

Large-time Behavior of Solutions to the Equations of a Viscous Polytopic Ideal Gas (*).

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Abstract. – *First we prove for the equations of a viscous polytopic ideal gas in bounded annular domains in \mathbb{R}^n ($n = 2, 3$) that (generalized) spherically symmetric solutions decay to a constant state exponentially as time goes to infinity. Then we show that solutions of the Cauchy problem in \mathbb{R} are asymptotically stable if the initial specific volume is close to a constant in L^∞ and weighted L^2 , the initial velocity is small in weighted $L^2 \cap L^4$, and the initial temperature is close to a constant in weighted L^2 .*

1. – Introduction.

In this paper we study the asymptotic behavior of solutions to the following equations in the domain G_n ($1 \leq n \leq 3$):

$$(1.1) \quad \partial_t \varrho + \partial_r(\varrho v) + \frac{(n-1)}{r} \varrho v = 0,$$

$$(1.2) \quad \varrho(\partial_t v + v \partial_r v) = (\lambda + 2\mu) \left(\partial_r^2 v + \frac{(n-1)}{r} \partial_r v - \frac{(n-1)}{r^2} v \right) - R \partial_r(\varrho \theta),$$

$$(1.3) \quad c_V \varrho(\partial_t \theta + v \partial_r \theta) = \kappa \partial_r^2 \theta + \kappa \frac{(n-1)}{r} \partial_r \theta - R \varrho \theta \left(\partial_r v + \frac{(n-1)}{r} v \right) + \\
 + \lambda \left(\partial_r v + \frac{(n-1)}{r} v \right)^2 + 2\mu (\partial_r v)^2 + 2\mu \frac{(n-1)}{r^2} v^2,$$

$r \in G_n, t > 0,$

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where $G_n = \mathbb{R}$ for $n = 1$ and $G_n = (a, b)$ ($a > 0$) for $n = 2, 3$; $R, c_V, \kappa, \lambda, \mu$ are constants satisfying $R, c_V, \kappa, \mu > 0, \lambda + 2\mu/n \geq 0$. For (1.1)-(1.3) we will consider the Cauchy problem in the case of $n = 1$ and the following initial boundary value problem in the case of $n = 2, 3$:

$$(1.4) \quad \varrho(r, 0) = \varrho_0(r), \quad v(r, 0) = v_0(r), \quad \theta(r, 0) = \theta_0(r), \quad r \in G_n \quad \text{for } 1 \leq n \leq 3,$$

$$(1.5) \quad v(a, t) = v(b, t) = 0, \quad \theta_r(a, t) = \theta_r(b, t) = 0, \quad t \geq 0 \quad \text{for } n = 2 \text{ or } 3.$$

The equations (1.1)-(1.3) describe the motion of a viscous polytropic ideal gas in \mathbb{R} in the case of $n = 1$, or the spherically symmetric motion of a viscous polytropic ideal gas in the annular domain $\{x \in \mathbb{R}^n \mid a < |x| < b\}$ in the case of $n = 2, 3$ (cf. [1, 5, 10]), where ϱ, v, θ are the density, the velocity, and the absolute temperature, respectively; λ and μ are the constant viscosity coefficients, R, c_V , and κ are the gas constant, the specific heat capacity, and the thermal conductivity, respectively.

In two or three dimensions the global existence and large-time behavior of smooth solutions to the equations of a viscous polytropic ideal gas have been investigated for general domains only in the case of sufficiently small initial data (see e.g. [2, 3], [16]-[20], [27, 28], where more general constitutive equations were considered). For large initial data the global existence of (generalized) solutions was shown in [4, 5, 25] resp. in [10] for the spherically symmetric motion in a bounded annular domain resp. in an exterior domain. The asymptotic behavior of the (spherically symmetric) solutions in the bounded annular domain, however, is not discussed in [4, 5, 25] (in [10] some large-time behavior of ϱ, v was discussed only for the case $n = 3$).

In one dimension it is well known that global solutions exist. Moreover, for initial boundary value problems in bounded domains a solution converges to a steady state (exponentially) as $t \rightarrow \infty$ (see [1, 8, 9], [21]-[24]). For the Cauchy problem the large-time behavior of solutions is investigated only for small initial data. In [12, 15] (also cf. [6]) decay rates of solutions were studied for the initial data sufficiently small at least in $H^3(\mathbb{R})$. Kanel, Kawashima, Nishida, and Okada [11, 13, 26] proved that if the $H^1(\mathbb{R})$ -norm of the initial data is sufficiently small, then a (smooth) solution converges to a constant steady state as $t \rightarrow \infty$.

In the present paper first we prove the exponential decay of (generalized) solutions of (1.1)-(1.5) for $n = 2$ or 3 . Then we show that in the case of $n = 1$ (generalized) solutions of the Cauchy problem (1.1)-(1.4) converge to a constant steady state as $t \rightarrow \infty$ provided that the initial specific volume is close to a constant in L^∞ and weighted L^2 , the initial velocity is small in weighted $L^2 \cap L^4$, and the initial temperature is close to a constant in weighted L^2 .

To show the time-asymptotic behavior it is convenient to transform the system (1.1)-(1.3) to that in Lagrangian coordinates. The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$(1.6) \quad r(\xi, t) = r_0(\xi) + \int_0^t \tilde{v}(\xi, \tau) d\tau,$$

where $\tilde{v}(\xi, t) := v(r(\xi, t), t)$, and

$$r_0(\xi) := \eta^{-1}(\xi), \quad \eta(r) := \int_{d_n}^r s^{n-1} \varrho_0(s) ds, \quad r \in G_n; \quad d_n := \begin{cases} 0, & n = 1, \\ a, & n = 2, 3. \end{cases}$$

It should be noted that if $\inf \{ \varrho_0(s); s \in \overline{G}_n \} > 0$ (which will be assumed later), then η as a function of $r \in \overline{G}_n$ is invertible. Denote $L := \int_a^b s^{n-1} \varrho_0(s) ds > 0$. Using the equation (1.1), (1.6), and (1.5), we obtain $\partial_t \int_{d_n}^{r(\xi, t)} s^{n-1} \varrho(s, t) ds = \delta_{n1} v(0, t) \varrho(0, t)$ with δ_{ij} being the Kronecker delta, which by integration turns into

$$\int_{d_n}^{r(\xi, t)} s^{n-1} \varrho(s, t) ds = \int_{d_n}^{r_0(\xi)} s^{n-1} \varrho_0(s) ds + \delta_{n1} \int_0^t (v\varrho)(0, \tau) d\tau = \xi + \delta_{n1} \int_0^t (v\varrho)(0, \tau) d\tau.$$

Thus, under the assumption $\inf \{ \varrho(s, t); s \in \overline{G}_n, t \geq 0 \} > 0$ (which is posteriori justified) we see that G_n is transformed to Ω_n with $\Omega_n = \mathbb{R}$ if $n = 1$ and $\Omega_n = (0, L)$ if $n = 2, 3$. Moreover we have

$$(1.7) \quad \partial_\xi r(\xi, t) = [r(\xi, t)^{n-1} \varrho(r(\xi, t), t)]^{-1}.$$

For a function $\varphi(r, t)$ we write $\tilde{\varphi}(\xi, t) := \varphi(r(\xi, t), t)$. By virtue of (1.6) and (1.7),

$$(1.8) \quad \begin{cases} \partial_t \tilde{\varphi}(\xi, t) = \partial_t \varphi(r, t) + v \partial_r \varphi(r, t), \\ \partial_\xi \tilde{\varphi}(\xi, t) = \partial_r \varphi(r, t) \partial_\xi r(\xi, t) = \frac{1}{r^{n-1} \varrho(r, t)} \partial_r \varphi(r, t). \end{cases}$$

Without danger of confusion we denote $(\tilde{\varrho}, \tilde{v}, \tilde{\theta})$ still by (ϱ, v, θ) and (ξ, t) by (x, t) . We use $u := 1/\varrho$ to denote the specific volume. Therefore, by virtue of (1.7)-(1.8), the equations (1.1)-(1.5) in the new variables (x, t) read:

$$(1.9) \quad u_t = (r^{n-1} v)_x,$$

$$(1.10) \quad v_t = r^{n-1} \left[\beta \frac{(r^{n-1} v)_x}{u} - R \frac{\theta}{u} \right]_x, \quad x \in \Omega_n, t > 0,$$

$$(1.11) \quad c_V \theta_t = \kappa \left[\frac{r^{2n-2} \theta_x}{u} \right]_x + \frac{1}{u} [\beta (r^{n-1} v)_x - R\theta] (r^{n-1} v)_x - 2\mu(n-1)(r^{n-2} v^2)_x$$

together with

$$(1.12) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega_n, \quad 1 \leq n \leq 3,$$

$$(1.13) \quad v(0, t) = v(L, t) = 0, \quad \theta_x(0, t) = \theta_x(L, t) = 0, \quad t \geq 0, \quad n = 2, 3.$$

Here $\Omega_n = \mathbb{R}$ if $n = 1$ and $\Omega = (0, L)$ if $n = 2, 3$; $u_0 = 1/\varrho_0$, $\beta = \lambda + 2\mu$, and by virtue of (1.6), $r \equiv r(x, t)$ is determined by

$$(1.14) \quad \begin{cases} r(x, t) = r_0(x) + \int_0^t v(x, \tau) d\tau, & x \in [0, L], \quad t \geq 0, \\ r_0(x) := \left\{ (d_n)^n + n \int_0^x u_0(y) dy \right\}^{1/n}. \end{cases}$$

For the formulation of the main result we introduce the following notation: H^m and $\|\cdot\|_{H^m}$ ($m \geq 0$ integer) denote $H^m(\Omega_n)$ and its norm ($1 \leq n \leq 3$), respectively. $\|\cdot\|$ and $\|\cdot\|_{L^p}$ denote the norms in $L^2(\Omega_n)$ and $L^p(\Omega_n)$ ($1 \leq p \leq \infty$), respectively. Q_T stands for the domain $\Omega_n \times (0, T)$ ($1 \leq n \leq 3$). For a vector valued function $f = (f_1, \dots, f_m)$ we put $\|f\| := \|f_1\| + \dots + \|f_m\|$, where $\|\cdot\|$ denotes a norm.

As mentioned in the introduction the global existence of (generalized) solutions to (1.9)-(1.14) has been established. In the case of $n = 1$ Kazhikhov [1, 14] proved that if for some positive constants $\bar{u}, \bar{\theta}, u_0 - \bar{u}, v_0, \theta_0 - \bar{\theta} \in H^1$, and $u_0(x), \theta_0(x) > 0$ on \mathbb{R} , then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ with positive u and θ to the Cauchy problem (1.1)-(1.12) on $\mathbb{R} \times [0, \infty)$ such that for every $T > 0$

$$(1.15) \quad u - \bar{u} \in L^\infty([0, T], H^1), \quad v, \theta - \bar{\theta} \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2),$$

$$u_t, v_t, \theta_t \in L^2(Q_T).$$

In the case of $n = 2, 3$ Nikolaev [25] (also cf. [4, 5]) showed that if $u_0, v_0, \theta_0 \in H^1$, $u_0(x), \theta_0(x) > 0$ on $[0, L]$ and the initial data are compatible with the boundary conditions (1.13), then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ with positive u and θ to (1.9)-(1.14) on $[0, L] \times [0, \infty)$ such that for every $T > 0$

$$(1.16) \quad u \in L^\infty([0, T], H^1), \quad v, \theta \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2),$$

$$u_t, v_t, \theta_t \in L^2(Q_T).$$

Denote

$$(1.17) \quad \begin{cases} u^* := \frac{1}{L} \int_0^L u_0(x) dx, \\ \theta^* := \frac{1}{c_V L} \int_0^L \left\{ c_V \theta_0 + \frac{v_0^2}{2} \right\} (x) dx; \\ r^*(x) := (a^n + nu^* x)^{1/n}, \quad x \in [0, L]. \end{cases}$$

We assume for $n = 2$ or 3 that λ and μ satisfy

$$(1.18) \quad n\lambda + 2\mu > 0.$$

Then the main result of the paper reads:

THEOREM 1.1. - (i) *Let $n = 2$ or 3 . Assume that (1.18) is satisfied. Let $\{u(x, t), v(x, t), \theta(x, t)\}$ be a solution of (1.9)-(1.14) in the function class indicated in (1.16). Then there are positive constants α, T_0, C , independent of t , such that*

$$(1.19) \quad \|(u - u^*, v, \theta - \theta^*)(t)\|_{H^1} + \|r(t) - r^*\|_{H^2} \leq Ce^{-\alpha t}, \quad \text{for any } t \geq T_0.$$

(ii) *Let $n = 1$ and $\{u(x, t), v(x, t), \theta(x, t)\}$ be a solution of (1.9)-(1.12) in the function class indicated in (1.15). Denote*

$$e_0^2 := \|u_0 - \bar{u}\|_{L^\infty}^2 + \int_{\mathbb{R}} (1 + x^2)^\gamma \{ (u_0 - \bar{u})^2 + v_0^2 + (\theta_0 - \bar{\theta})^2 + v_0^4 \} dx$$

where $\gamma > 1/2$ is an arbitrary but fixed constant. Then there is a constant $\varepsilon \in (0, 1]$, independent of u_0, v_0, θ_0 , such that if $e_0 \leq \varepsilon$, then

$$(1.20) \quad \|(u - \bar{u}, v, \theta - \bar{\theta})(t)\|_{L^\infty} + \|(u_x, v_x, \theta_x)(t)\| \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

REMARK 1.1. - The same techniques work and a result analogous to Theorem 1.1 (i) holds when (1.13) is replaced by $v|_{\partial\Omega_n} = 0, \theta|_{\partial\Omega_n} = 1$.

We will prove (i) and (ii) of Theorem 1.1 in Sections 2 and 3, respectively.

2. - Proof of Theorem 1.1-(i).

In this section the same letter C (sometimes used as C_1, C_2) denotes various positive constants which are in particular independent of t and x . The proof of Theorem 1.1 (i) is essentially based on a careful examination of a priori estimates which are shown to be independent of t . The difficulties arise from the dependence on the time and spatial variables of the coefficients in the equations (1.9)-(1.11), but can be overcome in our approach by modifying an idea of Kazhikhov [1,14] for the one-dimensional case. The proof will be partitioned into several steps.

The first observation is that, by virtue of (1.7) and (1.14),

$$(2.1) \quad r_t(x, t) = v(x, t), \quad r^{n-1}(x, t) r_x(x, t) = u(x, t), \quad x \in [0, L], t \geq 0.$$

By (1.13)-(1.14) and (2.1) we obtain $r_x(0, t) = r^{1-n}(0, t) u(0, t) = a^{1-n} u(0, t) > 0$ for all $t \geq 0$. Thus, if $r_x(x, t) > 0$ is violated on $[0, L] \times [0, \infty)$, there are $y \in (0, L)$ and $\tau \in [0, \infty)$ such that $r_x(x, t) > 0$ for $0 \leq x < y, 0 \leq t \leq \tau$, but $r_x(y, \tau) = 0$. So by continuity, $r_x(x, t) \geq 0$ for $x \in [0, y]$ and $t \in [0, \tau]$, and we have $r(y, \tau) \geq r(0, \tau) = a > 0$. From (2.1) we get $0 = r_x(y, \tau) = r^{1-n}(y, \tau) u(y, \tau) > 0$ which is a contradiction. This shows

$r_x(x, t) > 0$ for $0 \leq x \leq L$, $t \geq 0$. Therefore,

$$(2.2) \quad a = r(0, t) \leq r(x, t) \leq r(L, t) = b \quad \text{for } x \in [0, L], t \geq 0.$$

The following estimate embodies the dissipative character of viscosity and thermal diffusion and is motivated by the second law of thermodynamics.

LEMMA 2.1. – *There is a positive constant c_0 , independent of t , such that*

$$(2.3) \quad \int_0^L U(x, t) dx + \int_0^t \int_0^L \left(\frac{v_x^2}{u\theta} + \frac{\theta_x^2}{u\theta^2} \right) dx ds \leq c_0, \quad \forall t \geq 0,$$

where

$$(2.4) \quad U(x, t) := \{v^2/2 + R(u - \log u - 1) + c_V(\theta - \log \theta - 1)\}(x, t).$$

PROOF. – Using (1.9)-(1.11), we obtain after a straightforward calculation that

$$(2.5) \quad U_t + \frac{\beta}{u\theta} (r^{n-1}v)_x^2 + \frac{\kappa}{u\theta^2} (r^{n-1}\theta_x)^2 = \left[r^{n-1}v \left(\frac{\beta}{u} (r^{n-1}v)_x - R \frac{\theta}{u} \right) \right]_x + R(r^{n-1}v)_x + \kappa \left[\left(1 - \frac{1}{\theta} \right) \frac{r^{2n-2}\theta_x}{u} \right]_x - 2(n-1)\mu \left(1 - \frac{1}{\theta} \right) (r^{n-2}v^2)_x.$$

Recalling $2\mu + n\lambda$, $2\mu + (n-1)\lambda > 0$, we utilise (2.1) to arrive at

$$(2.6) \quad \frac{\beta}{u\theta} (r^{n-1}v)_x^2 - 2\mu(n-1) \frac{(r^{n-2}v^2)_x}{\theta} = \frac{1}{u\theta} \left\{ (n-1)(2\mu + (n-1)\lambda) \left(r^{-1}uv + \frac{\lambda r^{n-1}v_x}{2\mu + (n-1)\lambda} \right)^2 + \frac{2\mu(2\mu + n\lambda)}{2\mu + (n-1)\lambda} r^{2n-2}v_x^2 \right\} \geq \frac{2\mu(2\mu + n\lambda)}{(2\mu + (n-1)\lambda)} \frac{r^{2n-2}v_x^2}{u\theta}.$$

By virtue of Taylor's theorem, $\int_0^L U(x, 0) dx \leq C(1 + \|(u_0, v_0, \theta_0)\|^2)$. So If we integrate (2.5) over $[0, L] \times [0, t]$ ($t \geq 0$), use (1.13) and (2.6), we obtain (2.3). ■

As a corollary of Lemma 2.1 we have

LEMMA 2.2. – *There are positive constants α_1, α_2 , independent of t , such that*

$$(2.7) \quad \alpha_1 \leq \int_0^L \theta(x, t) dx \leq \alpha_2 \quad \forall t \geq 0,$$

and for each $t \geq 0$ there is an $a(t) \in [0, L]$ satisfying

$$(2.8) \quad \alpha_1 \leq \theta(a(t), t) \leq \alpha_2.$$

PROOF. – (2.3) implies

$$(2.9) \quad c_V \int_0^L (\theta(x, t) - \log \theta(x, t) - 1) dx \leq c_0, \quad t \geq 0.$$

Therefore by virtue of the mean value theorem, for each $t \geq 0$ there is an $a(t) \in [0, L]$ such that $\theta(a(t), t) - \log \theta(a(t), t) - 1 \leq (c_V L)^{-1} c_0$, from which it follows that $\xi_1 \leq \theta(a(t), t) \leq \xi_2$ with ξ_1, ξ_2 being two (positive) roots of the equation: $y - \log y - 1 = (c_V L)^{-1} c_0$. If we use (2.9) and apply Jensen’s inequality to the convex function $y - \log y - 1$, we obtain:

$$\int_0^L \theta(x, t) dx - \log \int_0^L \theta(x, t) dx - 1 \leq c_V^{-1} c_0, \quad t \geq 0.$$

Therefore $0 < \xi_3 \leq \int_0^L \theta(x, t) dx \leq \xi_4$ for $t \geq 0$, where ξ_3, ξ_4 are two (positive) roots of the equation: $y - \log y - 1 = c_V^{-1} c_0$. Taking $\alpha_1 := \min \{ \xi_1, \xi_3 \}$ and $\alpha_2 := \max \{ \xi_2, \xi_4 \}$, we obtain (2.7)-(2.8). ■

Next we adapt and modify an idea of Kazhikhov [14] (also cf. [1]) for the one-dimensional case to give a representation for u .

Let

$$(2.10) \quad \sigma(x, t) := \beta \frac{(r^{n-1} v)_x}{u} - R \frac{u}{\theta},$$

$$(2.11) \quad \phi(x, t) := \int_0^t \sigma(x, s) ds + \int_0^x r_0^{-(n-1)}(y) v_0(y) dy + (n-1) \int_0^t \int_0^L r^{-n}(y, s) v^2(y, s) dy ds.$$

Then by (1.10), a partial integration in the variable t , and (2.1),

$$(2.12) \quad \phi_x(x, t) = r^{-(n-1)}(x, t) v(x, t).$$

Note that in view of (2.1) ϕ satisfies

$$(2.13) \quad \phi_t = \beta \frac{(r^{n-1} v)_x}{u} - R \frac{\theta}{u} + \frac{(n-1)}{n} \frac{(r^n)_x}{u} \int_x^L r^{-n} v^2 dy.$$

Multiplying (2.13) by u , using (1.9) and (2.12), we arrive at

$$(2.14) \quad (u\phi)_t - (r^{n-1}v\phi)_x = -v^2 - R\theta + \beta(r^{n-1}v)_x + \frac{(n-1)}{n} (r^n)_x \int_x^L r^{-n} v^2 dy = \\ = -\frac{v^2}{n} - R\theta + \beta(r^{n-1}v)_x + \frac{(n-1)}{n} \left[r^n \int_x^L r^{-n} v^2 dy \right]_x.$$

Keeping in mind that v vanishes on the boundary and $r(0, t) = a$, we integrate (2.14) over $[0, L] \times [0, t]$ to infer

$$(2.15) \quad \int_0^L (u\phi)(x, t) dx = \int_0^L u_0(x) \phi_0(x) dx - \int_0^t \int_0^L \left(\frac{v^2}{n} + R\theta \right) dx ds - \\ - \frac{(n-1)}{n} a^n \int_0^t \int_0^L r^{-n} v^2 dx ds,$$

where $\phi_0(x) := \phi(x, 0)$. It follows from integration of (1.9) over $[0, L] \times [0, t]$ and use of (1.13) that

$$(2.16) \quad \int_0^L u(x, t) dx = \int_0^L u_0(x) dx \equiv u^* \quad \text{for } t \geq 0.$$

Note that $u > 0$. If we apply the mean value theorem to (2.15) and use (2.16), we conclude that for each $t \geq 0$ there is an $x_0(t) \in [0, L]$ such that

$$(2.17) \quad \phi(x_0(t), t) = \frac{1}{u^*} \int_0^L \phi(x, t) u(x, t) dx.$$

Therefore from (2.11), (2.15), and (2.17) we get

$$(2.18) \quad \int_0^t \sigma(x_0(t), s) ds = \phi(x_0(t), t) - \int_0^{x_0(t)} r_0^{-(n-1)} v_0 dy - (n-1) \int_0^t \int_{x_0(t)}^L r^{-n} v^2 dy ds = \\ = -\frac{1}{u^*} \int_0^t \int_0^L \left(\frac{v^2}{n} + R\theta \right) dx ds - \frac{(n-1) a^n}{nu^*} \int_0^t \int_0^L r^{-n} v^2 dx ds - \\ - (n-1) \int_0^t \int_{x_0(t)}^L r^{-n} v^2 dx ds + \frac{1}{u^*} \int_0^L u_0 \phi_0 dx - \int_0^{x_0(t)} r_0^{-(n-1)} v_0 dy$$

for any $t \geq 0$. Using (2.18), we can show

LEMMA 2.3. – *We have the following representation*

$$(2.19) \quad u(x, t) = \frac{D(x, t)}{B(x, t)} \left\{ 1 + \frac{R}{\beta} \int_0^t \frac{\theta(x, s) B(x, s)}{D(x, s)} ds \right\}, \quad x \in [0, L], t \geq 0,$$

where

$$(2.20) \quad D(x, t) := u_0(x) \exp \left\{ \frac{1}{\beta} \left[\frac{1}{u^*} \int_0^L u_0 \phi_0 dx - \int_0^x r_0^{-(n-1)} v_0 dy + \int_{x_0(t)}^x r^{-(n-1)} v dy \right] \right\},$$

$$(2.21) \quad B(x, t) := \exp \left\{ \frac{1}{\beta} \left[\frac{1}{u^*} \int_0^t \int_0^L \left(\frac{v^2}{n} + R\theta \right) dx ds + \right. \right. \\ \left. \left. + \frac{(n-1) a^n}{nu^*} \int_0^t \int_0^L r^{-n} v^2 dy dx ds + (n-1) \int_0^t \int_x^L r^{-n} v^2 dy ds \right] \right\},$$

and $x_0(t) \in [0, L]$ is the same as in (2.17).

PROOF. – Using (1.9) we may write (1.10) in the form

$$(2.22) \quad r^{-(n-1)} v_t = \beta [\log u]_{xt} - R \left[\frac{\theta}{u} \right]_x \quad (\Leftrightarrow r^{-(n-1)} v_t = \sigma_x).$$

Integrate (2.22) over $[0, t]$, then integrate over $[x_0(t), x]$ with respect to x . If we integrate by parts with respect to t , utilise (2.1) and (2.18), we infer

$$\beta \log u - R \int_0^t \frac{\theta}{u} ds = \beta \log u_0 + \int_0^t \sigma(x_0(t), s) ds + \int_{x_0(t)}^x \int_0^t r^{-(n-1)} v_t ds dy = \\ = \beta \log u_0 - \frac{1}{u^*} \int_0^t \int_0^L \left(\frac{v^2}{n} + R\theta \right) dx ds - \frac{(n-1) a^n}{nu^*} \int_0^t \int_0^L r^{-n} v^2 dx ds - \\ - (n-1) \int_0^t \int_x^L r^{-n} v^2 dy ds + \int_{x_0(t)}^x r^{-(n-1)} v dy + \frac{1}{u^*} \int_0^L u_0 \phi_0 dx - \int_0^x r_0^{-(n-1)} v_0 dy,$$

which, when the exponentials are taken, turns into

$$(2.23) \quad \frac{B(x, t)}{D(x, t)} = \frac{1}{u(x, t)} \exp \left(\frac{R}{\beta} \int_0^t \frac{\theta(x, s)}{u(x, s)} ds \right).$$

Multiplying (2.23) by $R\theta/\beta$ and integrating over $[0, t]$, we arrive at

$$\exp\left(\frac{R}{\beta} \int_0^t \frac{\theta(x, s)}{u(x, s)} ds\right) = 1 + \frac{R}{\beta} \int_0^t \frac{\theta(x, s) B(x, s)}{D(x, s)} ds.$$

Substituting this into (2.23), we obtain the lemma. ■

Now we are able to derive bounds on $u(x, t)$ by using the representation (2.19).

LEMMA 2.4. – *There are positive constants \underline{u} and \bar{u} , independent of t , such that*

$$(2.24) \quad \underline{u} \leq u(x, t) \leq \bar{u} \quad \text{for any } x \in [0, L], t \geq 0.$$

PROOF. – Recalling the definition of $D(x, t)$, we have by (2.2), Cauchy-Schwarz's inequality, and Lemma 2.1 that

$$(2.25) \quad 0 < C^{-1} \leq D(x, t) \leq C, \quad \forall x \in [0, 1], t \geq 0.$$

Noting that $u > 0$, we get from (2.2) and (2.7) that

$$(2.26) \quad \frac{B(x, s)}{B(x, t)} \leq \exp\left\{-\frac{R}{\beta u^*} \int_s^t \int_0^L \theta(x, s) dx ds\right\} \leq \exp\left\{-\frac{R\alpha_1(t-s)}{\beta u^*}\right\}, \quad t \geq s \geq 0.$$

Similarly,

$$(2.27) \quad B(x, s)/B(x, t) \geq Ce^{-C_1(t-s)}, \quad t \geq s \geq 0; \quad e^{Ct} \geq B(x, t) \geq 1, \quad t \geq 0$$

with C_1 being independent of t , where we have used (2.2)-(2.3) and (2.7).

It is easy to see by (2.2) and (2.7) that

$$\begin{aligned} |\theta^{1/2}(x, t) - \theta^{1/2}(a(t), t)| &\leq \int_0^L \theta^{-1/2}(x, t) |\theta_x(x, t)| dx \leq \\ &\leq \left(\int_0^L \frac{\theta_x^2}{u\theta^2} dx\right)^{1/2} \left(\int_0^L \theta u dx\right)^{1/2} \leq C \left[\int_0^L \left(\frac{\theta_x^2}{u\theta^2}\right)(x, t) dx\right]^{1/2} \max_{[0, L]} u^{1/2}(\cdot, t), \end{aligned}$$

which together with (2.8) gives

$$(2.28) \quad \frac{\alpha_1}{2} - C \max_{[0, L]} u(\cdot, t) \int_0^L \frac{\theta_x^2}{u\theta^2} dx \leq \theta(x, t) \leq \\ \leq 2\alpha_2 + C \max_{[0, L]} u(\cdot, t) \int_0^L \frac{\theta_x^2}{u\theta^2} dx \quad \forall x \in [0, L], t \geq 0.$$

Hence it follows from (2.19) and (2.25)-(2.28) that

$$(2.29) \quad u(x, t) \leq C + C \int_0^t \left(1 + \max_{[0, L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2} dx \right) e^{-(t-s)/C} ds \leq \\ \leq C + C \int_0^t \max_{[0, L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2} dx ds.$$

Applying Gronwall's inequality to (2.29) and utilising (2.3), one gets $u(x, t) \leq \bar{u} \forall x \in [0, L] \forall t \geq 0$ for some positive constant \bar{u} independent of t and x .

To complete the proof it remains to show the lower boundedness of u . To this end we make use of (2.3), (2.19), (2.25), (2.27), and (2.28) to infer

$$(2.30) \quad u(x, t) \geq \frac{RD(x, t)}{\beta} \int_0^t \frac{\theta(x, s) B(x, s)}{D(x, s) B(x, t)} ds \geq \\ \geq C_2 \int_0^t \left(\frac{\alpha_1}{2} - C \max_{[0, L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2} dx \right) e^{-C_1(t-s)} ds \geq \\ \geq \frac{C_2 \alpha_1}{2C_1} (1 - e^{-C_1 t}) - C e^{-C_1 t/2} \int_0^t \int_0^L \frac{\theta_x^2}{u\theta^2} dx ds - C \int_{t/2}^t \int_0^L \frac{\theta_x^2}{u\theta^2} dx ds \geq \frac{C_2 \alpha_1}{4C_1} > 0$$

for all $t \geq T_0$ and some (large) $T_0 > 0$, where C_2 is independent of t . Furthermore, from (2.19), (2.25), and (2.27) we get $u(x, t) \geq D(x, t)/B(x, t) \geq C^{-1} e^{-Ct}$ for all $x \in [0, L]$ and $t \geq 0$. This combined with (2.30) shows that u is bounded from below. The proof is complete. ■

In the sequel we derive Sobolev-norm estimates of derivatives for u, v, θ by applying the energy method.

Recalling (2.10), using (1.9)-(1.10), we may write the equation (1.11) as follows

$$(2.31) \quad \left[c_V \theta + \frac{v^2}{2} \right]_t = \left[\kappa \frac{r^{2n-2} \theta_x}{u} + \sigma r^{n-1} v - 2(n-1) \mu r^{n-2} v^2 \right]_x.$$

Multiply (2.31) by $c_V \theta + v^2/2$ and integrate. If we integrate by parts with respect to x ,

and make use of (2.1)-(2.2), Cauchy-Schwarz's inequality, and (2.24), we obtain that

$$(2.32) \quad \frac{1}{2} \int_0^L \left[c_V \theta + \frac{v^2}{2} \right]^2 (x, t) dx \leq \\ \leq C - \frac{c_V K}{2} \int_0^t \int_0^L \frac{r^{2n-2} \theta_x^2}{u} dx ds + C \int_0^t \int_0^L (r^{2n-2} v_x^2 v^2 + v^4 + \theta^2 v^2) dx ds .$$

To bound the term $\int_0^t \int_0^L r^{2n-2} v_x^2 v^2 dx ds$, we multiply (1.10) by v^3 , integrate over $[0, L] \times [0, t]$, integrate by parts with respect to x , and utilise (2.1)-(2.2), Cauchy-Schwarz's inequality, and (2.24) to get

$$(2.33) \quad \frac{1}{4} \int_0^L v^4(x, t) dx \leq C - \frac{\beta}{\bar{u}} \int_0^t \int_0^L r^{2n-2} v^2 v_x^2 dx ds + C \int_0^t \int_0^L (v^4 + v^2 \theta^2) dx ds .$$

We multiply (2.32) by $\beta/(2 \bar{u} C)$ and add the resulting inequality to (2.33) to obtain, with the help of (2.2)-(2.3) and (2.24), the result

$$(2.34) \quad \int_0^L (\theta^2 + v^4)(x, t) dx + \int_0^t \int_0^L (v^2 v_x^2 + \theta_x^2) dx ds \leq \\ \leq C + C \int_0^t \max_{[0, L]} v^2(\cdot, s) ds + C \int_0^t \max_{[0, L]} v^2(\cdot, s) \int_0^L \theta^2(x, s) dx ds .$$

On the other hand, by (2.2)-(2.3), (2.7) and (2.24),

$$(2.35) \quad \int_0^t \max_{[0, L]} v^2(\cdot, s) ds \leq \int_0^t \left(\int_0^L |v_x| dx \right)^2 ds \leq \int_0^t \int_0^L \frac{v_x^2}{u \theta} dx \int_0^L u \theta dx ds \leq C, \quad t \geq 0 .$$

In view of (2.35), we apply Gronwall's inequality to (2.34) to obtain

LEMMA 2.5.

$$(2.36) \quad \int_0^L (\theta^2 + v^4)(x, t) dx + \int_0^t \int_0^L (v^2 v_x^2 + \theta_x^2) dx ds \leq C, \quad t \geq 0 .$$

LEMMA 2.6.

$$(2.37) \quad \int_0^L u_x^2(x, t) dx + \int_0^t \int_0^L (v_x^2 + u_x^2 + \theta u_x^2) dx ds \leq C, \quad t \geq 0.$$

PROOF. - By virtue of (2.1)-(2.2) and (2.24),

$$(2.38) \quad (r^{n-1}v)_x^2 = (r^{n-1}v_x + (n-1)r^{-1}rv)^2 \geq r^{2n-2}v_x^2/2 - Cv^2 \geq a^{2n-2}v_x^2/2 - Cv^2.$$

So multiplying (1.10) by v and integrating, we integrate by parts with respect to x , use Cauchy-Schwarz's inequality, (2.7), (2.24), and (2.35)-(2.36), to deduce

$$(2.39) \quad \begin{aligned} \frac{1}{2} \int_0^L v^2(x, t) dx + \frac{\beta a^{2n-2}}{2\bar{u}} \int_0^t \int_0^L v_x^2 dx ds &\leq \\ &\leq C + C\delta^{-1} \int_0^t \int_0^L (\theta_x^2 + v^2 + \theta v^2) dx ds + \delta \int_0^t \int_0^L \theta u_x^2 dx ds \leq \\ &\leq C\delta^{-1} + \delta \int_0^t \int_0^L \theta u_x^2 dx ds, \quad (0 < \delta < 1 \text{ constant}). \end{aligned}$$

With the help of (1.9), we may write (1.10) in the form $\beta[u_x/u]_t = r^{-(n-1)}v_t + R[\theta_x/u - \theta u_x/u^2]$. Multiply this by u_x/u and integrate. After utilising (2.3), (2.24), and (2.36), we infer

$$(2.40) \quad \begin{aligned} \frac{\beta}{2} \int_0^L \left[\frac{u_x}{u} \right]^2(x, t) dx + \frac{R}{2} \int_0^t \int_0^L \frac{\theta u_x^2}{u^3} dx ds &\leq \\ &\leq C + \int_0^t \int_0^L r^{-(n-1)}v_t \frac{u_x}{u} dx ds + C \int_0^t \int_0^L \frac{\theta^2}{u} \left(1 + \frac{1}{\theta^2} \right) dx ds. \end{aligned}$$

Noting that $[u_x/u]_t = [u_t/u]_x$, the second term on the right hand side of (2.40) can be estimated, with the help of integration by parts, and (1.9), (2.1)-(2.3), (2.24) and (2.35), as follows:

$$(2.41) \quad \begin{aligned} \int_0^t \int_0^L r^{-(n-1)}v_t \frac{u_x}{u} dx ds &= \int_0^L r^{-(n-1)}v \frac{u_x}{u} dx \Big|_0^t + \\ &+ (n-1) \int_0^t \int_0^L r^{-n}v^2 \frac{u_x}{u} dx ds - \int_0^t \int_0^L r^{-(n-1)}v \left[\frac{u_t}{u} \right]_x dx ds \leq \\ &\leq C + \frac{\beta}{4} \int_0^L \left[\frac{u_x}{u} \right]^2(x, t) dx + C \int_0^t \max_{[0, L]} v^2 \int_0^L u_x^2 dx ds + \frac{2}{\bar{u}} \int_0^t \int_0^L v_x^2 dx ds. \end{aligned}$$

Substituting (2.41) into (2.40), taking (2.2)-(2.3), (2.24), and (2.36) into account, one gets

$$(2.42) \quad \frac{\beta}{4\bar{u}^2} \int_0^L u_x^2(x, t) dx + \frac{R}{2\bar{u}^3} \int_0^t \int_0^L \theta u_x^2 dx ds \leq \\ \leq C + C \int_0^t \max_{[0, L]} v^2 \int_0^L u_x^2 dx ds + \frac{2}{\underline{u}} \int_0^t \int_0^L v_x^2 dx ds .$$

Multiplying (2.42) by $\underline{u}\beta a^{2n-2}/(8\bar{u})$, and adding the resulting inequality to (2.39), we obtain for an appropriately small but fixed $\delta \in (0, 1)$ that

$$\int_0^L u_x^2(x, t) dx + \int_0^t \int_0^L (v_x^2 + \theta u_x^2) dx ds \leq C + C \int_0^t \max_{[0, L]} v^2 \int_0^L u_x^2 dx ds, \quad t \geq 0 .$$

In view of (2.35), we apply Gronwall's inequality to the above inequality to obtain

$$(2.43) \quad \int_0^L u_x^2(x, t) dx + \int_0^t \int_0^L (v_x^2 + \theta u_x^2) dx ds \leq C, \quad \forall t \geq 0 .$$

Finally, it follows from (2.24), (2.28), (2.43), and (2.3) that

$$\frac{\alpha_1}{2} \int_0^t \int_0^L u_x^2 dx ds \leq \int_0^t \int_0^L \theta u_x^2 dx ds + C \int_0^t \int_0^L \frac{\theta_x^2}{u\theta^2} dx \int_0^L u_x^2 dx ds \leq C, \quad t \geq 0 ,$$

from which and (2.43), (2.37) follows. This completes the proof. ■

In the following lemma we bound v_t in the $L^2((0, L) \times (0, \infty))$ -norm.

LEMMA 2.7.

$$(2.44) \quad \int_0^L v_x^2(x, t) dx + \int_0^t \int_0^L v_t^2 dx ds \leq C, \quad \forall t \geq 0 ,$$

$$(2.45) \quad |v(x, t)| \leq C, \quad \forall x \in [0, L], t \geq 0 .$$

PROOF. – We first note that by (2.8) and Cauchy-Schwarz's inequality,

$$(2.46) \quad \max_{[0, L]} \theta(\cdot, t) \leq C + C \max_{[0, L]} |\theta(\cdot, t) - \theta(a(t), t)| \leq \\ \leq C + C \int_0^L |\theta_x| dx \leq C + C \|\theta_x(t)\|, \quad t \geq 0 .$$

Multiply (1.10) by v_t and integrate over $[0, L] \times [0, t]$. Integrating by parts, using (2.1)-(2.3), (2.24), (2.35)-(2.37), and (2.46), taking into account that $(r^{n-1}v_t)_x =$

$= (r^{n-1}v)_{xt} - (n-1)(r^{n-2}v^2)_x$ and $|(r^{n-2}v^2)_x| \leq C\{v^2 + (r^{n-1}v_x^2)\}$, we obtain

$$\begin{aligned}
 (2.47) \quad & \frac{1}{2} \int_0^t \|v_t\|^2 ds + \frac{\beta}{2\bar{u}} \|(r^{n-1}v)_x(t)\|^2 \leq \\
 & \leq C + C \int_0^t \int_0^L \left\{ \frac{|(r^{n-1}v)_x|}{u} (v^2 + (r^{n-1}v_x^2)) + \theta_x^2 + \theta^2 u_x^2 \right\} dx ds \leq \\
 & \leq C + C \int_0^t \left\{ \max_{[0, L]} \left| \frac{(r^{n-1}v)_x}{u} \right| + \max_{[0, L]} v^2 \right\} \|(r^{n-1}v)_x\|^2 ds .
 \end{aligned}$$

Here $\max |(r^{n-1}v)_x/u|$ can be bounded as follows, using (2.10), Sobolev's imbedding theorem ($H^1 \hookrightarrow L^\infty$), and (2.24), (1.10), (2.1)-(2.2), and (2.46)

$$\begin{aligned}
 (2.48) \quad & \beta \left| \frac{(r^{n-1}v)_x}{u} \right| (x, t) \leq |\sigma| + R \frac{\theta}{u} \leq C \left(1 + \|\sigma\| + \|\sigma_x\| + \max_{[0, L]} \theta \right) \leq \\
 & \leq C \left(1 + \max_{[0, L]} v^2 + \|v_x\|^2 + \|\theta_x\|^2 + \|v_t\| \right), \quad \forall x \in [0, L], t \geq 0 .
 \end{aligned}$$

Inserting (2.48) into (2.47) and recalling that $|(r^{n-1}v_x^2)| \leq C(v^2 + v_x^2)$, we get from (2.24), (2.35), and (2.37) that

$$\begin{aligned}
 (2.49) \quad & \frac{1}{4} \int_0^t \|v_t\|^2 ds + \frac{\beta}{2\bar{u}} \|(r^{n-1}v)_x(t)\|^2 \leq \\
 & \leq C + C \int_0^t \left(\max_{[0, L]} v^2 + \|v_x\|^2 + \|\theta_x\|^2 \right) \|(r^{n-1}v)_x\|^2 ds .
 \end{aligned}$$

Applying Gronwall's inequality to (2.49) and taking account of (2.35)-(2.37), we conclude that $\int_0^t \|v_t\|^2 ds + \|(r^{n-1}v)_x(t)\|^2 \leq C$ for $t \geq 0$, which combined with (2.3) and (2.37)-(2.38) yields (2.44). Finally, (2.45) follows from Sobolev's inequality, (2.3), and (2.44). The proof is complete. ■

As a result of Lemma 2.7 we have

$$(2.50) \quad \int_0^t \int_0^L (u_{xt}^2 + v_{xx}^2)(x, s) dx ds + \int_0^t \max_{[0, L]} v_x^2(\cdot, s) ds \leq C, \quad t \geq 0 .$$

In fact, by virtue of Sobolev's imbedding theorem ($W^{1,1} \hookrightarrow L^\infty$), $\max v_x^2(\cdot, t) \leq C\varepsilon^{-1} \|v_x(t)\|^2 + \varepsilon \|v_{xx}(t)\|^2$ ($0 < \varepsilon < 1$), we get from (2.1)-(2.2), (2.24), and (2.35)-(2.37),

(1.10), (2.46), and (2.44) that

$$\begin{aligned} \int_0^t \int_0^L (u_{xt}^2 + v_{xx}^2) dx ds &\leq C \int_0^t \int_0^L [(r^{n-1}v)_{xx}^2 + v_x^2 + u_x^2 v^2 + v^2] dx ds \leq \\ &\leq C + C \int_0^t \int_0^L \left[\frac{(r^{n-1}v)_x}{u} \right]_x^2 dx ds + C \int_0^t \int_0^L u_x^2 v_x^2 dx ds \leq \\ &\leq C + C \int_0^t \int_0^L v_t^2 dx ds + C \int_0^t \max_{[0, L]} \theta^2 \int_0^L u_x^2 dx ds + C \int_0^t \max_{[0, L]} v_x^2 ds \leq C + \frac{1}{2} \int_0^t \|v_{xx}\|^2 ds, \end{aligned}$$

which implies (2.50).

We multiply (1.11) by θ_t and integrate, we obtain by the same arguments as used in Lemma 2.7 and (2.50) that

$$(2.51) \quad \int_0^L \theta_x^2(x, t) dx + \max_{[0, L]} |\theta(\cdot, t)| + \int_0^t \int_0^L \left(\theta_t^2 + \theta_{xx}^2 + \max_{[0, L]} \theta^2 \right) (s) ds \leq C, \quad \forall t \geq 0.$$

Now we are able to prove Theorem 2.1. By (2.37), (2.44), (2.50)-(2.51), and the identities

$$\int_0^L v_x v_{xt} dx = - \int_0^L v_{xx} v_t dx, \quad \int_0^L \theta_x \theta_{xt} dx = - \int_0^L \theta_{xx} \theta_t dx,$$

we see that

$$\int_0^\infty \left\{ \left| \frac{d}{dt} \|u_x(t)\|^2 \right| + \left| \frac{d}{dt} \|v_x(t)\|^2 \right| + \left| \frac{d}{dt} \|\theta_x(t)\|^2 \right| \right\} dt \leq C,$$

which together with (2.36)-(2.37) implies

$$(2.52) \quad \|u_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2 \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

From (2.16) and Poincaré’s inequality we get $\|u(t) - u^*\|_{H^1} + \|v(t)\|_{H^1} \rightarrow 0$ as $t \rightarrow \infty$. Recalling (1.13) and the definition of θ^* , we integrate (2.31) over $[0, L] \times [0, t]$ to infer

$$\int_0^L \{ (c_V \theta + v^2/2)(x, t) - c_V \theta^* \} dx = 0, \quad t \geq 0,$$

from which it follows with the help of Poincaré’s inequality and (2.45) that as $t \rightarrow \infty$

$$\|\theta(t) - \theta^*\| \leq C \|c_V \theta(t) + v^2(t)/2 - c_V \theta^*\| + C \|v^2(t)\| \leq C (\|\theta_x\| + \|v(t)\|_{H^1}) \rightarrow 0.$$

To show $r(x, t) \rightarrow (a^n + nu^*x)^{1/n}$ as $t \rightarrow \infty$ we note that by (2.1) and (1.14),

$$(2.53) \quad r^n(x, t) = r^n(0, t) + n \int_0^x u(y, t) dy = [r^*(x)]^n + n \int_0^x (u(y, t) - u^*) dy,$$

where $r^*(x)$ is defined by (1.17). It follows from (2.2) and (2.53) that $\|r(t) - r^*\| \leq C\|u(t) - u^*\|, t \geq 0$. Therefore, differentiating (2.53) with respect to x and recalling (2.2), we find that $\|r(t) - r^*\|_{H^2} \leq C\|u(t) - u^*\|_{H^1} \rightarrow 0$ as $t \rightarrow \infty$. We have known that for large $t\{u(x, t) - u^*, v(x, t), \theta(x, t) - \theta^*\}$ and $r(x, t) - r^*(x)$ become small in the H^1 - and H^2 -norms respectively, thus we can apply arguments similar to those used in [26, Theorem 2.2] to obtain (1.19) in Theorem 1.1 (the exponential decay). This completes the proof of Theorem 1.1 (i).

3. - Proof of Theorem 1.1-(ii).

We use and modify an idea of Hoff [7] for barotropic fluids to prove Theorem 1.1 (ii) for the system (1.9)-(1.12) in the case of $n = 1$. Let $e_0 \leq 1$ be satisfied in this section. In what follows C or \tilde{C} denotes a generic constant (≥ 1) which may depend at most on $\bar{u}, \bar{\theta}, \beta, R, c_v, \kappa,$ and γ .

Define $\phi(t) := \min\{1, t\}$. We first assume that u, θ satisfy

$$(3.1) \quad |u(x, t) - \bar{u}|, \quad \phi(t) |\theta(x, t) - \bar{\theta}| \leq \min\{\bar{u}, \bar{\theta}\}/2 \quad \text{for all } x \in \mathbb{R}, t \geq 0.$$

In the sequel we derive a priori estimates for u, v, θ under (3.1).

Following the same procedure as in the proof of Lemma 2.1 (recalling $n = 1$), applying (3.1) and the mean value theorem, we can show

$$(3.2) \quad \int_{\mathbb{R}} U(x, t) dx + \bar{\theta} \int_1^t \int_{\mathbb{R}} \left(\beta \frac{v_x^2}{u\theta} + \kappa \frac{\theta_x^2}{u\theta^2} \right) dx ds = \int_{\mathbb{R}} U(x, 1) dx \leq C \int_{\mathbb{R}} \{v^2 + (u - \bar{u})^2 + (\theta - \bar{\theta})^2\}(x, 1) dx, \quad \forall t \geq 1,$$

where

$$(3.3) \quad U(x, t) := \left\{ \frac{v^2}{2} + R\bar{\theta} \left(\frac{u}{\bar{u}} - \log \frac{u}{\bar{u}} - 1 \right) + c_v \left(\theta - \bar{\theta} \log \frac{\theta}{\bar{\theta}} - \bar{\theta} \right) \right\} (x, t).$$

Now we estimate $\{u - \bar{u}, v, \theta - \bar{\theta}\}$ in a weighted L^2 -norm for $0 \leq t \leq 1$. For simplicity we denote $\psi(x) := (1 + x^2)^\gamma$ with γ being the same as in Theorem 1.1. Multiply (1.10) by $2\psi v$ (recalling $n = 1$) and integrate over $\mathbb{R} \times (0, t)$ ($t \in [0, 1]$). We integrate

by parts to arrive at

$$(3.4) \quad \int_{\mathbb{R}} \psi v^2(x, t) dx + \int_0^t \int_{\mathbb{R}} \psi v_x^2 dx ds \leq \\ \leq C e_0^2 + C \int_0^t \int_{\mathbb{R}} \psi((u - \bar{u})^2 + v^2 + (\theta - \bar{\theta})^2) dx ds, \quad t \in [0, 1].$$

Multiplying (1.9) by $2\psi(u - \bar{u})$ and integrating, we easily see that

$$\int_{\mathbb{R}} \psi(u(x, t) - \bar{u})^2 dx \leq e_0^2 + C \int_0^t \int_{\mathbb{R}} \psi(u - \bar{u})^2 dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \psi v_x^2 dx ds,$$

which together with (3.4) gives

$$(3.5) \quad \int_{\mathbb{R}} \psi((u - \bar{u})^2 + v^2)(x, t) dx + \int_0^t \int_{\mathbb{R}} \psi v_x^2 dx ds \leq \\ \leq C e_0^2 + C \int_0^t \int_{\mathbb{R}} \psi((u - \bar{u})^2 + v^2 + (\theta - \bar{\theta})^2) dx ds$$

for all $t \in [0, 1]$.

Let us denote $h(t) := \sup_{0 \leq s \leq t} \int_{\mathbb{R}} \psi\{v^2 + (\theta - \bar{\theta})^2\}(x, s) dx$. Utilising (3.1), we obtain

by the same arguments as used for (2.31)-(2.23) that

$$(3.6) \quad \int_{\mathbb{R}} \psi((\theta - \bar{\theta})^2 + v^4)(x, t) dx + \int_0^t \int_{\mathbb{R}} \psi(v^2 v_x^2 + \theta_x^2) dx ds \leq \\ \leq C e_0^2 + C h(t) \int_0^t \max_{\mathbb{R}}(\theta - \bar{\theta})^2(\cdot, s) ds + \int_0^t \int_{\mathbb{R}} \psi((\theta - \bar{\theta})^2 + v^4 + v^2) dx ds \leq \\ \leq C(e_0^2 + h^3(t)) + C \int_0^t \int_{\mathbb{R}} \psi((\theta - \bar{\theta})^2 + v^4 + v^2) dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \psi \theta_x^2 dx ds, \quad t \in [0, 1],$$

where we have also used the inequality $\|\cdot\|_{L^\infty} \leq C \|\cdot\|_{\partial_x \cdot}$ for $\max(\theta - \bar{\theta})^2(\cdot, s)$.

Applying the generalized Gronwall inequality to (3.5) and (3.6), we find that for all $t \in [0, 1]$

$$(3.7) \quad \int_{\mathbb{R}} \psi((u - \bar{u})^2 + v^2 + v^4 + (\theta - \bar{\theta})^2)(x, t) dx + \\ + \int_0^t \int_{\mathbb{R}} \psi(v_x^2 + v^2 v_x^2 + \theta_x^2) dx ds \leq C(e_0^2 + h^3(t)).$$

By the definition of $h(t)$ and (3.7) we have $h(t) \leq C(e_0^2 + h^3(t))$ for all $t \in [0, 1]$, which gives $h(t) \leq 2Ce_0^2 \leq e_0$ for all $t \in [0, 1]$ provided $e_0 \leq 1/(2C)$. Therefore, in view of (3.7) we conclude

$$(3.8) \quad \int_{\mathbb{R}} \psi((u - \bar{u})^2 + v^2 + v^4 + (\theta - \bar{\theta})^2)(x, t) dx + \int_0^t \int_{\mathbb{R}} \psi(v_x^2 + v^2 v_x^2 + \theta_x^2) dx ds \leq Ce_0^2, \quad t \in [0, 1]$$

provided $e_0 \leq 1/(2C)$. Using (3.1) and the mean value theorem, we get from (3.8) and (3.2) that

$$(3.9) \quad \int_{\mathbb{R}} \{v^2 + (u - \bar{u})^2 + (\theta - \bar{\theta})^2\}(x, t) dx + \int_0^t \int_{\mathbb{R}} (v_x^2 + \theta_x^2) dx ds \leq Ce_0^2, \quad \forall t \geq 0$$

provided $e_0 \leq 1/(2C)$.

Next we derive Sobolev-norm estimates for u, v, θ . We define

$$(3.10) \quad A(t) := \sup_{0 \leq s \leq t} \{ \|u - \bar{u}\|_{L^\infty}^2 + \phi^2 \|v_x\|^2 + \phi^4 \|\theta_x\|^2 \}(s) + \int_0^t \{ \phi^2 \|v_t\|^2 + \phi^4 \|\theta_t\|^2 + \|v_x\|^2 \}(s) ds.$$

Multiply (1.10) by $\phi^2 v_t$ and integrate. We integrate by parts, utilise (3.1), (3.9), and Cauchy-Schwarz's inequality to infer

$$\begin{aligned} \phi^2 \int_{\mathbb{R}} v_x^2 dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2 dx ds &\leq \\ &\leq Ce_0^2 + C \int_0^t \int_{\mathbb{R}} \phi^2 |v_x|^3 dx ds + C \left| \int_0^t \int_{\mathbb{R}} \phi^2 \left[\left(\frac{1}{u} - \frac{1}{\bar{u}} \right) \theta \right] v_t dx ds \right| \leq \\ &\leq Ce_0^2 + C \int_0^t \int_{\mathbb{R}} \phi^4 v_x^4 dx ds + C \int_{\mathbb{R}} |u - \bar{u}| \phi |v_x| dx + \\ &+ C \int_0^1 \int_{\mathbb{R}} \phi |u - \bar{u}| \theta |v_x| dx ds + C \left| \int_0^t \int_{\mathbb{R}} \phi^2 \left[\left(\frac{1}{u} - \frac{1}{\bar{u}} \right) \theta \right] v_x dx ds \right| \leq \\ &\leq Ce_0^2 + C \int_0^t \int_{\mathbb{R}} \phi^4 v_x^4 dx ds + C \int_0^t \int_{\mathbb{R}} |u - \bar{u}|^2 \phi^4 \theta_t^2 dx ds + \frac{1}{2} \phi^2 \int_{\mathbb{R}} v_x^2 dx, \end{aligned}$$

whence

$$(3.11) \quad \phi^2 \int_{\mathbb{R}} v_x^2 dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2 dx ds \leq C e_0^2 + C \int_0^t \int_{\mathbb{R}} \phi^4 v_x^4 dx ds + A^2(t), \quad t \geq 0,$$

where the second term on the right-hand side of (3.11) can be bounded as follows, using (3.1), (3.9), and $\|\cdot\|_{L^\infty}^2 \leq C \|\cdot\|_{H^1}^2$, (1.10) and (3.2)

$$(3.12) \quad \int_0^t \int_{\mathbb{R}} \phi^4 v_x^4 dx ds \leq C \int_0^t \phi^4 \max_{\mathbb{R}} v_x^2 \int_{\mathbb{R}} v_x^2 dx ds \leq \\ \leq C e_0^2 + C \int_0^t \phi^4 \max_{\mathbb{R}} \left(\beta \frac{v_x}{u} - R \frac{\theta}{u} + R \frac{\bar{\theta}}{\bar{u}} \right)^2 \int_{\mathbb{R}} v_x^2 dx ds \leq \\ \leq C e_0^2 + C \int_0^t \phi^4 (\|v_x\|^2 + \|u - \bar{u}\|^2 + \|\theta - \bar{\theta}\|^2 + \|v_t\|^2) \|v_x\|^2 ds \leq C(e_0^2 + A^2(t)).$$

Inserting (3.12) into (3.11), one obtains

$$(3.13) \quad \phi^2 \int_{\mathbb{R}} v_x^2 dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2 dx ds \leq C(e_0^2 + A^2(t)), \quad \forall t \geq 0.$$

Multiplying (1.11) by $\phi^4 \theta_t$ and integrating, following the same arguments as used for (3.11)-(3.13), we deduce that

$$(3.14) \quad \phi^4(t) \|\theta_x(t)\|^2 + \int_0^t \|\theta_t\|^2 \phi^4 ds \leq C e_0^2 + C \int_0^t \int_{\mathbb{R}} (\phi^4 v_x^4 + \phi^4 |v_x| \theta_x^2) dx ds \leq \\ \leq C(e_0^2 + A^2(t)) + C \int_0^t \phi^8 \max_{\mathbb{R}} v_x^2 \int_{\mathbb{R}} \theta_x^2 dx ds \leq C(e_0^2 + A^2(t)), \quad \forall t \geq 0.$$

We are now able to derive pointwise bounds for $u - \bar{u}$. We may write (1.10) in the form (recalling $n = 1$): $v_t = \beta [\log(u/\bar{u})]_{tx} - R[\theta/u - \bar{\theta}/\bar{u}]_x$. Integrating this over $(-\infty, x) \times (0, t)$ ($t \in [0, 1]$) and then taking the absolute value, making use of (3.1) and (3.8)-(3.9), we see that

$$(3.15) \quad |u - \bar{u}| \leq C \left| \log \frac{u}{\bar{u}} \right| \leq \\ \leq C |u_0 - \bar{u}| + C \int_{-\infty}^x (|v| + |v_0|) dy + C \int_0^t (|u - \bar{u}| + |q - \bar{\theta}|) ds \leq$$

$$\begin{aligned} &\leq C e_0 + \|\psi^{1/2} v\| \|\psi^{-1/2}\| + C \int_0^t |u - \bar{u}| ds + C \int_0^t \|\theta - \bar{\theta}\|_{H^1} ds \leq \\ &\leq C e_0 + C \int_0^t |u - \bar{u}| ds + C \left(\int_0^t \|\theta - \bar{\theta}\|_{H^1}^2 ds \right)^{1/2} \leq C e_0 + C \int_0^t |u - \bar{u}| ds, \quad t \in [0, 1]. \end{aligned}$$

An application of Gronwall’s inequality to (3.15) yields

$$(3.16) \quad |u(x, t) - \bar{u}| \leq C e_0, \quad \forall x \in \mathbb{R}, t \in [0, 1].$$

To estimate $u - \bar{u}$ for $t \geq 1$ we denote $F := \beta[v_x/u] - R[u/\theta] + R[\bar{u}/\bar{\theta}]$ and find that

$$\beta \left[\frac{1}{u} - \frac{1}{\bar{u}} \right]_t + \frac{R\theta}{u} \left[\frac{1}{u} - \frac{1}{\bar{u}} \right] = -\frac{F}{u} - \frac{R(\theta - \bar{\theta})}{u\bar{u}}.$$

Multiplying this by $1/u - 1/\bar{u}$, using (3.1), (3.10), and (3.9), we get

$$\begin{aligned} \left[\frac{1}{u} - \frac{1}{\bar{u}} \right]_t^2 + C^{-1} \left[\frac{1}{u} - \frac{1}{\bar{u}} \right]_t^2 &\leq C(\|F\|_{L^\infty}^2 + \|\theta - \bar{\theta}\|_{L^\infty}^2) \leq \\ &\leq C(\|F\|^2 + \|F_x\|^2 + \|\theta - \bar{\theta}\|_{H^1}^2) \leq C e_0^2 + \|(v_x, v_t, \theta_x)\|^2, \quad t \geq 1, \end{aligned}$$

which together with (3.9) and (3.13) gives

$$(3.17) \quad |u(x, t) - \bar{u}|^2 \leq C e_0^2 + C |u(x, 1) - \bar{u}|^2 + C \int_1^t \|(v_x, v_t, \theta_x)\|^2 ds \leq C(e_0^2 + A^2(t)), \quad \forall x \in \mathbb{R}, t \geq 1.$$

Combining (3.9), (3.13)-(3.14), and (3.16)-(3.17), we obtain $A(t) \leq \tilde{c}\{e_0^2 + A^2(t)\}$ for $t \geq 0$, where $\tilde{c} \geq 1$ depends at most on $\bar{u}, \bar{\theta}, \beta, R, c_v, \kappa,$ and γ . Hence we have

$$(3.18) \quad A(t) \leq 2\tilde{c}e_0^2, \quad \text{for all } t \geq 0$$

provided $e_0 < \min\{1/(2\tilde{c}), 1/(2C)\}$.

From (3.18), (3.10), and (3.9) we conclude that

$$(3.19) \quad |u(x, t) - \bar{u}| + \phi(t) |\theta(x, t) - \bar{\theta}| \leq A^{1/2}(t) + C \|\theta - \bar{\theta}\|^{1/2} \phi \|\theta_x\|^{1/2} \leq A^{1/2}(t) + C e_0 A^{1/4}(t) \leq \sqrt{2\tilde{c}} e_0 (1 + \tilde{C} \sqrt{e_0}) < \min\{\bar{u}, \bar{\theta}\}/3, \quad \forall x \in \mathbb{R}, t \geq 0$$

provided $e_0 < \min\{\bar{u}/(6\sqrt{\tilde{c}}), \bar{\theta}/(6\sqrt{\tilde{c}}), 1/(3\tilde{C})^2, 1/(2\tilde{c}), 1/(2C)\} =: \varepsilon$.

For the initial data satisfying $e_0 < \varepsilon$ we thus have proved that under (3.1) the estimate (3.19) holds. Since (3.19) is valid for $t = 0$, by virtue of the continuity of u and θ , (3.19) remains valid for all $t \geq 0$. Hence (3.9), (3.18) hold for all $t \geq 0$. We now multiply (1.10) by u_x/u and integrate over $\mathbb{R} \times (1, t)$. We make use of (1.9), (3.19), (3.9), and

(3.18) to deduce (cf. (2.40))

$$(3.20) \quad \int_{\mathbb{R}} u_x^2(x, t) dx + \int_1^t \|u_x\|^2 ds \leq C(1 + \|u_x(1)\|^2) < \infty, \quad t \geq 1.$$

Similarly, if we multiply (1.10) resp. (1.11) by v_{xx} resp. by θ_{xx} and integrate over $\mathbb{R} \times (1, t)$, utilise (3.20), we have

$$(3.21) \quad \int_1^t (\|v_{xx}\|^2 + \|\theta_{xx}\|^2) ds \leq C(1 + \|u_x(1)\|^2) < \infty, \quad t \geq 1.$$

From (3.18) and (3.20)-(3.21) we get by the same argument as used for (2.52) that $\|u_x(t)\| + \|v_x(t)\| + \|\theta_x(t)\| \rightarrow 0$ as $t \rightarrow \infty$. From this, (3.9), and $\|\cdot\|_z^\infty \leq C\|\cdot\|\|\partial_x \cdot\|$, (1.20) follows immediately. The proof is complete.

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