}{\partial t} = -\sum_i D_i B^i(X, u) + B^0(X, u, Du)\,. \tag{8.22}$$

Let us suppose that the derivatives $\partial a^i/\partial p_k^j$, and so the matrices $A_{ij}(X, u, p)$, are uniformly continuous in $\bar{\Lambda}$. Since they are also bounded (see (1.5)) it follows that there exists a non-decreasing, bounded, continuous and concave function $\omega(\sigma)$, defined for $\sigma \geqslant 0$ with $\omega(\sigma) = 0$, such that $\forall (X, u, p), (Y, v, \bar{p}) \in \Lambda$

$$\sum_{ij} \|A_{ij}(X, u, p) - A_{ij}(Y, v, \bar{p})\|^2 \leqslant \omega\big(d^2(X, Y) + \|u - v\|^2 + \|p - \bar{p}\|^2\big)\,. \tag{8.23}$$

Fix $Q(X_0, 2\sigma) \subset\subset Q$, with $\sigma \leqslant 1$, and, for the sake of simplicity, set

$$\bar{A}_{ij} = A_{ij}\big(X_0, u_{Q(\sigma)}, (Du)_{Q(\sigma)}\big)\,. \tag{8.24}$$

Reasoning like in the case $n > 2$ of section 7, in $Q(\sigma)$ we write $u = v + w$, where $w \in L^2\big(t_0 - \sigma^2, t_0, H_0^1(B(\sigma))\big)$ is the solution of C.D. problem (7.38), while $v \in L^2\big(t_0 - \sigma^2, t_0, H^1(B(\sigma))\big)$ is a solution of system (7.39).

Vectors v and w must fulfil respectively inequalities (7.40) and (7.41), which enable us to conclude that, $\forall \lambda \in (0, 1)$ and $\forall \varepsilon \in (0, \alpha n)$

$$\begin{aligned}\tilde{\Phi}(u, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, \sigma)\{\lambda^{n+2-\varepsilon} + o(\sigma)\} &+ \\ + c(v, M)\int_{Q(\sigma)} \sum_{ij} &\|A_{ij}(X, u, Du) - \bar{A}_{ij}\|^2 \cdot \|Du\|^2\, dX\end{aligned} \tag{8.25}$$

where $o(\sigma)$ goes to zero in respect of σ.

On the other hand, taking into account (8.23), lemma 7.I, the boundedness and concavity of ω, by reasoning as in (8.14), we obtain that

$$\begin{aligned}\int_{Q(\sigma)} \sum_{ij} &\|A_{ij}(X, u, Du) - \bar{A}_{ij}\|^2 \cdot \|Du\|^2\, dX \leqslant \\ &\leqslant C(u)\,\Phi(u, X_0, 2\sigma)\left[\omega\left(c\sigma^{-n}\,\Phi(u, X_0, \sigma) + \fint_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dX\right)\right]^{1-2/q}.\end{aligned} \tag{8.26}$$

From (8.25), (8.26) we get that $\forall\lambda \in (0, 1)$ and $\forall\varepsilon \in (0, \alpha n)$

$$\tilde{\Phi}(u, X_0, \lambda\sigma) \leqslant c\Phi(u, X_0, 2\sigma)\{\lambda^{n+2-\varepsilon} + o(\sigma) + [\omega(...)]^{1-2/q}\} . \tag{8.27}$$

This inequality is trivially true for $1 \leqslant \lambda < 2$, moreover to the left-hand side we may add the integral

$$\int_{Q(\lambda\sigma)} 1 + \|u\|^{\circ} dX$$

for the motivations we pleaded in the proof of theorem 7.I.

We conclude with the following theorem

THEOREM 8.III. – *If u is a solution of system* (8.1), *under the hypotheses* (1.5), (1.10), (8.2), (8.3), (8.4) *and if the derivatives $\partial a^i/\partial p^j_k$ are uniformly continuous in $\bar{A}$, then $\forall Q(X_0, \sigma) \subset\subset Q$, with $\sigma \leqslant 2$, $\forall\lambda \in (0, 1)$ and $\forall\varepsilon \in (0, \alpha n)$*

$$\Phi(u, X_0, \lambda\sigma) \leqslant \tag{8.28}$$
$$\leqslant A\Phi(u, X_0, \sigma)\left\{\lambda^{n+2-\varepsilon} + o(\sigma) + \left[\omega\left(c\sigma^{-n}\tilde{\Phi}(u, X_0, \sigma) + \fint_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dX\right)\right]^{1-2/q}\right\}$$

where $o(\sigma) \to 0$ when $\sigma \to 0$.

From the previous theorem, the partial Hölder continuity in Q, of the vector u, follows.

Set

$$Q_1 = \left\{X \in Q : \lim_{\sigma \to 0}{}'' \fint \|Du - (Du)_{Q(\sigma)}\|^2\, dY > 0\right\} \tag{8.29}$$

$$Q_2 = \left\{X \in Q : \lim_{\sigma \to 0}{}' \sigma^{-n}\Phi(u, X, \sigma) > 0\right\} . \tag{8.30}$$

It turns out that

$$\text{meas } Q_1 = 0 \tag{8.31}$$

and (see [9], theorem 2)

$$H_n(Q_2) = 0 . \tag{8.32}$$

Reasoning exactly as in theorem 5.I of [2] we prove that

THEOREM 8.IV. – *If u is a solution of system* (8.1), *under the hypotheses* (1.5), (1.10), (8.2), (8.3), (8.4) *and, moreover, if the derivatives $\partial a^i/\partial p^j_k$ are uniformly con-*

tinuous in $\bar{\Lambda}$, *then there exists a set* Q_0, *closed in* Q,

$$Q_2 \subset Q_0 \subset Q_1 \cup Q_2 \quad (\textit{hence } \operatorname{meas} Q_0 = 0) \tag{8.33}$$

such that

$$u \in C^{0,\mu}(Q \setminus Q_0, d)\,, \qquad \forall \mu < 1\,. \tag{8.34}$$

REFERENCES

[Q] S. CAMPANATO, *Sistemi ellittici in forma divergenza. Regolarità all'interno*, « Quaderni » Ann. Scuola Norm. Sup. Pisa, 1980.

[1] S. CAMPANATO, *Hölder continuity of the solutions of some nonlinear elliptic systems*, Advances in Math., **42** (1983).

[2] S. CAMPANATO, *Hölder continuity and partial Hölder continuity results for* $H^{1,q}$*-solutions of non-linear elliptic systems with controlled growth*, to appear in Rend. di Milano.

[3] S. CAMPANATO, L^p*-regularity for weak solutions of parabolic systems*, Ann. Scuola Norm. Sup. Pisa, **7** (1980).

[4] S. CAMPANATO, *Partial Hölder continuity of solutions of quasi-linear parabolic systems of second order with linear growth*, Rend. Sem. Mat. Univ. Padova, **64** (1981).

[5] S. CAMPANATO, L^p*-regularity and partial Hölder continuity for solutions of second order parabolic systems with strictly controlled growth*, Ann. Mat. Pura Appl., **182** (1980).

[6] S. CAMPANATO, *Equazioni paraboliche del secondo ordine e spazi* $\mathfrak{L}^{2,\theta}(\Omega, \delta)$, Ann. Mat. Pura Appl., **73** (1966).

[7] P. CANNARSA, *Second order non variational parabolic systems*, Boll. Un. Mat. Ital., Analisi Funz. e Appl., **82**, C.N. 1 (1981).

[8] G. DA PRATO, *Spazi* $\mathfrak{L}^{(p,\theta)}(\Omega, \delta)$ *e loro proprietà*, Ann. Mat. Pura Appl., **69** (1965).

[9] M. GIAQUINTA - E. GIUSTI, *Partial regularity for the solutions to non-linear parabolic systems*, Ann. Mat. Pura Appl., **97** (1973).

[10] O. A. LADYZENSKAJA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Amer. Mathem. Soc. Translations of Mathem. Monographs, 1968.

[11] J. L. LIONS, *Equations differentielles operationnelles*, Springer, 1961.

[12] J. L. LIONS - E. MAGENES, *Problèmes aux limites non homogènes et applications*, Dunod, 1968.

[13] M. MARINO - A. MAUGERI, L^p *theory and partial Hölder continuity for quasilinear parabolic systems of higher order with strictly controlled growth*, to appear.

[14] A. MAUGERI, *Partial Hölder continuity for the derivatives of order* $(m-1)$ *of solutions to* $2m$ *order quasilinear parabolic systems with linear growth*, to appear.