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The Preparation Theorem on Banach Spaces (*).

FERRANTE BALBONI

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Abstract. – *We give a generalization, for smooth Fredholm maps between Banach spaces, of the Preparation Theorem known in finite dimension. As an application we obtain the Prepared Form Theorem which is a basic tool in singularity theory.*

Introduction.

In this paper we generalize, for smooth Fredholm maps between Banach spaces, the well-known Preparation Theorem (see e.g. [G-G], Chap. 4, Thm. 3.6) and a related important result, i.e. the Prepared Form Theorem. It is maybe worthwhile to emphasize the significance of the finite-dimensional Preparation Theorem (FPT in short) in order to understand the motivations behind our study of the Banach Space version (BPT in short). In fact the FPT has been successfully used in differential topology: just to give two examples we recall the Mather stability theory for smooth maps and the Morin singularities classification in singularity theory. In the last case the FPT is the basic tool for the proof of the Normal Form Theorems which show the equivalence of a suitable class of smooth maps, near a singular point, to a more simple *canonical form* up to local changes of coordinates (see e.g. [G-G]). In analogy with the finite-dimensional case the BPT or better the Prepared Form Theorem, as we call one of its remarkable consequences, is essential for the proof of the Normal Form Theorem for *k-singularities*. These are direct generalizations of the Morin singularities, for smooth maps between Banach spaces, which occur in a natural way in the study of some non-linear boundary value problems. The theory of k-singularities and some concrete and significant examples will be presented in the forthcoming paper [Ba-D].

We recall that an infinite-dimensional statement of the Preparation Theorem has already been formulated, with only a sketch of proof, in [B-C-T]. As we point out below

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Indirizzo dell'A.: Dipartimento di Matematica, Università di Roma «Tor Vergata», Via della Ricerca Scientifica, 00133 Roma, Italy.

in Remark 2.5, that version is less general than ours which closely follows the finite-dimensional statement (see again [G-G], loc. cit.). Moreover to the author's knowledge no complete proof of the BPT has been given up to now and so we believe useful to fill the gap. We refer to Section 2 for a brief historical outline of the *preparation mathematics* (see also [W]). As an application of the BPT and since we are mostly interested in the singularities classification we also chose to prove the Prepared Form Theorem. Our proof of this important result was in part suggested by that given in [B-C-T] and it seems to be simpler. In a Remark at the end of Section 4 we point out the differences.

This paper, which improves the Appendix in the author thesis [Ba], is organized as follows. In Section 1 we describe the algebraic machinery which is needed for the statement and the proof of the Preparation Theorem. This section being very technical, at a first reading one can just retain the definitions and statements, omitting the proofs. Some of these follow in part similar ones presented in [G-G]. In Section 2 we recall the Division Theorem due to P. MICHOR [Mi] which is the direct infinite-dimensional version of the well-known result of B. MALGRANGE [Ma]; this theorem is fundamental in the proof of the BPT. Since in the sequel we also need the Local Representation Theorem we state it in a suitable form after a brief recall about double-splitting operators. Then we introduce some algebraic-geometric notions and finally we can state the BPT. The whole Section 3 is devoted to the proof of the BPT. In Section 4 we state and prove the Prepared Form Theorem which, as we already said, gives the concrete way to use the BPT for the singularities classification considered in [Ba-D].

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1 - Algebraic and analytical preliminaries.

We shall suppose as known the notions of ring (always commutative and with unit 1), ideal of a ring, quotient ring (of the initial ring by one of its ideals) and homomorphism, or morphism, of rings.

If R is a ring we recall that $r \in R$ is *invertible* if there exists $r' \in R$ such that $r' r = 1$ (so $rr' = 1$ too); when r' does exist it is unique and it will be denoted by r^{-1}, the *inverse* of r. A *field* is a ring where each element different from zero has an inverse (as a particular case the null ring $\{0\}$ is a field).

For every ring R there exist the ideals $\{0\}$ and R: it may happen that $\{0\}$ and R are the only ideals of R. In fact we note that when R is a ring and $\mathfrak{I} \neq R$ is an ideal, then $\mathfrak{I} \subseteq \{$not invertible elements of $R\}$. This is because if $q \in \mathfrak{I}$ is invertible, then $1 = q^{-1} q \in \mathfrak{I}$ and so $r = r1 \in \mathfrak{I}$, $\forall r \in R$, that is $\mathfrak{I} = R$, a contradiction. As an immediate consequence a ring R which is a field has only ideals $\{0\}$ and R (the reverse statement is also true).

An ideal $\mathfrak{I}$ of a ring R is called *maximal* if $\mathfrak{I} \neq R$ and for every ideal $\mathfrak{I}'$ of R such that $\mathfrak{I} \subseteq \mathfrak{I}' \subseteq R$ then $\mathfrak{I} = \mathfrak{I}'$ or $\mathfrak{I}' = R$.

If $\mathfrak{I}$ is an ideal of R, $\mathfrak{I} \neq R$, then there exists a maximal ideal of R containing $\mathfrak{I}$. This can be proved by a standard application of Zorn's lemma to the family $\{\mathfrak{I}' : \mathfrak{I}'$ ideal of R, $\mathfrak{I} \subseteq \mathfrak{I}' \neq R\}$, partially ordered by inclusion. The required ideal is a maximal element of such a family. In particular we have that every ring R, $R \neq \{0\}$, has at least a maximal ideal.

A ring R is a *local ring* when it only contains one maximal ideal $\mathfrak{I}_0(R) \equiv \mathfrak{I}_0$. Therefore, by definition, a local ring is always not trivial, that is $R \neq \{0\}$; moreover for any proper ideal $\mathfrak{I}$, i.e. $\mathfrak{I} \neq R$, we have $\mathfrak{I} \subseteq \mathfrak{I}_0$. In fact this is the unique maximal ideal containing $\mathfrak{I}$ by Zorn's Lemma. A field R different from zero is an obvious example of local ring, and in this case $\mathfrak{I}_0 = \{0\}$, as we saw before. In the sequel we shall show a more significant example of local ring.

If R is a ring and $\mathfrak{I}$ is a maximal ideal of R, it is easy to verify that the quotient ring $R/\mathfrak{I}$ is a field. In particular, when R is a local ring with (unique) maximal ideal $\mathfrak{I}_0(R) \equiv \equiv \mathfrak{I}_0$, we shall call $R/\mathfrak{I}_0$ the *residual field* of R.

Finally, when R and R' are local rings with maximal ideals $\mathfrak{I}_0$ and $\mathfrak{I}_0'$ respectively, a morphism $\varphi: R' \to R$ is called *local* if $\varphi(\mathfrak{I}_0') \subseteq \mathfrak{I}_0$.

1.1. REMARK. – If R is a local ring then

$$\mathfrak{I}_0 = \{\text{not invertible elements of } R\}.$$

In fact, since $\mathfrak{I}_0 \neq R$, the inclusion $\subseteq$ follows as seen before. Vice versa if $q \in R$ is not invertible, we consider the set of R defined by $\langle q \rangle := \{rq\colon r \in R\}$, which is an ideal. Then $\langle q \rangle \neq R$, otherwise $1 \in \langle q \rangle = R$, and this implies that it should exist $r \in R$ such that $1 = = rq$ or equivalently $r = q^{-1}$. Hence $\langle q \rangle$ is a proper ideal and so $\langle q \rangle \subseteq \mathfrak{I}_0$; since $q = 1q \in \langle q \rangle$ it follows $q \in \mathfrak{I}_0$.

A *module* A over a ring R, or in short an *R-module*, is an abelian group A with a *product*, or (left) *action* of R on A, defined as a map

$$R \times A \to A\,,$$

$$(r, a) \mapsto r \cdot a \equiv ra\,,$$

which is compatible with the operations of R and A: that is, $\forall r, r' \in R$ and $\forall a, a' \in A$ we have

$$(rr')\,a = r(r'a), \qquad (r + r')\,a = ra + r'a\,, \qquad r(a + a') = ra + ra'\,.$$

Moreover we will require the module has to be *unital* i.e. $1a = a$, $\forall a \in A$.

Given a subset B of A, we say that B is a *submodule* of A if it is a subgroup and it is closed under the product on A, that is $rb \in B$, $\forall r \in R$ and $\forall b \in B$.

In a similar way we can define the *quotient module* A/B. It has a natural structure as a group and the action of R on A/B is defined by $r[\cdot][a] := [r \cdot a] \equiv [ra]$, $r \in R$, $a \in A$. Here $[a]$ denotes the equivalence class of $a \in A$ with respect to the quotient relation and $[\cdot]$ denotes the new product. This product is well-posed and it makes A/B an R-module: for the sake of simplicity, since there is not fear of confusion, we shall denote it by the notation $r[a] \equiv r[\cdot][a] = [ra]$.

Finally, we say that an R-module is *finitely generated* (f.g. in short) over R if there exist $e_1, \ldots, e_p \in A$ such that $a \in A$ may be written in the form $a = \sum_{j=1}^{p} r_j e_j$, $r_j \in R$.

Let $\varphi: R' \to R$ be a morphism between the rings R' and R and let A be an R-module with product $r \cdot a \equiv ra$, $r \in R$, $a \in A$. An action of R' on A is defined in the following way:

$$R' \times A \to A\,,$$

$$(r', a) \mapsto \varphi(r')\, a\,.$$

This action is compatible with the operations of R' and A. Since in this paper we shall suppose that every ring homomorphism is *unitary*, that is $\varphi(1') = 1$ where $1, 1'$ are the units of R, R', the above defined action is actually a product on A. Thus A becomes an R'-module according to the definition.

We shall denote this new product on A with the notation $\dot{\varphi}$ in order to remember that it follows from the old product $\cdot$ by means of the morphism φ. Hence, by definition, $r' \dot{\varphi} a := \varphi(r') \cdot a \equiv \varphi(r')\, a$, $\forall r' \in R'$, $\forall a \in A$. As a matter of notations we shall write A_φ if we consider A with the new module structure induced by φ, i.e. if A is considered as an R'-module. Instead we shall continue to write A when we consider A as a set without additional structure (for example in the notation $a \in A$), or when it has the primitive structure as R-module.

1.2. REMARK. – Let R, R', R'' be rings, $\varphi\colon R' \to R$, $\varphi'\colon R'' \to R'$ ring morphisms and let A be an R-module with product $r \cdot a \equiv ra$, $r \in R$, $a \in A$. Then ${}_{\varphi \circ \varphi'}^{\;\cdot} = \underset{\varphi'}{(\dot{\varphi})}$ holds. For the meaning of this identity let us consider the product ${}_{\varphi \circ \varphi'}^{\;\cdot}$ on A related to the initial product $\cdot$ on A by $\varphi \circ \varphi'\colon R'' \to R$ and then the product $\underset{\varphi'}{(\dot{\varphi})}$ on A_φ derived, by means of the morphism φ', from the product $\dot{\varphi}$ (in its turn related, by means of φ, to the product $\cdot$ on A). Then these products on A are equal: indeed, $\forall r'' \in R''$ and $\forall a \in A$, we have

$$r'' \underset{\varphi \circ \varphi'}{\cdot} a = (\varphi \circ \varphi')(r'') \cdot a = \varphi(\varphi'(r'')) \cdot a = \varphi'(r'') \dot{\varphi} a = r'' \underset{\varphi'}{(\dot{\varphi})}\, a\,.$$

This can be summarized by writing $A_{\varphi \circ \varphi'} = (A_\varphi)_{\varphi'}$. In fact $A_{\varphi \circ \varphi'}$ is the set A with the R''-module structure deriving from the primitive structure of A as an R-module by the morphism $\varphi \circ \varphi'$. Analogously, A_φ is the set A with the R'-module structure induced by φ from the R-module structure of A. Finally consider $(A_\varphi)_{\varphi'}$, i.e. the R''-module structure of A_φ derived by φ' from the structure of A_φ as an R'-module. Then these structures on A, as an R''-module, are equal.

We recall that if A is an R-module and B is a subset of A, $\langle B \rangle$ denotes the *R-module generated* by B that is $\langle B \rangle := \bigcap_{B' \in \mathcal{B}} B'$, where $\mathcal{B} = \{$submodules B' of A: $B \subseteq B'\}$. $\langle B \rangle$ is a non-empty submodule of A (note that $A \in \mathcal{B}$) and it is the smallest submodule of A containing B. It is easy to see that $\langle B \rangle = \{$finite sums $\sum_j r_j b_j$: $r_j \in R$, $b_j \in B\}$.

In particular we note that a ring R is a module over itself and a submodule of R is just an ideal and vice versa. Then if S is a subset of R and $\langle S \rangle$ is defined as above we have that $\langle S \rangle = \bigcap_{\mathfrak{I} \in \mathcal{S}} \mathfrak{I} = \{$finite sums $\sum_j r_j s_j$: $r_j \in R$, $s_j \in S\}$, where $\mathcal{S} = \{$ideals $\mathfrak{I}$ of

$R\colon S\subseteq\mathfrak{I}\}$. Therefore $\langle S\rangle$ is the smallest ideal of R which contains S and it is named the *ideal generated* by S.

Now we state Nakayama's Lemma in one of the several known versions for this famous algebraic result.

1.3. PROPOSITION (Nakayama's Lemma). – *Let R be a local ring and $\mathfrak{I}_0$ its maximal ideal, let A be a* f.g. R-module *and let us suppose $A=\mathfrak{I}_0A$, where $\mathfrak{I}_0A$ denotes the submodule of A generated by $\{qa\colon q\in\mathfrak{I}_0,\ a\in A\}$. Then $A=\{0\}$.*

PROOF. – According to the above characterization of a submodule generated by a set, it is easy to see that $\mathfrak{I}_0A=\{\text{finite sums } \sum_j q_ja_j\colon q_j\in\mathfrak{I}_0,\ a_j\in A\}$. Let $e_1,\dots,e_p$ be generators for A: by hypothesis $A=\mathfrak{I}_0A$ and so $e_p=\sum_{j=1}^{k}q_ja_j$, $q_j\in\mathfrak{I}_0$, $a_j\in A$. On the other hand there exist $r_{ij}\in R$, $i=1,\dots,p$, $j=1,\dots,k$, such that $a_j=\sum_{i=1}^{p}r_{ij}e_i$ and so $e_p=\sum_{j=1}^{k}q_j(\sum_{i=1}^{p}r_{ij}e_i)=\sum_{i=1}^{p}(\sum_{j=1}^{k}q_jr_{ij})\,e_i$. Setting $q_i':=\sum_{j=1}^{k}q_jr_{ij}$ we obtain $e_p=\sum_{i=1}^{p}q_i'\,e_i$ and thus $(1-q_p')\,e_p=\sum_{i=1}^{p-1}q_i'\,e_i$ holds. We claim that $(1-q_p')$ is invertible. Were it not so then we should have $(1-q_p')\in\mathfrak{I}_0$, by Remark 1.1. Note that $q_i'\in\mathfrak{I}_0$, $i=1,\dots,p$, and thus $(1-q_p')+q_p'=1\in\mathfrak{I}_0$; therefore, always by 1.1, it would follow that 1 is not invertible, a contradiction.

Hence $e_p=(1-q_p')^{-1}\sum_{i=1}^{p-1}q_i'\,e_i$ so that $e_1,\dots,e_{p-1}$ generate A. By arguing in the same way for $e_1,\dots,e_{p-1}$ we obtain that $(1-q)\,e_1=0$, $q\in\mathfrak{I}_0$, e_1 generator of A. Like above $(1-q)$ is invertible and so $e_1=(1-q)^{-1}(1-q)\,e_1=(1-q)^{-1}\,0=0$, that is $A=\{0\}$. ■

We already observed that when A is an R-module and B is a submodule then A/B is an R-module equipped with the product $r[a]=[ra]$, where $[a]$ denotes the equivalence class of a in A with respect to the quotient relation by B. As seen before, when $\mathfrak{I}$ is an ideal of R then it is defined the submodule of A, $\mathfrak{I}A=\langle\{qa\colon q\in\mathfrak{I},\ a\in A\}\rangle$. In such a way $A/\mathfrak{I}A$ becomes an R-module. Really we have also that $A/\mathfrak{I}A$ is a module over $R/\mathfrak{I}$, where $R/\mathfrak{I}$ has the natural quotient ring structure. The new product is defined by $\{r\}_{\{[\cdot]\}}[a]:=[r\cdot a]\equiv[ra]$, with $\{\cdot\}$, $[\cdot]$ corresponding to the equivalence classes in R and A respectively. It is easily seen that this action of $R/\mathfrak{I}$ on $A/\mathfrak{I}A$ is well defined and it is a product. Since the contest will make our meaning clear, this rigorous but cumbersome product notation will be replaced by $\{r\}[a]\equiv\{r\}_{\{[\cdot]\}}[a]=[ra]$.

In particular, we obtain that if $\mathfrak{I}$ is a maximal ideal of R then $A/\mathfrak{I}A$ is a vector space over the field $R/\mathfrak{I}$.

1.4. COROLLARY. – *Let R be a local ring, $\mathfrak{I}_0$ the maximal ideal and A a* f.g. R-module. *Let $e_1,\dots,e_p\in A$: then $e_1,\dots,e_p$ generate A over R if and only if $[e_1],\dots,[e_p]$ generate $A/\mathfrak{I}_0A$ over $R/\mathfrak{I}_0$.*

PROOF. – The «only if» part is immediate: indeed if $a \in A$ one has

$$a = \sum_{j=1}^{p} r_j e_j, \quad r_j \in R, \quad \text{thus } [a] = \left[\sum_{j=1}^{p} r_j e_j\right] = \sum_{j=1}^{p} [r_j e_j] = \sum_{j=1}^{p} \{r_j\}[e_j].$$

We will now prove the «if» part. Let B be the submodule of A generated by $e_1, \ldots, e_p$, that is $B := \langle e_1, \ldots, e_p \rangle$: then we have to prove that $A = B$. Let us denote by $[\cdot]'$ the equivalence classes in A with respect to the quotient by B, so that A/B is an R-module with the product $r[a]' = [ra]'$, $r \in R$, $a \in A$.

Firstly we note that $A = B + \Im_0 A$. In fact, since $[e_1], \ldots, [e_p]$ generate $A/\Im_0 A$ over $R/\Im_0$, $\forall a \in A$ we have $[a] = \sum_{j=1}^{p} \{r_j\}[e_j]$, $r_j \in R$. Hence $[a] = [\sum_{j=1}^{p} r_j e_j]$ and from this it follows that $a = \sum_{j=1}^{p} r_j e_j + c$ where $\sum_{j=1}^{p} r_j e_j \in B$ and $c \in \Im_0 A$.

We claim that $A/B = \Im_0(A/B)$. Since the inclusion $\Im_0(A/B) \subseteq A/B$ is obvious we only need to verify that $A/B \subseteq \Im_0(A/B)$. If $a \in A$ we can write $a = b + c$ with $b \in B$, $c \in \Im_0 A$, as seen above. Since $c = \sum_{i=1}^{k} q_i a_i$, $q_i \in \Im_0$, $a_i \in A$ and in A/B one has $[a]' = [c]'$, we can conclude that $[a]' = [\sum_{i=1}^{k} q_i a_i]' = \sum_{i=1}^{k} [q_i a_i]' = \sum_{i=1}^{k} q_i [a_i]' \in \Im_0(A/B)$ and hence $A/B = \Im_0(A/B)$. Since A/B is f.g., we can use Nakayama's Lemma and conclude that $A/B = 0$, that is $A = B$. ■

1.5. PROPOSITION. – *Let R, R' be local rings with maximal ideals $\Im_0$ and $\Im_0'$ respectively and let $\varphi \colon R' \to R$ be a local morphism. Let A be a* f.g. R-module *and let us suppose there exist $e_1, \ldots, e_p \in A$ such that $[e_1]', \ldots, [e_p]'$ generate $A_\varphi / \Im_0' \dot{\varphi} A_\varphi$ over $R'/\Im_0'$, $[\cdot]$ denoting the equivalence classes of the R'*-module *A_φ with respect to the quotient by the submodule $\Im_0' \dot{\varphi} A_\varphi$. Then each element a of A has the form $a = \sum_{j=1}^{p} (\varphi(r_j') + q_j) e_j$, where $r_j' \in R'$ and $q_j \in \langle \varphi(\Im_0') \rangle$, the ideal of R generated by $\varphi(\Im_0')$, $j = 1, \ldots, p$. In other words, for a suitable integer k, $q_j = \sum_{i=1}^{k} a_{ij} \varphi(q_i')$, $a_{ij} \in R$, $q_i' \in \Im_0'$, $j = 1, \ldots, p$, $i = 1, \ldots, k$.*

PROOF. – Since $[e_1]', \ldots, [e_p]'$ generate $A_\varphi / \Im_0' \dot{\varphi} A_\varphi$ over $R'/\Im_0'$, then $\forall a \in A$ $[a]' = \sum_{j=1}^{p} \{r_j'\}' [e_j]'$, $r_j' \in R'$, $\{\cdot\}'$ denoting the equivalence classes in R' with respect to the quotient relation by $\Im_0'$. Hence, by definition, $[a]' = \sum_{j=1}^{p} [r_j' \dot{\varphi} e_j]' = [\sum_{j=1}^{p} r_j' \dot{\varphi} e_j]'$ and so $a = \sum_{j=1}^{p} r_j' \dot{\varphi} e_j + b$ with $b \in \Im_0' \dot{\varphi} A_\varphi$. Again by definition $b = \sum_{i=1}^{k} q_i' \dot{\varphi} a_i$, $q_i' \in \Im_0'$, $a_i \in A$, that is $b = \sum_{i=1}^{k} \varphi(q_i') a_i$. But $\varphi(q_i') \in \varphi(\Im_0') \subseteq \Im_0$ and so $b \in \Im_0 A$. Now, denoting by $[\cdot]$ and $\{\cdot\}$ the equivalence classes in $A/\Im_0 A$ and $R/\Im_0$ respectively, it will follow that $[a] = [\sum_{j=1}^{p} r_j' \dot{\varphi} e_j] = [\sum_{j=1}^{p} \varphi(r_j') e_j] = \sum_{j=1}^{p} [\varphi(r_j') e_j] = \sum_{j=1}^{p} \{\varphi(r_j')\}[e_j]$. Hence $[e_1], \ldots, [e_p]$ generate $A/\Im_0 A$ over $R/\Im_0$ and, by the previous corollary, we have that $e_1, \ldots, e_p$ are generators of A over R. We are now able to conclude. Indeed, as already seen, if $a \in A$ then $a = \sum_{j=1}^{p} r_j' \dot{_\varphi} e_j + b = \sum_{j=1}^{p} \varphi(r_j')\, e_j + b$ with $r_j' \in R'$ and $b = \sum_{i=1}^{k} \varphi(q_i')\, a_i$, $q_i' \in \Im_0'$. But for every $i = 1, \ldots, k$, we have $a_i = \sum_{j=1}^{p} \alpha_{ij} e_j$, $\alpha_{ij} \in R$, and so $b = \sum_{i=1}^{k} \varphi(q_i')(\sum_{j=1}^{p} \alpha_{ij} e_j) = \sum_{j=1}^{p} (\sum_{i=1}^{k} \alpha_{ij} \varphi(q_i'))\, e_j$. If we set $q_j := \sum_{i=1}^{k} \alpha_{ij} \varphi(q_i')$ then, by construction, we have that $q_j \in \langle \varphi(\Im_0') \rangle$. ■

We give the following:

1.6. DEFINITION. – Let $\varphi \colon R' \to R$ an homomorphism between the rings R', R. We shall say that φ is a *Malgrange-Mather* or *M-M morphism*, in short φ is M-M, if the following properties hold:

– R, R' are local rings and φ is a local morphism,

– let $\Im_0'$ be the maximal ideal of R'. Then, for every f.g. R-module A, we have that

if $A_\varphi / \Im_0' \dot{_\varphi} A_\varphi$ is a f.g. vector space over the residual field $R'/\Im_0' \Rightarrow$

$\Rightarrow A_\varphi$ is a f.g. module over R'.

The vice versa of the above implication is trivially true. We introduced this definition for pure convenience, in order to simplify the next sections. In this way we may easily state the following two results, which are the basic algebraic tools for the proof of the Preparation and Prepared Form Theorems.

1.7. PROPOSITION. – *Let* $\varphi \colon R' \to R$ *be an* M-M morphism, A *a* f.g. R-module *and let us suppose that the vector space* $A_\varphi / \Im_0' \dot{_\varphi} A_\varphi$ *is* f.g. *over the residual field* $R'/\Im_0'$: *then* A_φ *is a* f.g. R'-module. *Moreover if* $e_1, \ldots, e_p \in A$ *and* $[e_1], \ldots, [e_p]$ *generate* $A_\varphi / \Im_0' \dot{_\varphi} A_\varphi$ *over* $R'/\Im_0'$ *then* $e_1, \ldots, e_p$ *generate* A_φ *over* R'.

PROOF. – If $A_\varphi / \Im_0' \dot{_\varphi} A_\varphi$ is f.g. it follows, by the very definition of M-M morphism, that A_φ is a f.g. R'-module. Thus, if $[e_1], \ldots, [e_p]$ generate $A_\varphi / \Im_0' \dot{_\varphi} A_\varphi$, we can apply Corollary 1.4 and conclude that $e_1, \ldots, e_p$ generate A_φ over R'.

1.8. PROPOSITION. – *Suppose there exists a commutative diagram*

$$\begin{array}{ccc} R'' & & \\ \varphi' \downarrow & \searrow \psi & \\ R' & \xrightarrow[\varphi]{} & R \end{array}$$

of local morphisms φ, φ', ψ *between the local rings* R, R', R'' *and let* φ' *be an* M-M morphism. *Finally suppose either*

(1) φ *is surjective*

or

(2) φ *is an* M-M morphism.

Then ψ *is also an* M-M morphism.

PROOF. – Let A be a f.g. R-module, $\Im_0''$ the maximal ideal of R'' and suppose that $A_\psi/\Im_0''\dot{_\psi}A_\psi$ is f.g. over $R''/\Im_0''$. We have to verify that A_ψ is f.g. over R''.

Note that this is true when A_φ is f.g. over R'. In fact by Remark 1.2, and since $\psi=\varphi\circ\varphi'$, we have

$$A_\psi/\Im_0''\dot{_\psi}A_\psi=A_{\varphi\circ\varphi'}/\Im_0''{}_{\varphi\dot{\circ}\varphi'}A_{\varphi\circ\varphi'}=(A_\varphi)_{\varphi'}/\Im_0''\underset{\varphi'}{(\dot{\varphi})}(A_\varphi)_{\varphi'}\,.$$

By hypothesis φ' is an M-M morphism and since $(A_\varphi)_{\varphi'}/\Im_0''\underset{\varphi'}{(\dot{\varphi})}(A_\varphi)_{\varphi'}$ is f.g. over $R''/\Im_0''$, if A_φ is f.g. over R' then, by definition of M-M morphism, we shall have that $(A_\varphi)_{\varphi'}=A_\psi$ is f.g. over R''.

Hence it suffices to show that A_φ is f.g. over R'. We prove this by considering separately the hypotheses (1) and (2):

(1) Let $e_1,\dots,e_p\in A$ be generators of A over R: thus, $\forall a\in A$, we have $a=\sum_{j=1}^p r_je_j$, $r_j\in R$. Since φ is surjective we get $r_j=\varphi(r_j')$, $r_j'\in R'$, and thus $a=\sum_{j=1}^p\varphi(r_j')\,e_j=\sum_{j=1}^p r_j'\dot{_\varphi}e_j$. Therefore $e_1,\dots,e_p$ generate A_φ as an R'-module , i.e. A_φ is f.g. over R'.

(2) Let now φ be an M-M morphism. Denoting by $\Im_0'$ the maximal ideal of R', it will be enough to check that $A_\varphi/\Im_0'\dot{_\varphi}A_\varphi$ is f.g. over $R'/\Im_0'$. Indeed if this is true, and since A is f.g. over R, we may invoke Proposition 1.7 to conclude that A_φ is f.g. over R'.

Hence it remains to verify that $A_\varphi/\Im_0'\dot{_\varphi}A_\varphi$ is f.g.. For this it is convenient to denote by $[\cdot]'$, $\{\cdot\}'$, $[\cdot]''$, $\{\cdot\}''$ the elements of $A_\varphi/\Im_0'\dot{_\varphi}A_\varphi$, $R'/\Im_0'$, $A_\psi/\Im_0''\dot{_\psi}A_\psi$, $R''/\Im_0''$, respectively.

Since, by hypothesis, $A_\psi/\Im_0''\dot{_\psi}A_\psi$ is f.g. over $R''/\Im_0''$ there exist $e_1,\dots,e_p\in A$ such that $[e_1]'',\dots,[e_p]''$ generate $A_\psi/\Im_0''\dot{_\psi}A_\psi$ over $R''/\Im_0''$. Then, $\forall a\in A$, one has $[a]''=\sum_{j=1}^p\{r_j''\}''[e_j]''$, $r_j''\in R''$, so that $a=\sum_{j=1}^p r_j''\dot{_\psi}e_j+b$, $b\in\Im_0''\dot{_\psi}A_\psi$. Hence $b=\sum_{i=1}^k q_i''\dot{_\psi}a_i$, $q_i''\in\Im_0''$, $a_i\in A$, that is $b=\sum_{i=1}^k q_i''{}_{\varphi\dot{\circ}\varphi'}a_i=\sum_{i=1}^k q_i''\underset{\varphi'}{(\dot{\varphi})}a_i=\sum_{i=1}^k\varphi'(q_i'')\dot{_\varphi}a_i$. Moreover $\varphi'(\Im_0'')\subseteq\Im_0'$ since φ' is a local morphism.

Therefore $\varphi'(q_i'')\in\Im_0'$ $i=1,\dots,k$, and thus $b\in\Im_0'\dot{_\varphi}A_\varphi$. Since $a=\sum_{j=1}^p r_j''\dot{_\psi}e_j+b$ it follows that $[a]'=[\sum_{j=1}^p r_j''\dot{_\psi}e_j]'$. Finally, as seen above for b, we may write $[a]'=$

$= [\sum_{j=1}^{p} \varphi'(r_j'') \dot{_\varphi} e_j]' = \sum_{j=1}^{p} [\varphi'(r_j'') \dot{_\varphi} e_j]' = \sum_{j=1}^{p} \{\varphi'(r_j'')\}'[e_j]'$ and thus $[e_1]', \ldots, [e_p]'$ are generators for $A_\varphi / \mathfrak{I}_0' \cdot A_\varphi$ over $R'/\mathfrak{I}_0'$. Hence $A_\varphi/\mathfrak{I}_0' \dot{_\varphi} A_\varphi$ is f.g. over $R'/\mathfrak{I}_0'$, as it was to be shown. ∎

We conclude the algebraic part of the section with another notion which will be useful in the next section.

Let R be a commutative ring (with unit 1): we denote the set of all $n \times m$ *matrices*, whose entries are elements of R, by $M(n, m, R)$. We also indicate the matrix $A \in M(n, m, R)$ by the set of its values, i.e. $A = (A_{ij})_{\substack{i=1, \ldots, n \\ j=1, \ldots, m}}$, $A_{ij} \in R$. If $A \in M(n, m, R)$ and $B \in M(m, p, R)$ we can associate with them the *product matrix* $AB \in M(n, p, R)$: it is given by the usual rows by columns multiplication. In particular when $A, B \in M(n, R) := M(n, n, R)$ then $AB \in M(n, R)$. It is easily seen that $M(n, R)$ is a ring (which is non-commutative in general) with unit equal to the matrix $\mathbf{1}$, defined as $(\mathbf{1})_{ij} = \delta_{ij}$, $i, j = 1, \ldots, n$. Note that, $\forall r \in R$, $\forall A = (A_{ij})_{i,j=1,\ldots,n} \in M(n, R)$ it is defined the matrix $rA \in M(n, R)$ by $(rA)_{ij} = rA_{ij}$, $i, j = 1, \ldots, n$. Hence $M(n, R)$ becomes an *algebra* over R, i.e. a ring which is also an R-module and such that the ring and R-module operations are compatible, that is $r(AB) = (rA)B = A(rB)$, $\forall r \in R$, $\forall A, B \in M(n, R)$. Let us denote by R^n the set of all ordered n-tuples of elements of R: it is an abelian group in a obvious way. Moreover, it is an $M(n, R)$-module with the action of $M(n, R)$ on R^n defined as follows. Consider $R^n \equiv M(n, 1, R)$, the set of «column» vectors with n components: then, as seen above, $\forall A \in M(n, R)$, $\forall \underline{r} \in R^n$ the rows by columns product $A\underline{r} \in R^n$ is defined.

For $A \in M(n, R)$ we define the *determinant* of A, $\det A \in R$, to be the element $\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) A_{1\sigma(1)} \cdot \ldots \cdot A_{n\sigma(n)}$. Here S_n is the group of permutations on n elements, say $\{1, \ldots, n\}$. Moreover, $\forall \sigma \in S_n$, $\varepsilon(\sigma)$ is the sign of permutation σ which is either $+1$ or -1 according to it is even or odd (i.e. if it is obtained by product of an even or odd number of transpositions, which are permutations that interchange two only elements and leave fixed the remaining).

If $A \in M(n, R)$ and $i, j \in \{1, \ldots, n\}$ we define the matrix $A(i, j) \in M(n-1, R)$ as the $(n-1) \times (n-1)$ matrix which is obtained from A by omitting the i-th row and the j-th column of A. We introduce the *(i, j)-cofactor* of A, $\widehat{A}(i, j) \in R$, by $\widehat{A}(i, j) := (-1)^{i+j} \det \widehat{A}(i, j)$. Lastly, we define the *adjoint* of A, $\operatorname{adj} A \in M(n, R)$, by $(\operatorname{adj} A)_{ij} := \widehat{A}(j, i)$, $i, j = 1, \ldots, n$. We have:

1.9. PROPOSITION (Cramer's Rule). – For every $A \in M(n, R)$ it results

$$A(\operatorname{adj} A) = (\operatorname{adj} A) A = (\det A)\, \mathbf{1}\,.$$

We do not give here the proof of this known result. For a thorough treatment of the whole subject we refer to [S], Chap. 7, § 7.

We close this section by introducing an elementary but useful analytical tool. For this we recall that a *multi-index* α with n components is an ordered n-tuple of integers α_j, i.e. $\alpha = (\alpha_1, \ldots, \alpha_n)$, such that $\alpha_j \geqslant 0$, $j = 1, \ldots, n$. We shall indicate by $|\alpha|$ the *length* of α defined as $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and denote the *factorial* of α by $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_n!$.

If $\mathbb{R}^n = \{x = (x_1, \dots, x_n): x_j \in \mathbb{R}, j = 1, \dots, n\}$ and α is a multi-index we call *α-th power* of $x \in \mathbb{R}^n$ the number $x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Let Y be a B-space, Ω an open subset of $\mathbb{R}^n \times Y$ and $f: \Omega \to \mathbb{R}$ a smooth function: we set, $\forall (x, y) \in \Omega$,

$$D_x^\alpha f(x, y) := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}} (x, y).$$

1.10. PROPOSITION (Remainder Formula). – *Let U be an open convex subset of R^n, V an open subset in the B-space Y and $f: U \times V \subseteq \mathbb{R}^n \times Y \to \mathbb{R}$ a smooth function. Then, for any integer $k \geq 0$ and $x_0 \in U$, $\forall (x, y) \in U \times V$ we have the formula*

$$f(x, y) = \sum_{0 \leq |\alpha| \leq k} q_\alpha(y)(x - x_0)^\alpha + \sum_{|\alpha| = k+1} Q_\alpha(x, y)(x - x_0)^\alpha$$

where $q_\alpha \in C^\infty(V, \mathbb{R})$ $\forall \alpha: 0 \leq |\alpha| \leq k$, $Q_\alpha \in C^\infty(U \times V, \mathbb{R})$, $\forall \alpha: |\alpha| = k+1$.
Moreover $q_\alpha(y) = (1/\alpha!) D_x^\alpha f(x_0, y)$, $\forall y \in V$.

PROOF. – It suffices to consider $x_0 = 0 \in U$ and proceed by induction on k. For every $x \in U$ the set $\{sx: s \in [0, 1]\}$ is contained in U for convexity. Hence, $\forall y \in V$, we can define $h_y \in C^\infty([0, 1], \mathbb{R})$ as $h_y(s) := f(sx, y)$, $s \in [0, 1]$.
Therefore

$$f(x, y) - f(0, y) = h_y(1) - h_y(0) = \int_0^1 \frac{dh_y}{ds}(s)\, ds = \int_0^1 \nabla_x f(sx, y) \cdot x\, ds =$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(sx, y)\, x_i\, ds = \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}(sx, y)\, ds \right) x_i .$$

Given the multi-indices with n components $\alpha_0 = (0, \dots, 0)$, $\alpha_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the i-th place, $i = 1, \dots, n$, we define the functions $q_{\alpha_0}(y) := f(0, y)$, $y \in V$, and $Q_{\alpha_i}(x, y) := \int_0^1 \partial f / \partial x_i (sx, y)\, ds$, $(x, y) \in U \times V$. It can be readily verified they are smooth functions (see, for example, [A-M-R], 2.4.16). It follows that, for $k = 0$, the result is true. Let us suppose it holds for $k \geq 0$ and prove it for $k+1$.
Since it is true for k, then

$$(*) \qquad f(x, y) = \sum_{0 \leq |\alpha| \leq k} q_\alpha(y)\, x^\alpha + \sum_{|\alpha| = k+1} Q_\alpha(x, y)\, x^\alpha ,$$

where $(x, y) \in U \times V$, q_α and Q_α being suitable smooth functions. Again by the formula for $k = 0$ we obtain, $\forall \alpha$ such that $|\alpha| = k+1$, that $Q_\alpha(x, y) = Q_{\alpha, 0}(y) + \sum_{i=1}^n Q_{\alpha, i}(x, y)\, x_i$, $(x, y) \in U \times V$, with $Q_{\alpha, 0}(y)$, $Q_{\alpha, i}(x, y)$ smooth functions which we do not need to specify. Let $q_\alpha(y) := Q_{\alpha, 0}(y)$, $y \in V$, $|\alpha| = k+1$: by construction q_α, $|\alpha| =$

$= k+1$, is smooth and we may write

$$f(x, y) = \sum_{0 \leq |\alpha| \leq k+1} q_\alpha(y)\, x^\alpha + \sum_{|\alpha| = k+1} \left(\sum_{i=1}^{n} Q_{\alpha, i}(x, y)\, x_i \right) x^\alpha, \quad (x, y) \in U \times V .$$

For every multi-index $\beta = (\beta_1, \ldots, \beta_n)$, $|\beta| = k+2$, and for every $i = 1, \ldots, n$, we define, for $\beta_i \geq 1$, the smooth functions $Q_\beta := Q_{\alpha, i}$, where $\alpha = (\beta_1, \ldots, \beta_i - 1, \ldots, \beta_n)$. Notice that $|\alpha| = k+1$ and we have

$$\sum_{|\alpha| = k+1} \left(\sum_{i=1}^{n} Q_{\alpha, i}(x, y)\, x_i \right) x^\alpha = \sum_{|\beta| = k+2} Q_\beta(x, y)\, x^\beta ,$$

thus the Remainder Formula for $k+1$ holds.

Finally, remark that

$$D_x^\alpha x^\beta(0) = \begin{cases} 0 & \text{if } \alpha \neq \beta , \\ \alpha_1! \cdot \ldots \cdot \alpha_n! & \text{if } \alpha = \beta , \end{cases}$$

$\forall \alpha, \beta$ multi-indices. Apply now D_x^α, $|\alpha| \leq k$, to both sides of the Remainder Formula $(*)$ for the integer k and evaluate in 0. We obtain $D_x^\alpha f(0, y) = \alpha_1! \cdot \ldots \cdot \alpha_n! q_\alpha(y) = \alpha! q_\alpha(y)$, as it was to be proved.

2. – The division and preparation theorems on Banach spaces.

There are deep mathematical results which are called *division theorems*. This is because they remember us the division algorithm with remainder for polynomials. These results and some of their corollaries are also named *preparation theorems* since they are often formulated in an algebraic version which is more suitable for the study of other fundamental problems. We followed this usage and called our Theorem 2.4 the Preparation Theorem.

For an interesting critical and historical review on *preparation mathematics* we refer to [W]. Here we only recall that the first result in this direction was the Weierstrass Preparation Theorem for holomorphic maps which in fact «prepared» the study of the zeroes' set of holomorphic functions. It was then extended by H. SPATH to the more general Division Theorem for holomorphic maps. The question arised whether a similar statement held for smooth maps (on euclidean spaces). Such a (local) theorem was proved by B. MALGRANGE in 1962, [Ma]. A global version of the Division Theorem for smooth maps was then given by J. MATHER in 1968, [M1]. Finally in 1980 P. MICHOR, [Mi], generalized the Malgrange Theorem to smooth maps on Banach spaces.

Now we state, without proof, the Malgrange-Michor Division Theorem.

2.1. THEOREM (Division Theorem). – *Let* X, Y *be* real Banach spaces *and let* $d: \mathbb{R} \times X \to \mathbb{R}$ *be a smooth function, defined near* $\underline{0} = (0, 0)$ *such that* $d(t, \underline{0}) = \bar{d}(t)\, t^k$ *for some* $k \geq 0$, *where* $\bar{d}: \mathbb{R} \to \mathbb{R}$ *is smooth, defined near* 0 *and such that* $\bar{d}(0) \neq 0$. *Then given any smooth function* $f: \mathbb{R} \times X \to Y$ *defined near* $\underline{0}$, *there are smooth functions,* *defined near* $\underline{0}$, $q\colon \mathbb{R}\times X\to Y$, $r_i\colon X\to Y$, $i=0, 1, \ldots, k-1$, *such that* $f(t, x) = q(t, x)\, d(t, x) + \sum_{i=0}^{k-1} r_i(x)\, t^i$.

The complete and detailed proof of this deep theorem can be found in [Mi].

In the sequel we also need the Local Representation Theorem. For this purpose we recall that when X, Y are Banach spaces and $A\in L(X, Y)$, then A is a *double-splitting* operator if $N(A)$ splits in X and $R(A)$ splits in Y, i.e. they are closed subspaces and have closed complements in X and Y respectively. In such a case we shall write $A\in DS(X, Y)$. For $A\in L(X, Y)$ we set $\mathrm{nul}(A) := \dim N(A)$, $\mathrm{def}(A) := \mathrm{codim}\, R(A) = \dim Y/R(A)$ and we say A is a DSF *operator* (or A is DSF) if $A\in DS(X, Y)$ and either $\mathrm{nul}(A) < +\infty$ or $\mathrm{def}(A) < +\infty$. In this case we write $A\in DSF(X, Y)$ and define the *index* of A by $\mathrm{ind}(A) := \mathrm{nul}(A) - \mathrm{def}(A)$. This is a generalization of Fredholm operators and so when $A\in DSF(X, Y)$ and $\mathrm{ind}(A) = i\in\mathbb{Z}\cup\{\pm\infty\}$ we will say that $A\in DSF_i(X, Y)$.

For a *local commutative diagram* (l.c.d. in short)

$$\begin{array}{ccc} x_0\in U\subseteq X & \xrightarrow{F} & V\subseteq Y \\ \alpha\downarrow & & \downarrow\beta \\ x_0'\in U'\subseteq X' & \xrightarrow[\Phi]{} & V'\subseteq Y', \end{array}$$

we mean that:

- open subsets U, U', V, V' are assigned in the B-spaces X, X', Y, Y' respectively;
- smooth mappings F, Φ are defined on some neighbourhood of x_0, x_0' in U, U' respectively. Moreover F, Φ map these neighbourhoods into V, V' respectively;
- there exist (smooth) diffeomorphisms α, β defined on some neighbourhoods of x_0, $F(x_0)$ respectively such that $\alpha(x_0) = x_0'$ and the above diagram locally commutes, i. e. near x_0 one has $\beta F = \Phi\alpha$.

2.2. THEOREM (Local Representation Theorem). – *Let* $F\colon U\subseteq X\to V\subseteq Y$ *be a smooth map between open subsets* U, V *of the* B-spaces X, Y *and let* F *be* double-splitting *in* $x_0\in U$, *that is* $F'(x_0)\in DS(X, Y)$. *Let* X_0, Y_0 *be closed complements of* $N(F'(x_0))$, $R(F'(x_0))$ *in* X, Y *respectively and let* p, π *be the projections of* $X = N(F'(x_0))\oplus X_0$ *on* $N(F'(x_0))$ *and* $Y = Y_0\oplus R(F'(x_0))$ *on* $R(F'(x_0))$ *respectively. Then there exists a* local commutative diagram

$$\text{(D)}\qquad \begin{array}{ccc} x_0\in U\subseteq X & \xrightarrow{F} & V\subseteq Y \\ \alpha\downarrow & & \downarrow\beta \\ (px_0, \pi F(x_0))\in N(F'x_0))\times R(F'(x_0)) & \xrightarrow[\Phi]{} & Y_0\times R(F'(x_0)), \end{array}$$

with α, β, Φ *depending on* X_0, Y_0 *and such that* Φ *has the form* $\Phi(n, r) = (f(n, r), r)$, *for* (n, r) *near* $(px_0, \pi F(x_0)) = \alpha(x_0)$, *where* $f: N(F'(x_0)) \times R(F'(x_0)) \to Y_0$ *is a suitable smooth map defined near* $\alpha(x_0)$ *and such that* $f'(\alpha(x_0)) = 0$.

For a proof of this standard result we refer to [B-Z-S], Thm. 1.1.

Let X be a Banach space (or more generally a smooth Banach manifold) and let $x \in X$. Let us consider the set I of all smooth functions defined on some neighbourhood of x in X, that is $I := \{f \in C^\infty(U, \mathbb{R}): x \in U$ open subset of $X\}$. The functions $f, g \in I$ are *equivalent* if they coincide on a suitable neighbourhood of x. Indeed this is an equivalence relation and the corresponding quotient set will be indicated with $C_x^\infty(X)$. It can be made a ring in the following way. Let us denote by $[\cdot]$ the equivalence classes in $C_x^\infty(X)$ and let $f, g \in I$ be defined on open subsets U, V respectively. Then on $U \cap V$ we define the functions $f+g$, fg by $(f+g)(y) := f(y) + g(y)$, $(fg)(y) := f(y)\,g(y)$, $y \in U \cap V$: it is not hard to see they are smooth, i.e. $f+g, fg \in I$. In $C_x^\infty(X)$ we set $[f] + [g] := [f+g]$, $[f][g] := [fg]$. These operations are well defined, because they do not depend on the choice of the representatives in $[f]$, $[g]$. The above operations make $C_x^\infty(X)$ a commutative ring with unit $[\mathbf{1}]$, where $\mathbf{1}$ is the function on X which is identically equal to $1 \in \mathbb{R}$. Since the ring structure does not depend on the representatives in I, in the sequel we simply indicate the elements of $C_x^\infty(X)$ by $f, g, \ldots$ where $f, g, \ldots \in I$.

We consider now the set $\mathfrak{I}_0 \equiv \mathfrak{I}_0(C_x^\infty(X)) := \{f \in C_x^\infty(X): f(x) = 0\}$. As it is easily seen this is an ideal of $C_x^\infty(X)$ and, obviously, $\mathfrak{I}_0 \neq C_x^\infty(X)$. Let $\mathfrak{I}$ be another ideal, $\mathfrak{I} \neq C_x^\infty(X)$, and let us show that $\mathfrak{I} \subseteq \mathfrak{I}_0$. Suppose on the contrary that $\mathfrak{I} \backslash \mathfrak{I}_0 \neq \emptyset$ and let $g \in \mathfrak{I} \backslash \mathfrak{I}_0$. Then $g(x) \neq 0$ and near x the smooth function g^{-1} is defined, i.e. there exists $g^{-1} \in C_x^\infty(X)$. Let $h \in C_x^\infty(X)$: by definition of ideal, it follows that $h = (hg^{-1})\,g \in \mathfrak{I}$ and hence $\mathfrak{I} = C_x^\infty(X)$, a contradiction. Therefore $\mathfrak{I} \backslash \mathfrak{I}_0 = \emptyset$, that is $\mathfrak{I} \subseteq \mathfrak{I}_0$. This shows two facts:

- $\mathfrak{I}_0$ is a maximal ideal;
- there are not other maximal ideals.

Thus $\mathfrak{I}_0$ is the only maximal ideal and we have shown that $C_x^\infty(X)$ is a local ring (see §1). Consider the residual field $C_x^\infty(X)/\mathfrak{I}_0$: we have a natural isomorphism $C_x^\infty(X)/\mathfrak{I}_0 \approx \mathbb{R}$. In fact note that the ring homomorphism

$$C_x^\infty(X) \to \mathbb{R},$$

$$f \mapsto f(p),$$

is surjective and its kernel is $\mathfrak{I}_0$. Hence it remains defined the isomorphism

$$C_x^\infty(X)/\mathfrak{I}_0 \to \mathbb{R},$$

$$\{f\} \mapsto f(p),$$

where $f \in C_x^\infty(X)$ and $\{\cdot\}$ denotes the equivalence classes in $C_x^\infty(X)/\mathfrak{I}_0$. The inverse one is given by $r \in \mathbb{R} \mapsto \{\mathbf{r}\} \in C_x^\infty(X)/\mathfrak{I}_0$, $\mathbf{r}$ being the function on X which is identically equal to $r \in \mathbb{R}$. From now on, we shall write r instead of $\mathbf{r}$ for such a constant function, without fear of confusion.

2.3. REMARK. – Let $F: U \subseteq X \to V \subseteq Y$ be a smooth map between open subsets U, V of the Banach spaces X, Y. Let $x \in U$, $y \in V$ such that $F(x) = y$: then the map

$$F^*: C_y^\infty(Y) \to C_x^\infty(X),$$

$$g \mapsto F^*(g) := g \circ F,$$

is a ring homomorphism and it is called the *induced morphism* by F (or *pull-back morphism* via F). It is evident that $*$ is «functorial» (in a controvariant way), that is if $H: V \subseteq Y \to W \subseteq Z$ is a smooth map between open subsets V, W of the Banach spaces Y, Z and $H(y) = z \in W$ then

$$(HF)^* = F^* H^*: C_z^\infty(Z) \to C_x^\infty(X).$$

Moreover $F^*(\mathfrak{I}_0(C_y^\infty(Y))) \subseteq \mathfrak{I}_0(C_x^\infty(X))$: in fact if $g \in \mathfrak{I}_0(C_y^\infty(Y))$, i.e. $g(y) = 0$, then $F^*(g)(x) = (g \circ F)(x) = g(F(x)) = g(y) = 0$, that is $F^*(g) \in \mathfrak{I}_0(C_x^\infty(X))$. Thus we have shown that F^* is a local morphism (see § 1).

When $F: U \subseteq X \to V \subseteq Y$ is a smooth map such that $F(x) = y$ then, as seen above, $F^*: C_y^\infty(Y) \to C_x^\infty(X)$ is a local morphism between local rings. These features of the pull-back morphism lead in a natural way to the following formulation of the Preparation Theorem.

2.4. THEOREM (Preparation Theorem). – *Let X, Y be* B-spaces, *U and V open subsets of X and Y respectively, $x_0 \in U$ and $F: U \subseteq X \to V \subseteq Y$ a smooth map such that $F'(x_0) \in DSF_i(X, Y)$, $i \in \mathbb{Z} \cup \{-\infty\}$,* i.e. *$F'(x)$ is double-splitting and* $\dim N(F'(x_0)) < +\infty$. *Then $F^*: C_{F(x_0)}^\infty(Y) \to C_{x_0}^\infty(X)$ is an* M-M *morphism.*

As we said in the introduction this theorem extends the finite-dimensional Preparation Theorem which holds for smooth maps between euclidean finite-dimensional spaces X, Y. Indeed any linear map $T: X \to Y$ is a Fredholm operator since $T \in DSF_i(X, Y)$, $i = \dim X - \dim Y$. We refer to [G-G], Chap. IV, Thm. 3.6, for the finite-dimensional case. Note that there the thesis is differently, but equivalently, formulated.

We recall that, in the finite-dimensional case, the Preparation Theorem was stated by MALGRANGE [Ma] for a particular class of $C_{x_0}^\infty(X)$-modules, that is quotients of $C_{x_0}^\infty(X)$. Later it was generalized by MATHER [M2] for every $C_{x_0}^\infty(X)$-module, see also [W]. For this reason we introduced the *Malgrange-Mather morphisms* (some authors say these morphisms have the *Weierstrass property*, see [P]).

2.5. REMARK. – The Preparation Theorem amounts to say that, by taking $R := C_{x_0}^\infty(X)$, $R' := C_{F(x_0)}^\infty(Y)$ and $\varphi := F^*$, then for every f.g. R-module A we have that $A_\varphi / \mathfrak{I}'_0 \dot{\varphi} A_\varphi$ f.g. over $R'/\mathfrak{I}'_0 \approx \mathbb{R}$ implies that A_φ is f.g. over R', where $\mathfrak{I}'_0$ is the maximal ideal of R'. One can state an apparently more general version of the Preparation Theorem 2.4 in the following way:

(GPT) *in the same hypotheses of* 2.4, *if there exists an ideal $\mathfrak{I}' \subseteq R'$, $\mathfrak{I}' \neq R'$, such that $A_\varphi / \mathfrak{I}' \dot{\varphi} A_\varphi$ is* f.g. *over $R'/\mathfrak{I}'$ then A_φ is* f.g. *over R'.*

This can be proved as in the classical case, where $\Im' = \Im_0'$, adapting some of our results in a simple and direct way. However, we explicitly note that such a version can also be deduced from the classical one. In fact suppose $A_\varphi/\Im' \dot{_\varphi} A_\varphi$ f.g. over $R'/\Im'$ and let $e_1, \ldots, e_p \in A$ such that $[e_1], \ldots, [e_p]$ generate $A_\varphi/\Im' \dot{_\varphi} A_\varphi$. Hence, for all $a \in A$, we have $[a] = \sum_{j=1}^{p} \{r_j'\}[e_j] = \sum_{j=1}^{p} [r_j' \dot{_\varphi} e_j]$, $r_j' \in R'$, i.e. $a = \sum_{j=1}^{p} r_j' \dot{_\varphi} e_j + b$, where $b \in \Im' \dot{_\varphi} A_\varphi$. Thus b has the form $b = \sum_{i=1}^{k} q_i' \dot{_\varphi} a_i$, $q_i' \in \Im'$, $a_i \in A$, and since $R' = C^\infty_{F(x_0)}(Y)$ is a local ring, that is $\Im' \subseteq \Im_0'$, we obtain that $b \in \Im_0' \dot{_\varphi} A_\varphi$. Hence, denoting by $[[\cdot]]$ the equivalence classes in $A_\varphi/\Im_0' \dot{_\varphi} A_\varphi$, we have shown that $[[a]] = \sum_{j=1}^{p} \{r_j'\}[[e_j]]$, that is $[[e_1]], \ldots, [[e_p]]$ generate $A_\varphi/\Im_0' \dot{_\varphi} A_\varphi$. Since $A_\varphi/\Im_0' \dot{_\varphi} A_\varphi$ is f.g. then, from the Preparation Theorem 2.4, we can conclude that A_φ is f.g. over R'.

It is maybe worthwhile to point out that (GPT) allows us to prove the statement of Preparation Theorem given in Lemma 2.6 of [B-C-T] where a suitable ideal $\Im'$ is considered. Such a statement is given in a form which is comparable to ours by means of the Local Representation Theorem (as it is shown by the diagram $(\overline{\mathrm{D}})$ considered in PART 2 of the next section). However in [B-C-T] it is assumed, as an extra-hypothesis, that $f(0, z) = 0$ for each z near $\mathbf{0}$, where f is given in $(\overline{\mathrm{D}})$ below. It would be interesting to know whether the result in [B-C-T] could be used to prove (GPT) or Theorem 2.4, with or without the mentioned additional hypothesis.

3. – Proof of the preparation theorem.

The proof of the Preparation Theorem on Banach spaces is divided in two parts: in the first one we show the theorem in a particular and very simple case by using the Division Theorem 2.1. Here our method of proof is inspired by that one of the finite-dimensional case, (e.g. see [G-G], Chap. IV, Thm. 3.6). In the second part we achieve the proof thanks to the Local Representation Theorem 2.2. In fact the more important difference with respect to the finite-dimensional case, but natural in the context of Fredholm maps, is the use of the diagram $(\overline{\mathrm{D}})$ there introduced. This will allows us to deal with the involved maps like finite-dimensional maps.

PART 1. – We shall prove the following weak form of the theorem: let Z be a B-space, let the map

$$\pi\colon \mathbb{R} \times Z \to Z\,,$$

$$(t, z) \mapsto z\,,$$

be the natural projection where we denote by 0 the origin of $\mathbb{R}$, by $\mathbf{0}$ the origin of Z and with $\underline{0} = (0, \mathbf{0})$ the origin of $\mathbb{R} \times Z$. Then $\pi^*\colon C_{\mathbf{0}}^\infty(Z) \to C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ is an M-M morphism.

By definition we have to prove the following assertion:

let A be a f.g. $C_{\underline{0}}^\infty(\mathbb{R} \times Z)$-module, $\Im_0 \equiv \Im_0(C_{\mathbf{0}}^\infty(Z))$ the maximal ideal of $C_{\mathbf{0}}^\infty(Z)$ and suppose $A_{\pi^*}/\Im_0 {}_{\pi^*}^{\cdot} A_{\pi^*}$ is f.g. over the field $C_0^\infty(Z)/\Im_0 \approx \mathbb{R}$, where

$$\pi^*: C_0^\infty(Z) \to C_{\underline{0}}^\infty(\mathbb{R} \times Z),$$

$$g \mapsto \pi^*(g) = g \circ \pi,$$

is the ring morphism induced by π and A_{π^*} denotes A as a module over $C_0^\infty(Z)$ via π^*. Then it is true that A_{π^*} is f.g., that is A is f.g. as a module over $C_0^\infty(Z)$.

To this end let $e_1, \dots, e_p \in A$ be such that $[e_1], \dots, [e_p]$, the equivalence classes of $e_1, \dots, e_p$ in $A_{\pi^*}/\Im_0 {}_{\pi^*}^{\cdot} A_{\pi^*}$, generate $A_{\pi^*}/\Im_0 {}_{\pi^*}^{\cdot} A_{\pi^*}$ over $C_0^\infty(Z)/\Im_0 \approx \mathbb{R}$.

From Remark 2.3 the map π^* is a local morphism. Therefore, by Proposition 1.5, it follows that every $a \in A$ has the form $a = \sum_{j=1}^{p} (\pi^*(g_j) + f_j) e_j$ where, $\forall j = 1, \dots, p$, $g_j \in C_0^\infty(Z)$ and $f_j \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ has the form $f_j = \sum_{i=1}^{k} h_{ij} \pi^*(q_i) = \sum_{i=1}^{k} h_{ij}(q_i \circ \pi)$, $h_{ij} \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$, $q_i \in \Im_0$, $j = 1, \dots, p$, $i = 1, \dots, k$.

On the other hand $\pi^*(g_j) = \pi^*(g_j(0)) + \pi^*(g_j - g_j(0))$, where $g_j(0)$ is the function which is identically equal to $g_j(0)$ near $0 \in Z$, that is $g_j(0) \in C_0^\infty(Z)$. By abuse of language, let us denote again by $g_j(0)$ the constant function $g_j(0)$ near $(0, 0) \in \mathbb{R} \times Z$, i.e. $g_j(0) \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$. It is evident that $\pi^*(g_j(0)) = g_j(0) \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$: in fact, by definition, $\pi^*(g_j(0))(t, z) = (g_j(0) \circ \pi)(t, z) = g_j(0)(\pi(t, z)) = g_j(0)(z) = g_j(0)$. Hence $\pi^*(g_j) = g_j(0) + \pi^*(g_j - g_j(0))$ holds.

From this we obtain that every $a \in A$ can be written as

$$a = \sum_{j=1}^{p} [g_j(0) + (\pi^*(g_j - g_j(0)) + f_j)] e_j .$$

Since $q_i \in \Im_0$ we have $q_i(0) = 0$ and thus

$$(\pi^*(g_j - g_j(0)) + f_j)(t, 0) = \left(\pi^*(g_j - g_j(0)) + \sum_{i=1}^{k} h_{ij} \pi^*(q_i)\right)(t, 0) =$$

$$= ((g_j - g_j(0)) \circ \pi)(t, 0) + \sum_{i=1}^{k} h_{ij}(t, 0)(q_i \circ \pi)(t, 0) =$$

$$= (g_j - g_j(0))(\pi(t, 0)) + \sum_{i=1}^{k} h_{ij}(t, 0)\, q_i(\pi(t, 0)) =$$

$$= (g_j - g_j(0))(0) + \sum_{i=1}^{k} h_{ij}(t, 0)\, q_i(0) = g_j(0) - g_j(0) = 0 .$$

Therefore we have proved that each a in A has the form $a = \sum_{j=1}^{p} (\gamma_j + \phi_j) e_j$, where $\gamma_j \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ are suitable constant functions and $\phi_j \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ are such that, near $\underline{0}$, $\phi_j(t, 0) = 0$, $j = 1, \dots, p$.

In particular, denoting by $p \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ the projection

$$p: \mathbb{R} \times Z \to \mathbb{R},$$

$$(t, z) \mapsto t .$$

we have that the products $pe_j \in A$, $j = 1, \ldots, p$, can be written as

$$(*) \qquad pe_j = \sum_{i=1}^{p} (\gamma_{ij} + f_{ij})\, e_i\,,$$

where $\gamma_{ij} \in C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$ are constant functions and $f_{ij} \in C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$ are such that $f_{ij}(t, 0) = 0$ for $i, j = 1, \ldots, p$.

This suggest to consider the commutative ring $\mathcal{R} := M(p, C_{\underline{0}}^{\infty}(\mathbb{R} \times Z))$ of the $p \times p$ square matrices with elements in $C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$. As we have seen in Section 1, for all $H = (h_{ij})_{i,j=1,\ldots,p} \in \mathcal{R}$, $h_{ij} \in C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$, $i, j = 1, \ldots, p$, it is defined $\det H \in C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$, the determinant of H. For any (t, z) near $\underline{0}$ we introduce the $p \times p$ real matrix $H(t, z)$ defined by $(H(t, z))_{ij} := h_{ij}(t, z)$, $i, j = 1, \ldots, p$. Hence there exists $\det H(t, z) \in \mathbb{R}$: by definition of determinant, it is evident that $(\det H)(t, z) = \det H(t, z)$ for all (t, z).

Let now G be the constant matrix with elements $G_{ij} := \gamma_{ij}$ and F the matrix with elements $F_{ij} := f_{ij}$. Let us consider the «column» vectors over A with p components

$$\underline{e} := (e_i)_{i=1,\ldots,p}\,, \qquad \underline{0} := (0)_{i=1,\ldots,p}\,.$$

As we saw in Section 1, $\underline{e}, \underline{0} \in M(n, 1, \mathcal{R}) \equiv A^n$ which is an $\mathcal{R}$-module with the action of $\mathcal{R}$ on A^n given by the usual rows by columns product. Thus we may write $(*)$ in the matricial form $(p\mathbf{1}_{\mathcal{R}} - G - F)\,\underline{e} = \underline{0}$, where $\mathbf{1}_{\mathcal{R}}$ is the identity matrix. Taking into account that the action of $\mathcal{R}$ on A^n is associative and Cramer's Rule we have

$$\underline{0} = (\operatorname{adj}(p\mathbf{1}_{\mathcal{R}} - G - F))\,\underline{0} = (\operatorname{adj}(p\mathbf{1}_{\mathcal{R}} - G - F))[(p\mathbf{1}_{\mathcal{R}} - G - F)\,\underline{e}] =$$

$$= [(\operatorname{adj}(p\mathbf{1}_{\mathcal{R}} - G - F))(p\mathbf{1}_{\mathcal{R}} - G - F)]\,\underline{e} = [(\det(p\mathbf{1}_{\mathcal{R}} - G - F))\,\mathbf{1}_{\mathcal{R}}]\,\underline{e} =$$

$$= (\det(p\mathbf{1}_{\mathcal{R}} - G - F))[\mathbf{1}_{\mathcal{R}}\underline{e}] = (\det(p\mathbf{1}_{\mathcal{R}} - G - F))\,\underline{e}\,,$$

that is the identity $(\det(p\mathbf{1}_{\mathcal{R}} - G - F))\,e_j = 0$, $j = 1, \ldots, p$, holds in A.

For the sake of brevity let $d := \det(p\mathbf{1}_{\mathcal{R}} - G - F) \in C_{\underline{0}}^{\infty}(\mathbb{R} \times Z)$ so that in A we have the identity $de_j = 0$, $j = 1, \ldots, p$.

As seen above $f_{ij}(t, 0) = 0$, $i, j = 1, \ldots, p$, hence the $p \times p$ real matrix $F(t, 0)$ is the null matrix. Thus, indicating by $\mathbf{1}_p$ the $p \times p$ real identity matrix, we get

$$d(t, 0) = (\det(p\mathbf{1}_{\mathcal{R}} - G - F))(t, 0) = \det((p\mathbf{1}_{\mathcal{R}} - G - F)(t, 0)) =$$

$$= \det((p\mathbf{1}_{\mathcal{R}}(t, 0) - G) = \det(t\mathbf{1}_p - G)$$

where we have used the definition of p and that the matrix G is constant. Therefore, by construction,

$$d(t, 0) = \det(t\mathbf{1}_p - G) = t^p + \alpha_{p-1}t^{p-1} + \ldots + \alpha_1 t + \alpha_0\,, \qquad \alpha_j \in \mathbb{R}\,,\ j = 0, \ldots, p-1\,,$$

is the characteristic polynomial of G. Then there exists an integer k, $p \geq k \geq 0$, such that

$$\frac{\partial^j d}{\partial t^j}(0, 0) = 0, \qquad j = 1, \dots, k-1, \qquad \text{and} \qquad \frac{\partial^k d}{\partial t^k}(0, 0) \neq 0.$$

By the remainder formula, Proposition 1.10, it follows that in some neighbourhood of 0 we can write $d(t, 0) = \bar{d}(t)\, t^k$, where $\bar{d}$ is a smooth function such that $\bar{d}(0) \neq 0$.

We may now conclude the proof. If $a \in A$ then, as seen, it has the form

$$a = \sum_{j=1}^{p} (\pi^*(g_j) + f_j)\, e_j, \qquad g_j \in C_0^\infty(Z) \quad \text{and} \quad f_j \in C_{\underline{0}}^\infty(\mathbb{R} \times Z), \qquad j = 1, \dots, p.$$

The Division Theorem 2.1, for $Y = \mathbb{R}$, yields $f_j = q_j d + \sum_{i=0}^{k-1} r_{ij} t^i$, $q_j \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$, $r_{ij} \in C_0^\infty(Z)$, $i = 0, 1, \dots, k-1$, $j = 1, \dots, p$, where, with abuse of notation, we denote by $t^i \in C_{\underline{0}}^\infty(\mathbb{R} \times Z)$ the functions $(t, z) \in \mathbb{R} \times Z \mapsto t^i \in \mathbb{R}$, $i = 0, 1, \dots, k-1$. By definition $\pi^*(r_{ij})(t, z) = r_{ij}(\pi(t, z)) = r_{ij}(z)$ and we may deduce the identity $f_j = q_j d + \sum_{i=0}^{k-1} \pi^*(r_{ij})\, t^i$. Since $de_j = 0$, $j = 1, \dots, p$, it follows that

$$f_j e_j = (q_j d)\, e_j + \left(\sum_{i=0}^{k-1} \pi^*(r_{ij})\, t^i\right) e_j = q_j (de_j) + \sum_{i=0}^{k-1} (\pi^*(r_{ij})\, t^i)\, e_j =$$

$$= \sum_{i=0}^{k-1} \pi^*(r_{ij})(t^i e_j) = \sum_{i=0}^{k-1} r_{ij}{}_{\dot{\pi}^*}(t^i e_j).$$

Hence

$$a = \sum_{j=1}^{p} (\pi^*(g_j)\, e_j + f_j e_j) = \sum_{j=1}^{p} \left(g_{j\,\dot{\pi}^*} e_j + \sum_{i=0}^{k-1} r_{ij}{}_{\dot{\pi}^*}(t^i e_j) \right).$$

Since $t^0 e_j = 1 e_j = e_j$ we can write

$$a = \sum_{j=1}^{p} (g_j + r_{0j})_{\dot{\pi}^*} e_j + \sum_{j=1}^{p} \sum_{i=1}^{k-1} r_{ij}{}_{\dot{\pi}^*}(t^i e_j),$$

that is $e_1, \dots, e_p, te_1, \dots, te_p, \dots, t^{k-1} e_1, \dots, t^{k-1} e_p$ are generators of A over $C_0^\infty(Z)$ so that A_{π^*} is f.g., as we had to show. ∎

PART 2. – Let us now suppose that the hyphoteses in 2.4 are verified: for X, Y B-spaces and U, V open subsets of X, Y, let $x_0 \in U$ and $F: U \subseteq X \to V \subseteq Y$ be a smooth map such that $F'(x_0) \in DSF_i(X, Y)$, $i \in \mathbb{Z} \cup \{-\infty\}$. We have to prove that $F^*: C^\infty_{F(x_0)}(Y) \to C^\infty_{x_0}(X)$ is an M-M morphism. Since $\operatorname{ind}(F'(x_0)) = i < +\infty$, i.e. $n := \operatorname{nul}(F'(x_0)) < +\infty$, the Local Representation Theorem 2.2 allows us to say there exists a l.c.d.

$$\begin{array}{ccc} x_0 \in U \subseteq X & \xrightarrow{\ F\ } & V \subseteq Y \\ \bar{\varphi} \downarrow & & \downarrow \bar{\psi} \\ \underline{0} \in \mathbb{R}^n \times Z & \xrightarrow[\ \bar{F}\ ]{} & Y_0 \times Z, \end{array}$$

where Z, Y_0 are B-spaces, $\bar{F}$ is a smooth map defined near the origin $\underline{0} = (0, 0)$ of $\mathbb{R}^n \times$

$\times Z$ having the form $\bar{F}(t, z) = (f(t, z), z)$ for $(t, z) \in \mathbb{R}^n \times Z$, with $f: \mathbb{R}^n \times \underline{Z} \to Y_0$ a smooth map defined near $\underline{0}$ and such that $f(0, 0) = \underline{0}$ the origin of Y_0, i.e. $\bar{F}(0, 0) = (\underline{0}, 0)$. This diagram is easily obtained from that given in the statement of Theorem 2.2: it suffices to put $Z = R(F'(x_0))$, to translate $\alpha(x_0)$, $\beta(x_0)$ in the respective origins, to fix an isomorphism $R(F'(x_0) \approx \mathbb{R}^n$ and hence to consider $\bar{\varphi}$, $\bar{\psi}$, $\bar{F}$ which are trivially determinated by α, β, Φ. For a reason which will soon be evident it is also convenient to define the isomorphism

$$\sigma: Y_0 \times Z \to Z \times Y_0,$$

$$(y_0, z) \mapsto (z, y_0),$$

and to consider the l.c.d.

$$(\bar{\mathrm{D}}) \qquad \begin{array}{ccc} x_0 \in U \subseteq X & \xrightarrow{F} & V \subseteq Y \\ \bar{\varphi}\downarrow & & \downarrow \sigma\bar{\psi} \\ \underline{0} \in \mathbb{R}^n \times Z & \xrightarrow[\tilde{F}]{} & Z \times Y_0, \end{array}$$

where $\tilde{F}$ has now the form $\tilde{F}(t, z) = (z, f(t, z))$ for $(t, z) \in \mathbb{R}^n \times Z$ and $\tilde{F}(0, 0) = (0, 0)$.

By functoriality of $*$, Remark 2.3, it is an easy matter to show that the local rings and the induced morphisms are invariant, up to isomorphisms, under changes of coordinates, i.e. local diffeomorphisms. Thus, by the above diagram, it is sufficient to prove the Preparation Theorem for $\tilde{F}$ i.e. it will be enough to show that

$$\tilde{F}^*: C_0^\infty(Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z) \text{ is an M-M morphism}.$$

In the sequel, for the sake of simplicity, we shall denote by 0 the origin of all product spaces which will be considered. Moreover we shall always write $\mathbb{R}^{n-j} = \{(t_{j+1}, \ldots, t_n): t_i \in \mathbb{R}, i = j+1, \ldots, n\}$, for $j = 1, \ldots, n$.

We define the (smooth) projections

$$\pi_n: \mathbb{R}^n \times Z \times Y_0 \to \mathbb{R}^{n-1} \times Z \times Y_0,$$

$$(t_1, \ldots, t_n, z, y_0) \mapsto (t_2, \ldots, t_n, z, y_0),$$

$$\pi_{n-1}: \mathbb{R}^{n-1} \times Z \times Y_0 \to \mathbb{R}^{n-2} \times Z \times Y_0,$$

$$(t_2, \ldots, t_n, z, y_0) \mapsto (t_3, \ldots, t_n, z, y_0),$$

$$\vdots$$

$$\pi_1: \mathbb{R} \times Z \times Y_0 \to Z \times Y_0,$$

$$(t_n, z, y_0) \mapsto (z, y_0),$$

and the smooth maps

$$i_n\colon \mathbb{R}^n \times Z \to \mathbb{R}^n \times Z \times Y_0\,,$$

$$(t_1, \ldots, t_n, z) \mapsto (t_1, \ldots, t_n, z, f(t_1, \ldots, t_n, z))\,,$$

$$i_{n-1}\colon \mathbb{R}^n \times Z \to \mathbb{R}^{n-1} \times Z \times Y_0\,,$$

$$(t_1, \ldots, t_n, z) \mapsto (t_2, \ldots, t_n, z, f(t_1, \ldots, t_n, z))\,,$$

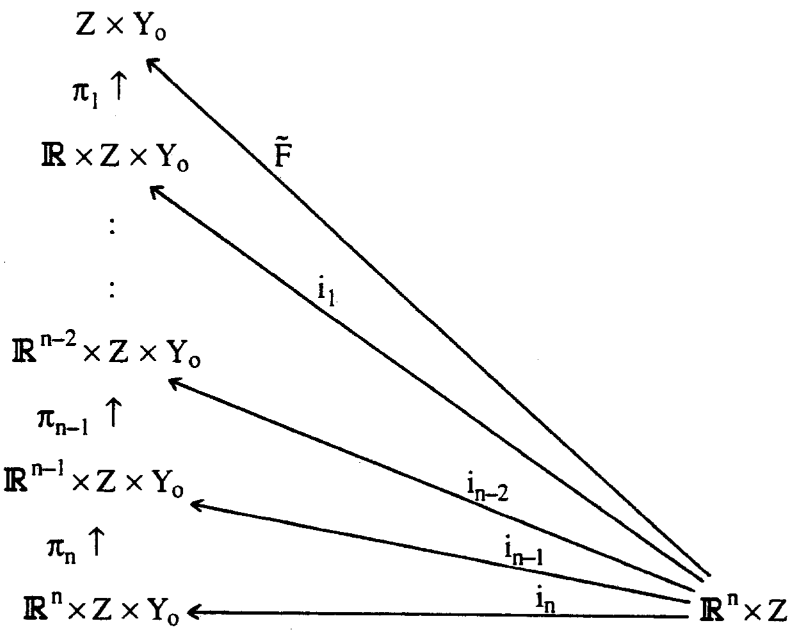
$$\vdots$$

$$i_1\colon \mathbb{R}^n \times Z \to \mathbb{R} \times Z \times Y_0\,,$$

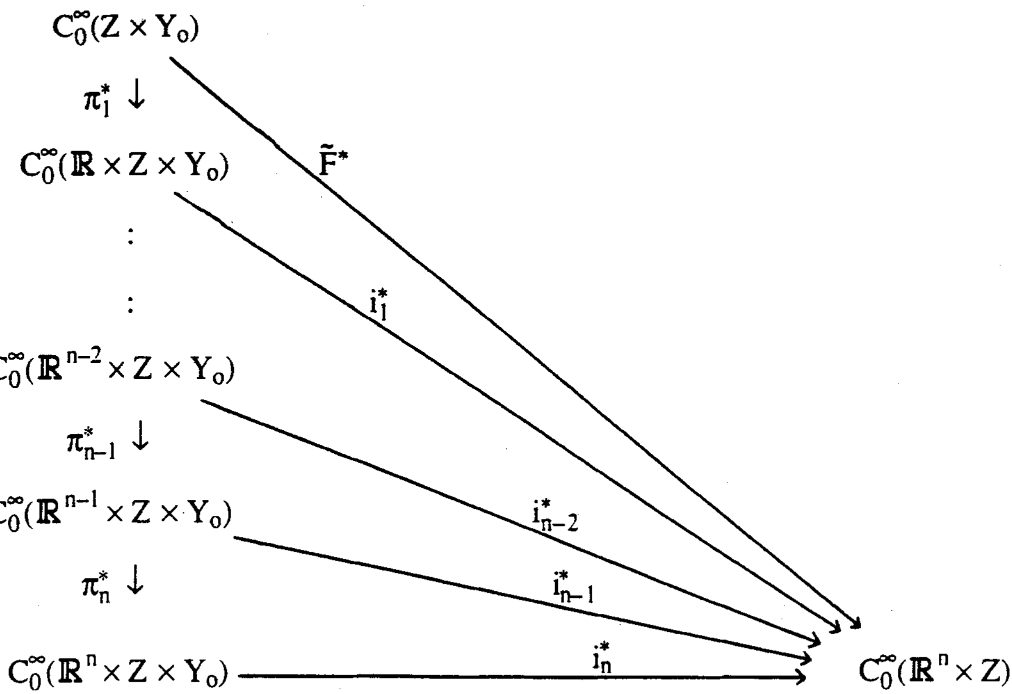
$$(t_1, \ldots, t_n, z) \mapsto (t_n, z, f(t_1, \ldots, t_n, z))\,.$$

It is obvious that for all $k = 1, \ldots, n$ we have $\pi_k(0) = 0$ and $i_k(0) = 0$. Moreover, as we said in Remark 2.3, the induced homomorphisms π_k^* and i_k^* are local morphism. Now we show that, by the controvariant functoriality of $*$, the injectivity of the map i_n implies that the morphism $i_n^*\colon C_0^\infty(\mathbb{R}^n \times Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z)$ is surjective. Indeed let $g \in C_0^\infty(\mathbb{R}^n \times Z)$ and define $h \in C_0^\infty(\mathbb{R}^n \times Z \times Y_0)$ via $h(t_1, \ldots, t_n, z, y_0) := g(t_1, \ldots, t_n, z)$. Then $i_n^*(h) = g$ because $i_n^*(h)(t_1, \ldots, t_n, z) = (h \circ i_n)(t_1, \ldots, t_n, z) = h(t_1, \ldots, t_n, z, f(t_1, \ldots, t_n, z)) = g(t_1, \ldots, t_n, z)$.

It is also easy to verify that the following diagram is commutative:



By functoriality, or as a direct verification, the following diagram is also commutative:



We are now able to show that $\tilde{F}^*: C_0^\infty(Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z)$ is an M-M morphism, and thus to prove the Preparation Theorem. Firstly remark that from PART 1 of the proof it follows that $\pi_n^*: C_0^\infty(\mathbb{R}^{n-1} \times Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z \times Y_0)$ is an M-M morphism. Next, since i_n^* is surjective then, by Proposition 1.8, hypothesis 1), also $i_{n-1}^*: C_0^\infty(\mathbb{R}^{n-1} \times Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z)$ is M-M. Lastly, always by PART 1, we know that $\pi_{n-1}^*: C_0^\infty(\mathbb{R}^{n-2} \times Z \times Y_0) \to C_0^\infty(\mathbb{R}^{n-1} \times Z \times Y_0)$ is M-M. Then, since i_{n-1}^*, π_{n-1}^* are M-M, by Proposition 1.8, hypothesis 2), one obtains that $i_{n-2}^*: C_0^\infty(\mathbb{R}^{n-2} \times Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z)$ is an M-M morphism. By continuing as in this last step we can conclude that $\tilde{F}^*: C_0^\infty(Z \times Y_0) \to C_0^\infty(\mathbb{R}^n \times Z)$ is an M-M morphism. ■

4. – The prepared form theorem.

We are now able to state and prove the Prepared Form Theorem. This means the possibility to formulate a characteristic polynomial identity for a particular class of smooth maps on *B*-spaces. Such an identity is called the *prepared form* of the considered map. As usual this name indicates that the map is «prepared» for future developments, for instance connected to the study of the singularities of maps (see e.g. [B-C-T], [Ba-D]).

4.1. THEOREM (Prepared form). – *Let* Z *be a* B-space, $\mathbb{R}^n = \{x = (x_1, \ldots, x_n): x_i \in \mathbb{R}, i = 1, \ldots, n\}$ *and let* $f: \mathbb{R}^n \times Z \to \mathbb{R}$ *be a smooth function defined near*

$\underline{0} = (0, \ldots, 0, 0)$, *the origin of* $\mathbb{R}^n \times Z$. *Let* $k \geq 0$ *be an integer such that*

$$\frac{\partial^i f}{\partial x_1^i}(0, \ldots, 0, z) = 0, \qquad \text{for } z \text{ near } 0 \in Z \text{ and } i = 0, \ldots, k,$$

$$\frac{\partial^{k+1} f}{\partial x_1^{k+1}}(0, \ldots, 0, 0) \neq 0,$$

If we consider the smooth map

$$F: \mathbb{R}^n \times Z \to \mathbb{R}^n \times Z,$$

$$(x_1, \ldots, x_n, z) \mapsto (f(x_1, \ldots, x_n, z), x_2, \ldots, x_n, z),$$

defined near the origin $\underline{0}$ *of* $\mathbb{R}^n \times Z$ *and such that* $F(\underline{0}) = \underline{0}$ (*since* $f(\underline{0}) = 0$), *then there exist smooth maps* $a_i: \mathbb{R}^n \times Z \to \mathbb{R}$, $i = 0, \ldots, k$, *defined near* $\underline{0}$ *and such that the* prepared form *for* F *holds, i.e. the identity*

$$\text{(FP)} \quad x_1^{k+1} = a_k(F(x, z))\,x_1^k + \ldots + a_1(F(x, z))\,x_1 + a_0(F(x, z)),$$

is valid for $(x, z) = (x_1, \ldots, x_n, z)$ *near* $\underline{0}$.

PROOF. – For the sake of more clarity it is convenient to indicate the local ring $C_{\underline{0}}^\infty(\mathbb{R}^n \times Z)$ with the index $'$ or $''$. This indexes are used when we consider $\underline{0}$ either in the domain or in the range of F respectively, which is $\mathbb{R}^n \times Z$ in both cases. We emphasize that it is not an algebraic distinction, but it is only done for the reader's convenience.

Thus we may define the rings homomorphism (see the previous section)

$$F^*: C_{\underline{0}}^\infty(\mathbb{R}^n \times Z)'' \to C_{\underline{0}}^\infty(\mathbb{R}^n \times Z)',$$

$$g \mapsto F^*(g) := g \circ F;$$

moreover it is easy to show that F is a Fredholm map of index zero. We sketch the proof: write $F'(x_1, \ldots, x_n, z)$: $\mathbb{R}^n \times Z \to \mathbb{R}^n \times Z$ in a matricial form. By definition of F it is readily seen that $F'(x_1, \ldots, x_n, z)$ is an isomorphism for every $(x_1, \ldots, x_n, z) \in \mathbb{R}^n \times Z$, unless $(\partial f/\partial x_1)(x_1, \ldots, x_n, z) = 0$. In the last case $N(F'(x_1, \ldots, x_n, z)) = \{(t_1, 0, \ldots, 0, 0): t_1 \in \mathbb{R}\}$, which has dimension 1, and

$$R(F'(x_1, \ldots, x_n, z)) = \left\{\left(t_2 \frac{\partial f}{\partial x_2}(x_1, \ldots, x_n, z) + \ldots + t_n \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n, z) + \frac{\partial f}{\partial z}(x_1, \ldots, x_n, z)\,v, t_2, \ldots, t_n, v\right): t_2, \ldots, t_n \in \mathbb{R}, v \in Z\right\},$$

which has codimension 1, since $\{(t_1, 0, \ldots, 0, 0): t_1 \in \mathbb{R}\}$ is a complementary subspace.

Since F is Fredholm it follows, by the Preparation Theorem, that F^* is an M-M morphism.

Note that the set, or better the abelian group, $A := C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)$ has a natural structure of f.g. $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)'$-module: for $g \in C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)'$, $h \in A$, the product is defined by $g \cdot h := gh$ the pointwise product function, and moreover the function 1 is the generator. Thus it is defined in the usual way the $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$-module A_{F^*}, i.e. the set A considered as a module over $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$ via F^*: here the product is defined by $g_{F^*} h := F^*(g)\, h$, $g \in C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$, $h \in A_{F^*}$.

We shall indicate by $[\cdot]$, $\{\cdot\}$ the elements in the quotient module (real vector space) $A_{F^*}/\mathfrak{I}_{0\,F^*}A_{F^*}$ and in the residual field $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''/\mathfrak{I}_0 \approx \mathbb{R}$ respectively. As usual $\mathfrak{I}_0 \equiv \mathfrak{I}_0(C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)'') \subsetneq C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$ is the maximal ideal.

Now suppose we have already shown the following statement:

denote, with abuse of notations, by x_1^i, $i = 0, \ldots, k+1$, the functions

$(x_1, \ldots, x_n, z) \in \mathbb{R}^n \times Z \mapsto x_1^i \in \mathbb{R}$, so that $x_1^i \in A_{F^*}$: then

a) $[1], [x_1], \ldots, [x_1^k]$ generate $A_{F^*}/\mathfrak{I}_{0\,F^*}A_{F^*}$ over $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''/\mathfrak{I}_0 \approx \mathbb{R}$.

If this happens the Theorem is proved because by Proposition 1.7 it will follow that 1, $x_1, \ldots, x_1^k$ generate A_{F^*} over $C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$. In particular, since $x_1^{k+1} \in A_{F^*}$, there exist $a_i \in C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)''$, $i = 0, \ldots, k$, such that

$$x_1^{k+1} = \sum_{i=0}^{k} a_{i\,F^*} x_1^i = \sum_{i=0}^{k} F^*(a_i)\, x_1^i = \sum_{i=0}^{k} (a_i \circ F)\, x_1^i .$$

In pointwise notation this means, for (x, z) near $\underline{0}$,

$$x_1^{k+1} = \sum_{i=0}^{k} (a_i \circ F)(x, z)\, x_1^i = \sum_{i=0}^{k} a_i(F(x, z))\, x_1^i$$

that is the prepared form for F.

Hence we have only to prove statement *a*). Let $g \in C_{\underline{0}}^{\infty}(\mathbb{R}^n \times Z)$: by the remainder formula 1.10 the identity

$$g(x, z) = \sum_{0 \le |\alpha| \le k+1} g_\alpha(z)\, x^\alpha + \sum_{|\alpha| = k+2} G_\alpha(x, z)\, x^\alpha$$

holds near $\underline{0}$ with g_α, G_α suitable smooth functions. Since $\alpha = (\alpha_1, \ldots, \alpha_n)$, we can write

$$g(x, z) = \sum_{\substack{0 \le |\alpha| \le k+1 \\ \alpha_1 = |\alpha|}} g_\alpha(z)\, x^\alpha + \sum_{\substack{0 < |\alpha| \le k+1 \\ \alpha_1 < |\alpha|}} g_\alpha(z)\, x^\alpha +$$

$$+ \sum_{\substack{|\alpha| = k+2 \\ \alpha_1 = |\alpha|}} G_\alpha(x, z)\, x^\alpha + \sum_{\substack{|\alpha| = k+2 \\ \alpha_1 < |\alpha|}} G_\alpha(x, z)\, x^\alpha .$$

By defining, for $i = 0, \ldots, k+1$, $g_i := g_{(i, 0, \ldots, 0)}$ and $G_0 := G_{(k+2, 0, \ldots, 0)}$ we obtain

$$\sum_{\substack{0 \le |\alpha| \le k+1 \\ \alpha_1 = |\alpha|}} g_\alpha(z)\, x^\alpha = \sum_{i=0}^{k+1} g_i(z)\, x_1^i , \qquad \sum_{\substack{|\alpha| = k+2 \\ \alpha_1 = |\alpha|}} G_\alpha(x, z)\, x^\alpha = G_0(x, z)\, x_1^{k+2}$$

and thus we get

$$g(x, z) = \sum_{i=0}^{k} g_i(z)\, x_1^i + x_1^{k+1}(g_{k+1}(z) + G_0(x, z)\, x_1) +$$
$$+ \sum_{\substack{0<|\alpha|\leq k+1\\ \alpha_1<|\alpha|}} g_\alpha(z)\, x^\alpha + \sum_{\substack{|\alpha|=k+2\\ \alpha_1<|\alpha|}} G_\alpha(x, z)\, x^\alpha .$$

For the given function f a similar expression is also true:

$$f(x, z) = \sum_{i=0}^{k} f_i(z)\, x_1^i + x_1^{k+1}(f_{k+1}(z) + F_0(x, z)\, x_1) +$$
$$+ \sum_{\substack{0<|\alpha|\leq k+1\\ \alpha_1<|\alpha|}} f_\alpha(z)\, x^\alpha + \sum_{\substack{|\alpha|=k+2\\ \alpha_1<|\alpha|}} F_\alpha(x, z)\, x^\alpha ,$$

with f_i, $i = 0, \ldots, k+1$, F_0, f_α, F_α smooth functions. Moreover, by the remainder formula and the hypotheses on f, we have

$$f_i(z) = f_{(i,0,\ldots,0)}(z) = (i!)^{-1}\, \frac{\partial^i f}{\partial x_1^i}(0, \ldots, 0, z) = 0\,, \quad \text{for } z \text{ near } 0 \in Z\,, \quad i = 0, \ldots, k\,,$$

$$f_{k+1}(0) = f_{(k+1,0,\ldots,0)}(0) = ((k+1)!)^{-1}\, \frac{\partial^{k+1} f}{\partial x_1^{k+1}}(0, \ldots, 0, 0) \neq 0\,.$$

Note that the function $f_{k+1}(z) + F_0(x, z)\, x_1$ at the point $\underline{0} = (0, \ldots, 0, 0)$ takes on the value $((k+1)!)^{-1} f_{k+1}(0) \neq 0$. Then, by continuity, we can write

$$x_1^{k+1} = (f_{k+1}(z) + F_0(x, z)\, x_1)^{-1}\left[f(x, z) - \sum_{\substack{0<|\alpha|\leq k+1\\ \alpha_1<|\alpha|}} f_\alpha(z)\, x^\alpha + \sum_{\substack{|\alpha|=k+2\\ \alpha_1<|\alpha|}} F_\alpha(x, z)\, x^\alpha \right],$$

for (x, z) near $\underline{0}$.

Let us now define, in some neighbourhood of $\underline{0}$, the smooth function h_1 as $h_1(x, z) := (f_{k+1}(z) + F_0(x, z)\, x_1)^{-1}(g_{k+1}(z) + G_0(x, z)\, x_1)$. Replacing the above expression of x_1^{k+1} in the remainder formula for g we deduce

$$g(x, z) = \sum_{i=0}^{k} g_i(z)\, x_1^i + h_1(x, z)\, f(x, z) + \sum_{\substack{0<|\alpha|\leq k+1\\ \alpha_1<|\alpha|}} [g_\alpha(z) - h_1(x, z) f_\alpha(z)]\, x^\alpha +$$
$$+ \sum_{\substack{|\alpha|=k+2\\ \alpha_1<|\alpha|}} [G_\alpha(x, z) - h_1(x, z)\, F_\alpha(x, z)]\, x^\alpha .$$

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n)$ such that $0 < |\alpha| \leq k+2$ and $\alpha_1 < |\alpha|$, there exists α_j, $j > 1$, such that $\alpha_j > 0$ and thus in the monomial x^α we can isolate the variable x_j for some $j > 1$. Hence it is clear that we may define smooth functions

$h_j(x, z)$, $j = 2, \ldots, n$, such that

$$g(x, z) = \sum_{i=0}^{k} g_i(z)\, x_1^i + h_1(x, z) f(x, z) + \sum_{j=2}^{n} h_j(x, z)\, x_j .$$

Let $\tilde{g}_i$ be the trivial extensions of the functions g_i to $\mathbb{R}^n \times Z$, i.e. $\tilde{g}_i(x, z) := g_i(z)$, for $(x, z) = (x_1, \ldots, x_n, z)$ near $\underline{0}$, $i = 0, \ldots, k$, and let π_j, $j = 1, \ldots, n$, be the projections

$$\pi_j \colon \mathbb{R}^n \times Z \to \mathbb{R}^n$$

$$(x_1, \ldots, x_n, z) \mapsto x_j .$$

By construction

$$\tilde{g}_i(F(x, z)) = \tilde{g}_i(f(x, z), x_2, \ldots, x_n, z) = g_i(z), \qquad i = 0, \ldots, k,$$

$$\pi_1(F(x, z)) = \pi_1(f(x, z), x_2, \ldots, x_n, z) = f(x, z),$$

$$\pi_j(F(x, z)) = \pi_j(f(x, z), x_2, \ldots, x_n, z) = x_j, \qquad j = 2, \ldots, n .$$

Thus we can write

$$g(x, z) = \sum_{i=0}^{k} \tilde{g}_i(F(x, z))\, x_1^i + \sum_{j=1}^{n} h_j(x, z)\, \pi_j(F(x, z)),$$

that is, in functional notation,

$$g = \sum_{i=0}^{k} (\tilde{g}_i \circ F)\, x_1^i + \sum_{j=1}^{n} (\pi_j \circ F)\, h_j = \sum_{i=0}^{k} F^*(\tilde{g}_i)\, x_1^i + \sum_{j=1}^{n} F^*(\pi_j)\, h_j =$$

$$= \sum_{i=0}^{k} \tilde{g}_{i\,\dot{F^*}}\, x_1^i + \sum_{j=1}^{n} \pi_{j\,\dot{F^*}}\, h_j .$$

We note that, for $j = 1, \ldots, n$, $h_j \in A_{F^*}$ and $\pi_j \in \Im_0 \subsetneq C^\infty_{\underline{0}}(\mathbb{R}^n \times Z)''$, being $\pi_j(\underline{0}) = 0$. Then $\sum_{j=1}^{n} \pi_{j\,\dot{F^*}} h_j \in \Im_{0\,\dot{F^*}} A_{F^*}$ and therefore $[g] = [\sum_{i=0}^{k} \tilde{g}_{i\,\dot{F^*}} x_1^i]$ in $A_{F^*}/\Im_{0\,\dot{F^*}} A_{F^*}$, i.e. $[g] = \sum_{i=0}^{k} [\tilde{g}_{i\,\dot{F^*}} x_1^i] = \sum_{i=0}^{k} \{\tilde{g}_i\}[x_1^i]$, $\tilde{g}_i \in C^\infty_{\underline{0}}(\mathbb{R}^n \times Z)''$. Since this is true for every $g \in A_{F^*}$ we have thus showed that $[1]$, $[x_1], \ldots, [x_1^k]$ generate $A_{F^*}/\Im_{0\,\dot{F^*}} A_{F^*}$ over $C^\infty_{\underline{0}}(\mathbb{R}^n \times Z)''/\Im_0 \approx \mathbb{R}$. ■

4.2. REMARK. – The statement of Theorem 4.1 was in part suggested by Lemma 2.7 in [B-C-T]. Part (2) of this lemma states, under the same hypotheses for the function f, that $1, x_1, \ldots, x_1^k$ generate A_{F^*} over $C^\infty_{\underline{0}}(\mathbb{R}^n \times Z)''$. We proved this by means of the assertion

a) $[1], [x_1], \ldots, [x_1^k]$ generate $A_{F^*}/\Im_{0\,\dot{F^*}} A_{F^*}$ over $C^\infty_{\underline{0}}(\mathbb{R}^n \times Z)''/\Im_0 \approx \mathbb{R}$

and then by using Proposition 1.7.

The idea at the beginning of the proof of $a)$, that is the development of g and f according to the remainder formula, is in the proof of Part (1) of Lemma 2.7 in [B-C-T], whose statement is analogous to $a)$. In [B-C-T] Nakayama's Lemma and a result similar to Proposition 1.7 are then used to prove that $1, x_1, \ldots, x_1^k$ generate A_{F^*} over $C_0^\infty(\mathbb{R}^n \times Z)''$ while, by a further direct computation, we can easily proceed to show $\bar{a})$.

The use of Nakayama's Lemma, as in [B-C-T], is inspired by what happens in finite dimension. We recall that the finite dimensional prepared form theorem is true under weaker assumptions on f. It is in fact sufficient to require that the function f satisfies $\partial^i f / \partial x_1^i (0, \ldots, 0, 0) = 0$, $i = 0, \ldots, k$, with $Z \approx \mathbb{R}^m$, $m \geq 0$, to obtain the same conclusions as in 4.1 (see e.g. [G-G], Cap. V, Corol. 3.11, and Cap. VI § 2). This is precisely what we obtain from Theorem 4.1 by considering $\mathbb{R}^{n+m}$ and $Z = \{0\}$; hence our approach can also be used to prove the finite dimensional case.

We do not know whether these weaker conditions could be used for any B-space Z, too. This would be an advantage in the proof of the normal form theorem for higher order singularities (see e.g. [Ba-D]) though this procedure seems intimately related to the existence of a finite number of variables.

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