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A Representation Theorem for the Group of Autoprojectivities of an Abelian p-Group of Finite Exponent (*).

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Dedicated to M. Suzuki on the occasion of his 70th birthday

Abstract. – *Given the abelian p-group* $M = \langle a \rangle \oplus \langle b \rangle \oplus C$, *where* $|a| = p^n \geq |b| = p^m > \exp C = p^s > 1$, *set* $R(M) = \{\varphi \in P(M) \mid H^\varphi = H,\ \varphi \mid \Omega_s(M) = 1\}$. *Our main result is the existence of a well determined isomorphism of* $R(M)$ *onto a well defined subgroup of* $\prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m)$.

Introduction.

Let M be an abelian p-group of finite exponent, p^n say, and $P(M)$ its group of autoprojectivities, that is the group of automorphisms of the subgroup lattice $l(M)$ of M. In case a basis of M contains at least 3 elements of order p^n, a well known result of R. Baer [1] states that every autoprojectivity of M is *linear*, that is it is induced by a group automorphism; on the other hand if the rank of M is less than 3 the elements of $P(M)$ have been completely described in terms of automorphisms of the poset $\mathcal{C}(M)$ of the cyclic subgroups of M [2].

The purpose of the present paper is to bridge the gap, that is to give a description of the autoprojectivities of M in the case M has the following structure: $M = H \oplus C$ where $H = \langle a \rangle \oplus \langle b \rangle$ with $p^n = |a| \geq |b| = p^m \geq p^s = \exp C \neq 1$, and where either $|a| > |b|$ or $|a| = |b|$ and $s < n$. For a fixed prime p, we shall call such a p-group an (n, m, s)-group, or simply an (n, s)-group in case $m = n$.

In dealing with the group $P(M)$, to begin with, we show that $P(M)$ is a product of two permutable subgroups, $P(M) = RPA(M)$, with $R \cap PA(M) = 1$, where $PA(M)$ is

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the group of linear autoprojectivities of M. The main concern in this paper will be to establish a very useful representation theorem of R as well determined subgroup of $P = \prod_{k=0}^{n-m} PR(p^{n-k-m}R_{n-k}) \times PR(pR_m)$, where $R_v = \mathbb{Z}/p^v\mathbb{Z}$ and $PR(p^w R_v)$ is the group of automorphisms of the poset $\mathcal{R}(p^w R_v)$ of the cosets of $p^w R_v$.

The analysis needed to reach our conclusion is quite involved and complex. The paper is divided in five sections.

In the first section we establish the above mentioned factorization, derive some useful propositions and show that R can be embedded in $P(H)$. In the second section it is shown that the subgroup $P_{\langle p^{m-1}b\rangle}(H)$ of $P(H)$ which stabilizes $\langle p^{m-1}b\rangle$ is isomorphic to P. In the third section we introduce a convenient subgroup Φ of P and show that R embeds in Φ. In the fourth section we deal with the case $n = m$ and show that the embedding is actually an isomorphism. Finally in the fifth section we prove that in general the embedding is an isomorphism.

Our notation is standard, relying essentially on [3], [4] and [5]. If $X \leqslant Y \leqslant G$, $X \lessdot Y$ means that X is a maximal subgroup of Y.

$\mathcal{U}(R_n)$ is the group of units of the ring $R_n = \mathbb{Z}/p^n\mathbb{Z}$.

1. – Preliminaries.

Given an abelian p-group M of finite exponent p^n, it will be convenient to view M as a $\mathbb{Z}$-module as well as an R_n-module.

For later references, let us define some specific subgroups of $P(M)$ for a given abelian p-group M. For $X \leqslant M$ and a given integer s we set respectively

$$P_X(M) = \{\varrho \in P(M) \mid X^\varrho = X\}, \qquad R_s(M) = \{\varrho \in P(M) \mid \varrho \mid \Omega_s(M) = 1\}.$$

In the case $M = H \oplus C$ is an (n, m, s)-group, $R(M)$ denotes the group $\{\varrho \in R_s(M) \mid H^\varrho = H\}$. If (a, b) is a basis of H, $\mathcal{A} = (\langle a\rangle, \langle b\rangle)$ will be called the frame associated to (a, b), and $u = \langle p^{n-m}a + b\rangle$ a unit point; we set

$$R_{\mathcal{A}}(M) = \{\varrho \in R(M) \mid \mathcal{A}^\varrho = \mathcal{A}\}, \qquad R_{\mathcal{A},u}(M) = \{\varrho \in R_{\mathcal{A}}(M) \mid u^\varrho = u\}.$$

We begin with a statement whose proof is straightforward.

(1) Let $M = A + B$ be an abelian p-group of finite exponent, with $\exp B = \exp(A \cap B) = p^s$, and $p^{s-1}A$ not cyclic. Then for given α, β in $PA(M)$ we have $\alpha = \beta$ if and only if $\alpha \mid A = \beta \mid A$ and $\alpha \mid B = \beta \mid B$. ■

1.1. THEOREM. – *Let $M = H \oplus C$ be an (n, m, s)-group, (a, b) a basis of H with associated frame $\mathcal{A}$ and unit point u. Then*

$$P(M) = R_{\mathcal{A},u}(M)\, PA(M), \qquad R_{\mathcal{A},u}(M) \cap PA(M) = 1 .$$

PROOF. – Let (c_i) be a basis of C; then for a given φ in $P(M)$ there exists α in $PA(M)$ such that for $\tau = \varphi\alpha$ we get $\mathcal{A}^\tau = \mathcal{A}$, $u^\tau = u$, $\langle c_i\rangle^\tau = \langle c_i\rangle$. Now $\tau \mid \Omega_s(M) \in PA(\Omega_s(M))$, so $\tau \mid \Omega_s(M)$ is induced by an automorphism of the form $1 \oplus \gamma$, where γ is in Aut C. Set $\beta = 1 \oplus \gamma^{-1}$ in Aut M; then (with the obvious abuse of notation) $\tau\beta = \varphi\alpha\beta$ lies in

$R_{\mathfrak{a},u}(M)$, and φ is in $R_{\mathfrak{a},u}(M)PA(M)$. If now $\varrho \in R_{\mathfrak{a},u}(M) \cap PA(M)$, it is clear that $\varrho | H = 1$ and $\varrho | \Omega_s(M) = 1$. Hence $\varrho = 1$ by (1). ■

From (1.1) also follows the relation

(2)
$$R(M) = R_{\mathfrak{a},u}(M)(PA(M) \cap R(M)) .$$

Given an abelian p-group M, $\mathcal{P}(M)$ shall denote the set of all maximal cyclic subgroups of M. Assume $\langle a \rangle \leqslant A \leqslant M$, we define

$$\sqrt{\langle a \rangle}_A := \langle X | X \in \mathcal{P}(A), \langle a \rangle \leqslant X \rangle .$$

So $\exp \sqrt{\langle a \rangle}_A = |a| p^{h(a)}$, $h(a)$ being the height of a in A [3], hence also $\langle a \rangle = p^{h(a)} \sqrt{\langle a \rangle}_A$.

It follows that

(3) for φ in $P(M)$, $\langle a \rangle^\varphi = \langle a \rangle$ if $\sqrt{\langle a \rangle}_A^\varphi = \sqrt{\langle a \rangle}_A$, and conversely provided $A^\varphi = A$. In the case $A = M$, we simply write $\sqrt{\langle a \rangle}$.

We recall that if H is a homocyclic abelian p-group of rank 2, then $\delta : \langle x \rangle \mapsto \sqrt{\langle x \rangle}$ can be extended (in a unique way) to an autoduality of H [6]; we shall refer to it as the *expanding autoduality* of H. A useful property of δ is the following one.

1.2. PROPOSITION. – *Let H be a homocyclic abelian p-group of* rank 2, *δ its expanding autoduality and χ in $P(H)$. Then $\chi\delta = \delta\chi$.*

PROOF. – $\langle x \rangle \delta\chi = (\sqrt{\langle x \rangle})^\chi = \sqrt{\langle x \rangle^\chi} = \langle x \rangle \chi\delta$. ■

It follows in particular that

(4) for given $t > 0$, $\chi | \Omega_t(H)$ and $\chi | H/p^t H$ are equivalent since $\Omega_t(H)$ is dual to $H/p^t H$.

Next we give two criteria for extending autoprojectivities.

1.3. PROPOSITION ([4]). – *Let M be an abelian p-group of exponent p^n with $p^{n-1}M$ of order p and (φ_1, φ_2) in $P(\Omega_{n-1}(M)) \times P(M/p^{n-1}M)$. Then there exists a unique φ in $P(M)$ such that $\varphi | \Omega_{n-1}(M) = \varphi_1$ and $\varphi | M/p^{n-1}M = \varphi_2$ if and only if*

$$\varphi_1 | \Omega_{n-1}(M)/p^{n-1}M = \varphi_2 | \Omega_{n-1}(M)/p^{n-1}M . \quad ■$$

1.4. PROPOSITION. – *Let M be an abelian p-group of exponent p^n, with $p^{n-1}M$ of order p^2, and $(\varphi_0, \varphi_1, \ldots, \varphi_p, \varrho)$ be in $\prod_{i=0}^{p} P(\sqrt{X_i}) \times P(M/p^{n-1}M)$, where $X_0, \ldots, X_p$ are the minimal subgroups of $p^{n-1}M$. Then there exists a unique φ in $P(M)$ such that*

$$\varphi | \sqrt{X_i} = \varphi_i \quad \textit{and} \quad \varphi | M/p^{n-1}M = \varrho$$

if and only if

$$(*) \qquad \varphi_i \,|\, \Omega_{n-1}(M) = \varphi_j \,|\, \Omega_{n-1}(M),$$

$$(**) \qquad v_i \,|\, \sqrt{X_i}/p^{n-1}M = \varrho \,|\, \sqrt{X_i}/p^{n-1}M .$$

PROOF. – Necessity. For $i \neq j$ we get $\sqrt{X_i} \cap \sqrt{X_j} = \Omega_{n-1}(M)$. Hence $\varphi_i \,|\, \Omega_{n-1}(M) = \varphi | \Omega_{n-1}(M) = \varphi_j \,|\, \Omega_{n-1}(M)$; moreover $\varphi_i \,|\, \sqrt{X_i}/p^{n-1}M = \varphi | \sqrt{X_i}/p^{n-1}M = \varrho \,|\, \sqrt{X_i}/p^{n-1}M$.

Sufficiency. Given $(\varphi_0, \varphi_1, \ldots, \varphi_p, \varrho)$ define

$$(2) \qquad T^{\varphi} = \begin{cases} T^{\varphi_i} & \text{if } T \leqslant \sqrt{X_i} \text{ for some } i\,, \\ T^{\varrho} & \text{otherwise}\,. \end{cases}$$

Similarly for $(\varphi_0^{-1}, \varphi_1^{-1}, \ldots, \varphi_p^{-1}, \varrho^{-1})$ define φ'.

a) φ and φ' are well defined bijections: if $T \leqslant \sqrt{X_i}$, $T \leqslant \sqrt{X_j}$ for $i \neq j$, then $T \leqslant \Omega_{n-1}(M)$, so $T^{\varphi_i} = T^{\varphi_j}$. If $T \nleqslant \sqrt{X_i}$ for every i, then $p^{n-1}M \leqslant T$. Similarly for φ'. But now $\varphi\varphi' = \varphi'\varphi = 1$, so that φ and φ' are bijections.

b) φ and φ' preserve inclusions. Let $T_1 < T_2 \leqslant M$.

*b*₁) $T_2 \leqslant \sqrt{X_i}$. Then $T_1^{\varphi} = T_1^{\varphi_i} < T_2^{\varphi_i} = T_2^{\varphi}$;

*b*₂) $T_1 \leqslant \sqrt{X_i}$ and $p^{n-1}M \leqslant T_2$; then $T_1 \leqslant T_1 + p^{n-1}M \leqslant T_2$ hence

$$T_1^{\varphi} = T_1^{\varphi_i} \leqslant (T_1 + p^{n-1}M)^{\varphi_i} = (T_1 + p^{n-1}M)^{\varrho} \leqslant T_2^{\varrho} = T_2^{\varphi};$$

*b*₃) $p^{n-1}M \leqslant T_1$; then

$$T_1^{\varphi} = T_1^{\varrho} < T_2^{\varrho} = T_2^{\varphi}\,.$$

Similarly for φ'.

Therefore φ is an autoprojectivity with $\varphi | \sqrt{X_i} = \varphi_i$ and $\varphi | M/p^{n-1}M = \varrho$. ∎

We end this paragraph with an analogue to (1).

1.5. THEOREM. – *Let $M = H \oplus C$ be an (n, m, s)-group and η, ϑ in $P(M)$. Then $\eta = \vartheta$ if and only if $\eta | H = \vartheta | H$ and $\eta \,|\, \Omega_s(M) = \vartheta \,|\, \Omega_s(M)$.*

PROOF. – Set $\varrho = \eta\vartheta^{-1}$. We have to show that $\varrho = 1$. Set $r = r(M) = n - s$; we begin with $r = 1$.

a) $|p^{n-1}M| = p$. We have $\varrho \,|\, \Omega_{n-1}(M) = 1$, and by (1), with $A = H/p^{n-1}M$ and $B = \Omega_{n-1}(M)/p^{n-1}M$, $\varrho | M/p^{n-1}M = 1$; so by 1.3 $\varrho = 1$.

b) $|p^{n-1}M| = p^2$. Let $0 < X < p^{n-1}M$; by *a*), $\varrho \,|\, \sqrt{X} = 1$, hence $\varrho = 1$.

We now assume $r > 1$ and use induction.

a) $|p^{n-1}M| = p$. Since $r(\Omega_{n-1}(M)) = r - 1$, by induction $\varrho \,|\, \Omega_{n-1}(M) = 1$. Since $\Omega_s(M/p^{n-1}M) \leqslant \Omega_{n-1}(M)/p^{n-1}M$, $\varrho \,|\, \Omega_s(M/p^{n-1}M) = 1$; thus by induction $\varrho | M/p^{n-1}M = 1$ and so, by 1.3, $\varrho = 1$.

b) $|p^{n-1}M| = p^2$. $\varrho|\Omega_{n-1}(M) = 1$, by induction, hence $\varrho|\Omega_{n-1}(\sqrt{X}) = 1$ for $0 < X \leq p^{n-1}M$. Since $\Omega_s(\sqrt{X}/X) \leq \Omega_{n-1}(\sqrt{X})/X$, $\varrho|\Omega_s(\sqrt{X}/X) = 1$; thus by induction $\varrho|\sqrt{X}/X = 1$, hence $\varrho|\sqrt{X} = 1$ by 1.3; therefore $\varrho = 1$. ■

1.6. COROLLARY. – *Let $M = H \oplus C$ be an (n, m, s)-group. Then the restriction map*

$$\psi \colon R(M) \to R_s(H),\ \varphi \mapsto \varphi|H$$

is a monomorphism. ■

Proposition 1.5 tells us that a φ in $P(H)$ can be lifted to a $\tilde{\varphi}$ in $P(M)$ satisfying prescribed values on $\Omega_s(M)$ in at most one way. Actually our main purpose is to characterize the image of $R(M)$ under ψ.

2. – A description of the autoprojectivities of $R_s(H)$.

Let H be a 2-generated abelian p-group. We then know that if $\mathcal{C}(H)$ is the poset of all cyclic subgroups of H, the map

$$f \colon P(H) \to \operatorname{Aut} \mathcal{C}(H), \qquad \chi \mapsto \chi|\mathcal{C}(H) \tag{5}$$

defines an isomorphism [2]; this canonical identification of the two groups will be understood whenever it turns out convenient. We also recall the known fact

(6) if $K \leq H$ and H is homocyclic, then any χ in $\operatorname{Aut}\mathcal{C}(K)$ extends to a $\tilde{\chi}$ in $\operatorname{Aut}\mathcal{C}(H)$ [2].

Given an (n, m, s)-group $M = H \oplus C$, to describe for a given φ in $R(M)$ its action on H, it will be convenient to consider M embedded as a subgroup in a (n, s)-group $\tilde{M} = \tilde{H} \oplus C$. Given a basis (a, b) of H, where we assume $|a| = p^n$, we choose one $(\tilde{a}, \tilde{b})$ for $\tilde{H}$ such that $(a, b) = (\tilde{a}, p^{n-m}\tilde{b})$, while for $\Omega_{n-k}(\tilde{H})$, $0 \leq k \leq n-m$, we pick $(a_k, b_k) = (p^k\tilde{a}, p^k\tilde{b})$.

Let $\pi_k \colon R_{n-k} \to R_{n-k-1}$ be the canonical epimorphism, β_k the canonical module endomorphism of R_{n-k} defined by $x \mapsto px$. Since $\ker \pi_k = \ker \beta_k = p^{n-(k+1)}R_{n-k}$, we have the canonical factorization $\pi_k = \beta_k \gamma_k$, so that via γ_k the module pR_{n-k} can be identified with $R_{n-(k+1)}$, as well as $PR(pR_{n-k})$ with $PR(R_{n-(k+1)})$ via the induced isomorphism $\tilde{\gamma}_k \colon \varrho \mapsto \gamma_k^{-1}\varrho\gamma_k = \varrho'$. Since, for a given i in pR_{n-k} we have $\langle a_k + ib_k \rangle = \langle a_k + (i'p)b_k \rangle = \langle a_k + i'\pi_k pb_k \rangle = \langle a_k + i\gamma_k pb_k \rangle$, it follows that

$$\begin{cases} \langle a_k + i\varrho b_k \rangle = \langle a_k + i\gamma_k \varrho'(pb_k) \rangle, \\ \langle i\varrho a_k + b_k \rangle = \langle (i\gamma_k)\varrho' pa_k + b_k \rangle \end{cases} \tag{7}$$

holds for every i in pR_{n-k}.

Some further observations concerning R_n are in order.

We know a given element i of the local ring R_n can be uniquely represented in its p-adic expansion $i = i_0 + i_1 p + \ldots + i_{n-1}p^{n-1}$, where $i_j \in \{0, 1, \ldots, p-1\} \subset R_n$; obviously $i \in p^s R_n$ if and only if $i_0 = \ldots = i_{s-1} = 0$, $i \in \mathcal{U}(R_n)$ if and only if $i_0 \neq 0$. Modulo the obvious identifications we have

$$\text{(8)}\qquad \begin{cases} i\pi_0 = i_0 + i_1 p + \ldots + i_{n-2} p^{n-2}, \\ (i_1 p + \ldots + i_{n-1} p^{n-1})\gamma_0 = i_1 + \ldots + i_{n-1} p^{n-2}, \end{cases}$$

while ν_0: $i_0 + i_1 p + \ldots + i_{n-2} p^{n-2} \mapsto i_0 + i_1 p + \ldots + i_{n-2} p^{n-2} + 0 p^{n-1}$ defines an injection of R_{n-1} into R_n such that $i\pi_0 \nu_0 = i_0 + i_1 p + \ldots + i_{n-2} p^{n-2}$.

Set $\mathcal{P}_k = \{\langle x \rangle \leqslant \Omega_{n-k}(H) \mid |x| = p^{n-k}\}$ and $C(\mathcal{P}_k) = \{p^t \langle x \rangle \mid \langle x \rangle \in \mathcal{P}_k,\ 0 \leqslant t \leqslant m\}$; we have $\mathcal{C}(H) = \bigcup_{k=0}^{n-m} C(\mathcal{P}_k)$.

To describe the action of $\chi \in P(H)$ on $C(\mathcal{P}_k)$ it will be convenient to introduce coordinates in the set $\mathcal{P}_k$ with reference to the basis (a_k, b_k) of $\Omega_{n-k}(\widetilde{H})$.

$$\begin{cases} \mathcal{P}_k = \{\langle a_k + i b_k \rangle \mid i \in p^{n-k-m} R_{n-k}\} & 0 \leqslant k < n-m, \\ \mathcal{P}_{n-m} = U \cup V = \{\langle a_{n-m} + i b_{n-m} \rangle \mid i \in R_m\} \cup \{\langle i a_{n-m} + b_{n-m} \rangle \mid i \in R_m\}, \end{cases}$$

$$U \cap V = \{\langle a_{n-m} + i b_{n-m} \rangle \mid i \in \mathcal{U}(R_m)\},$$

$$U' = \{\langle a_{n-m} + i b_{n-m} \rangle \mid i \in pR_m\}, \qquad V' = \{\langle i a_{n-m} + b_{n-m} \rangle \mid i \in pR_m\}.$$

Now the maps

$$\begin{cases} \delta_k\colon p^t \langle a_k + i b_k \rangle \mapsto i + p^{n-k-m-t} R_{n-k}, & 0 \leqslant k \leqslant n-m,\quad 0 \leqslant t \leqslant m, \\ \delta'_{n-m}\colon p^t \langle i a_{n-m} + b_{n-m} \rangle \mapsto i + p^{m-t} R_m, & 0 \leqslant t \leqslant m, \end{cases}$$

define antiisomorphisms between the posets $C(\mathcal{P}_k)$, $0 \leqslant k < n-m$, $C(U)$, $C(V)$ and $\mathcal{R}(p^{n-k-m} R_{n-k})$. By restricting i to pR_m, we see that δ_{n-m} and δ'_{n-m} define antiisomorphisms between $C(U')$, $C(V')$ and $\mathcal{R}(pR_m)$. It follows that the map

$$\prod_{k=0}^{n-m-1} \operatorname{Aut} C(\mathcal{P}_k) \times \operatorname{Aut} C(U) \times \operatorname{Aut} C(V) \to \prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(R_m),$$

$\chi_k \mapsto \delta_k^{-1} \chi_k \delta_k = \sigma_k$, $\chi_{n-m} \mapsto \delta_{n-m}^{-1} \chi_{n-m} \delta_{n-m} = \sigma_{n-m}$, $\chi'_{n-m} \mapsto \delta'^{-1}_{n-m} \chi'_{n-m} \delta'_{n-m} = \tau_{n-m}$ is an isomorphism.

By restriction we get the isomorphism

$$\text{(9)}\quad \varepsilon\colon \prod_{k=0}^{n-m-1} \operatorname{Aut} C(\mathcal{P}_k) \times \operatorname{Aut} C(U) \times \operatorname{Aut} C(V') \to \prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m).$$

2.1. REMARK. – The meaning of the automorphisms σ_k in $PR(p^{n-k-m} R_{n-k})$, τ_{n-m} in $PR(R_m)$ are best described for a given χ in $P(H)$ (provided $U^\chi = U$, $V^\chi = V$ if $n = m$) by the relations

$$\text{(10)}\qquad \begin{cases} \langle a_k + i b_k \rangle^\chi = \langle a_k + i\sigma_k b_k \rangle, & i \in p^{n-k-m} R_{n-k}, \\ \langle i a_{n-m} + b_{n-m} \rangle^\chi = \langle i\tau_{n-m} a_{n-m} + b_{n-m} \rangle, & i \in R_m, \end{cases}$$

for i in $\mathcal{U}(R_m)$, $i\tau_{n-m} = (i^{-1}\sigma_{n-m})^{-1}$. In particular, in the case $n = m$, σ and τ as elements of $PR(R_n)$ must satisfy anologous relations. ■

2.1. LEMMA. – *Let $H = \langle a \rangle \oplus \langle b \rangle$ be a group with $p^n = |a| \geq |b| = p^m$ and $1 \leq s \leq m$. Then*

$$\vartheta\colon P(H)_{\langle p^{n-1}b\rangle} \to \prod_{k=0}^{n-m-1} \operatorname{Aut} C(\mathcal{P}_k) \times \operatorname{Aut} C(U) \times \operatorname{Aut} C(V') = T\,,$$

$\chi \mapsto (\chi_0, \ldots, \chi_{n-m-1}, \chi_U, \chi_{V'})$, *where* $\chi_k = \chi | C(\mathcal{P}_k)$, $\chi_U = \chi | C(U)$, $\chi_{V'} = \chi | C(V')$, *is a monomorphism. A ϱ in T lies in $P(H)^{\vartheta}_{\langle p^{n-1}b\rangle}$ if and only if*

$$\text{(11)}\qquad \begin{cases} \chi_k = \chi_{k+1} & \text{on } C(\mathcal{P}_k) \cap C(\mathcal{P}_{k+1}) \;\; 0 \leq k \leq n-m-2\,, \\ \chi_{n-m-1} = \chi_U & \text{on } C(\mathcal{P}_{n-m-1}) \cap C(U)\,, \end{cases}$$

A ϱ in $P(H)^{\vartheta}_{\langle p^{n-1}b\rangle}$ lies in $R_s(H)^{\vartheta}$ if and only if

$$\text{(12)}\qquad \chi_U | p^{m-s} U = 1\,, \qquad \chi_{V'} | p^{m-s} V' = 1\,.$$

PROOF. – Clearly ϑ is an embedding and (11) and (12) are to be satisfied. Conversely the compatatibility conditions (11) give rise to an automorphism of $\mathcal{C}(H)$, so by (5) to an element χ of $P(H)$ which clearly fixes $\langle p^{n-1}b\rangle$. Moreover χ belongs to $R_s(H)$ if (12) is satisfied. ■

We may now claim that for a given basis (a, b) of the group H where $p^n = |a| \geq |b| = p^m$ and $1 \leq s \leq m$ the map

$$\text{(13)}\qquad \eta = \vartheta\varepsilon\colon R_s(H) \to \prod_{k=0}^{n-m} PR(p^{n-k-m}R_{n-k}) \times PR(pR_m)$$

is a monomorphism.

2.2. THEOREM. – *Let $\varrho = (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ be an element of*

$$\prod_{k=0}^{n-m} PR(p^{n-k-m}R_{n-k}) \times PR(pR_m)\,.$$

Then ϱ lies in $P(H)^{\eta}_{\langle p^{n-1}b\rangle}$ if and only if

a) $\sigma_k \pi_k = \pi_k \sigma_{k+1}$, $0 \leq k < n-m$.

A ϱ in $P(H)^{\eta}_{\langle p^{n-1}b\rangle}$ lies in $R_s(H)^{\eta}$ if and only if

b) $i\sigma_k \equiv i\ p^s R_{n-k}$, $0 \leq k \leq n-m$, $i\tau_{n-m} \equiv i\ p^s R_m$.

PROOF. – *a)* Given ϱ, assume there exists a χ in $P(H)_{\langle p^{n-1}b\rangle}$ such that $\chi^{\eta} = \varrho$. Then using Remark 2.1 we get $\langle a_{k+1} + i\pi_k\sigma_{k+1}b_{k+1}\rangle = \langle a_{k+1} + i\pi_k b_{k+1}\rangle^{\chi} = p\langle a_k + ib_k\rangle^{\chi} = \langle a_{k+1} + i\sigma_k\pi_k b_{k+1}\rangle$, that is $\pi_k\sigma_{k+1} = \sigma_k\pi_k$. On the other hand, given ϱ satisfying *a)*, by (9) there exists a $\xi = (\chi_0, \ldots, \chi_{n-m-1}, \chi_U, \chi_{V'})$ such that $\xi^{\varepsilon} = \varrho$. But now $p\langle a_k + ib_k\rangle^{\chi_k} = \langle a_{k+1} + i\sigma_k\pi_k b_{k+1}\rangle = \langle a_{k+1} + i\pi_k\sigma_{k+1}b_{k+1}\rangle = \langle a_{k+1} + i\pi_k b_{k+1}\rangle^{\chi_{k+1}}$, hence by 2.1 there exists a χ in $P(H)_{\langle p^{n-1}b\rangle}$ such that $\chi^{\eta} = \varrho$.

b) χ lies in $R_s(H)$ if and only if:

$$p^{n-k-s}\langle a_k + ib_k\rangle = p^{n-k-s}\langle a_k + ib_k\rangle^{\chi} = p^{n-k-s}\langle a_k + i\sigma_k b_k\rangle$$

$$p^{m-s}\langle ia_{n-m} + b_{n-m}\rangle = p^{m-s}\langle ia_{n-m} + b_{n-m}\rangle^{\chi} = p^{m-s}\langle i\tau_{n-m} a_{n-m} + b_{n-m}\rangle,$$

i.e. if and only if $i\sigma_k \equiv i\ p^s R_{n-k}$, $i\tau_{n-m} \equiv i\ p^s R_m$. ■

We shall denote this mutual relationship via η between the elements χ of $R_s(H)$ and those $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ of $\prod_{k=0}^{n-m} PR(p^{n-k-m}R_{n-k}) \times PR(pR_m)$ as expressed in 2.2 by writing $\chi \overset{\eta}{\leftrightarrow} (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ or simply by $\chi \leftrightarrow (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ if η is clear from the context.

In case $n = m$ the situation in 2.2 becomes very simple; in fact

2.3. COROLLARY. – *Let H be a homocyclic group of exponent p^n, (a, b) a basis of it and $1 \leq s \leq n$. Then η defines an isomorphism of $R_s(H)$ onto the group $\{(\sigma, \tau) \in PR(R_n) \times PR(pR_n) \mid i\sigma \equiv i\ p^s R_n,\ i\tau \equiv i\ p^s R_n\}$.* ■

2.4. PROPOSITION. – *Let $\tilde{H}$ be a homocyclic group of exponent p^n, $1 \leq s \leq n$, $(\tilde{a}, \tilde{b})$ a basis of $\tilde{H}$, $\tilde{\chi}$ in $R_s(\tilde{H})$, $\tilde{\chi} \leftrightarrow (\tilde{\sigma}, \tilde{\tau})$, $H = \langle a\rangle \oplus \langle b\rangle$ a subgroup of $\tilde{H}$ and $\varrho_k = \pi_0 \ldots \pi_{k-1}\colon R_n \to R_{n-k}$ the canonical epimorphism.*

a) if $(a, b) = (\tilde{a}, p^{n-m}\tilde{b})$, $1 \leq m < n$, then

$$H^{\tilde{\chi}} = H \text{ if and only if } (p^{n-m}R_n)^{\tilde{\sigma}} = p^{n-m}R_n;$$

b) let $\chi \leftrightarrow (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$, $1 \leq s \leq m$; then

$$\chi = \tilde{\chi} \mid H \text{ if and only if } \begin{cases} \varrho_k \sigma_k = \tilde{\sigma} \varrho_k, & \text{on } p^{n-k-m}R_n, \\ \varrho_{n-m}\tau_{n-m} = \tilde{\tau}\varrho_{n-m}, & \text{on } pR_m; \end{cases}$$

c) if $(a, b) = (a_k, b_k)$, $1 \leq s \leq n-k$, for χ_k in $R_s(p^k\tilde{H})$, let $\chi_k \leftrightarrow (\sigma_k, \tau_k)$. Then

$$\chi_k = \tilde{\chi} \mid p^k H \text{ if and only if } \varrho_k\sigma_k = \tilde{\sigma}\varrho_k,\ \varrho_k\tau_k = \tilde{\tau}\varrho_k;$$

d) if $(\bar{a}_k, \bar{b}_k) = (\tilde{a} + \Omega_{n-k}(\tilde{H}), \tilde{b} + \Omega_{n-k}(\tilde{H}))$, $\chi_k = \tilde{\chi} \mid \overline{H}_k = \tilde{H}/\Omega_{n-k}(\tilde{H}))$, $\bar{\chi} \in R_s(\overline{H}_k)$ with $\bar{\chi} \leftrightarrow (\bar{\sigma}_k, \bar{\tau}_k)$; then

$$\chi_k = \bar{\chi} \text{ if and only if } \varrho_k\bar{\sigma}_k = \tilde{\sigma}\varrho_k,\ \varrho_k\bar{\tau}_k = \tilde{\tau}\varrho_k .$$

PROOF. – *a)* $\langle \tilde{a} + i\tilde{b}\rangle \leq H$ if and only if $i \in p^{n-m}R_n$. Hence $\langle \tilde{a} + i\tilde{b}\rangle^{\tilde{\chi}} = \langle \tilde{a} + i\tilde{\sigma}\tilde{b}\rangle \leq H$ if and only if $i\tilde{\sigma} \in p^{n-m}R_n$.

b) $\chi = \tilde{\chi} \mid H$ if and only if $\langle a_k + i\varrho_k\sigma_k b_k\rangle = \langle a_k + i\varrho_k b_k\rangle^{\chi_k} = p^k\langle \tilde{a} + i\tilde{b}\rangle^{\tilde{\chi}} = p^k\langle \tilde{a} + i\tilde{\sigma}\tilde{b}\rangle = \langle a_k + i\tilde{\sigma}\varrho_k b_k\rangle$; similarly for τ_{n-m}.

c) $\chi_k = \tilde{\chi} \mid p^k\tilde{H}$ if and only if for $i \in R_n$, $\langle a_k + i\varrho_k\sigma_k b_k\rangle = \langle a_k + i\varrho_k b_k\rangle^{\chi_k} = p^k\langle \tilde{a} + i\tilde{b}\rangle^{\tilde{\chi}} = p^k\langle \tilde{a} + i\tilde{\sigma}\tilde{b}\rangle = \langle a_k + i\tilde{\sigma}\varrho_k b_k\rangle$; similarly for τ_k;

d) Follows from *c*) and (4).

2.5. PROPOSITION. – *Let H be a homocyclic group of exponent p^n, $1 \leq s \leq n$, (a, b) a basis of H and χ in $PA(H)$. χ is induced by the automorphism $a \mapsto a + zb$, $b \mapsto \lambda b$, represented by the matrix $\begin{pmatrix} 1 & z \\ 0 & \lambda \end{pmatrix}$, λ in $\mathcal{U}(R_n)$, z in R_n if and only if for $\chi^\eta = (\sigma, \tau) \in PR(R_n) \times PR(pR_n)$ we have σ: $i \mapsto i\lambda + z$, τ: $i \mapsto i(\lambda + iz)^{-1}$.*

In this case we have

a) σ^{-1}: $i \mapsto i\lambda^{-1} - z\lambda^{-1}$, τ^{-1}: $i \mapsto \lambda i(1 - iz)^{-1}$; *moreover* $j\sigma - i\sigma = (j - i)\lambda$, $j\tau - i\tau = (j - i)\lambda_1$, *where* $\lambda_1 = \lambda((\lambda + iz)(\lambda + jz))^{-1}$.

b) $\chi | \Omega_s(H) = 1$ *if and only if* $\lambda \equiv 1 \ p^s R_n$, $z \equiv 0 \ p^s R_n$.

PROOF. – χ is induced by $\begin{pmatrix} 1 & z \\ 0 & \lambda \end{pmatrix}$ in $\mathrm{Aut}\, H$, if and only if $\langle a + i\sigma b \rangle = \langle a + ib \rangle^\chi = \langle a + (i\lambda + z)b \rangle$, $\langle i\tau a + b \rangle = \langle ia + b \rangle^\chi = \langle ia + (\lambda + iz)b \rangle = \langle i(\lambda + iz)^{-1} a + b \rangle$ holds.

a) A straightforward computation.

b) $i\lambda + z \equiv i \ p^s R_n$ for every i if and only if $\lambda \equiv 1 \ p^s R_n$, $z \equiv 0 \ p^s R_n$; but then $(\lambda + iz)^{-1} \equiv 1 \ p^s R_n$, i.e. $i\tau \equiv i \ p^s R_n$. ■

2.6. PROPOSITION. – *Let H be a homocyclic group of exponent p^n, $1 \leq s \leq n$, (a, b) a basis of H, χ in $P(H)$ such that $U^\chi = U$ and $\chi \leftrightarrow (\sigma, \tau)$. Let $(\tilde{a}, \tilde{b})$ be the basis of H with $(\tilde{a}, \tilde{b}) = (b, a)$ and $\chi \leftrightarrow (\tilde{\sigma}, \tilde{\tau})$. Then*

$$i\tilde{\sigma} = \begin{cases} (i^{-1}\sigma)^{-1} & \text{if } i \in \mathcal{U}(R_n), \\ i\tau & \text{if } i \in pR_n, \end{cases} \qquad \tilde{\tau} = \sigma | pR_n .$$

PROOF. – $R_n = \mathcal{U}(R_n) \dot{\cup} pR_n$. For i in $\mathcal{U}(R_n)$: $\langle \tilde{a} + i\tilde{\sigma}\tilde{b} \rangle = \langle \tilde{a} + i\tilde{b} \rangle^\chi = \langle a + i^{-1} b \rangle^\chi = \langle (i^{-1}\sigma)^{-1} a + b \rangle = \langle \tilde{a} + (i^{-1}\sigma)^{-1} \tilde{b} \rangle$; for i in pR_n: $\langle \tilde{a} + i\tilde{\sigma}\tilde{b} \rangle = \langle \tilde{a} + i\tilde{b} \rangle^\chi = \langle b + i\tau a \rangle = \langle \tilde{a} + i\tau\tilde{b} \rangle$; for i in pR_n $\langle i\tilde{a} + \tilde{b} \rangle = \langle a + ib \rangle^\chi = \langle a + i\sigma b \rangle = \langle i\sigma\tilde{a} + \tilde{b} \rangle$. ■

3. – Congruence relations associated to autoprojectivities.

Let us, as usual, denote with $M = H \oplus C$ an (n, m, s)-group, (a, b) a basis of H with $|a| = p^n$. Due to Baer's result [1], if $K \oplus C \leq M$, where $K \leq H$ and $p^{s-1}K$ is non-cyclic, for any φ in $R(M)$ such that $K^\varphi = K$, $\varphi' = \varphi | K \oplus C/p^s K$ lies in $PA(K \oplus C/p^s K)$, hence it is induced by an automorphism $\alpha \oplus \mu$ (determined up to a multiplication), where $\alpha \in \mathrm{Aut}\, K/p^s K$ and $\mu \in \mathcal{U}(R_n)$, determined modulo $p^s R_n$, is a multiplication on $p^s K \oplus \oplus C/p^s K$; by abuse of notation, for simplicity, we shall write $\varphi' = (\alpha; \mu)$. We start by gathering some information about α and μ in some specific situations relevant to us. We shall also denote by (λ_1, λ_2), for $\lambda_i \in \mathcal{U}(R_n)$, the dilatation $a \mapsto a\lambda_1$, $b \mapsto b\lambda_2$, and with trz, for z in R_n the transvection $a \mapsto a + zb$, $b \mapsto b$.

3.1. LEMMA. – *Let $M = H \oplus C$ be an (n, m, s)-group, $1 \leq s < m$ and φ in $R(M)$. Let $\varphi_t = \varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t}$. Then*

$$\varphi_t = \begin{cases} (\lambda_t; 1) & \lambda_t \in \mathcal{U}(R_n) \text{ with } \lambda_1 \equiv 1 \ p^{s-1}R_n, \ 0 \leq t \leq m-s, \\ (\alpha_t; 1) & \alpha_t \in PA\Omega_{s+t}(H)/p^s\Omega_{s+t} \\ & \quad \text{fixing at least a cyclic subgroup of order } p^s, \ \ m-s < t \leq n-s. \end{cases}$$

PROOF. – By (4) $\varphi \mid \Omega_{s+t}(H)/p^s \Omega_{s+t} = 1$ for $0 \leq t \leq m-s$; hence $\varphi_t = (\lambda_t; 1)$. Since $\varphi_1 \mid \Omega_s(M)/\Omega_1(H) = 1$, we must have $\lambda_1 \equiv 1 p^{s-1} R_n$. Let $H = \langle a \rangle \oplus \langle b \rangle$ with $|b| = p^m$; consider $X = \langle p^{m-s} b \rangle$; then by (3) $(\sqrt{X})^\varphi = \sqrt{X^\varphi} = \sqrt{X}$, hence $\langle \sqrt{X}, p^s H \rangle / p^s H$ is a fixed cyclic subgroup of order p^s, and now the conclusion follows easily. ■

3.2. LEMMA. – *Let $M = H \oplus C$ be a (n, s)-group with $n = s+1$, $0 < X < H$, (a, b) a basis of H such that $X < \langle a \rangle$ and φ in $P(M)$ such that $\varphi \mid \Omega_s(H) = 1$. Then the following statements are equivalent*

i) $H^\varphi = H$ *and* $\varphi \mid C = 1$;

ii) $\varphi \mid \Omega_s(M) = (\lambda_1; 1)$, $\varphi \mid M/p^s M = (\lambda_2; 1)$ *with* $\lambda_i \in \mathcal{U}(R)n$ *(determined modulo $p^s Rn$)*;

iii) $\varphi \mid \sqrt{X}/X = \left(\begin{pmatrix} \lambda_2 & z_X \\ 0 & \lambda_1 \end{pmatrix}; 1 \right)$ *with* $C^\varphi = C$.

PROOF. – By 1.3 $\varphi \mid H/p^s H = 1$, hence i) implies ii); that ii) implies i) is clear. ii) implies iii): since $\varphi \mid \sqrt{X} \cap H/X$ has $\langle pb, X \rangle / X$ as fixed point being $\varphi \mid \Omega_s(H) = 1$, $\varphi \mid \sqrt{X}/X = \left(\begin{pmatrix} \lambda_2' & z_X \\ 0 & \lambda_1' \end{pmatrix}; 1 \right)$. Now $\varphi \mid \sqrt{X}/\langle X, pb \rangle = (\lambda_2'; 1) = (\lambda_2; 1)$, $\varphi \mid \langle pb, C \rangle = (\lambda_1'; 1) = (\lambda_1; 1)$ hence $\lambda_i' \equiv \lambda_i p^s R_n$. iii) implies ii): $\varphi \mid \Omega_s(M) = (\lambda; 1)$ since $\varphi \mid C = 1$ and $\varphi \mid \Omega_s(H) = 1$; moreover $\varphi \mid \langle pb, C \rangle = (\lambda_1; 1)$. But now by (1) $\lambda \equiv \lambda_1 p^s R_n$; similarly $\varphi \mid M/p^s M = (\lambda_2; 1)$. ■

3.1. REMARK. – In case in 3.2 we have $\lambda_1 \not\equiv \lambda_2 \, pR_n$, clearly by a proper choice of a with $X < \langle a \rangle$ one can reduce z_X to 0.

3.3. LEMMA. – *Let $M = H \oplus C$ be an (n, s)-group and φ in $R(M)$. By* 3.1

$$\varphi \mid \Omega_{s+1}(M)/p^s \Omega_{s+1} = (\lambda; 1) \quad \text{where } \lambda \in \mathcal{U}(R_n) \text{ and } \lambda \equiv 1 \ p^{s-1} R_n .$$

Then

$$\varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t} = (\lambda^t; 1)$$

for every $0 \leq t \leq n-s$. *Moreover for* $0 < X < H$,

$$\varphi \mid (\sqrt{X})_{\Omega_{s+t}(M)} / p^s (\sqrt{X})_{\Omega_{s+t}} = \left(\begin{pmatrix} \lambda^t & z_t \\ 0 & \lambda^{t-1} \end{pmatrix}; 1 \right)$$

(*with reference to a basis* (a, b) *with* $X < \langle a \rangle$).

PROOF. – By 3.1 $\varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t} = (\lambda_t; 1)$ and $\lambda = \lambda_1 \equiv 1 \ p^{s-1} R_n$. Assume $\lambda \not\equiv 1 \ p^s R_n$; since by 3.2 and Remark 3.1 $\varphi \mid (\sqrt{X})_{\Omega_{s+1}(H)}/p^s(\sqrt{X})_{\Omega_{s+1}} = (\lambda, 1)$, via the expanding autoduality of $(\sqrt{X})_{\Omega_{s+t}(H)}/X$ we get $\varphi \mid (\sqrt{X})_{\Omega_{s+t}(H)}/p^s(\sqrt{X})_{\Omega_{s+t}} = (\lambda, 1)$. Moreover 3.2 applied to $\Omega_{s+t}(M)/p^s \Omega_{s+t-1}$ gives us by induction $\varphi \mid (\sqrt{X})_{\Omega_{s+t}(M)}/p^s(\sqrt{X})_{\Omega_{s+t}} = (\lambda_t, \lambda^{t-1}; 1)$ for $0 < t \leq n-s$; hence $\lambda_t \lambda^{-(t-1)} \equiv \lambda \ p^s R_n$, that is $\lambda_t \equiv \lambda^t p^s R_n$, and using 3.2 one gets that $\varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t} = (\lambda^t; 1)$ $0 \leq t \leq n-s$. Assume now there is a $t > 1$ such that $\lambda_t \not\equiv 1 \ p^s R_n$ while $\lambda_{t-1} \equiv 1 \ p^s R_n$; then by 3.2 $\varphi \mid (\sqrt{X})_{\Omega_{s+t}(M)}/p^s(\sqrt{X})_{\Omega_{s+t}} = (\lambda_t, 1; 1)$ while $\varphi \mid (\sqrt{X})_{\Omega_{s+1}(M)}/X = (trz; 1)$ which is a contradiction to the expanding autoduality. The conclusion now follows using again 3.2 applied to $\varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t-1}$. ■

3.2. REMARK. – Let $M = H \oplus C$ be an (n, m, s)-group, $s < m$, (a, b) a basis of H, with $|a| = p^n$, $\mathcal{A}$ the associated frame and φ in $R(M)$. If one decomposes φ according to (2): $\varphi = \varphi_1(\alpha; 1)$, where $\varphi_1 \in R_{\mathcal{A}, u}(M)$, $\alpha \in PA(H)$ such that $\alpha \mid \Omega_s(H) = 1$, then

$$\varphi \mid \Omega_{s+1}(M)/p^s \Omega_{s+1} = (\lambda; 1) = \varphi_1 \mid \Omega_{s+1}(M)/p^s \Omega_{s+1};$$

moreover, as a consequence of (3)

$$\langle b, p^s \Omega_{s+t} \rangle = \langle b, p^s \Omega_{s+t} \rangle^{\varphi} \quad \text{for } t > m-s\,. \qquad (14)$$

■

3.4. THEOREM. – *Let* $M = H \oplus C$ *be an* (n, m, s)*-group with* $s < m$, (a, b) *a basis of* H *with* $p^n = |a|$, $\mathcal{A}$ *the associated frame and* φ *in* $R(M)$. *By* 3.1 $\varphi \mid \Omega_{s+1}(M)/p^s \Omega_{s+1} = (1; \mu)$, μ *in* $\mathcal{U}(R_n)$ *determined modulo* $p^s R_n$ *and* $\mu \equiv 1 \ p^{s-1} R_n$. *Then there exists* $\alpha \in PA(H) \cap R_s(H)$ *such that*

$$\varphi \mid \Omega_{s+t}(M)/p^s \Omega_{s+t} = \begin{cases} (1; \mu^t), & 0 \leq t \leq m-s\,, \\ (1, \mu^{t-(m-s)}; \mu^t)(\alpha_t; 1), & m-s < t \leq n-s\,, \end{cases}$$

where $\alpha_t = \alpha \mid \Omega_{s+t}(M)/p^s \Omega_{s+t}$ *has* $\langle b, p^s \Omega_{s+t} \rangle / p^s \Omega_{s+t}$ *as a fixed point.*

PROOF. – Set $r = r(M) = n - m$. If $r = 0$ the conclusion holds by 3.3 for $\mu = \lambda^{-1}$. Assume now $r > 0$. According to Remark 3.2 we write $\varphi = \varphi_1(\alpha; 1)$, $\alpha \in PA(M)$ and $\varphi_1 \in R_{\mathcal{A}, u}(M)$. As a conseguence of (14), α_t has the required property. Set $\overline{M} = M/\langle p^{n-1} a \rangle$. By 3.2 for $\mu = \lambda^{-1}$ we get $\varphi_1 \mid \Omega_s(\overline{M}) = (1, \mu; \mu)$, $\varphi_1 \mid \Omega_{s+1}(\overline{M}) = (1, \mu; \mu^2)$. Thus for $\varrho = \varphi_1(1, \mu^{-1}; \mu^{-1})$, $\varrho \mid \Omega_s(\overline{M}) = 1$, $\varrho \mid \Omega_{s+1}(\overline{M}) = (1; \mu)$. If now $r = 1$, by 3.3 $(1; \mu^{n-1-s}) = \varrho \mid \overline{M}/p^s \overline{M} = \varrho \mid M/p^s M$, hence $\varphi_1 \mid M/p^s M = (1, \mu; \mu^{n-s})$, that is for $r = 1$ 3.4 holds.

Assume $r > 1$. Since $r(\overline{M}) = r - 1$, by induction $\varrho \mid \overline{M}/p^s \overline{M} = (1, \mu^{n-1-m}; \mu^{n-1-s})$;

but then $\varphi_1 | M/p^s M = \varphi_1 | \overline{M}/p^s \overline{M} = (1, \mu^{n-m}; \mu^{n-s})$, and the conclusion follows. ∎

Following the notation introduced in n. 2 we have

3.5. LEMMA. – *Let $M = H \oplus C$ be an (n, m, s)-group, $s < m$, (a, b) a basis of H with $p^n = |a|$, φ in $R(M)$ and μ in $\mathcal{U}(R_n)$ as given in 3.1. If now, according to* (13), $(\varphi | H)^\eta = (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ *in* $\prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m)$, *for any c in C we have*

$$\langle a_k + ib_k + c \rangle^\varphi = \langle a_k + i\sigma_k b_k + \mu^{n-k-s} c \rangle, \qquad 0 \leqslant k \leqslant n-s,$$

$$\langle ia_{n-m} + b_{n-m} + c \rangle^\varphi = \langle i\tau_{n-m} a_{n-m} + b_{n-m} + \mu^{m-s} c \rangle.$$

PROOF. – (10) implies

$$\text{(15)} \qquad \begin{cases} \langle a_k + ib_k + c \rangle^\varphi = \langle a_k + i\sigma_k b_k + lc \rangle, & 0 \leqslant k \leqslant n-s, \\ \langle ia_{n-m} + b_{n-m} + c \rangle^\varphi = \langle i\tau_{n-m} a_{n-m} + b_{n-m} + l'c \rangle. \end{cases}$$

On the other hand, using 3.4 we have

$$\text{(16)} \qquad \begin{cases} \langle a_k + ib_k + c \rangle^\varphi = \langle a_k + i' b_k + \mu^{n-k-s} c + p^s(ya_k + zb_k) \rangle \\ \langle ia_{n-m} + b_{n-m} + c \rangle^\varphi = \langle i' a_{n-m} + b_{n-m} + \mu^{m-s} c + p^s(y' a_{n-m} + z' b_{n-m}) \rangle \end{cases}$$

$$\begin{cases} = \langle (1 + yp^s) a_k + (i' + zp^s) b_k + \mu^{n-k-s} c \rangle = \langle a_k + i'' b_k + \mu^{n-k-s} c \rangle, \\ = \langle (i' + p^s y') a_{n-m} + (1 + z' p^s) b_{n-m} + \mu^{m-s} c \rangle = \langle i'' a_{n-m} + b_{n-m} + \mu^{m-s} c \rangle. \end{cases}$$

Comparing (15) with (16) and picking a c in C of order p^s, one concludes. ∎

We are now in the position to establish the announced congruence relations.

3.6. THEOREM. – *Let $M = H \oplus C$ be an (n, m, s)-group, $s < m$, (a, b) a basis of H with $p^n = |a|$, $\mathcal{A}$ the associated frame and φ in $R(M)$. We know from 3.1 that $\varphi | \Omega_{s+1}(M)/p^s \Omega_{s+1} = (1; \mu)$, where μ lies in $\mathcal{U}(R_n)$ with $\mu \equiv 1 \; p^{s-1} R_n$. According to* (10), $(\varphi | H)^\eta = (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m})$ *lies in* $\prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m)$. *Then the followig relations hold*

a) $\sigma_k \pi_k = \pi_k \sigma_{k+1}$, $0 \leqslant k \leqslant n-m-1$,

b) $i\sigma_k \equiv i \; p^s R_{n-k}$, $0 \leqslant k \leqslant n-m$, $i\tau_{n-m} \equiv i \; p^s R_m$.

$$\text{(17}a\text{)} \qquad j \equiv i \quad p^f R_{n-k} \Rightarrow j\sigma_k - i\sigma_k \equiv (j-i)\mu^f \quad p^{s+f} R_{n-k},$$

for every $0 \leqslant k \leqslant n-m$, $n-m-k \leqslant f \leqslant n-s-k$;

$$\text{(17}b\text{)} \qquad j \equiv i \quad p^f R_m \Rightarrow j\tau_{n-m} - i\tau_{n-m} \equiv (j-i)\mu^f \quad p^{s+f} R_m,$$

for every $0 \leqslant f \leqslant m-s$.

PROOF. – 2.4 takes care of $a)$ and $b)$; moreover, according to Remark 2.1 ii), we have to deal only with σ_k, since we derive the congruence relations for τ_{n-m} from those for σ_{n-m}. We remark that due to limitations $0 \leqslant k \leqslant n-m$ and $n-m-k \leqslant f \leqslant n-s-k$, for any h in R_{n-k}, $\langle hp^f b_k + c \rangle$ is contained in $\Omega_m(M)$. Thus, if for $h \neq 0$ we decompose $h = p^e h'$, h' in $\mathcal{U}(R_{n-k})$, from 3.4 we get, for some $w \in p^s \Omega_{n-k-(f+e)}(M)$,

$$\text{(18)} \qquad \langle hp^f b_k + c \rangle^\varphi = \begin{cases} \langle hp^f b_k + \mu^{n-k-(f+e+s)} c + w \rangle, & \text{if } f < n-k-(e+s), \\ \langle hp^f b_k + c \rangle, & \text{if } f \geqslant n-k-(e+s). \end{cases}$$

If we choose a c in C of order p^s, by 3.5 we have

$$\text{(19)} \qquad \langle a_k + (i + hp^f)\, b_k + c \rangle^\varphi = \langle a_k + (i + hp^f)\, \sigma_k b_k + \mu^{n-k-s} c \rangle .$$

On the other hand $\langle a_k + (i + hp^f)\, \sigma_k b_k + \mu^{n-k-s} c \rangle \leqslant \langle a_k + i\sigma_k b_k \rangle + \langle hp^f b_k + c \rangle^\varphi$. Taking into account (18) and (19) one gets

$$\langle a_k + (i + hp^f)\, \sigma_k b_k + \mu^{n-k-s} c \rangle =$$

$$= \begin{cases} \langle a_k + i\sigma_k b_k + vhp^f b_k + v\mu^{n-k-(f+e+s)} c + vw \rangle, & \text{if } f < n-k-(e+s), \\ \langle a_k + i\sigma_k b_k + v' hp^f b_k + v' c \rangle & \text{if } f \geqslant n-k-(e+s). \end{cases}$$

Thus

$$v \equiv \mu^{f+e} \quad p^s R_{n-k}, \qquad v' \equiv \mu^{n-k-s} \quad p^s R_{n-k} .$$

But then

$$\text{(20)} \qquad \langle a_k + (i + hp^f)\, \sigma_k b_k + \mu^{n-k-s} c \rangle =$$

$$= \begin{cases} \langle a_k + i\sigma_k b_k + \mu^{f+e} hp^f b_k + \mu^{n-k-s} c + \mu^{f+e} p^{s+f+e} (ya_k + zb_k) \rangle \\ \hfill \text{if } f < n-k-(e+s), \\ \langle a_k + i\sigma_k b_k + \mu^{n-k-s} hp^f b_k + \mu^{n-k-s} c \rangle \quad \text{if } f \geqslant n-k-(e+s), \end{cases}$$

$$= \begin{cases} \langle a_k (1 + p^{s+f+e} y\mu^{f+e}) + (i\sigma_k + p^f h\mu^{f+e})\, b_k + \mu^{f+e} p^{s+f+e} zb_k + \mu^{n-k-s} c \rangle \\ \hfill \text{if } f < n-k-(e+s), \\ \langle a_k + i\sigma_k b_k + \mu^{n-k-s} hp^f b_k + \mu^{n-k-s} c \rangle \quad \text{if } f \geqslant n-k-(e+s), \end{cases}$$

$$= \begin{cases} \langle a_k + \varepsilon (i\sigma_k + p^f h\mu^{f+e})\, b_k + \varepsilon \mu^{f+e} p^{s+f+e} zb_k + \mu^{n-k-s} c \rangle \\ \hfill \text{if } f < n-k-(e+s), \\ \langle a_k + i\sigma_k b_k + \mu^{n-k-s} hp^f b_k + \mu^{n-k-s} c \rangle \quad \text{if } f \geqslant n-k-(e+s). \end{cases}$$

Since $\varepsilon \equiv 1 \; p^{s+f+e} R_{n-k}$, comparing the coefficients of b_k in (20) and taking into account that $\mu^e \equiv 1 \; p^{s-1} R_{n-k}$, setting $j = i + hp^s$, the conclusion follows for σ_k. ■

3.3. REMARK. – i) (17a) and $0\sigma_k \equiv 0 \; p^s R_n$ implies $i\sigma_k \equiv i \; p^s R_n$ for every i;

ii) $(17a)$ and σ_k in $\operatorname{Sym} p^{n-k-m} R_{n-k}$ implies σ_k in $PR(p^{n-k-m} R_{n-k})$.

Similarly for τ_{n-m}. ■

At this stage we set $\mathcal{U}(n, s) = \{[\mu] \in \mathcal{U}(R_n/p^s R_n) \mid \mu \equiv 1 p^{s-1} R_n\}$. This is a cyclic group of order $p-1$ if $s=1$, of order p if $s \geq 2$. Now we introduce a particular subgroup $\Phi_{n,m,s}$ of $\prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m) \times \mathcal{U}(n, s)$: $\Phi_{n,m,s} = \{(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu]) \mid$ the relations $a)$, $b)$, $(17a)$ and $(17b)$ of 3.6 hold$\}$. When $n = m$ we simply write $\Phi_{n,s}$. That this group is relevant to our investigations stems from the fact that 3.6 gives us an embedding of $R(M)$ into $\Phi_{n,m,s}$. Let us be more precise. With the help of the monomorphism η in (13) define (with abuse of notation) the monomorphism

$$(21) \qquad \eta\colon R_s(H) \times \mathcal{U}(n, s) \to \prod_{k=0}^{n-m} PR(p^{n-k-m} R_{n-k}) \times PR(pR_m) \times \mathcal{U}(n, s),$$

$(\chi, [\mu]) \mapsto (\chi^\eta, [\mu])$. Then Theorem 2.2 tells us that $\Phi_{n,m,s} \leq (R_s(H) \times \mathcal{U}(n, s))^\eta$ and using 1.6

$$(22) \quad \omega\colon R(M) \to R_s(H) \times \mathcal{U}(n, s), \quad \varphi \mapsto (\varphi | H, [\mu]) \text{ and } (1; \mu) = \varphi | \Omega_{s+1}(M)/p^s \Omega_{s+1},$$

is a monomorphism (μ in $\mathcal{U}(R_n)$ determined modulo $p^s R_n$).

Thus by 3.6

$$(23) \qquad j = \omega\eta\colon R(M) \to \Phi_{n,m,s}, \qquad \varphi \mapsto ((\varphi | H)^\eta, [\mu])) \text{ is a monomorphism}.$$

The main result of n. 5 will be the statement that actually j is an isomorphism onto $\Phi_{n,m,s}$, giving us a very handy representation of $R(M)$. The identification of $\Phi_{n,m,s}^{\eta^{-1}} \leq R(H) \times \mathcal{U}(n, s)$ via η with $\Phi_{n,m,s}$ (uniquely determined modulo a basis of H) will be understood whenever we shall need it, and for it again a notation like $(\chi, [\mu]) \leftrightarrow (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$ will be used.

In case M is an (n, s)-group, as already pointed out in 2.3, the situation in 3.6 becomes simpler as expressed in the following

3.7. COROLLARY. – *Let $M = H \oplus C$ be an (n, s)-group, (a, b) a basis of H and φ in $R(M)$. We know that $\varphi | \Omega_{s+1}(M)/p^s \Omega_{s+1} = (1; \mu)$, where μ lies in $\mathcal{U}(R_n)$ with $\mu \equiv 1\ p^{s-1} R_n$. According to (10), $(\varphi | H)^\eta = (\sigma, \tau)$ lies in $PR(R_n) \times PR(pR_n)$. Then the following relations hold*

b) $i\sigma \equiv i\ p^s R_n$, $i\tau \equiv i\ p^s R_n$,

$$(24) \qquad \begin{cases} j \equiv i \quad p^f R_n \Rightarrow j\sigma - i\sigma \equiv (j-i)\mu^f \quad p^{s+f} R_n, & 0 \leq f \leq n-s, \\ j \equiv i \quad p^f R_n \Rightarrow j\tau - i\tau \equiv (j-i)\mu^f \quad p^{s+f} R_n, & 0 \leq f \leq n-s. \end{cases} \quad \blacksquare$$

We end this paragraph with

3.8. PROPOSITION. – *Let $\widetilde{M} = \widetilde{H} \oplus C$ be an (n, s)-group, $M = H \oplus C$ an (n, m, s)-subgroup of $\widetilde{M}$ with $s < m$, and $\tilde{\varphi}$ in $R(\widetilde{M})$. Pick a basis $(\tilde{a}, \tilde{b})$ of $\widetilde{H}$ and suppose $(a, b) = (\tilde{a}, p^{n-m}\tilde{b})$ is a basis of H. According to (23), let $\tilde{\varphi}^j = (\tilde{\sigma}, \tilde{\tau}, [\mu])$ be in $\Phi_{n,s}$.*

Then $M^{\tilde{\varphi}} = M$ *if and only if* $(p^{n-m}R_n)^{\tilde{\sigma}} = p^{n-m}R_n$. *Moreover for a given* $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu'])$ *in* $\Phi_{n,m,s}$, *we have* $(\tilde{\varphi}\,|M)^{j} = (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu'])$ *if and only if the following relations hold:* $[\mu'] = [\mu]$, $\varrho_k\sigma_k = \tilde{\sigma}\varrho_k$ *on* $p^{n-k-m}R_n$ *for* $0 \leqslant k \leqslant n-m$ *and* $\varrho_{n-m}\tau_{n-m} = \tilde{\tau}\varrho_{n-m}$ *on* pR_n.

PROOF. – Straightforward using 24 *a*) and *b*). ■

4. – A representation theorem of $R(M)$, M an (n, s)-group.

In this number we begin with the homocyclic case. The general situation will be dealt with in n. 5.

4.1. LEMMA. – *Let* H *be a homocyclic group of exponent* p^n, $1 \leqslant s < n$, (a, b) *and* $(\tilde{a}, \tilde{b})$ *bases of* H *and for* χ *in* $R_s(H)$, *according to* (13), *let* $\chi^{\eta} = (\sigma, \tau)$, $\chi^{\tilde{\eta}} = (\tilde{\sigma}, \tilde{\tau})$. *Then*

$$(\sigma, \tau, [\mu]) \in \Phi_{n,s} \text{ if and only if } (\tilde{\sigma}, \tilde{\tau}, [\mu]) \in \Phi_{n,s}.$$

PROOF. – For symmetry reasons it is sufficient to prove one implication. Notice that $\mathcal{U}(R_n)^{\sigma} = \mathcal{U}(R_n)$, $(pR_n)^{\sigma} = pR_n$ since $i\sigma \equiv i\ p^sR_n$.

1) $(\tilde{a}, \tilde{b}) = (b, a)$.

Using 2.6, one gets

$$i\tilde{\sigma} = (i^{-1}\sigma)^{-1} \equiv i\,p^sR_n \quad \text{for } i \in \mathcal{U}(R_n), \qquad i\tilde{\sigma} = i\tau \equiv i\,p^sR_n \quad \text{for } i \in pR_n,$$

$$i\tilde{\tau} = i\sigma \equiv i \quad p^sR_n \quad \text{for } i \in pR_n.$$

Hence for $j \equiv i\ p^0R_n$, we have $j\tilde{\sigma} - i\tilde{\sigma} \equiv (j-i)\mu^0\ p^sR_n$, $j\tilde{\tau} - i\tilde{\tau} \equiv (j-i)\mu^0\ p^sR_n$. Assume now $j \equiv i\ p^fR_n$, $0 < f \leqslant n-s$, and observe that here $j \in \mathcal{U}(R_n)$ if and only if $i \in \mathcal{U}(R_n)$. Using again 2.6, a straightforward computation leads to conclude that $(\tilde{\sigma}, \tilde{\tau}, [\mu]) \in \Phi_{n,s}$.

2) $(\tilde{a}, \tilde{b}) = (a, \lambda b)$, $\lambda \in \mathcal{U}(R_n)$.

By 2.5 $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}^{\eta} = (\sigma_1, \tau_1) \in PR(R_n) \times PR(pR_n)$, where $\sigma_1\colon i \mapsto i\lambda$, $\tau_1\colon i \mapsto i\lambda^{-1}$. Since $\tilde{\sigma} = \sigma_1\sigma\sigma_1^{-1}$, $\tilde{\tau} = \tau_1\tau\tau_1^{-1}$, taking into account 2.5 *a*), one easily sees that $(\tilde{\sigma}, \tilde{\tau}, [\mu]) \in \Phi_{n,s}$.

3) $(\tilde{a}, \tilde{b}) = (a+b, b)$.

By 2.5 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\eta} = (\sigma_1, \tau_1) \in PR(R_n) \times PR(pR_n)$, where $\sigma_1\colon i \mapsto i+1$, $\tau_1\colon i \mapsto i(1+i)^{-1}$. Since $\tilde{\sigma} = \sigma_1\sigma\sigma_1^{-1}$, by 2.5 *a*) one concludes that for $j \equiv i\ p^fR_n$, $j\tilde{\sigma} - i\tilde{\sigma} \equiv$

$\equiv (j-i)\mu^f\ p^{s+f}R_n$, $0 \leq f \leq n-s$. Since again $\tilde{\tau} = \tau_1 \tau \tau_1^{-1}$ we get

$$j'\tilde{\tau} - i'\tilde{\tau} = \frac{j\tau - i\tau}{(1-j\tau)(1-i\tau)} \equiv \frac{j-i}{(1-j\tau)(1-i\tau)}\mu^f \equiv$$

$$\equiv \frac{j-i}{(1-j)(1-i)}\mu^f = \left(\frac{j}{1-j} - \frac{i}{1-i}\right)\mu^f = (j'-i')\,\mu^f\ \ p^{s+f}R_n\,,$$

being

$$\mu^f(j-i)\frac{(1-i)(1-j)-(1-i\tau)(1-j\tau)}{(1-i\tau)(1-j\tau)} \equiv 0\ \ p^{s+f}R_n\,.$$

It follows that $(\tilde{\sigma}, \tilde{\tau}, [\mu]) \in \Phi_{n,s}$.

Since any basis $(\tilde{a}, \tilde{b})$ of H can be obtained from (a, b) by applying successively elementary transformations, the conclusion follows.

4.2. LEMMA. – *Let H be a homocyclic group of exponent p^n, $1 \leq s < n-1$, (a, b) a basis of H and $(\sigma, \tau, [\mu])$ in $\Phi_{n,s}$. According to* (21) *there is χ in $R_s(H)$ such that $(\chi, [\mu])^\eta = (\sigma, \tau, [\mu])$.*

a) Set $H_1 = pH$, and choose the basis $(a_1, b_1) = (pa, pb)$. If $(\chi | H_1, [\mu]) \mapsto$ $\mapsto (\sigma_1, \tau_1, [\mu])$, then $(\sigma_1, \tau_1, [\mu])$ lies in $\Phi_{n-1,s}$.

b) Set $\overline{H} = H/\Omega_1(H)$, and choose the basis $(\bar{a}, \bar{b}) = (a + \Omega_1(H), b + \Omega_1(H))$. If $(\chi | \overline{H}, [\mu]) \mapsto (\bar{\sigma}, \bar{\tau}, [\mu])$, then $(\bar{\sigma}, \bar{\tau}, [\mu])$ lies in $\Phi_{n-1,s}$.

PROOF. – *a*) By 2.4 *c*)

$$\begin{cases} i\pi_0\sigma_1 = i\sigma\pi_0 \equiv i\pi_0 p^s R_{n-1}\,, \\ i\pi_0\tau_1 = i\tau\pi_0 \equiv i\pi_0 p^s R_{n-1}\,. \end{cases}$$

Moreover

$$\begin{cases} j\pi_0\sigma_1 - i\pi_0\sigma_1 = (j\sigma - i\sigma)\,\pi_0 \equiv (j\pi_0 - i\pi_0)\,\mu^f\ \ p^{s+f}R_{n-1}\,, \\ j\pi_0\tau_1 - i\pi_0\tau_1 = (j\tau - i\tau)\,\pi_0 \equiv (j\pi_0 - i\pi_0)\,\mu^f\ \ p^{s+f}R_{n-1}\,, \end{cases}$$

that is $(\sigma_1, \tau_1, [\mu]) \in \Phi_{n-1,s}$.

b) Using 2.4 *d*) one again concludes. ∎

We remark that due to 4.1 the conclusions of 4.2 are independent of the choice of the basis (a, b) of H.

4.3. LEMMA. – *Let H be a homocyclic group of exponent p^n, $1 \leq s < n$ and $0 \lessdot X <$ $< H$. Let (a, b) be a basis of H such that $X = \langle p^{n-1}a \rangle$. Choose the basis $(\bar{a}, \bar{b}) = (a +$ $+ X, pb + X)$ for $\sqrt{X}/X$, and let $(\sigma, \tau, [\mu])$ be in $\Phi_{n,s}$. Let χ in $R_s(H)$ be such that*

$(\chi,[\mu])^\eta=(\sigma,\tau,[\mu])$, and put $\chi_X=\chi\,|\sqrt{X}/X$. If $\chi_X^\eta=(\sigma_X,\tau_X)$ in $PR(R_{n-1})\times$ $\times PR(pR_{n-1})$, then

b) $i\sigma_X\equiv i\ p^{s-1}R_{n-1}\quad i\tau_X\equiv i\ p^{s+1}R_{n-1}$,

$$c)\quad\begin{cases} j\equiv i\ p^fR_{n-1}\Rightarrow j\sigma_X-i\sigma_X\equiv(j-i)\,\mu^{f+1}\ \ p^{s+f}R_{n-1}\,, & 0\le f\le n-1-s\,,\\ j\equiv i\ p^fR_{n-1}\Rightarrow j\tau_X-i\tau_X\equiv(j-i)\mu^{f-1}\ \ p^{s+f}R_{n-1}\,, & 0\le f\le n-1-s\,.\end{cases}$$

PROOF. – Using Remark 2.1 and (7) one gets: $\langle\bar a+i\sigma_X\bar b\rangle=\langle\bar a+i\bar b\rangle^{\chi_X}=$ $=\langle a+i\gamma_0^{-1}b\rangle^\chi/X=\langle a+i\gamma_0^{-1}\sigma b\rangle/X=\langle a+i\gamma_0^{-1}\sigma\gamma_0 pb\rangle/X=\langle\bar a+i\sigma'\bar b\rangle$, hence $\sigma_X=$ $=\sigma'\in PR(R_{n-1})$ where $\sigma'=\gamma_0^{-1}\sigma\gamma_0$. Notice that $j\equiv i\ p^fR_{n-1}$ if and only if $j\gamma_0^{-1}\equiv i\gamma_0^{-1}p^{f+1}R_n$; but then: $i\sigma_X=(i\gamma_0^{-1})\,\sigma\gamma_0\equiv i\ \ p^{s-1}R_{n-1}$ and $j\sigma_X-i\sigma_X=$ $=(j\gamma_0^{-1}\sigma-i\gamma_0^{-1}\sigma)\,\gamma_0\equiv(j-i)\,\mu^{f+1}\ p^{s+f}R_{n-1}$.

We proceed now to determine τ_X. Having in mind Remark 2.1, we may consider τ as an element of $PR(R_n)$: $\langle i\pi_0\tau_X\bar a+\bar b\rangle=\langle i\pi_0\bar a+\bar b\rangle^{\chi_X}=\langle ia+pb\rangle^\chi+X/X=$ $=\langle i'(pa)+pb\rangle^\chi+X/X=\langle i'\tau p\pi_0\bar a+\bar b\rangle$, that is $i\pi_0\tau_X=i'\tau p\pi_0$. It follows that $i\pi_0\tau_X=i'\tau p\pi_0\equiv i'p\pi_0=i\pi_0p^{s+1}R_{n-1}$. Observe that $j=pj'\equiv i=pi'p^fR_n$ if and only if $j'\equiv i'p^{f-1}R_n$, $1\le f$; but then $j\pi_0\tau_X-i\pi_0\tau_X=j'\tau p\pi_0-i'\tau p\pi_0\equiv$ $\equiv(j'-i')\mu^{f-1}p\pi_0=(j\pi_0-i\pi_0)\,\mu^{f-1}\ p^{s+f}R_{n-1}$. ■

We are now in the position to prove the main result of this paragraph.

4.4. THEOREM. – *Let $(\sigma,\tau,[\mu])$ be an element of $\Phi_{n,s}$, $M=H\oplus C$ an (n,s)-group and (a,b) a basis of H. Define on $\mathcal{C}(M)$ the following map φ:*

$$\begin{cases}\langle p^k(a+ib)+c\rangle^\varphi=\langle p^k(a+i\sigma b)+\mu^{n-s-k}c\rangle, & i\in R_n,\quad 0\le k\le n-s\,,\\ \langle p^k(ia+b)+c\rangle^\varphi=\langle p^k(i\tau a+b)+\mu^{n-s-k}c\rangle, & i\in pR_n,\quad 0\le k\le n-s\,,\\ \varphi=1\,, & \text{on } \mathcal{C}(\Omega_s(M))\,,\end{cases}$$

Then there exists a unique $\tilde\varphi$ in $R(M)$ such that $\tilde\varphi\,|\,\mathcal{C}(M)=\varphi$.

PROOF. – The uniqueness is clear since an autoprojectivity is uniquely determined by its action on the cyclic subgroups. We remark that if $\tilde\varphi$ exists, then it lies in $R(M)$ and $\tilde\varphi\,|\,\Omega_{s+t}(M)/p^s\Omega_{s+t}=(1;\mu^t)$, being $i\sigma\equiv i\ p^sR_n$, $i\tau\equiv i\ p^sR_n$.

Let $\tilde\chi$ in $R_s(H)$ be such that $(\tilde\chi,[\mu])^\eta=(\sigma,\tau,[\mu])$. To prove the existence of $\tilde\varphi$ we shall use induction on $r=n-s$.

a) $r=1$.

Pick any minimal subgroup X_i of H. According to 4.1, without loss of generality we may assume $X_i=\langle p^{n-1}a\rangle$. By 4.3 *c)*, for $\chi_i=\tilde\chi\,|\sqrt{X_i}\cap H/X_i$ we have (notice that here $f=0$) $i\sigma_{X_i}=i\mu+0\sigma_{X_i}$, and since $\langle pb\rangle^{\chi_i}=\langle pb\rangle$, χ_i is induced by the automorphism $\begin{pmatrix}1 & z_i\\ 0 & \mu\end{pmatrix}$, where $z_i=0\sigma_{X_i}\equiv 0\,p^{s-1}R_s$, $\mu\in\mathcal{U}(R_n)$, $\mu\equiv 1\,p^{s-1}R_n$. It follows that $\varphi\,|\sqrt{X_i}/X_i$ is induced by the automorphism $\left(\begin{pmatrix}1 & z_i\\ 0 & \mu\end{pmatrix};\ \mu\right)$; hence by 1.3 φ defines an autoprojectivity

φ_i on $\sqrt{X_i}$. Since $\varphi|\mathcal{C}(M/p^s M)$, as it has been defined, is induced by $(1;\mu)$, one checks that conditions $(*)$ and $(**)$ of 1.4 are satisfied by $(\varphi_0, \ldots, \varphi_p, \varrho)$, where ϱ is the autoprojectivity of $M/p^s M$ induced by the automorphism $(1;\mu)$. But now the existence of $\tilde{\varphi}$ is assured by 1.4.

b) $r > 1$.

b_1) From *a)* we know that $\varphi|\mathcal{C}(\Omega_s(\sqrt{X_\nu}/X_\nu))$ is induced by the automorphism

$$\left(\begin{pmatrix} 1 & 0\sigma_{X_\nu} \\ 0 & \mu \end{pmatrix}; \mu\right).$$

Consider on $\sqrt{X_\nu}/X_\nu$ the automorphism

$$\alpha_\nu = \left(\begin{pmatrix} 1 & -0\sigma_{X_\nu}\mu^{-1} \\ 0 & \mu^{-1} \end{pmatrix}; 1\right);$$

then, by 2.5, we have

$$\text{(25)} \qquad \begin{cases} \sigma_\nu\colon\ i \mapsto i\mu^{-1} - 0\sigma_{X_\nu}\mu^{-1} & \text{for } i \in R_{n-1}, \\ \tau_\nu\colon\ i \mapsto i\mu(1 - i0\sigma_{X_\nu})^{-1} \equiv i \quad p^s R_{n-1} & \text{for } i \in pR_{n-1}, \end{cases}$$

being $0\sigma_{X_\nu} \equiv 0 \ \ p^{s-1}R_{n-1}$ by 4.3 *b)*, and $\mu \equiv 1 p^{s-1}R_{n-1}$.

But then for $i \in R_{n-1}$, $i\sigma_{X_\nu} = 0\sigma_{X_\nu} + i\mu + jp^s$ for some $j \in R_{n-1}$ by 4.3 *c)*. Hence $i\sigma_{X_\nu}\sigma_\nu = (0\sigma_{X_\nu} + i\mu + jp^s)\mu^{-1} - 0\sigma_{X_\nu}\mu^{-1} = i + jp^s\mu^{-1} \equiv i \ \ p^s R_{n-1}$. On the other hand, for $i \in pR_{n-1}$ by 4.3 *b)* and (25) we get $i\tau_{X_\nu}\tau_\nu \equiv i\tau_{X_\nu} \equiv i \ \ p^s R_{n-1}$. We have therefore proved

$$\text{(26)} \qquad \begin{cases} i\sigma_{X_\nu}\sigma_\nu \equiv i \quad p^s R_{n-1} & \text{for } i \in R_{n-1}, \\ i\tau_{X_\nu}\tau_\nu \equiv i \quad p^s R_{n-1} & \text{for } i \in pR_{n-1}. \end{cases}$$

Finally,, taking into account 2.5 *a)*, 4.3 *c)* and (26), we have for $j \equiv i \ \ p^f R_{n-1}$

$$\text{(27)} \qquad j\sigma_{X_\nu}\sigma_\nu - i\sigma_{X_\nu}\sigma_\nu = (j\sigma_{X_\nu} - i\sigma_{X_\nu})\mu^{-1} \equiv (j-i)\mu^f \quad p^{s+f}R_{n-1},$$

$j\tau_{X_\nu}\tau_\nu - i\tau_{X_\nu}\tau_\nu = (j\tau_{X_\nu} - i\tau_{X_\nu})\mu\varepsilon^{-1}$, where $\varepsilon = (1 - i0\sigma_{X_\nu})(1 - j0\sigma_{X_\nu}) \equiv 1 \ \ p^s R_{n-1}$. Hence $(j\tau_{X_\nu} - i\tau_{X_\nu})\mu\varepsilon^{-1} \equiv (j\tau_{X_\nu} - i\tau_{X_\nu})\mu \quad p^{s+f}R_{n-1}$; but $(j\tau_{X_\nu} - i\tau_{X_\nu})\mu \equiv$ $\equiv (j-i)\mu^f \ \ p^{s+f}R_{n-1}$. Therefore

$$\text{(28)} \qquad j\tau_{X_\nu}\tau_\nu - i\tau_{X_\nu}\tau_\nu \equiv (j-i)\mu^f \quad p^{s+f}R_{n-1}.$$

We conclude from (26), (27) and (28) that $(\sigma_{X_\nu}\sigma_\nu, \tau_{X_\nu}\tau_\nu, [\mu])$ lies in $\Phi_{n-1,s}$. Hence by induction $\varphi\tau_\nu|\mathcal{C}(\sqrt{X_\nu}/X_\nu)$ defines an autoprojectivity on $\sqrt{X_\nu}/X_\nu$: hence φ determines an autoprojectivity φ'_ν on $\sqrt{X_\nu}/X_\nu$.

b_2) By using 2.4 *c)*, a straightforward verification shows that $(\sigma_1, \tau_1, [\mu])$ lies in $\Phi_{n-1,s}$, where $\tilde{\chi}|\Omega_{n-1}(H) \leftrightarrow (\sigma_1, \tau_1)$. Hence by induction $\varphi|\mathcal{C}(\Omega_{n-1}(M))$ determines an autoprojectivity ψ. But then, by (1, 3), (ψ, φ'_ν) defines an autoprojectivity φ_ν on $\sqrt{X_\nu}$ induced by the map φ.

b_3) Using 2.4 d), a direct computation shows that $(\bar\sigma_1, \bar\tau_1, [\mu]) \in \Phi_{n-1, s}$, where $\tilde\chi \,|\, H/\Omega_1(H) \leftrightarrow (\bar\sigma_1, \bar\tau_1)$, so it defines an autoprojectivity on $M/\Omega_1(H)$ which modulo the automorphism $(1; \mu)$ induces the same map as φ on $\mathcal{C}(M/\Omega_1(H))$. Call ϱ the autoprojectivity of $M/\Omega_1(H)$ induced by φ. Applying 1.4 to $(\varphi_0, \ldots, \varphi_p, \varrho)$ one concludes that there is $\tilde\varphi$ in $P(M)$ such that $\tilde\varphi \,|\, \mathcal{C}(\sqrt{X_\nu}) = \varphi \,|\, \mathcal{C}(\sqrt{X_\nu})$. Hence $\tilde\varphi$ is the autoprojectivity of M determined by φ. ■

4.5. COROLLARY (representation theorem). – *Let $M = H \oplus C$ be an (n, s)-group and (a, b) a basis of H. Then the monomorphism $j\colon R(M) \to \Phi_{n,s}$ of* (23) *is an isomorphism.*

PROOF. – This follows from 4.4. ■

4.1. REMARK. – Define $\Phi^{\mathfrak{a}}_{n,s} = \{(\sigma, \tau, [\mu]) \in \Phi_{n,s} \,|\, 0\sigma = 0,\ 0\tau = 0\}$. Then $j \,|\, R_{\mathfrak{a}}(M)$ defines an isomorphism of $R_{\mathfrak{a}}(M)$ onto $\Phi^{\mathfrak{a}}_{n,s}$, while $j \,|\, R_{\mathfrak{a},u}(M)$ defines an isomorphism of $R_{\mathfrak{a},u}(M)$ onto $\Phi^{\mathfrak{a},u}_{n,s} = \{(\sigma, \tau, [\mu]) \in \Phi^{\mathfrak{a}}_{n,s} \,|\, 1\sigma = 1\}$.

5. – The general representation theorem.

Let $M = H \oplus C$ be an (n, m, s)-group. We begin with the case $m = s$. We treat this case separately since it is radically different from the case $m > s$.

5.1. LEMMA. – *Let M be an (n, s, s)-group, (a, b) a basis of H with $|a| = p^n$, and φ in $R_{\mathfrak{a}}(M)$. Then $\varphi \,|\, \Omega_{s+t}(M)/p^s\Omega_{s+t} = \left(\begin{pmatrix} 1 & 0 \\ 0 & \mu_t \end{pmatrix}; \mu_t\right)$, where $\mu_t \equiv 1\ p^{s-1}R_n$, $0 < t \leqslant n - s$.*

PROOF. – Set $\varphi_t = \varphi \,|\, \Omega_{s+t}(M)/p^s\Omega_{s+t} \in PA(\Omega_{s+t}(M)/p^s\Omega_{s+t})$. Since $\varphi \in R_{\mathfrak{a}}(M)$, it follows that φ_t is induced by an automorphism of the form $\left(\begin{pmatrix} 1 & 0 \\ 0 & \mu_t \end{pmatrix}; \mu_t\right)$. To conclude we only need to assume $s \neq 1$. Since $1 \neq \Omega_{s+t-1}(M)/p^s\Omega_{s+t} < \Omega_{s+t}(M)/p^s\Omega_{s+t}$ we get $\mu_t \equiv \mu_{t-1}\ p^{s-1}R_n$.

Now assume $t = 1$ and write $X = \langle p^{n-1}a\rangle$. Let γ in $\mathrm{Aut}(\Omega_{s+1}(M)/X)$ induce φ_1. Now $\langle p^{n-s}a + c + X\rangle = \langle p^{n-s}a + c + X\rangle^\gamma = \langle p^{n-s}a + \mu c + X\rangle$. Since $|p^{n-s}a + X| = p^{s-1}$, choosing an element c in C of order p^{s-1} one concludes that $\mu_1 \equiv 1\ p^{s-1}R_n$. ■

5.2. THEOREM. – *Let* $M = H \oplus C$ *be an* (n, s, s)*-group,* (a, b) *a basis of* H *with* $|a| = p^n$, μ *in* $\mathcal{U}(R_n)n$ *with* $\mu \equiv 1\, p^{s-1}R_n$, *and* φ *in* $R_{\mathfrak{a}}(\Omega_{n-1}(M))$. *Then there exists* $\tilde{\varphi}$ *in* $R_{\mathfrak{a}}(M)$ *such that* $\tilde{\varphi}\,|M/p^sM = \left(\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix};\ \mu\right)$.

PROOF. – Set $r = n - s$.

1) $r = 1$.

Here $\varphi = 1$ and for M/p^sM pick $\varphi' = \left(\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix};\ \mu\right)$. Then $\varphi'\,|\Omega_s(M)/p^sM = 1$, so by (1.3) (φ, φ') defines a $\tilde{\varphi}$ in $R_{\mathfrak{a}}(M)$ with the required properties.

2) $r > 1$.

We use induction on r. There exists $\beta = \left(\begin{pmatrix}1 & 0\\ 0 & \mu'\end{pmatrix};\ \mu'\right) \in \mathrm{Aut}\,(M/p^{n-1}M)$ such that $\psi = \varphi\beta\,|\Omega_{n-1}(M)/p^{n-1}M$ lies in $R_{\mathfrak{a}}(\Omega_{n-1}(M)/p^{n-1}M)$. Now by induction ψ extends to a $\tilde{\psi} \in R_{\mathfrak{a}}(M/p^{n-1}M)$ such that $\tilde{\psi}\,|M/p^sM = \left(\begin{pmatrix}1 & 0\\ 0 & \mu\mu'\end{pmatrix};\ \mu\mu'\right)$. But now $\tilde{\psi}\beta^{-1}\,|\Omega_{n-1}(M)/p^{n-1}M = \varphi$, hence by 1.3 $(\varphi, \tilde{\psi}\beta^{-1})$ defines an element $\tilde{\varphi}$ of $R_{\mathfrak{a}}(M)$ and one can check that it has the required properties. ■

5.3. EXTENSION LEMMA. – *Let* $s < m < n$ *be positive integers. If* $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$ *lies in* $\Phi_{n,m,s}$, *then there exists a* $(\tilde{\sigma}_0, \ldots, \tilde{\sigma}_{n-m-1}, \tilde{\tau}_{n-m-1}, [\mu])$ *in* $\Phi_{n,m+1,s}$ *such that for* $0 \le k \le n-m-1$, $\sigma_k = \tilde{\sigma}_k\,|p^{n-k-m}R_{n-k}$, $\tilde{\sigma}_{n-m-1}\pi_{n-m-1} = \pi_{n-m-1}\sigma_{n-m}$, $\tilde{\tau}_{n-m-1}\pi_{n-m-1} = \pi_{n-m-1}\tau_{n-m}$.

PROOF. – Let $i \in R_n$, and let $i = i_0 + \ldots + i_{n-1}p^{n-1}$ be its p-adic expansion as introduced in n. 2. Set $r = n - m$. We begin with

a_1) $r = 1$.

Define

$$i\tilde{\sigma} = \begin{cases} i\sigma_0 & \text{if } i \in pR_n\,,\\ (i_0 + \ldots + i_{n-s-1}p^{n-s-1})\,\pi_0\sigma_1\nu_0 + (i_{n-s}p^{n-s} + \ldots + i_{n-1}p^{n-1})\,\mu^{n-s} & \\ & \text{if } i \in \mathcal{U}(R_n)\,,\end{cases}$$

$$i\tilde{\tau} = (i_1p + \ldots + i_{n-s-1}p^{n-s-1})\,\pi_0\tau_1\nu_0 + (i_{n-s}p^{n-s} + \ldots + i_{n-1}p^{n-1})\,\mu^{n-s} \qquad \text{if } i \in pR_n\,.$$

Clearly $(\tilde{\sigma}, \tilde{\tau})$ lies in $\mathrm{Sym}\,R_n \times \mathrm{Sym}\,pR_n$, and we have

$$i\tilde{\sigma}\pi_0 = i\sigma_0\pi_0 = i\pi_0\sigma_1 \quad \text{for } i \in pR_n\,, \tag{29}$$

by 3.6 a).

Take now $i \in \mathcal{U}(R_n)$; then for $j = i_0 + \ldots + i_{n-s-1}\, p^{n-s-1}$ we have $j \equiv i\ p^{n-s} R_n$, so that

$$(30)\qquad j\pi_0\sigma_1 - i\pi_0\sigma_1 = (j\pi_0 - i\pi_0)\,\mu^{n-s} = -(i_{n-s}p^{n-s} + \ldots + i_{n-1}p^{n-1})\,\pi_0\mu^{n-s}.$$

It follows from (30) that

$$(31)\qquad i\tilde{\sigma}\pi_0 = j\pi_0\sigma_1 + i\pi_0\sigma_1 - j\pi_0\sigma_1 = i\pi_0\sigma_1 .$$

In particular for $i \in R_n$ we have $i\tilde{\sigma} \equiv i\ p^s R_n$; moreover by (29) and (31) we get

$$(32)\qquad \tilde{\sigma}\pi_0 = \pi_0\sigma_1 .$$

Since $i \equiv i\pi_0\nu_0\ p^{n-1}R_n$, using (32) we obtain for j, i in R_n

$$(33)\qquad j\tilde{\sigma} - i\tilde{\sigma} \equiv j\pi_0\sigma_1\nu_0 - i\pi_0\sigma_1\nu_0\ p^{n-1}R_n .$$

(33) with $(17a)$ shows that

$$j\tilde{\sigma} - i\tilde{\sigma} \equiv (j-i)\,\mu^f\, p^{s+f}R_n \qquad \text{for } j \equiv i\ p^f R_n,\ 0 \leqslant f < n-s .$$

Morever, for $f = n - s$ by the definition we get $j\tilde{\sigma} - i\tilde{\sigma} = (j-i)\mu^{n-s}$. Since $0\,\tilde{\sigma} = 0\sigma_0 \equiv 0p^s R_n$, by Remark 3.3 we conclude that $\tilde{\sigma}$ lies in $PR(R_n)$. With a similar procedure one can deal with $\tilde{\tau}$.

We come now to the case

$a_2)$ $r > 1$.

We observe that $(\sigma_1, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$ lies in $\Phi_{n-1,m,s}$. By induction there exists a $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-(m+1)}, \tilde{\tau}_{n-(m+1)}, [\mu])$ in $\Phi_{n-1,m+1,s}$ such that for $1 \leqslant k \leqslant n-(m+1)$, $\tilde{\sigma}_k \in PR(p^{n-k-m-1}R_{n-k})$ with $\sigma_k = \tilde{\sigma}_k\,|p^{n-k-m}R_{n-k}$, $\tilde{\sigma}_{n-m-1}\pi_{n-m-1} = \pi_{n-m-1}\sigma_{n-m}$, $\tilde{\tau}_{n-m-1}\pi_{n-m-1} = \pi_{n-m-1}\tau_{n-m}$.

Since $p^{n-m-1}R_n = p^{n-m}R_n \,\dot{\cup}\, p^{n-m-1}\mathcal{U}(R_n)$, similarly to $a_1)$ we introduce $\tilde{\sigma}_0$ on $p^{n-m-1}R_n$, defining

$$i\tilde{\sigma}_0 = \begin{cases} i\sigma_0 & \text{if } i \in p^{n-m}R_n ,\\ (i_{n-m-1}p^{n-m-1} + \ldots + i_{n-s-1}p^{n-s-1})\,\pi_0\sigma_1\nu_0 + & \\ \qquad + (i_{n-s}p^{n-s} + \ldots + i_{n-1}p^{n-1})\,\mu^{n-s} & \text{if } i \in p^{n-m-1}\mathcal{U}(R)\,n . \end{cases}$$

A similar routine checking as in $a_1)$ leads us to recognize that $(\tilde{\sigma}_0, \tilde{\sigma}_1 \ldots, \tilde{\sigma}_{n-m-1}, \tilde{\tau}_{n-m-1}, [\mu])$ lies in $\Phi_{n,m+1,s}$, which concludes the proof. ■

5.4. THEOREM. – *Let $s < m < n$ be positive integers, $\varrho_k\colon R_n \to R_{n-k}$ the canonical epimorphism and $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$ in $\Phi_{n,m,s}$. Then there exists a $(\tilde{\sigma}, \tilde{\tau}, [\mu])$ in $\Phi_{n,s}$ such that for $0 \leqslant k \leqslant n-m$, $\tilde{\sigma}\varrho_k = \varrho_k\sigma_k$, $\tilde{\tau}\varrho_{n-m} = \varrho_{n-m}\tau_{n-m}$ hold on the obvious domains.*

PROOF. – Set $r = n - m$. If $r = 1$, the conclusion follows from 5.3. We assume now $r > 1$. By 5.3 there exists a $(\sigma'_0, \ldots, \sigma'_{n-m-1}, \tau'_{n-m-1}, [\mu])$ in $\Phi_{n,m+1,s}$ such that $\sigma'_k \mid p^{n-k-m} R_{n-k} = \sigma_k$, $\sigma'_{n-m-1} \pi_{n-m-1} = \pi_{n-m-1} \sigma_{n-m}$, $\tau'_{n-m-1} \pi_{n-m-1} = \pi_{n-m-1} \tau_{n-m}$. By induction, there exists a $(\tilde{\sigma}, \tilde{\tau}, [\mu])$ in $\Phi_{n,s}$ such that for $0 \leq k \leq n-m-1$ we have $\tilde{\sigma} \varrho_k = \varrho_k \sigma'_k$ on $p^{n-k-m-1} R_n$, $\tilde{\tau} \varrho_{n-m-1} = \varrho_{n-m-1} \tau'_{n-m-1}$ on pR_n.

Since $\sigma'_k \mid p^{n-k-m} R_{n-k} = \sigma_k$ and $\tau'_{n-m-1} \pi_{n-m-1} = \pi_{n-m-1} \tau_{n-m}$, the conclusion follows. ∎

5.5. THEOREM (the general representation theorem). – *Let $M = H \oplus C$ be an (n, m, s)-group, with $s < m$, (a, b) a basis of H with $p^n = |a|$. Then the monomorphism j of $R(M)$ into $\Phi_{n,m,s}$ as given in* (23) *is an isomorphism.*

PROOF. – Given a $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$ in $\Phi_{n,m,s}$ by 5.4 there exists a $(\tilde{\sigma}, \tilde{\tau}, [\mu])$ in $\Phi_{n,s}$ such that $\tilde{\sigma} \varrho_k = \varrho_k \sigma_k$ $0 \leq k \leq n-m$, $\tilde{\tau} \varrho_{n-m} = \varrho_{n-m} \tau_{n-m}$. Let $\widetilde{M} = \widetilde{H} \oplus C$ be an (n, s)-group with a basis $(\tilde{a}, \tilde{b})$ such that $a = \tilde{a}$, $b = p^{n-m} \tilde{b}$. By 4.5 we know that there exists a $\tilde{\varphi} \in R(\widetilde{M})$ such that $\tilde{\varphi}^j = (\tilde{\sigma}, \tilde{\tau}, [\mu])$; moreover by 3.8 $M^{\tilde{\varphi}} = M$ if and only if $(p^{n-m} R_n)^{\tilde{\sigma}} = p^{n-m} R_n$ which actually is the case, since $\tilde{\sigma} \mid p^{n-m} R_n = \sigma_0$. Finally, using 3.8 again, one concludes that $\tilde{\varphi} \mid M \in R(M)$ and $(\tilde{\varphi} \mid M)^j = (\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu])$.

5.6. COROLLARY. – *Let $M = H \oplus C$ be an (n, m, s)-group with $s < m$, (a, b) a basis of H with $p^n = |a|$, $u = \langle p^{n-m} a + b \rangle$ a unit point and $j \colon R(M) \to \Phi_{n,m,s}$ the isomorphism of* 5.5. *Then $(\sigma_0, \ldots, \sigma_{n-m}, \tau_{n-m}, [\mu]) \in R^j_{a,u}(M)$ if and only if*

$$0\sigma_0 = 0\,, \qquad 1\sigma_{n-m} = 1 \quad \textit{and} \quad 0\tau_{n-m} = 0\,.$$

PROOF. – In fact $\langle a \rangle^\varphi = \langle a \rangle$ if and only if $0\sigma_0 = 0$, $\langle p^{n-m} a + b \rangle^\varphi = \langle p^{n-m} a + b \rangle$ if and only if $1\sigma_{n-m} = 1$ and $\langle b \rangle^\varphi = \langle b \rangle$ if and only if $0\tau_{n-m} = 0$. ∎

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