Annali di Matematica pura ed applicata
(IV), Vol. CLXXIII (1997), pp. 373-389

An Equivalence Theorem on Properties A, B for Third Order Differential Equations (*).

M. CECCHI - Z. DOŠLÁ - M. MARINI

Abstract. – *Differential equations are often classified according to oscillatory/nonoscillatory properties of their solutions as equations having* property A *or* property B. *The aim of the paper is to state an equivalence theorem between property A and property B for third order differential equations. Some applications, to linear as well as to nonlinear equations, are given too. Particularly, we give integral criteria ensuring property A or B for nonlinear equations. Our only assumption on nonlinearity is its superlinearity in neighbourhood of infinity, hence our results apply also to Emden-Fowler type equations.*

1. – Introduction.

Consider the linear differential equations

$$(E\pm) \qquad x''' \pm q(t)x = 0$$

where q is a positive continuous function for $t \geqslant 0$.

It is well known that there is an analogy between the space of solutions of $(E+)$ and $(E-)$. For instance, by using the notion of equation of class I and II introduced by HANAN in [14], it is easy to show that $(E+)$ is nonoscillatory if and only if $(E-)$ is nonoscillatory. Another result in this direction is given in [24] (see also [22]) where it is proved that if there exists $\lambda > 0$ such that

$$\int^{\infty} t^{2-\lambda} q(t)\, dt = \infty ,$$

then $(E\pm)$ have both oscillatory and nonoscillatory solutions. In addition, every nonoscillatory solution x of $(E+)$ tends to zero as $t \to \infty$ and satisfies, for all large t,

(*) Entrata in Redazione il 18 novembre 1996.

Indirizzo degli AA.: M. CECCHI, M. MARINI, Depart. of Electr. Eng., University of Florence, Via S. Marta 3, 50139 Firenze, Italy; Z. DOŠLÁ, Depart. of Mathematics, Masaryk University, Janáčkovo nám. 2a, 66295 Brno, Czech Republic.

The second author wishes to thank C.N.R. of Italy and Grant Agency of Czech Republic (grant 201/96/0410) which made this research possible.

either the inequalities $x(t) > 0$, $x'(t) < 0$, $x(t) > 0$ or the inequalities $x(t) < 0$, $x'(t) > 0$, $x''(t) < 0$, while every nonoscillatory solution of $(E\,-)$ tends to infinity as $t \to \infty$ and satisfies, for all large t, either $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ or $x(t) < 0$, $x'(t) < 0$, $x''(t) < 0$.

Some authors referred to such property of $(E\,+)$ as *property A* and of $(E\,-)$ as *property B*. Both properties have been extended in several directions to linear and nonlinear equations of n-th order. Among the wide literature on this field we refer to [6-13, 16-20, 22, 23] and to the references contained therein. In most cases, these properties have been studied or proved separately. However, CHANTURIA [8] (see also [16, Th. 1.3] showed a certain analogy between both properties, namely, $(E\,+)$ has property A if and only if $(E\,-)$ has property B.

The aim of this paper is to extend this equivalence theorem to the linear equation of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0 \tag{L}$$

and to apply the obtained results to the nonlinear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(t)) = 0\,, \tag{N}$$

where

$$(H1) \qquad r, p, q \in C^0([0, \infty), \mathbf{R}), \qquad r(t) > 0\,, \qquad p(t) > 0\,, \qquad q(t) > 0 \text{ on } [0, \infty)$$

$$(H2) \qquad f \in C^0(\mathbf{R}), \quad f(0) = 0\,, \quad f(u)u > 0 \text{ for } u \in \mathbf{R}\backslash\{0\}, \quad \liminf_{|u| \to \infty} \frac{f(u)}{u} > 0\,.$$

When the functions p and/or r do not have a continuous first and/or second derivative, then (L) and (N) may be interpreted as a first order differential system for the vector $(x^{[0]}, x^{[1]}, x^{[2]})$ given by

$$x^{[0]} = x\,, \qquad x^{[1]} = \frac{1}{r}x'\,, \qquad x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}(x^{[1]})'\,, \tag{1}$$

where x is a solution of (L) or (N). Similarly, we denote with $x^{[3]} = (1/q)(x^{[2]})'$. The functions $x^{[i]}$ are called the quasiderivatives of x.

The plan of the paper is the following. In the first section we will prove an equivalence theorem on property A for (L) and on property B for the *adjoint equation*

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}u'(t)\right)'\right)' - q(t)u(t) = 0\,. \tag{L^a}$$

In the literature there are many papers devoted to property A or B independently. Our equivalence theorem enables us to apply criteria on property A to give criteria on property B and vice versa. In addition the oscillation of (L) and (L^a) is studied and new integral criteria in order for (L) $[(L^a)]$ to have property A [B] are stated.

In section 2 we derive some applications to the nonlinear case. In particular, by using a linearization device, we obtain a comparison theorem for properties A and B which is more suitable for application than others existing in the literature. Sufficient conditions ensuring properties A and B are also given. Such results are presented as integral criteria involving only the functions p, r, q.

We point out that, besides $(H2)$, other conditions on the forcing term f such as, for instance, monotonicity, superlinearity or sublinearity in a neighbourhood of zero or in the whole $\boldsymbol{R}$ are not assumed.

The obtained results will be compared with the ones existing in the literature in the framework of the paper.

Throughout the paper the following notation will be used

$$I(u_i) = \int_0^\infty u_i(t)\,dt\,, \qquad I(u_i, u_j) = \int_0^\infty u_i(t)\int_0^t u_j(s)\,ds\,dt\,, \qquad i, j = 1, 2\,,$$

$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t)\int_0^t u_j(s)\int_0^s u_k(\tau)\,d\tau\,ds\,dt\,, \qquad i, j, k = 1, 2, 3\,,$$

where u_i, $i = 1, 2, 3$, are continuous positive functions on $[0, \infty)$.

Part I. Linear equation.

I.1. *An equivalence theorem.*

As usual, we say that x is an *oscillatory solution* of (L) if it has infinitely arbitrarily large zeros. Otherwise this solution is said to be *nonoscillatory*. Equation (L) is said to be *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if all its solutions are nonoscillatory.

Equation (L) is said to have *property A* if any solution x of this equation is either oscillatory or satisfies

$$(2) \qquad |x^{[i]}(t)| \downarrow 0 \quad \text{as } t \to \infty\,, \qquad i = 0, 1, 2\,.$$

Equation (L^a) is said to have *property B* if any solution u of this equation is either oscillatory or satisfies

$$(3) \qquad |u^{[i]}(t)| \uparrow \infty \quad \text{as} \quad t \to \infty\,, \qquad i = 0, 1, 2,$$

where the notations $y(t) \downarrow 0$ and $y(t) \uparrow \infty$ mean that y monotonically decreases to zero as $t \to \infty$ or monotonically increases to infinity as $t \to \infty$, respectively.

As we already mentioned, there are some interesting relationships between the binomial equations $(E+)$ and $(E-)$, which involve the oscillation and properties A and B. Indeed it is known (see, e.g., [16]) that

(i) $(E+)$ is oscillatory if and only if $(E-)$ is oscillatory.

The same situation does not occur for the equation (L) and

$$(L-) \qquad \left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,x'(t)\right)'\right)' - q(t)\,x(t) = 0\,,$$

even if $\int^{\infty} p = \int^{\infty} r = \infty$, as the following example shows.

EXAMPLE 1. – Let $\varepsilon \in (0, 1)$ and $T > 1$. Consider the equation

$$(l\pm) \qquad \left(t\ln t\left(\frac{x'(t)}{\ln t}\right)'\right)' \pm \frac{x(t)}{t^2(\ln t)^{1+\varepsilon}} = 0\,, \qquad t \in [T, \infty)\,.$$

By [4, Theorem 8], $(l+)$ is oscillatory and, by [4, Theorem 5], $(l-)$ is nonoscillatory. Note that here $\int^{\infty} r = \int^{\infty} p = \infty$.

An analogous statement to (i) holds for (L) and the adjoint equation (L^{a}). Indeed it is known that (L) is oscillatory if and only if (L^{a}) is oscillatory (see, e.g., [14]). As regards the equivalence between properties A and B, it is stated by the following theorem which is our main result in this section.

THEOREM 1. – *(L) has property A if and only if (L^{a}) has property B.*

To prove this theorem, the following auxiliary results and notations will be needed.

Equation (L) is closely related to the following two linear equations obtained by means of an ordered cyclic permutation of functions p, r, q

$$(L^{c}) \qquad \left(\frac{1}{q(t)}\left(\frac{1}{p(t)}\,y'(t)\right)'\right)' + r(t)\,y(t) = 0$$

and

$$(L^{cc}) \qquad \left(\frac{1}{r(t)}\left(\frac{1}{q(t)}\,z'(t)\right)'\right)' + p(t)\,z(t) = 0\,,$$

as the following statements show.

PROPOSITION 1. – *If x is a solution of (L), then $x^{[1]}$ is a solution of (L^{c}) and $x^{[2]}$ is a solution of (L^{cc}).*

Similarly, if u is a solution of (L^{a}), then $u^{[1]}$ is a solution of

$$(L^{ac}) \qquad \left(\frac{1}{q(t)}\left(\frac{1}{r(t)}\,v'(t)\right)'\right)' - p(t)\,v(t) = 0$$

and $u^{[2]}$ is a solution of

$$(L^{acc}) \qquad \left(\frac{1}{p(t)}\left(\frac{1}{q(t)}\,w'(t)\right)'\right)' - r(t)\,w(t) = 0\,.$$

PROOF. – It follows by direct computation. ∎

Denote by $\mathcal{N}(L)$ and by $\mathcal{N}(L^{a})$ the set of all nontrivial nonoscillatory solutions of (L) and (L^{a}), respectively.

In [4] we have proved that (L) does not have weakly oscillatory solutions, i.e., solutions such that x is nonoscillatory and x' is oscillatory. In addition, if x is nonoscillatory, then $x^{[1]}$ is nonoscillatory and $x^{[2]}$ is nonoscillatory too. In view of this fact the set $\mathcal{N}(L)$ can be divided into the following four classes:

$$\mathcal{N}_0 = \{x \in \mathcal{N}(L),\ \exists T_x\colon\ x(t)\,x^{[1]}(t) < 0,\ x(t)\,x^{[2]}(t) > 0 \text{ for } t \geq T_x\},$$

$$\mathcal{N}_1 = \{x \in \mathcal{N}(L),\ \exists T_x\colon\ x(t)\,x^{[1]}(t) > 0,\ x(t)\,x^{[2]}(t) < 0 \text{ for } t \geq T_x\},$$

$$\mathcal{N}_2 = \{x \in \mathcal{N}(L),\ \exists T_x\colon\ x(t)\,x^{[1]}(t) > 0,\ x(t)\,x^{[2]}(t) > 0 \text{ for } t \geq T_x\},$$

$$\mathcal{N}_3 = \{x \in \mathcal{N}(L),\ \exists T_x\colon\ x(t)\,x^{[1]}(t) < 0,\ x(t)\,x^{[2]}(t) < 0 \text{ for } t \geq T_x\}$$

and the set $\mathcal{N}(L^{a})$ into the following four classes

$$\mathcal{M}_0 = \{u \in \mathcal{N}(L^{a}),\ \exists T_u\colon\ u(t)\,u^{[1]}(t) < 0,\ u(t)\,u^{[2]}(t) > 0 \text{ for } t \geq T_u\},$$

$$\mathcal{M}_1 = \{u \in \mathcal{N}(L^{a}),\ \exists T_u\colon\ u(t)\,u^{[1]}(t) > 0,\ u(t)\,u^{[2]}(t) < 0 \text{ for } t \geq T_u\},$$

$$\mathcal{M}_2 = \{u \in \mathcal{N}(L^{a}),\ \exists T_u\colon\ u(t)\,u^{[1]}(t) < 0,\ u(t)\,u^{[2]}(t) < 0 \text{ for } t \geq T_u\},$$

$$\mathcal{M}_3 = \{u \in \mathcal{N}(L^{a}),\ \exists T_u\colon\ u(t)\,u^{[1]}(t) > 0,\ u(t)\,u^{[2]}(t) > 0 \text{ for } t \geq T_u\}.$$

Obviously, if $x \in \mathcal{N}(L)$ satisfies (2), then x belongs to the class $\mathcal{N}_0$. Similarly, if $u \in \mathcal{N}(L^{a})$ satisfies (3), then u belongs to the class $\mathcal{M}_3$. This means that if (L) has property A, then $\mathcal{N}(L) = \mathcal{N}_0$ and if (L^{a}) has property B then $\mathcal{N}(L^{a}) = \mathcal{M}_3$.

In addition, if $x \in \mathcal{N}_0$, then the quasiderivatives $x^{[i]}$, $i = 0, 1, 2, 3$, have eventually an alternate sign and in the literature they are called *Kneser solutions.* If $u \in \mathcal{M}_3$, then the quasiderivatives $u^{[i]}$, $i = 0, 1, 2, 3$, have eventually the same sign and are called *strongly monotone solutions*. Their existence is ensured by the following result.

PROPOSITION 2. – *(L) has always a Kneser solution and (L^a) has always a strongly monotone solution.*

PROOF. – The result for (L) follows from results of Hartman/Wintner (see [15, p. 506]). The result for (L^a) follows from [17, Lemma 2]. (In this paper conditions $\int^\infty p = \int^\infty r = \infty$ are assumed, but in the proof of Lemma 2 such assumptions are not needed.) ■

REMARK 1. – It is easy to show that if $x \in \mathcal{N}_0$ then x verifies

$$x(t)x^{[1]}(t) < 0, \qquad x(t)x^{[2]}(t) > 0$$

not only eventually, but also for all $t \geqslant 0$. Indeed, let $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for $t \geqslant T$ and suppose that there exists $t_1 < T$ such that $x'(t_1) = 0$, $x(t) > 0$ for $t \in I = (t_1, T)$. Then $(x^{[2]}(t))' < 0$ on I, i.e., $x^{[2]}$ decreases on I. Because $x^{[2]}(T) > 0$, we have $x^{[2]}(t) > 0$ on I, which implies that $x^{[1]}$ is increasing on I. Because $x^{[1]}(T) < 0$, we obtain $x^{[1]}(t_1) = (1/r(t_1))x'(t_1) < 0$, which is a contradiction. Then $x(t)x^{[1]}(t) < 0$ for all t. By a similar argument and using Proposition 1 we obtain also $x(t)x^{[2]}(t) > 0$ for all t.

LEMMA 1. – *Let x, y be two linearly independent solutions of (L) $[(L^a)]$. Then*

$$u = x^{[1]}y - xy^{[1]} \tag{4}$$

is a solution of (L^a) $[(L)]$ and its quasiderivatives satisfy

$$u^{[1]} = x^{[2]}y - xy^{[2]}, \qquad u^{[2]} = x^{[2]}y^{[1]} - x^{[1]}y^{[2]}.$$

PROOF. – Let x, y be two linearly independent solutions of (L). By straightforward calculation we get that u is a solution of (L^a) and that

$$u^{[1]} = \frac{1}{p}u' = \frac{1}{p}(x^{[1]}y - xy^{[1]})' = x^{[2]}y - xy^{[2]}.$$

The remainder part of the statement follows by using a similar argument. ■

LEMMA 2. – *The following conditions are equivalent:*

(i) $\mathcal{N}(L) = \mathcal{N}_0$,

(ii) $\mathcal{N}(L^a) = \mathcal{M}_3$.

PROOF. – (ii) $\Rightarrow$ (i): By Proposition 2, $\mathcal{N}_0 \neq \emptyset$. Assume there exists $j \in \{1, 2, 3\}$ such that $\mathcal{N}_j \neq \emptyset$. Let $x \in \mathcal{N}_0$ and $y \in \mathcal{N}_j$. Wtihout loss of generality we may suppose $x(t) > 0$, $y(t) > 0$ for large t. Then the function u defined by (4) is a solution of (L^a) and satisfies, for large t,

$$u(t) < 0, \quad u^{[1]}(t) > 0 \qquad (\text{if } j = 1),$$

$$u(t) < 0, \quad u^{[2]}(t) > 0 \qquad (\text{if } j = 2),$$

$$u^{[1]}(t) > 0, \quad u^{[2]}(t) < 0 \quad (\text{if } j = 3).$$

This contradicts the fact that all nonoscillatory solutions of (L^a) are strongly monotonic solutions.

The claim (i) $\Rightarrow$ (ii) can be proved by using a similar argument as given in the first part. ∎

The following two statements generalize for $n = 3$ a result in [23, Theorem 7] which requires $I(r) = \infty$ and $I(p) = \infty$.

LEMMA 3. – *If there exists* $x \in \mathcal{N}_0$ *such that* $\lim_{t\to\infty} x^{[i]} = 0$, $i = 0, 1, 2$ *then*

$$I(q, p, r) = \infty\,, \qquad I(r, q, p) = \infty \qquad \text{and} \qquad I(p, r, q) = \infty\,. \tag{5}$$

PROOF. – Without loss of generality suppose that $I(q, p, r) < \infty$. Let x be an eventually positive solution of (L) which belongs to the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]} = 0$, $i = 0, 1, 2$. Integrating (L) three times in (t, ∞), t large, we obtain

$$x(t) = \int_t^\infty r(s) \int_s^\infty p(\sigma) \int_\sigma^\infty q(\tau)\, x(\tau)\, d\tau\, d\sigma\, ds \leq x(t) \int_t^\infty r(s) \int_s^\infty p(\sigma) \int_\sigma^\infty q(\tau)\, d\tau\, d\sigma\, ds\,.$$

Thus

$$1 \leq \int_t^\infty r(s) \int_s^\infty p(\sigma) \int_\sigma^\infty q(\tau)\, d\tau\, d\sigma\, ds\,.$$

Then, by interchanging the order of integration, we get a contradiction.

If $I(r, q, p) < \infty$ [$I(p, r, q) < \infty$], we use Proposition 1 and then considering (L^c) [(L^{cc})] we proceed by the same way. ∎

LEMMA 4. – i) *If there exists* $x \in \mathcal{N}_0$ *such that* $\lim_{t\to\infty} x \neq 0$, *then* $I(q, p, r) < \infty$.

ii) *If there exists* $x \in \mathcal{N}_0$ *such that* $\lim_{t\to\infty} x^{[1]} \neq 0$, *then* $I(r, q, p) < \infty$.

iii) *If there exists* $x \in \mathcal{N}_0$ *such that* $\lim_{t\to\infty} x^{[2]} \neq 0$, *then* $I(p, r, q) < \infty$.

PROOF. – Claim i). Let x be an eventually positive solution of (L) in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = x(\infty) > 0$. Hence there exists $T \geq 0$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Three cases are possible:

$$\text{I) } I(p) < \infty, \quad I(r) < \infty \qquad \text{II) } I(p) = \infty, \quad I(r) < \infty \qquad \text{III) } I(r) = \infty.$$

CASE I). Integrating (L) in (t, ∞), $t > T$, we obtain

$$x^{[2]}(t) = x^{[2]}(\infty) + \int_t^\infty q(s)x(s)\,ds \geq x^{[2]}(\infty) + x(\infty)\int_t^\infty q(s)\,ds.$$

Then $I(q) < \infty$ and so $I(q, p, r) < \infty$.

CASE II). First let us show that $I(p) = \infty$ implies $x^{[2]}(\infty) = 0$. Because $(x^{[2]}(t))' = -q(t)x(t) < 0$ for $t > T$, $x^{[2]}$ is an eventually positive decreasing function. Assume $x^{[2]}(\infty) > 0$. Then $x^{[2]}(t) > x^{[2]}(\infty)$ for $t > T$. By integrating we get

$$x^{[1]}(t) > x^{[1]}(T) + x^{[2]}(\infty)\int_T^t p(s)\,ds,$$

which gives a contradiction as $t \to \infty$, because $x^{[1]}$ is negative. Thus $x^{[2]}(\infty) = 0$. Integrating (L) twice in (t, ∞), $t > T$, we obtain

$$x^{[1]}(t) = x^{[1]}(\infty) - \int_t^\infty p(s)\int_s^\infty q(\sigma)x(\sigma)\,d\sigma\,ds \leq$$

$$\leq x^{[1]}(\infty) - x(\infty)\int_t^\infty p(s)\int_s^\infty q(\sigma)\,d\sigma\,ds = x^{[1]}(\infty) - x(\infty)\int_t^\infty q(\sigma)\int_t^\sigma p(s)\,ds\,d\sigma.$$

Then $I(q, p) < \infty$ and so $I(q, p, r) < \infty$.

CASE III). Using a similar argument as given in the case II), we get that $I(r) = \infty$ implies $x^{[1]}(\infty) = 0$. Integrating (L) three times in (t, ∞), $t > T$, we obtain

$$x(t) \geq x(\infty) + x^{[2]}(\infty)\int_t^\infty r(s)\int_s^\infty p(\sigma)\,d\sigma\,ds + x(\infty)\int_t^\infty r(s)\int_s^\infty p(\sigma)\int_\sigma^\infty q(\tau)\,d\tau\,d\sigma\,ds$$

and then, by interchanging order of integration, we get $I(q, p, r) < \infty$.

Finally claim ii) [iii)] may be proved by considering (L^c) [(L^{cc})] instead of (L) and by using a similar argument. ■

PROOF OF THEOREM 1. – First we prove that

(a) *if* (L) *has property* A, *then* (L^a) *has property* B.

Since (L) has property A, $\mathcal{N}(L) = \mathcal{N}_0$ and, in view of Lemma 2, $\mathcal{N}(L^{a}) = \mathcal{M}_3$. By Proposition 2, $\mathcal{M}_3 \neq \emptyset$. Let $u \in \mathcal{M}_3$, e.g. there exists $T \geq 0$ such that $u^{[i]}(t) > 0$ for $t > T$ and $i = 0, 1, 2$. Suppose that (L^{a}) does not have property B, i.e., there exists $i \in \{0, 1, 2\}$ such that $u^{[i]}$ is bounded.

First, let $u^{[2]}$ be bounded. By Proposition 1, $w = u^{[2]}$ is a solution of (L^{acc}), i.e.,

$$(L^{acc}) \qquad \left(\frac{1}{p(t)}\left(\frac{1}{q(t)}w'(t)\right)'\right)' - r(t)\,w(t) = 0\,,$$

and $w^{[1]} = u$, $w^{[2]} = u^{[1]}$. Then $w^{[i]}(t) > 0$ for $t \geq T$, $i = 0, 1, 2$ and w is bounded. Thus there exist $k_i > 0$, $i = 1, 2$, such that

$$(6) \qquad 0 < k_1 \leq w(t) \leq k_2 \qquad \text{for } t \geq T\,.$$

Integrating (L^{acc}) three times on $[T, t]$ we have

$$(7) \qquad w(t) = w(T) + w^{[1]}(T)\int_T^t q(s)\,ds + w^{[2]}(T)\int_T^t q(s)\int_T^s p(\sigma)\,d\sigma\,ds + \int_T^t q(s)\int_T^s p(\sigma)\int_T^\sigma r(\tau)\,w(\tau)\,d\tau\,d\sigma\,ds\,.$$

Because $u^{[i]}(T) > 0$, $i = 1, 2$, from (6) and (7) we obtain

$$(8) \qquad k_1\int_T^t q(s)\int_T^s p(\sigma)\int_T^\sigma r(\tau)\,d\tau\,d\sigma\,ds \leq w(t) \leq k_2\,.$$

From Proposition 2, (L) has a Kneser solution. Since (L) has property A, any Kneser solution satisfies (2). Thus, by Lemma 3, the integral on the left side of (8) is divergent, which gives a contradiction.

If $u^{[1]}$ $[u]$ is bounded, in view of Proposition 1, we consider (L^{ac}) $[(L^{a})]$ instead of (L^{acc}). Using a similar argument as above and Lemma 3 we get again a contradiction. Hence claim (a) is proved.

In order to complete the proof let us show that

(b) *if* (L^{a}) *has property B, then* (L) *has property A.*

Since (L^{a}) has property B then $\mathcal{N}(L^{a}) = \mathcal{M}_3$. From Lemma 2 and Proposition 2 $\mathcal{N}(L) = \mathcal{N}_0 \neq \emptyset$. Assume that there exists a Kneser solution x such that for some $i \in \{0, 1, 2\}$, $\lim_{t\to\infty} x^{[i]}(t) = c \neq 0$.

First suppose that $\lim_{t\to\infty} x(t) = c \neq 0$. Then by Lemma 4 we have

$$\int_0^\infty q(t) \int_0^t p(s) \int_0^s r(\tau)\, d\tau\, ds\, dt < \infty\,, \tag{9}$$

so also $I(q) < \infty$ and $I(q, p) < \infty$.

Let w be a nonoscillatory solution of (L^{ace}). Without loss of generality we may assume that w is eventually positive. In view of Proposition 1, there exists a solution u of (L^a) such that $w = u^{[2]}$. Since (L^a) has property B, u satisfies (3): hence w tends to infinity as $t \to \infty$ and there exists $T \geq 0$ such that $w^{[i]}(t) > 0$ for all $t \geq T$. In view of (9) we can take T such that

$$\int_T^\infty q(t) \int_T^t p(s) \int_T^s r(\sigma)\, d\sigma\, ds\, dt < 1\,. \tag{10}$$

For brevity denote

$$f(t) = w(T) + w^{[1]}(T) \int_T^t q(s)\, ds + w^{[2]}(T) \int_T^t q(s) \int_T^s p(\sigma)\, d\sigma\, ds\,,$$

$$g(t) = \int_T^t q(s) \int_T^s p(\sigma) \int_T^\sigma r(\tau)\, d\tau\, d\sigma\, ds\,.$$

Integrating (L^{ace}) three times on $[T, t)$ and taking into account that w is a nondecreasing function, it follows from (7)

$$w(t) \leq f(t) + w(t)\, g(t)\,.$$

Then

$$w(t) \leq \frac{f(t)}{1 - g(t)}\,.$$

From (10) and the boundedness of f we get that w is bounded, which yields a contradiction.

If $\lim x^{[1]}(t) = c \neq 0$ $[\lim x^{[2]}(t) = c \neq 0]$ then by Lemma 4 $I(r, q, p) < \infty$ $[I(p, r, q) < \infty]$. Taking w as a solution of (L^{ac}) $[(L^a)]$, and using a similar argument as above we obtain again a contradiction with the boundedness of w. Hence also claim b) is proved and the proof is now complete. ■

From Theorem 1 we obtain the following:

COROLLARY 1. – *If* (L) *has property A, then* (L^c), (L^{cc}) *have property A and* (L^a), (L^{ac}), (L^{acc}) *have property B.*

PROOF. – Let y be a nonoscillatory solution of (L^{c}). It is easy to show that $x = y^{[2]}$ is a nonoscillatory solution of (L) and $x^{[1]} = y$, $x^{[2]} = y^{[1]}$. Because (L) has property A, the function x verifies (2) and so also (L^{c}) has property A. A similar argument holds for (L^{cc}). Finally the remainder part of the assertion follows from Theorem 1. ∎

We remark that the argument, which is employed in the proof of Theorem 1, gives us also the following result:

COROLLARY 2. – *Every Kneser solution x of (L) satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$ if and only if every strongly monotone solution u of (L^{a}) satisfies $\lim_{t\to\infty} u^{[i]}(t) = \infty$ for $i = 0, 1, 2$.*

I.2. *Some applications.*

(i) Classification of equations having property A [B] involves both oscillatory properties of solutions and asymptotic behavior of nonoscillatory ones. The natural question which arises is whether or not property A [B] ensures the existence of both types of solutions occuring in its definition.

The existence of a nonoscillatory solution of (L) $[(L^{a})]$ occuring in definition of property A [B] is ensured by Proposition 2. As concerns the existence of oscillatory solutions, for the binomial equation we have (see, e.g., [16]):

(i) $(E+)$ is oscillatory if and only if $(E+)$ has property A,

(i') $(E-)$ is oscillatory if and only if $(E-)$ has property B.

The following example shows that this equivalence fails for the complete equation (L).

EXAMPLE 2. – Consider the equation $(l+)$ given in Example 1. Then $(l+)$ is oscillatory and by [23] it has a Kneser solution tending to a nonzero constant, thus $(l+)$ does not have property A, see also [3]. Similarly the equation $(T>1)$

$$(l^{a}) \qquad \left(\ln t\left(\frac{x'(t)}{t\ln t}\right)'\right)' - \frac{x(t)}{t^2(\ln t)^{1+\varepsilon}} = 0\,, \qquad t\in[T,\infty)$$

is oscillatory and, from Theorem 1, it does not have property B.

Proposition 2 and the following theorem show that property A [B] ensures existence of both types of solutions occurring in its definition.

THEOREM 2. – *If (L) has property A $[(L^{a})$ has property B$]$ then it is oscillatory.*

PROOF. – i) Let (L) have property A. Then any solution is either oscillatory or Kneser.

From this and Remark 1 it follows that a solution x with the initial condition

$$x(0)\,x'(0) > 0$$

is oscillatory.

ii) Let (L^a) have property B. Then by Theorem 1, (L) has property A and, from claim (i), (L) is oscillatory. Then its adjoint equation is oscillatory too. ■

REMARK 2. – If $\int^{\infty} r = \int^{\infty} p = \infty$, then Theorem 2 follows from Theorems 6 and 7 in [7].

(ii) In the literature there are many papers devoted to property A or B independently. Theorem 1 enables us to apply criteria on property A to obtain criteria on property B and vice versa. For example, by using criteria ensuring that (L) has property A, which we stated in [3], we immediately get the following criteria for property B:

THEOREM 3. – *Let one of the following conditions be satisfied:*

(i) $I(r) = I(p) = I(q, r) = I(q, p) = \infty$,

(ii) $I(p) = I(q) = I(r, p) = I(r, q) = \infty$,

(iii) $I(r) = I(q) = I(p, q) = I(p, r) = \infty$,

(iv) $I(p) = \infty$, $I(q, p) < \infty$ *and*

$$\int_0^{\infty} r(t)\left(\int_t^{\infty} q(s)\,ds\right)\left(\int_t^{\infty} p(s)\int_s^{\infty} q(\tau)\,d\tau\,ds\right)dt = \infty\,,$$

(v) $I(q) = \infty$, $I(r, q) < \infty$ *and*

$$\int_0^{\infty} p(t)\left(\int_t^{\infty} r(s)\,ds\right)\left(\int_t^{\infty} q(s)\int_s^{\infty} r(\tau)\,d\tau\,ds\right)dt = \infty\,,$$

(vi) $I(r) = \infty$, $I(p, r) < \infty$ *and*

$$\int_0^{\infty} q(t)\left(\int_t^{\infty} p(s)\,ds\right)\left(\int_t^{\infty} r(s)\int_s^{\infty} p(\tau)\,d\tau\,ds\right)dt = \infty\,.$$

Then (L^a) *has property B.*

PROOF. – It follows from Theorem 1 and [3, Theorems 4 and 5]. ■

REMARK 3. – If the condition (i) of Theorem 3 holds, then, by interchanging functions p, r in (L), we obtain that in this case $(L-)$ has also property B.

REMARK 4. – Other papers, where criteria for properties A, B were separately studied or proved, are, e.g., [17] (here property B is studied for general n) and the recent papers [18, 19].

It is also worth to note that our Theorem 3 is applicable in some cases in which the above quoted results fail. For instance, consider the equation

$$\left(\frac{1}{\ln t}\left(\frac{1}{\ln t}x'(t)\right)'\right)' \pm \frac{1}{t^2}x(t) = 0\,, \qquad t > 1\,. \tag{$e\pm$}$$

By Theorem 3-i), $(e+)$ has property A and $(e-)$ has property B, but Theorem 3.1 in [18] fails. Similarly, by Theorem 3-iv), the equation (L) with $p = 1$, $r = t^2$ and $q = 1/t^3$ has property A, but in this case Theorem 4.1 in [18] fails.

Part II. Nonlinear equation.

Here we use the results from Part I to study jointly property A and B for nonlinear equations

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,x'(t)\right)'\right)' + q(t)\,f(x(t)) = 0 \tag{N}$$

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\,u'(t)\right)'\right)' - q(t)\,f(u(t)) = 0 \tag{N^a}$$

when $(H1)$-$(H2)$ are assumed.

Any solution of (N) is said to be *proper*, if it is defined on R_+ and nontrivial in any neighbourhood of infinity. A proper solution is said to be *oscillatory* (*nonoscillatory*) if it has (does not have) arbitrarily large zeros.

Equation (N) is said to have *property A* if any proper solution x of this equation is either oscillatory or satisfies

$$|x^{[i]}(t)| \downarrow 0 \quad \text{as} \quad t \to \infty\,, \quad i = 0, 1, 2\,. \tag{1}$$

Equation (N^a) is said to have *property B* if any proper solution u of this equation is either oscillatory or satisfies

$$|u^{[i]}(t)| \uparrow \infty \quad \text{as} \quad t \to \infty\,, \quad i = 0, 1, 2\,. \tag{2}$$

It follows from $(H1)$, $(H2)$ and [1, Lemma 1 and Theorem 1] that all proper nonoscillatory solutions x of (N) belong to the four classes $\mathcal{N}_i$, $i = 0, 1, 2, 3$, and all proper nonoscillatory solutions u of (N^a) belong to $\mathcal{M}_i$, $i = 0, 1, 2, 3$, defined in Part I.

LEMMA 5. – *Assume* (*H*1) *and* (*H*2). *If* $I(r) = \infty$ $[I(p) = \infty]$, *then for every solution* u *of* (N^a) *such that* $u \in \mathcal{N}_3$ *we have* $u^{[1]}(\infty) = \infty$ $[u(\infty) = \infty]$.

PROOF. – Let u be a solution of (N^a) which belongs to the class $\mathcal{N}_3$. Without loss of generality we may assume that there exists T_u such that $u^{[i]}(t) > 0$ for all $t \geq T_u$, $i = 0, 1, 2$. Then, from (N^a), the functions $u^{[i]}$, $i = 0, 1, 2$, are increasing for all $t \geq T_u$. Because $u^{[2]}(t) = (1/r(t))[u^{[1]}(t)]'$, we have

$$u^{[1]}(t) = u^{[1]}(T_u) + \int_{T_u}^{t} r(s)\, u^{[2]}(s)\, ds \geq u^{[2]}(T_u) \int_{T_u}^{t} r(s)\, ds\,.$$

As $t \to \infty$ we get the first assertion.

Similarly it follows the second one. ∎

Using Lemma 5 we can prove the comparison theorem between the linear equation

$$(L_k) \qquad \left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\, x'(t)\right)'\right)' + kq(t)\, x(t) = 0\,, \qquad k > 0$$

and nonlinear equations (N), (N^a).

THEOREM 4. – *Assume* (*H*1), (*H*2) *and* $I(r) = I(p) = \infty$. *If* (L_k) *has property A for every* $k > 0$, *then* (N) *has property A and* (N^a) *has property B.*

PROOF. – *a*) Let us prove that (N) has property A.

Let x be a proper nonoscillatory solution of (N) defined on $[t_0, \infty)$. Suppose that x is eventually positive, i.e., there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$. Then, from a result in [22], we have

$$(11) \qquad x \in \mathcal{N}_0 \cup \mathcal{N}_2$$

and $x \in \mathcal{N}_0$ satisfies $x^{[i]}(\infty) = 0$, $i = 1, 2$.

Suppose that (N) does not have property A. Then either $x \in \mathcal{N}_2$ or $x \in \mathcal{N}_0$ satisfying $\lim_{t \to \infty} x(t) = l > 0$. Then there exists $c > 0$ such that

$$(12) \qquad x(t) \geq c > 0 \quad \text{for } t \text{ sufficiently large.}$$

Consider the following equation, which is obtained by linearization of (N),

$$(L_f) \qquad \left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\, v'(t)\right)'\right)' + q(t)\, \frac{f(x(t))}{x(t)}\, v = 0\,.$$

Then $v = x$ is a nonoscillatory solution of (L_f) for t sufficiently large.

In view of (12) and the continuity of f, there exists K such that

$$\frac{f((x(t))}{x(t)} \geqslant K \quad \text{for } x(t) > 0 \text{ and } t \text{ sufficiently large.} \tag{13}$$

Since (L_k) has property A for every $k > 0$, by a classical comparison theorem (see, e.g., [6]), also (L_f) has property A. It means that every solution v of (L_f) is either oscillatory or a Kneser solution tending to zero for $t \to \infty$. Being $v = x$ a nonoscillatory solution of (L_f), this contradicts (12).

b) Let us prove that (N^a) has property B.
Let u be a proper nonoscillatory solution of (N^a) defined on $[t_0, \infty)$. Suppose that u is eventually positive. Then, by a result in [22] (see also [16]), we have

$$u \in \mathcal{M}_1 \cup \mathcal{M}_3,$$

and $u \in \mathcal{M}_3$ satisfies $u^{[i]}(\infty) = \infty$, $i = 0, 1$. It means that u is eventually increasing and so $u(t) \geqslant c > 0$ for all large t. Suppose that (N^a) does not have property B, i.e., either $u \in \mathcal{M}_1$ or $u \in \mathcal{M}_3$ and $u^{[2]}(\infty)$ is bounded. Using the same linearization method as above and taking into account Theorem 1, we obtain a contradiction with the fact that (L_k) has property A for every $k > 0$. ■

REMARK 5. – Unlike other comparison results (see, e.g., [6], [9]), Theorem 4 does not require assumptions on growth of the nonlinearity in the whole $\boldsymbol{R}$.

Theorem 4 together with integral criteria ensuring property A for (L_k) gives the following result.

COROLLARY 4. – *Assume (H1) and (H2). Then (N) has property A and (N^a) has property B in the case any of the following conditions is satisfied:*

(i) $I(r) = I(p) = I(q, r) = I(q, p) = \infty$,

(ii) $I(p) = \infty$, $I(q, p) < \infty$ *and*

$$\int_0^\infty r(t) \left(\int_t^\infty q(s)\, ds \right) \left(\int_t^\infty p(s) \int_s^\infty q(\tau)\, d\tau\, ds \right) dt = \infty .$$

PROOF. – From Theorem 3 it follows that (L_k) has property A for every $k > 0$. Now Theorem 4 yields the assertion. ■

A typical example when Corollary 4 can be applied is the Emden-Fowler equation

$$\left(\frac{1}{p(t)} \left(\frac{1}{r(t)}\, x'(t) \right)' \right)' + q(t)|x|^\lambda \operatorname{sgn} x = 0\,, \qquad \lambda > 1\,. \tag{E-F}$$

COROLLARY 5. – *Let one of the conditions* (i)-(ii) *of Corollary* 4 *be satisfied. Then* (E-F) *has property A and equation*

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\,x'(t)\right)'\right)' - q(t)|x|^{\lambda}\operatorname{sgn} x = 0\,, \qquad \lambda > 1$$

has property B.

REFERENCES

[1] M. BARTUŠEK, *On the structure of solutions of a system of three differential inequalities*, Arch. Math., **30** (1994), pp. 117-130.

[2] M. CECCHI - M. MARINI - GABRIELE VILLARI, *On a cyclic disconjugate operator associated to linear differential equations*, Annali Mat. Pura Appl., IV, **CLXX** (1996), pp. 297-309.

[3] M. CECCHI - Z. DOŠLÁ - M. MARINI, *Some properties of third order differential operators*, Czech. Math. J. (1996).

[4] M. CECCHI - Z. DOŠLÁ - M. MARINI - GABRIELE VILLARI, *On the qualitative behavior of solutions of third order differential equations*, J. Math. Anal. Appl, **197** (1996), pp. 749-766.

[5] M. CECCHI - Z. DOŠLÁ - M. MARINI, *Comparison theorems for third order differential equations*, Proceedings of Dynamic Systems and Appl., **2** (1996), pp. 99-106.

[6] T. A. CHANTURIA, *Some comparison theorems for ordinary differential equations of higher order (Russian)*, Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys., **20** (1977), pp. 749-756.

[7] T. A. CHANTURIA, *On monotone and oscillatory solutions of ordinary differential equations*, Annal. Polon. Math., **37** (1980), pp. 93-111.

[8] T. A. CHANTURIA, *On the oscillation of solutions of higher order linear differential equations (Russian)*, Rep. Sem. I. N. Vekua Inst. Appl. Math., **16** (1982), pp. 3-72.

[9] T. A. CHANTURIA, *On oscillatory properties of system of nonlinear ordinary differential equations*, Proc. of I. N. Vekua Inst. of Appl. Math., Tbilisi , **14** (1983), pp. 163-203.

[10] J. DŽURINA, *Comparison theorems for functional differential equations with advanced argument*, Boll. U. M. I. (7), **7-A** (1993), pp. 461-470.

[11] M. GAUDENZI, *On the Sturm-Picone theorem for n-th-order differential equations*, Siam J. Math. Anal., **21** (1990), pp. 980-994.

[12] M. GREGUŠ, *Third Order Linear Differential Equation*, D. Reidel Publ. Comp., Dordrecht, Boston, Lancaster, Tokyo, 1987.

[13] M. GREGUŠ - M. GREGUŠ JR., *Asymptotic properties of solutions of a certain nonautonomous nonlinear differential equation of the third order*, Boll. U. M. I. (7), **7-A** (1993), pp. 341-350.

[14] M. HANAN, *Oscillation criteria for third-order linear differential equation*, Pacific J. Math., **11** (1961), pp. 919-944.

[15] P. HARTMAN, *Ordinary Differential Equations*, 2nd ed., Birkhäuser, Boston, 1982.

[16] I. T. KIGURADZE - T. A. CHANTURIA, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[17] T. KUSANO - M. NAITO - K. TANAKA, *Oscillatory and asymptotic behaviour of solutions of a class of linear ordinary differential equations*,Proc. Royal Soc. Edinburgh , **90** A (1981), pp. 25-40.
[18] J. OHRISKA, *Oscillation of differential equations and v-derivatives*, Czech. Math. J., **39** (114) (1989), pp. 24-44.
[19] J. OHRISKA, *Oscillatory and asymptotic properties of third and fourth order linear differential equations*, Czech. Math. J., **39** (114) (1989), pp. 215-224.
[20] J. OHRISKA, *Adjoint differential equations and oscillation*, J. Math. Anal. Appl., **195** (1995), pp. 778-796.
[21] C. A. SWANSON, *Comparison and Oscillation Theory of Linear Differential Equations*, Acad. Press, New York, 1968.
[22] M. ŠVEC, *Sur une propriété intégrale de l'équation* $y^{(n)} + Q(x)y = 0$, $n = 3, 4$, Czech. Math. J., 7 (1957), pp. 450-462.
[23] M. ŠVEC, *Behaviour of nonoscillatory solutions of some nonlinear differential equations*, Acta Math. Univ. Comenianae, 34 (1980), pp. 115-130.
[24] GAETANO VILLARI, *Contributi allo studio asintotico dell'equazione* $x'''(t) + p(t)x(t) = 0$, Ann. Mat. Pura Appl., IV, **LI** (1960), pp. 301-328.