Annali di Matematica pura ed applicata
(IV), Vol. CLXXIII (1997), pp. 279-298

A Vortex Free Boundary Problem: Existence and Uniqueness Results for the Physical Solution (*).

ALESSANDRO TORELLI

Summary. – *In this paper it is studied a vortex free boundary problem using some Complex Analysis and/or Harmonic Analysis techniques. It is obtained an existence and uniqueness result for the solution. A numerical method to approximate the problem is described.*

1. – Introduction.

The present paper is devoted to the study of a free boundary problem connected with the steady plane irrotational vortex motion for a non viscous and incompressible fluid. Let us now formally describe this problem (a precise formulation will be done in the next section).

Let us assume that Ω is the region of the complex plane occupied by the fluid in vortex motion and that $0 \in \Omega$ is the singularity of the vortex (see Figure 1). If $\Psi: \overline{\Omega} - \{0\} \to \mathbb{R}$ is the stream function defining the motion, then we have that Ψ is a harmonic function in $\Omega - \{0\}$ (since the motion is steady and irrotational). Near the singularity of the vortex we also have the following asymptotical behaviour:

$$\Psi(z) \sim -\log|z| \quad \text{as } z \to 0 . \tag{1.1}$$

We also assume that Ω is symmetric with respect to the imaginary axis and that Ω is «*like a ball*» which wraps around the singularity of the vortex in 0. Two conditions apply on the (free) boundary of Ω (c_1, c_2, g being real constants, the constant g being the gravity acceleration):

$$\Psi(z) = c_1 , \qquad \frac{1}{2}\,|(\nabla\Psi)(z)|^2 + g\,\mathrm{Im}\, z = c_2 ,\ z \in \partial\Omega . \tag{1.2}$$

(*) Entrata in Redazione il 28 settembre 1995.

Indirizzo dell'A.: Dipartimento di Matematica, Università di Pavia, Via Abbiategrasso 209, 27100 Pavia (Italy).

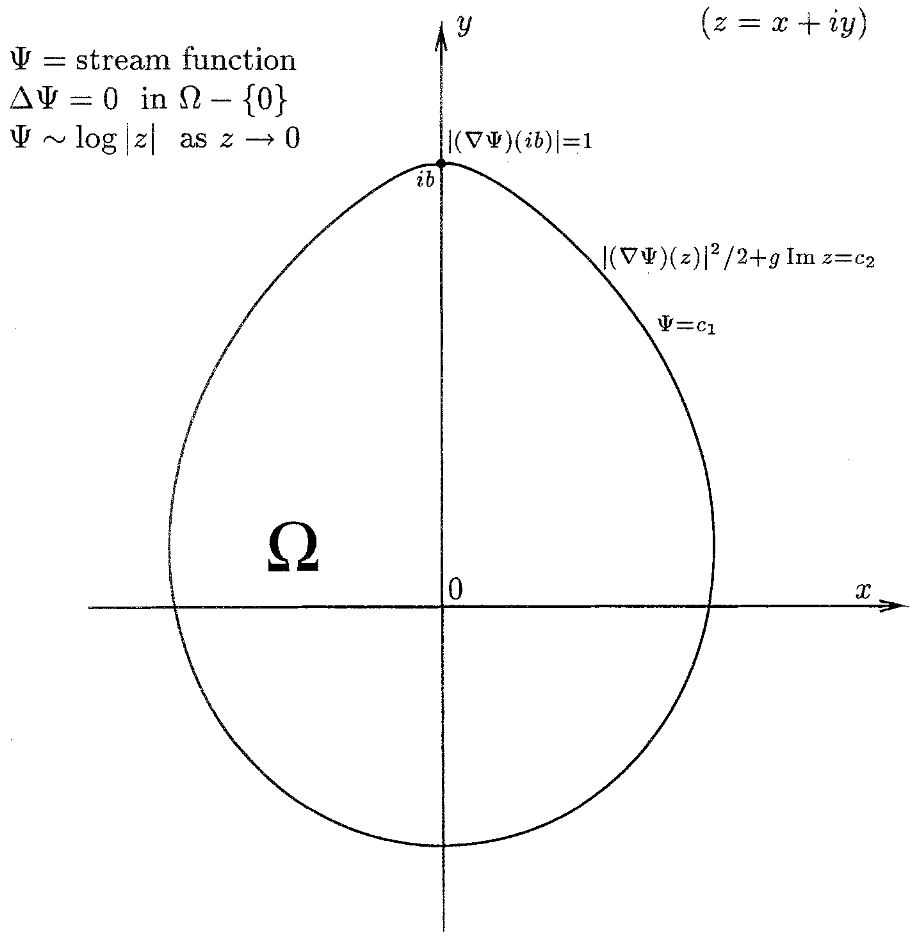

Fig. 1. – The vortex free boundary problem in terms of the stream function $\Psi(z) = \mathrm{Im}\,F(z) - \log z$

The former of such relations tells us that the boundary of Ω is a stream-line; the latter expresses Bernoulli law (assuming the external pressure as a constant). Notice that in (1.2), the constant c_1 is arbitrary, whereas the constant c_2 is an unknown quantity. To obtain a well-posed problem, we must add a further condition: for instance we can specify the value of the speed in an assigned point of $\overline{\Omega}$ (see Problem A later).

In what follows the previous problem is studied using elementary results of Harmonic Analysis or (equivalently) elementary results of Holomorphic Functions Theory in the unit disc of $\mathbb{C}$. This study is done in several steps.

I begin giving a precise complex mathematical formulation of the problem (see Problem A in section 2) and stating the main results of the paper.

In section 3 (by a suitable conformal transformation) I reduce the vortex problem to an equivalent problem (see Problem B in section 3) defined in the unit ball of the complex domain. Problem B (called *conformal problem*) consists in finding a conformal map defined in the unit ball of $\mathbb{C}$ verifying a suitable boundary condition. In such a way I can reduce a free boundary problem to a fixed boundary problem.

In section 4, I reduce Problem B to a new equivalent problem defined in the unit circle of the complex domain (which I call weak formulation of the problem). This result allows to represent the solution as the fixed point for a suitable operator: see Problem W.

Afterwords (section 5) an existence theorem is given for the weak formulation (for any value of $g \geqslant 0$), using a topological method. In section 6, a uniqueness theorem is also given (when $g \geqslant 0$ is small enough), by the use of the contraction mapping theorem. Consequently, using the equivalence between Problem A (the physical formulation) and Problem W (the weak formulation), I can obtain an existence result also for the physical formulation (for all non negative values of the gravity acceleration g) and an existence and uniqueness result when $g \geqslant 0$ is small enough. Moreover I shall prove that, if $g \geqslant 0$ is small enough, then the domain of motion Ω is a convex subset of the complex domain.

The formulation of Problem W is very simple and is *constructive*, that it is easy to approximate. In section 7 I briefly describe a method to compute an approximate solution obtained discretizing Problem W: the graph of figure 1, for example, is obtained by this method (see also later figure 2).

In a sequence of papers ([6], [10], [11]) and [12]) Lezzi and the author have already studied the vortex free boundary problem using a more complicated weak formulation. In [6] an existence and uniqueness theorem for the weak formulation when $g \geqslant 0$ is small enough is proved. In [10], [11] and [12], the author gives the idea to obtain an existence result for the weak formulation when $g \geqslant 0$ is arbitrary. In the present paper I prove the equivalence between the physical and the weak formulation and then I can give some existence and uniqueness result directly in the physical formulation. Moreover we determine a condition for the convexity of the domain of motion Ω.

In my opinion, the present problem is only a simple *model problem.* Later I think I will apply the present method to other problems.

Now we are working in a more general vortex free boundary problem: see paper [1] in which we study a vortex free boundary problem with an obstacle.

I refer to [6] for a detailed physical motivation of the present problem and for an extended bibliography on this subjet.

TERMINOLOGIC NOTE. – In what follows we use the notation:

$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}, \qquad \mathbb{T} = \partial\mathbb{D}. \qquad \blacksquare$$

2. – Precise formulation of the problem and main result.

Recalling that Ψ is a harmonic function in $\Omega - \{0\}$, relation (1.1) can be precised by assuming that there exists a function $B\colon \Omega \to \mathbb{R}$ with $\Delta B = 0$ in Ω, such that:

$$\Psi(z) = B(z) - \log|z| \quad \text{in } \Omega - \{0\}. \tag{2.1}$$

Since Ω is «like a ball», to the harmonic function B, we can associate a harmonic function $A\colon \Omega \to \mathbb{R}$ (with $A(0) = 0$) such that $A + iB$ is a holomorphic function on Ω. At this point we could consider the speed potential function $\Phi(z) = A(z) + \arg(z/i)$, where arg is the principal branch of the argument function. But we have that the complex potential function $\Phi + i\Psi$ is a holomorphic function only in the open set:

$$\Omega' = \Omega - \{z \in \Omega\colon \operatorname{Re} z = 0,\ \operatorname{Im} z \leqslant 0\}.$$

More suitable is the use of the holomorphic function $F\colon \Omega \to \mathbb{C}$ given by $A + iB$. Recalling (2.1), we obtain that $\Psi(z) = \operatorname{Im} F(z) - \log|z|$. Hence:

$$\frac{\partial \Psi}{\partial y}(z) + i\,\frac{\partial \Psi}{\partial x}(z) = F'(z) - \frac{i}{z}, \qquad z \in \Omega'. \tag{2.2}$$

Then, in terms of the function F, the problem described in section 1 can precisely be stated in the following way:

PROBLEM. A - *Physical formulation.* Given $g \geqslant 0$, we look for an open subset Ω of $\mathbb{C}$ with $0 \in \Omega$ and such that:

i) Ω is the inner domain of a C^∞-Jordan curve,

ii) Ω is symmetric and *balanced* (later we will explain what this means) with respect to the imaginary axis.

Moreover the function

$$F \in H(\Omega) \cap C^\infty(\overline{\Omega}) \tag{2.3}$$

characterized by:

$$\operatorname{Im} F(z) = \log|z|, \qquad z \in \partial\Omega, \qquad \text{with } \operatorname{Re} F(0) = 0, \tag{2.4}$$

verifies the supplementary boundary conditions (where $b = \sup\{y \in \mathbb{R}\colon iy \in \Omega\}$):

$$|F'(z) - i/z|^2 + 2g \operatorname{Im} z = \text{constant}, \qquad z \in \partial\Omega, \tag{2.5}$$

$$F'(ib) = (1 - b)/b. \quad \blacksquare \tag{2.6}$$

REMARK 2.1. – We call $\gamma\colon \mathbb{T} \to \mathbb{C}$ a C^∞-Jordan curve if $\gamma \in C^\infty(\mathbb{T})$ with $\gamma'(t) \neq 0$ ($t \in \mathbb{T}$) and γ is simple and positively oriented. With this notation, the inner domain of

γ is called *balanced* with respect to the imaginary axis if $\operatorname{Re}\gamma(t) < 0$ implies $\operatorname{Im}\gamma'(t) < 0$. A consequence of this relation is that the point ib is the top of $\overline{\Omega}$ (hence $b > 0$). ■

REMARK 2.2. – The relation $F \in H(\Omega) \cap C^\infty(\overline{\Omega})$ means that F is a holomorphic function in Ω and that all the derivatives $F^{(n)}$ have a continuous extension to $\overline{\Omega}$. ■

REMARK 2.3. – Relations (2.4) and (2.5) are the translations of (1.2) in terms of the function F (with $c_1 = 0$). By relation (2.2), we can easily verify that relation (2.6) simply prescribes that the speed in the highest point of $\partial\Omega$ is 1. Notice that the constant in (2.5) is not a datum of the problem. ■

REMARK 2.4. – From the physical point of view, the open set Ω is expected to be a convex set. Later we shall try to exhibit some condition connected with the case in which Ω is a convex set. ■

REMARK 2.5. – If $g = 0$ then $\Omega = \mathbb{D}$ is a solution of Problem A. ■

The main result contained in the present paper is the following:

THEOREM 2.6. – For all $g \geqslant 0$, there exists at least one solution of Problem A. If $g \geqslant 0$ is sufficiently small, there exists one and only one convex solution of Problem A. ■

3. – Transformation of the problem.

We now introduce:

PROBLEM B. – *Conformal formulation.* Given $g \geqslant 0$, we look for a conformal map $\mathcal{H} \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$ such that:

$$(\mathcal{H})'(w) \neq 0\,, \qquad w \in \overline{\mathbb{D}}\,, \tag{3.1}$$

$$\operatorname{Re}\mathcal{H}(1) = 0, \qquad \operatorname{Im}\mathcal{H}(1) > 0\,, \qquad \mathcal{H}'(1) = i\,, \qquad \mathcal{H}(0) = 0\,, \tag{3.2}$$

$$\overline{\mathcal{H}(w)} = -\mathcal{H}(\overline{w})\,, \qquad w \in \overline{\mathbb{D}}\,, \tag{3.3}$$

$$\operatorname{Im}\left[\frac{d}{d\vartheta}\,\mathcal{H}(e^{i\vartheta})\right] < 0\,, \qquad \vartheta \in]0, \pi[\,. \tag{3.4}$$

$$\frac{d}{d\vartheta}\,|\mathcal{H}'(e^{i\vartheta})|^{-2} + 2g\operatorname{Re}[\mathcal{H}'(e^{i\vartheta})e^{i\vartheta}] = 0\,, \qquad \vartheta \in [-\pi, \pi]\,. \tag{3.5}$$ ■

Problem B is equivalent to Problem A (the physical formulation) as stated by:

THEOREM 3.1. – Given $g \geq 0$, we have:

a) if Ω is a solution of Problem A, then the function

$$\Lambda : \overline{\Omega} \to \mathbb{C} \text{ defined by } \Lambda(z) = -iz \exp(iF(z)). \tag{3.6}$$

is a conformal mapping of $\overline{\Omega}$ onto $\overline{\mathbb{D}}$ and Λ^{-1} is a solution of Problem B;

b) if $\mathcal{H}$ is a solution of Problem B, then $\mathcal{H}(\mathbb{D})$ is a solution of Problem A.

PROOF. – We begin proving part a) of the theorem. Recalling relation (2.4), if $z \in \partial\Omega$ it follows that:

$$|\Lambda(z)| = |z| \exp(-\operatorname{Im} F(z)) = |z| \exp(-\log|z|) = 1,$$

which means that $\Lambda(\partial\Omega) \subset \partial\mathbb{D}$. Moreover we have that $\Lambda(0) = 0$ (with multeplicity 1). This fact (see, for instance, [8], Th. 1.9 and 2.6) proves that Λ is a conformal mapping of $\overline{\Omega}$ onto $\overline{\mathbb{D}}$. Using the C^∞-regularity of the Jordan curve $\partial\Omega$, we obtain that $\Lambda^{-1} \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$ and that relation (3.1) holds (see again [8], Th 3.5 and 3.6).

By the symmetry of Ω and the boundary Dirichlet behaviour of $\operatorname{Im} F$, it follows:

$$F(-\bar{z}) = -\overline{F(z)}, \qquad F'(-\bar{z}) = \overline{F'(z)}, \qquad z \in \overline{\Omega}. \tag{3.7}$$

Hence $\Lambda(-\bar{z}) = \overline{\Lambda(z)}$ $(z \in \overline{\Omega})$, which implies (3.3). Using again (3.7), we have that $\operatorname{Re} F(ib) = 0$. Hence, since $b > 0$ and recalling (2.4), it follows that:

$$\Lambda(ib) = b \exp(-\operatorname{Im} \mathrm{F(ib)}) = \mathrm{b} \exp(-\log(b)) = 1,$$

which proves that $\operatorname{Re} \Lambda^{-1}(1) = 0$ and $\operatorname{Im} \Lambda^{-1}(1) = b > 0$. By (3.6), we have that:

$$\Lambda'(z) = z[F'(z) - i/z] \exp(iF(z)), \qquad z \in \overline{\Omega} - \{0\}, \tag{3.8}$$

which implies, using (2.6), $\Lambda'(ib) = -i$, hence $(\Lambda^{-1})'(1) = i$. Also relation (3.2) is proved. By (3.2) and (3.3), already proved, $\mathcal{H} = \Lambda^{-1}$ maps the upper semidisk into $\overline{\Omega} \cap \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. We remark that $\Lambda^{-1} : \mathbb{T} \to \mathbb{C}$ is a parametrization of $\partial\Omega$ (the *conformal* parametrization): since the tangent direction of a curve does not depend on the parametrization, we obtain (3.4) as a consequence of the hypothesis that Ω is a balanced set. By (3.8) and (2.4) we have:

$$|\Lambda'(z)| = |z|\,|F'(z) - i/z| \exp(-\operatorname{Im} \mathrm{F(z)}) = |\mathrm{F}'(\mathrm{z}) - \mathrm{i/z}|, \qquad z \in \partial\Omega,$$

which, recalling (2.5), implies $|\Lambda'(z)|^2 + 2g \operatorname{Im} z = \text{constant}$ $(z \in \partial\Omega)$. Taking $z = \Lambda^{-1}(e^{i\vartheta})$, we obtain:

$$|(\Lambda^{-1})'(e^{i\vartheta})|^{-2} + 2g \operatorname{Im}[\Lambda^{-1}(e^{i\vartheta})] = \text{constant}, \qquad z \in \partial\Omega,$$

hence:

$$\frac{d}{d\vartheta}\,|(\Lambda^{-1})'(e^{i\vartheta})|^{-2} + 2g\,\mathrm{Im}\,[ie^{i\vartheta}(\Lambda^{-1})'(e^{i\vartheta})] = 0\,, \qquad z \in \partial\Omega\,,$$

which implies (3.5). Part *a*) is completely proved.

Now we shall prove part *b*). Taking $\gamma(t) = \mathcal{H}(e^{it})$ ($t \in [-\pi, \pi]$) and by (3.1), we obtain that $\Omega = \mathcal{H}(\mathbb{D})$ is the inner domain of a C^∞-Jordan curve, which is symmetric with respect to the imaginary axis (by (3.3)). Recalling (3.4), Ω is balanced with respect to the imaginary axis.

Let now $\Lambda = \mathcal{H}^{-1}$. We also have that $\Lambda \in H(\Omega)$. Since $\Lambda(z) = 0$ if and only if $z = 0$ (with multeplicity 1), there exists $F \in H(\Omega)$ such that:

$$e^{iF(z)} = i\Lambda(z)/z\,, \qquad z \in \Omega - \{0\}\,. \tag{3.9}$$

By (3.3) we have that $\Lambda(-\bar{z}) = \overline{\Lambda(z)}$ ($z \in \Omega$). Since $\mathrm{Re}\,\mathcal{H}(1) = 0$ and $\mathrm{Im}\,\mathcal{H}(1) > 0$, we obtain that $i\Lambda(z)/z > 0$ for all $z \in \Omega \cap \{\mathrm{Re}\,z = 0\}$. Then we can choose F such that $\mathrm{Re}\,F(iy) = 0$ for all $y \in \mathbb{R}$ verifying $iy \in \Omega$. Hence we have $\mathrm{Im}\,F(z) = -\log|\Lambda(z)/z|$, ($z \in \Omega - \{0\}$). Recalling that $|\Lambda(z)| \to 1$ as $z \to w \in \partial\Omega$, we obtain:

$$\lim_{z \to w} \mathrm{Im}\,F(z) = \log|w|\,, \qquad \forall w \in \partial\Omega$$

and, by the C^∞-regularity of $\partial\Omega$, it follows that $\mathrm{Im}\,F \in C^\infty(\overline{\Omega})$ and that (2.4) holds. We have also that $F \in H(\Omega) \cap C^\infty(\overline{\Omega})$ (and that $\Lambda \in H(\Omega) \cap C^\infty(\overline{\Omega})$). Since:

$$\mathrm{Re}\,[\mathcal{H}'(e^{i\vartheta})e^{i\vartheta}] = -\mathrm{Re}\left[i\,\frac{d}{d\vartheta}\,\mathcal{H}(e^{i\vartheta})\right] = \mathrm{Im}\left[\frac{d}{d\vartheta}\,\mathcal{H}(e^{i\vartheta})\right], \qquad \vartheta \in [-\pi, \pi]\,,$$

relation (3.5) can be written:

$$|\Lambda'(\mathcal{H}(e^{i\vartheta}))|^2 + 2g\,\mathrm{Im}\,\mathcal{H}(e^{i\vartheta}) = \text{constant}\,, \qquad \vartheta \in [-\pi, \pi]\,.$$

If we put $z = \mathcal{H}(e^{i\vartheta}) \in \partial\Omega$, this relation becomes:

$$|\Lambda'(z)|^2 + 2g\,\mathrm{Im}\,z = \text{constant}\,, \qquad z \in \partial\Omega\,. \tag{3.10}$$

Differentiating relation (3.9) we have:

$$e^{iF(z)}[1 + izF'(z)] = i\,\frac{\Lambda(z)}{z}\,[1 + izF'(z)] = i\Lambda'(z)\,, \qquad z \in \overline{\Omega} - \{0\}\,. \tag{3.11}$$

If $z \in \partial\Omega$ then $|\Lambda(z)| = 1$. Hence $|i/z - F'(z)| = |\Lambda'(z)|$ ($z \in \partial\Omega$). Thanks to relation (3.10), we obtain (2.5).

It only remains to prove relation (2.6). Since $\mathcal{H}(1)$ belongs to the positive imaginary axis, we can put: $\mathcal{H}(1) = ib$. By (3.4) ib is the top of $\partial\Omega$. Then we have: $\Lambda(ib) = 1$ and $\Lambda'(ib) = -i$. Using (3.11), we have $1/b - F'(ib) = 1$ and relation (2.6) follows. ■

4 – Weak formulations of the problem

a) Preliminary considerations.

NOTE 4.1. – A conformal map f defined in $\mathbb{D}$ and such that $f(0) = 0$ is called a *convex* (resp. *starlike*) function, if $f(\mathbb{D})$ is a convex subset of $\mathbb{C}$ (resp. a starlike subset of $\mathbb{C}$ with respect to the origin). It is well known that a function f is convex if and only if the function $wf'(w)$ is starlike (for more details see, for instance, [3], Section 2.5). ∎

Now we assume that $\mathcal{H}$ is a solution of Problem B and Ω the corresponding solution of Problem A (as stated in Theorem 3.1). We can introduce the following function:

$$\mathcal{L}: \overline{\mathbb{D}} \to \mathbb{C} \text{ defined by: } \mathcal{L}(w) = -i\mathcal{H}'(w). \tag{4.1}$$

The following result contains some preliminary properties of the function $\mathcal{L}$:

PROPOSITION 4.2. – We have that:

$$\mathcal{L} \in H(\mathbb{D}) \cap C^{\infty}(\overline{\mathbb{D}}), \tag{4.2}$$

$$\overline{\mathcal{L}(\overline{w})} = \mathcal{L}(w) \neq 0,\ w \in \overline{\mathbb{D}}, \tag{4.3}$$

$$\mathcal{L}(1) = 1. \tag{4.4}$$

Moreover we have that $\Omega = \mathcal{H}(\mathbb{D})$ is a convex set if and only if the function $w \in \mathbb{D} \to \to w\mathcal{L}(w)$ is starlike.

PROOF. – Relations (4.2)-(4.4) are easy consequences of the definition of $\mathcal{L}$. Using Theorem 3.1, we have that Ω is a convex set if and only if the function $\mathcal{H}$ is a convex function, and, by Note 4.1, this is equivalent to the condition that the function $w\mathcal{H}'(w)$ is a starlike function. ∎

Let us now consider the classical complex Poisson kernel:

$$H_r(\vartheta) = \frac{1 + re^{i\vartheta}}{1 - re^{i\vartheta}}, \qquad r \in [0, 1[, \quad \vartheta \in \mathbb{R}.$$

Put also $P_r(\vartheta) = \operatorname{Re} H_r(\vartheta)$ and $Q_r(\vartheta) = \operatorname{Im} H_r(\vartheta)$ which are (respectively) the ordinary Poisson kernel and the conjugate Poisson kernel. Then we can easily obtain:

PROPOSITION 4.3. – The function $\mathcal{L}$ may be represented as:

$$\mathcal{L}(re^{i\vartheta}) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(\vartheta - t)\log|\mathcal{L}(e^{it})|\,dt\right), (r, \vartheta) \in [0, 1[\times[-\pi, \pi]. \tag{4.5}$$ ∎

It is well known that (see for instance [5]):

$$\lim_{r \uparrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_r(\vartheta - t)\, h(t)\, dt = (\Xi h, \vartheta),\ \vartheta \in \mathbb{R},\ h \in C^{0,1}(\mathbb{T}), \tag{4.6}$$

where Ξ is the so-called *conjugate* operator defined by:

$$(\Xi h)(\vartheta) = (\Xi h, \vartheta) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h(\vartheta + t) - h(\vartheta - t)}{2\,\mathrm{tg}\,(t/2)}\, dt. \tag{4.7}$$

REMARK 4.4. – Notice that the definition of Ξ, suitably adapted by the use of a principal-value integral, can be extended to $h \in L^1(\mathbb{T})$ (see again [5]). ∎

Given $\alpha \in]0, 1[$ and $n \in \mathbb{N}$, let $C^{n,\alpha}(\mathbb{T})$ the space of the functions defined in $\mathbb{T}$ such that the derivatives $h^{(k)} \in C^{0,\alpha}(\mathbb{T})$ $(k \leq n)$, i.e. are Hölder continuous functions (with exponent α), with the norm:

$$\|h\|_{n,\alpha} = \sup\{|h(\vartheta)|,\ \vartheta \in [-\pi, \pi]\} + \tag{4.8}$$

$$+ \sum_{k=0}^{n} \sup\left\{ \frac{|h^{(k)}(\vartheta + t) - h^{(k)}(\vartheta)|}{|t|^\alpha},\ \vartheta, t \in [-\pi, \pi] \text{ with } t \neq 0 \right\}.$$

Taking into account of the extension of the conjugate operator described in Remark 4.4, if $h \in C^{0,\alpha}(\mathbb{T})$ the conjugate operator Ξh can be expressed as in (4.7) using an ordinary Lebesgue integral. We have that $(\alpha \in]0, 1[)$ Ξ: $C^{0,\alpha}(\mathbb{T}) \to C^{0,\alpha}(\mathbb{T})$ is a linear and continuous map. This is a result due to Fatou (see, for instance, [2] or [4]). It is elementary also to prove the following results $(\alpha \in]0, 1[)$:

$$\text{if } h \in C^{n,\alpha}(\mathbb{T}) \text{ then } \Xi h \in C^{n,\alpha}(\mathbb{T}) \text{ and } (\Xi h)^{(k)} = \Xi h^{(k)}\ (k \leq n) \tag{4.9}$$

$$\text{the map } \Xi\colon C^{n,\alpha}(\mathbb{T}) \to C^{n,\alpha}(\mathbb{T}) \text{ is linear and continuous}. \tag{4.10}$$

By (4.2)-(4.4) we can introduce the following function (as usual, identifying $\mathbb{T}$ with the interval $[-\pi, \pi]$):

$$\varphi\colon \mathbb{T} \to \mathbb{R} \text{ such that: } \varphi(0) = 0,\ \mathcal{L}(e^{i\vartheta}) = |\mathcal{L}(e^{i\vartheta})| \exp(i\varphi(\vartheta)), \tag{4.11}$$

that is φ is (a branch of) the argument of $\mathcal{L}(e^{i\vartheta})$.

PROPOSITION 4.5. – We have that $\varphi \in C^\infty(\mathbb{T})$. Moreover:

$$\varphi(-\vartheta) = -\varphi(\vartheta), \qquad \vartheta \in [-\pi, \pi], \tag{4.12}$$

$$0 < \vartheta + \varphi(\vartheta) < \pi \quad, \forall \vartheta \in]0, \pi[, \tag{4.13}$$

$$|\mathcal{L}(e^{i\vartheta})|^{-3} = 1 + 3g \int_0^\vartheta \sin(\varphi(t) + t)\, dt,\ \vartheta \in [-\pi, \pi], \tag{4.14}$$

$$\varphi(\vartheta) = (\Xi h, \vartheta),\ \vartheta \in [-\pi, \pi], \tag{4.15}$$

where the function h is defined by $h(\vartheta) = \log|\mathcal{L}(e^{i\vartheta})|$ $(\vartheta \in \mathbb{T})$. Moreover we have:

(4.16) $\mathcal{H}(\mathbb{D})$ is a convex set if and only if $\varphi'(\vartheta) \geqslant -1$, $\vartheta \in [0, \pi]$.

PROOF. – Since $\mathcal{L} \in C^\infty(\overline{\mathbb{D}})$ and since $\mathcal{L}$ never vanishes in $\overline{\mathbb{D}}$, it follows that $\varphi \in C^\infty(\mathbb{T})$. We also have that relation (4.3) implies (4.12). By (3.5) we have:

$$\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-2} - 2g\,\mathrm{Im}[e^{i\vartheta}\mathcal{L}(e^{i\vartheta})] = \frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-2} - 2g|\mathcal{L}(e^{i\vartheta})|\sin(\vartheta + \varphi(\vartheta)) = 0\,.$$

Since $\mathcal{L}(1) = 1$ and by the identity:

$$|\mathcal{L}(e^{i\vartheta})|^{-1}\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-2} = \frac{2}{3}\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-3}$$

we obtain relation (4.14). Taking the limit when $r \uparrow 1$ in relation 4.5, it follows:

$$|\mathcal{L}(e^{i\vartheta})|\exp(i\varphi(\vartheta)) = |\mathcal{L}(e^{i\vartheta})|\exp\left(\lim_{r\downarrow 1}\frac{i}{2\pi}\int_{-\pi}^{\pi} Q_r(\vartheta - t)\log|\mathcal{L}(e^{it})|\,dt\right),$$

hence (recalling that $\varphi(0) = 0$ and by the fact that $|\mathcal{L}(e^{it})|$ is an even function and the function $t \to Q_r(t)$ is an odd function):

$$\varphi(\vartheta) = \lim_{r\downarrow 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} Q_r(\vartheta - t)\log|\mathcal{L}(e^{it})|\,dt\,,$$

and by the property (4.6), we obtain relation (4.15). Since:

$$e^{i\vartheta}\mathcal{L}(e^{i\vartheta}) = -ie^{i\vartheta}\mathcal{H}'(e^{i\vartheta}) = -\frac{d}{d\vartheta}\left[\mathcal{H}(e^{i\vartheta})\right]$$

and, recalling relation (3.4), we obtain that $Im\ [e^{i\vartheta}\mathcal{L}(e^{i\vartheta})] > 0$ $(\vartheta \in]0, \pi[)$, which easily implies (4.13).

Recalling Proposition 4.2, we obtain that $\mathcal{H}(\mathbb{D})$ is a convex set if and only if the function $\vartheta \to \arg[e^{i\vartheta}\mathcal{L}(e^{i\vartheta})] = \vartheta + \varphi(\vartheta)$ is non decreasing. This fact concludes the proof of the present Proposition. ■

REMARK 4.6. – Relation (4.15) gives a representation of the argument of the boundary value of $\mathcal{L}$, that is the value of φ, in terms of the boundary value of $|\mathcal{L}|$. We also remark that relation (4.15), connected with equality (4.14), suggests a fixed point procedure to characterize the value of φ. Actually, if we propose a starting value of φ on $\mathbb{T} = \partial\mathbb{D}$, we can evaluate the value of $|\mathcal{L}(e^{i\vartheta})|$ (using relation (4.14)). Then replacing this value in (4.15), we must find again the starting value of φ. This fixed point procedure will be used to study the present problem (see later). This method will also be employed for the numerical treatment of the present problem. ■

b) Weak formulation.

Let us introduce the following linear space:

$$\mathcal{X} = \{\mu \in C^0(\mathbb{T}, \mathbb{R}) \colon \mu(\vartheta) = -\mu(-\vartheta), \vartheta \in [-\pi, \pi]\},$$

with the norm:

$$\|\mu\| = \max\{|\mu(\vartheta)|, \vartheta \in [-\pi, \pi]\}. \tag{4.17}$$

Then $(\mathcal{X}, \|\ \|)$ is a Banach space. Define now:

$$\mathcal{M} = \{\mu \in \mathcal{X} \colon 0 < \mu(t) + t < \pi, t \in]0, \pi[\},$$

$$\mathcal{M}_0 = \{\mu \in \mathcal{X} \colon \mu(t) - \mu(s) \geqslant s - t, 0 \leqslant s \leqslant t \leqslant \pi\}.$$

Later we will also need the set ($\lambda \geqslant 0$):

$$\mathcal{X}_\lambda = \{\mu \in \mathcal{X} \colon 0 \leqslant \vartheta + \mu(\vartheta) \leqslant \pi + \ell(\lambda), \vartheta \in [0, \pi]\},$$

where $\ell(0) = \pi/2$ and, if $\lambda > 0$, $\ell(\lambda) = \min\{\pi/2, 1/(6\lambda\pi)\}$. Notice that $\mathcal{M} \subset \mathcal{X}_\lambda \subset \mathcal{X}$ ($\lambda \geqslant 0$). Let us also introduce the space:

$$\mathcal{Y} = \{h \in C^1(\mathbb{T}, \mathbb{R}) \colon h(0) = 0, h(\vartheta) = h(-\vartheta), \vartheta \in [-\pi, \pi]\}$$

with the norm:

$$\|h\|_1 = \max\{|h'(\vartheta)|, \vartheta \in [-\pi, \pi]\}. \tag{4.18}$$

The space $\mathcal{Y}$, with this norm, is a Banach space. We can now introduce the following operator $S_\lambda \colon \mathcal{X}_\lambda \to \mathcal{Y}$ ($\lambda \geqslant 0$), defined by:

$$(S_\lambda \mu)(\vartheta) = -\frac{1}{3}\log\left(1 + 3\lambda\int_0^\vartheta \sin(\mu(t) + t)\,dt\right). \tag{4.19}$$

For every $\mu \in \mathcal{X}_\lambda$ the function $S_\lambda \mu$ has a meaning since the argument of log is always $\geqslant 1/2$. Since the function μ is an odd function, we have that $(S_g\mu)(\vartheta) = (S_g\mu)(\vartheta + 2\pi)$ ($\vartheta \in \mathbb{R}$). Hence $S_\lambda \colon \mathcal{X}_\lambda \to \mathcal{Y}$ is well defined. Moreover we have:

PROPOSITION 4.7. – Given $\lambda \geqslant 0$, for every $\mu \in \mathcal{X}_\lambda$ (hence for all $\mu \in \mathcal{M}$) we have that $\Xi S_\lambda \mu \in \mathcal{X}$.

PROOF. – Since the function μ is an odd function, we obtain that $(S_\lambda\mu)(\vartheta) = (S_\lambda\mu)(-\vartheta)$ ($\vartheta \in \mathbb{T}$). Hence $(\Xi S_\lambda\mu)(\vartheta) = -(\Xi S_\lambda\mu)(-\vartheta)$. ■

Let us now consider the following:

PROBLEM W. – *Weak formulation.* Given $g \geqslant 0$, we look for a function $\varphi \in \mathcal{M}$ such that $\Xi S_g \varphi = \varphi$. ■

Problem W and Problem B (or Problem A) are equivalent as stated by the following:

THEOREM 4.8. – We have:

i) if $\mathcal{H}$ is a solution of Problem B, then the function φ defined in (4.11) (the function $\mathcal{L}$ being introduced in (4.1)) is a solution of Problem W.

ii) given φ solution of Problem W, if we put:

$$\mathcal{L}(re^{i\vartheta}) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(\vartheta - t)S_g\varphi(t)\,dt\right], \qquad (r, \vartheta) \in [0, 1[\times[-\pi, \pi],$$

then the function $\mathcal{H} \in H(\mathbb{D})$, such that $\mathcal{H}' = i\mathcal{L}$ with $\mathcal{H}(0) = 0$, is a solution of Problem B.

iii) we have that $\mathcal{H}(\mathbb{D})$ is a convex set if and only if $\varphi \in \mathcal{M}_0$. ■

Part i) of Theorem 4.8 is an obvious consequence of Proposition 4.5. Part iii) is implied by parts i) and ii) and by (4.16). To prove part ii) we need some preliminary results.

LEMMA 4.9. – If $g \geq 0$ and if $\varphi \in \mathcal{X}_g$ with $\Xi S_g\varphi = \varphi$, then $\varphi \in C^\infty(\mathbb{T})$. Moreover there exists a sequence $k_n > 0$ such that, for all $\lambda \in [0, g]$ and for all $\varphi_\lambda \in \mathcal{X}_g$ verifying $\Xi S_\lambda \varphi_\lambda = \varphi_\lambda$, we have $\|\varphi_\lambda\|_{n, \alpha} < k_n$.

PROOF. – It is enough to prove that for all $n \in \mathbb{N}$, we have that φ_λ belongs to a bounded subset of $C^{n, \alpha}(\mathbb{T})$. Since $\varphi_\lambda \in \mathcal{X}_g$, then φ_λ belongs to a bounded subset of $C^0(\mathbb{T})$, hence $S_\lambda\varphi_\lambda$ belongs to a bounded subset of $\mathcal{Y}$ and then, by (4.10), we can determine $k_0 > 0$ such that $\|\varphi_\lambda\|_{0, \alpha} < k_0$.

By induction, we assume that there exists $k_n > 0$ such that $\|\varphi_\lambda\|_{n, \alpha} < k_n$. Then $S_\lambda\varphi_\lambda$ belongs to a bounded subset of $C^{n+1, \alpha}(\mathbb{T})$ and then, by (4.9) and (4.10), there exists $k_{n+1} > 0$ such that $\|\varphi_\lambda\|_{n+1, \alpha} < k_{n+1}$. ■

PROOF OF PART ii) OF THEOREM 4.8. – By Lemma 4.9, if φ is a solution of Problem W, it follows that $\varphi \in C^\infty(\mathbb{T})$. Using the fact that $\varphi \in \mathcal{X}$ and $t + \varphi(t) \in]0, \pi[$ $(t \in]0, \pi[)$, it follows that:

$$\int_0^{\vartheta} \sin(t + \varphi(t))\,dt \geq 0, \ \vartheta \in [-\pi, \pi],$$

hence we obtain that $S_g\varphi \in C^\infty(\mathbb{T})$. Put now:

$$\mathcal{K}(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(\vartheta - t)\,S_g\varphi(t)\,dt, \qquad (r, \vartheta) \in [0, 1[\times[-\pi, \pi],$$

we have that $\operatorname{Re}\mathcal{K}$ is a harmonic function in $\mathbb{D}$ which has C^∞ boundary value given by

$S_g\varphi$. Then $\operatorname{Re}\mathcal{K} \in C^\infty(\overline{\mathbb{D}})$ and so $\mathcal{K} \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$. This result implies that $\mathcal{L} = \exp(\mathcal{K}) \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$. We easily obtain that ($\vartheta \in [-\pi, \pi]$):

$$|\mathcal{L}(e^{i\vartheta})|^{-3} = \exp(-3S_g\varphi(\vartheta)) = 1 + 3g\int_0^\vartheta \sin(t + \varphi(t))\,dt \neq 0, \tag{4.20}$$

$$(\arg\mathcal{L})(e^{i\vartheta}) = \varphi(\vartheta),\ \vartheta \in [-\pi, \pi], \tag{4.21}$$

$$\mathcal{L}(\overline{w}) = \overline{\mathcal{L}(w)},\ w \in \overline{\mathbb{D}}. \tag{4.22}$$

If we introduce the function $\mathcal{H}$ by the conditions $\mathcal{H}' = i\mathcal{L}$ with $\mathcal{H}(0) = 0$, then we have that $\mathcal{H} \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$ and that (3.1) and (3.3) hold. By (4.20) and (4.21), we have $\mathcal{L}(1) = |\mathcal{L}(1)|\exp(i\varphi(0)) = 1$. Since $\mathcal{L}$ never vanishes, we obtain that $\mathcal{L}(r) > 0$ for all $r \in [-1, 1]$. Then $\mathcal{H}'(1) = i\mathcal{L}(1) = i$ and relation (3.2) is proved too. We also have:

$$\frac{d}{d\vartheta}\mathcal{H}(e^{i\vartheta}) = \mathcal{H}'(e^{i\vartheta})e^{i\vartheta}i = -\mathcal{L}(e^{i\vartheta})e^{i\vartheta},$$

hence (since $\varphi \in \mathcal{M}$):

$$\operatorname{Im}\left[\frac{d}{d\vartheta}\mathcal{H}(e^{i\vartheta})\right] = -\operatorname{Im}|\mathcal{L}(e^{i\vartheta})|e^{i(\vartheta+\varphi(\vartheta))} = -|\mathcal{L}(e^{i\vartheta})|\sin(\vartheta + \varphi(\vartheta)) < 0,\ \vartheta \in [0, \pi]$$

and (3.4) follows. By (4.20) we obtain:

$$3|\mathcal{L}(e^{i\vartheta})|^{-2}\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-1} = \frac{3}{2}|\mathcal{L}(e^{i\vartheta})|^{-1}\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-2} = 3g\sin(\varphi(\vartheta) + \vartheta),$$

hence:

$$\frac{d}{d\vartheta}|\mathcal{L}(e^{i\vartheta})|^{-2} = 2g|\mathcal{L}(e^{i\vartheta})|\sin(\varphi(\vartheta) + \vartheta) = 2g\operatorname{Im}[\mathcal{L}(e^{i\vartheta})e^{i\vartheta}].$$

Since $\mathcal{L}(e^{i\vartheta}) = -i\mathcal{H}'(e^{i\vartheta})$, it follows:

$$\frac{d}{d\vartheta}|\mathcal{H}'(e^{i\vartheta})|^{-2} = -2g\quad\operatorname{Im}[i\mathcal{H}'(e^{i\vartheta})e^{i\vartheta}],$$

which implies relation (3.5).

It only remains to prove that $\mathcal{H}$ is a conformal map on $\overline{\mathbb{D}}$. To this end, we begin proving that $\operatorname{Re}\mathcal{H}(w) < 0$ for all $w \in \mathbb{D}_+^* = \{w \in \overline{\mathbb{D}}\colon \operatorname{Im} w > 0\}$. By (3.3), already proved, we have that $\operatorname{Re}\mathcal{H}(x) = 0$ for all $x \in [-1, 1]$. By contradiction, if $\operatorname{Re}\mathcal{H}$ has a non negative maximum in $\mathbb{D}_+^*$, then (using the maximum principle for harmonic functions) there exists $\zeta \in]0, \pi[$ such that:

$$\operatorname{Re}\mathcal{H}(e^{i\zeta}) \geqslant 0;\quad \operatorname{Re}\mathcal{H}(e^{i\zeta}) \geqslant \operatorname{Re}\mathcal{H}(e^{i\vartheta}),\qquad \vartheta \in]0, \pi[.$$

By the Hops principle ([9], Th. II,7), we obtain that:

$$\frac{\partial \operatorname{Re} \mathcal{K}}{\partial r}(e^{i\xi}) = \frac{\partial Im \mathcal{K}}{\partial \vartheta}(e^{i\xi}) > 0\,,$$

which gives a contradiction compared with (3.4). Hence $\operatorname{Re} \mathcal{K}(w) < 0$ for all $w \in \mathbb{D}_+^*$. Similarly we can prove that $\operatorname{Re} \mathcal{K}(w) > 0$ for all $w \in \mathbb{D}_-^* = \{w \in \overline{\mathbb{D}}\colon \operatorname{Im} w < 0\}$. Hence:

$$\operatorname{Re} \mathcal{K}(e^{i\vartheta}) > 0 \text{ if } \vartheta \in]-\pi, 0[,\ \operatorname{Re} \mathcal{K}(e^{i\vartheta}) < 0 \text{ if } \vartheta \in]0, \pi[\,. \tag{4.23}$$

We can now prove that $\mathcal{K}$ is a conformal map. By (3.4), the map $\vartheta \in [0, \pi] \to \mathcal{K}(e^{i\vartheta})$ is an injective map. Similarly the map $\vartheta \in [-\pi, 0] \to \mathcal{K}(e^{i\vartheta})$ is an injective map too. Using relations (4.23), it follows that the map $\vartheta \in [-\pi, \pi] \to \mathcal{K}(e^{i\vartheta})$ is a Jordan closed curve. Since $\mathcal{K}(0) = 0$ (multeplicity 1), we can conclude that $\mathcal{K}$ is a conformal map in $\overline{\mathbb{D}}$ (see [8], Th. 1.9). ■

5. - An existence result for the solutions of Problem W.

We shall now prove the following existence result:

Theorem. 5.1. - For every $g \geq 0$ there exists at least a solution of Problem W. ■

REMARK 5.2. - Recalling Theorems 3.1 and 4.8, Theorem 5.1 states that, for every $g \geq 0$, there exists at least a physical solutions Ω defined by Problem A. ■

We now begin the proof of Theorem 5.1.

PROPOSITION 5.3. - Given $g > 0$ and $\lambda \in [0, g]$, if $\mu \in \mathcal{X}_g$ verifies relation $\Xi S_\lambda \mu = \mu$, then $\mu \in \mathcal{M}$. Moreover if $\lambda > 0$, then $\mu(\vartheta) > 0$ ($\vartheta \in]0, \pi[$). ■

If $\lambda = 0$ and $\Xi S_0 \mu = \mu$, it easily follows that $\mu \equiv 0$. Then the statement of the Proposition 5.3 is obvious in the case $\lambda = 0$. Hence we can assume $\lambda > 0$ in the proof of Proposition 5.3. We need now the following auxiliary function (where $\mu \in \mathcal{X}$ verifies the hypothesis of Proposition 5.3):

$$G(re^{i\vartheta}) = re^{i\vartheta} \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H_r(\vartheta - t)(S_\lambda \mu)(t)\, dt \right), (r, \vartheta) \in [0, 1[\times [-\pi, \pi]\,. \tag{5.1}$$

We can easily control (by Lemma 4.9) that:

$$G \in H(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}});\ G(\overline{w}) = \overline{G(w)},\ w \in \overline{\mathbb{D}}\,. \tag{5.2}$$

The first step of the proof of Proposition 5.3 is contained in:

LEMMA 5.4. – Given $\mu \in \mathcal{X}$ verifying the hypothesis of Proposition 5.3, we have that $\vartheta + \mu(\vartheta) < \pi$ $(\vartheta \in]0, \pi[)$.

PROOF. – By contradiction, we can assume that there exists $\vartheta^* \in]-\pi, \pi[$ such that:

$$\pi \leqslant \vartheta^* + \mu(\vartheta^*) \leqslant 3\pi/2\,, \qquad \vartheta^* + \mu(\vartheta^*) \geqslant \vartheta + \mu(\vartheta)\,, \qquad \vartheta \in [-\pi, \pi]\,. \tag{5.3}$$

We can now consider the restriction of the function G to the open set: $\mathbb{D}^+ = \{w \in \mathbb{D}\colon \operatorname{Im} w > 0\}$. Since G never vanishes in $\mathbb{D}^+$, we can consider a branch of the logarithm of G on $\mathbb{D}^+$, which is given by:

$$\log G(w) = \log|G(w)| + i \arg G(w)\,.$$

The function $\arg G$ can be chosen such that (extended to a smooth function in $\overline{\mathbb{D}^+} - \{0\}$):

$$\arg G(e^{i\vartheta}) = \vartheta + \mu(\vartheta)\,, \qquad \vartheta \in [-\pi, \pi]\,, \qquad \arg G(x) = \begin{cases} \pi\,, & \text{if } x \in [-1, 0[\,, \\ 0\,, & \text{if } x \in]0, 1]\,. \end{cases}$$

It is also easy to prove that:

$$\max \lim_{w \to \zeta} \arg G(w) \leqslant \pi,\ \zeta \in [-1, 1]\,. \tag{5.4}$$

The function $\arg G$ verifies:

$$\pi \leqslant \arg G(e^{i\vartheta^*}) = \vartheta^* + \mu(\vartheta^*)\,; \qquad \arg G(e^{i\vartheta^*}) \geqslant \arg G(w)\,, \qquad w \in \overline{\mathbb{D}^+} - \{0\}\,. \tag{5.5}$$

By the Hops principle ([9], Th. II,7), we obtain: $(\partial \arg G/\partial r)(e^{i\vartheta^*}) > 0$. Since the function $\arg G$ is the harmonic conjugate of the harmonic function $\log|G(w)|$, we obtain that $(d\log|G|/d\vartheta)(e^{i\vartheta^*}) < 0$. On the other hand, we easily obtain that: $\log|G(e^{i\vartheta})| = (S_\lambda \mu)(\vartheta)$. Hence:

$$\frac{d\log|G|}{d\vartheta}(e^{i\vartheta}) = -\frac{\lambda \sin(\vartheta + \mu(\vartheta))}{1 + \int_0^\vartheta \sin(t + \mu(t))\,dt}$$

and this relation implies (recalling (5.3)) that $(d\log|G|/d\vartheta)(e^{i\vartheta^*}) \geqslant 0$, which gives a contradiction. The proof of Lemma 5.4 is complete. ■

Let now:

$$\mathcal{M} = \{\mu \in \mathcal{X}\colon 0 \leqslant \mu(t) + t \leqslant \pi,\ t \in [0, \pi]\}\,.$$

LEMMA 5.5. – We have that:

i) if $\mu \in \mathfrak{M}$ and $\lambda > 0$, then $(\Xi S_\lambda \mu)(\vartheta) > 0$ $(\vartheta \in]-\pi, \pi[)$;

ii) if $\mu \in \overline{\mathfrak{M}}$ and $\lambda \geqslant 0$, then $(\Xi S_\lambda \mu)(\vartheta) \geqslant 0$ $(\vartheta \in [-\pi, \pi])$.

PROOF. – We only prove part i), since the proof of part ii) is similar. Using the hypotheses, we can easily verify that:

$$\left[\frac{d}{d\vartheta}(S_\lambda \mu)\right](\vartheta) = -\left[\frac{d}{d\vartheta}(S_\lambda \mu)\right](-\vartheta) < 0\,, \qquad \vartheta \in]0, \pi[\,,\ \ \lambda > 0\,,\ \ \mu \in \mathfrak{M}\,. \tag{5.6}$$

Consequently:

$$(S_\lambda \mu)(t_2) < (S_\lambda \mu)(t_1) < 0\,, \qquad \text{if } 0 < |t_1| < |t_2| < \pi\,,\ \ \lambda > 0\,,\ \ \mu \in \mathfrak{M}\,. \tag{5.7}$$

At last we have:

$$(S_\lambda \mu)(t) = (S_\lambda \mu)(|t|)\,, \qquad \text{if } |t| \leqslant \pi\,,\ \ \lambda > 0\,,\ \ \mu \in \mathfrak{M}\,. \tag{5.8}$$

We can prove now:

$$(S_\lambda \mu)(\vartheta + t) < (S_\lambda \mu)(\vartheta - t) < 0\,, \qquad \vartheta, t \in]0, \pi[\,,\ \ \lambda > 0\,,\ \ \mu \in \mathfrak{M}\,. \tag{5.9}$$

To prove this relation we must consider three cases:

First case: $\vartheta - t$, $\vartheta + t \in]-\pi, \pi[$. This case is obvious.

Second case: $\vartheta - t \in]0, \pi]$, $\vartheta + t \in [\pi, 2\pi[$. This fact implies $0 < \vartheta - t < \pi - t < 2\pi - \vartheta - t < \pi$. Hence (by the periodicity) $(S_g \mu)(\vartheta - t) > (S_g \mu)(2\pi - \vartheta - t) = (S_g \mu)(-\vartheta - t) = (S_g \mu)(\vartheta + t)$. The second case is proved.

Third case: $\vartheta - t \in]-\pi, 0]$, $\vartheta + t \in [\pi, 2\pi[$. This fact implies $0 < -(\vartheta - t) < 2\pi - \vartheta - t < \pi$. Hence: $(S_g \mu)(\vartheta - t) = (S_g \mu)(-(\vartheta - t)) > (S_g \mu)(2\pi - \vartheta - t) = (S_g \mu)(\vartheta + t)$. This completes the proof of (5.9).

By an easy calculation we can verify that the value $(\Xi S_g \mu)(\vartheta)$ can be written as:

$$(\Xi S_g \mu)(\vartheta) = -\frac{1}{\pi}\int_0^\pi \frac{(S_g \mu)(\vartheta + t) - (S_g \mu)(\vartheta - t)}{2 tg t/2}\, dt\,.$$

Using this representation of $(\Xi S_g \mu)(\vartheta)$ and recalling relation (5.9), we can complete the proof. ∎

CONCLUSION OF THE PROOF OF PROPOSITION 5.3. – If $\mu \in \mathfrak{X}$ satisfies the hypotheses of Proposition 5.3, thanks to Lemma 5.4 and 5.5, we obtain that:

$$\mu \in \overline{\mathfrak{M}}\,; \qquad \vartheta + \mu(\vartheta) < \pi\,, \quad \vartheta \in]0, \pi[\,.$$

Using Lemma 5.5, it follows that $\mu(\vartheta) \geqslant 0$ ($\vartheta \in [0, \pi]$), hence:

$$0 \leqslant \vartheta \leqslant \vartheta + \mu(\vartheta) < \pi\,, \qquad \vartheta \in]0, \pi[\,,$$

which implies that $\mu \in \mathcal{M}$. If $\vartheta \in]0, \pi[$ and $\lambda > 0$, it follows that $\mu(\vartheta) > 0$. Proposition 5.3 immediately follows. ■

To conclude the proof of Theorem 5.1 we use a topological method based on Schaefer's fixed point theorem (see, for instance, [7], Th. 4.4.11).

DEFINITION 5.6. – Let Z be a normed space and $\mathcal{N}$ a bounded subset of Z. A mapping T: $\mathcal{N} \to Z$ is called *compact* if: i) T is continuous; ii) $\overline{T(\mathcal{N})}$ is a compact subset of Z. ■

DEFINITION 5.7. – Let Z be a normed space and $\mathcal{N}$ a bounded subset of Z. The mapping $\mathcal{W}$: $[0, 1] \times \mathcal{N} \to Z$ is called a *homotopy of compact transformations* on $\mathcal{N}$ if:

i) for all $t \in [0, 1]$ the map $\eta \in \mathcal{N} \to \mathcal{W}(t, \eta) \in Z$ is compact.

ii) for all $\eta \in \mathcal{N}$ the map $t \in [0, 1] \to \mathcal{W}(t, \eta) \in Z$ is continuous. ■

THEOREM (SCHAEFER). – Let Z a normed space and $\mathcal{N}$ a bounded , closed, convex subset of Z containing the origin in its interior. Let $\mathcal{W}$: $[0, 1] \times \mathcal{N} \to Z$ be a homotopy of compact transformations such that:

$$\mathcal{W}(0, \partial\mathcal{N}) \subset \mathcal{N};\ \mathcal{W}(t, \eta) \neq \eta, (t, \eta) \in [0, 1[\times \partial\mathcal{N}\,. \tag{5.10}$$

Then there exists $\mu \in \mathcal{N}$ such that $\mu = \mathcal{W}(1, \mu)$. ■

If $g = 0$, Problem W has one and only one solution given by $\mu \equiv 0$. Then we can assume $g > 0$. Put now $Z = \mathcal{X} \cap C^1(\mathrm{T})$, which is a Banach space with the norm:

$$\|\eta\|_Z = \max\{\,|\eta'(\vartheta)|\,,\ \vartheta \in [-\pi, \pi]\}\,.$$

Put also (k_1 being the positive constant introduced in Lemma 4.9):

$$\mathcal{N} = \{\eta \in Z \cap \mathcal{X}_g\colon\ \|\eta\|_Z \leqslant k_1\}\,,$$

which is a bounded, closed, convex subset of Z containing the origin in its interior. Let also $\mathcal{W}$: $[0, 1] \times \mathcal{N} \to Z$ defined by $\mathcal{W}(t, \eta) = \Xi S_{tg}\eta$. Given $t \in [0, 1]$, the map $\eta \in \mathcal{N} \to$ $\to S_{tg}\eta \in C^2(\mathrm{T})$ is continuous. By (4.10) the map $\eta \in \mathcal{N} \to \Xi S_{tg}\eta \in Z$ is continuous and compact. Similarly we can prove that, given $\eta \in \mathcal{N}$, the map $t \in [0, 1] \to \Xi S_{tg}\eta \in Z$ is continuous. This means that the map $\mathcal{W}$: $[0, 1] \times \mathcal{N} \to Z$ is a homotopy of compact transformations. We also have that $\mathcal{W}(0, \mathcal{N}) = \{0\} \subset \mathcal{N}$, hence the former relation (5.10) is fulfilled. By Lemma 4.9 and Proposition 5.3, we also have that if $\mu \in \mathcal{N}$ verifies $\Xi S_{tg}\mu = \mu$, then μ belongs to the interior of $\mathcal{N}$. Then we have $\mathcal{W}(t, \eta) \neq \eta$ ($t \in]0, 1[$, $\eta \in \partial\mathcal{N}$) and so the latter relation (5.10) is satisfied too.

Using Schaefer's Theorem, we can conclude that there exists $\mu \in \mathcal{N}$ such that $\mu = \mathcal{W}(1, \mu) = \Xi S_g\mu$. This completes the proof of Theorem 5.1. ■

6. – An existence and uniqueness result for the solutions of Problem W.

Put now:

$$\mathcal{M}_1 = \{\mu \in \mathcal{X}\colon\ |\mu(t) - \mu(s)| \leqslant |t - s|\}.$$

We have:

PROPOSITION 6.1. – Given $g \geqslant 0$ sufficiently small, if φ is a solution of Problem W, then $\varphi \in \mathcal{M}_1$. ■

To prove this result we need:

LEMMA 6.2. – Given $g \geqslant 0$ and $\mu \in \mathcal{M}$, we have:

$$\alpha(\Xi S_g \mu) \leqslant 2g[(1 + 6g\pi)(\alpha(\mu) + 1) + 3g], \tag{6.1}$$

where:

$$\alpha(\mu) = \max\{|\mu(\vartheta) - \mu(\xi)|/|\vartheta - \xi|,\ \vartheta \neq \xi\}. \tag{6.2}$$

PROOF. – If $F(t) = (S_g\mu)(\vartheta + t) - (S_g\mu)(\xi + t)$, it follows that:

$$|(\Xi S_g\mu)(\vartheta) - (\Xi S_g\mu)(\xi)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \frac{F(t) - F(-t)}{2 tg t/2}\,dt\right| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |F'(\zeta(t))|\,dt,$$

where $|\zeta(t)| < |t|$. By an easy calculation we have:

$$|F'(\zeta(t))| \leqslant |\vartheta - \xi|[g(1 + 6g\pi)(\alpha(\mu) + 1) + 3g^2],$$

which completes the proof. ■

PROOF OF PROPOSITION 6.1. – Let φ a solution of Problem W. Thanks to Lemma 6.2, we obtain:

$$\alpha(\varphi) \leqslant 2g[(1 + 6g\pi)(\alpha(\varphi) + 1) + 3g].$$

Recalling that $g \geqslant 0$ is small enough, we can conclude the proof. ■

We can now prove the main result of the present section:

THEOREM 6.3. – If $g \geqslant 0$ is small enough, then Problem W admits one and only one solution $\varphi \in \mathcal{M}_1$.

PROOF. – Using Lemma 6.2 we easily obtain that $(\Xi S_g)(\mathcal{M}_1) \subset \mathcal{M}_1$. By a simple calculation we obtain:

$$\|S_g\eta - S_g\mu\|_1 \leqslant g(1 + 6g\pi)\|\eta - \mu\|,\ \eta, \mu \in \mathcal{M}_1,$$

where the norms $\|\ \|$ and $\|\ \|_1$ are defined in (4.17) and (4.18). Recalling (4.10), it follows that the map Ξ: $C^1([-\pi, \pi]) \to C^0([-\pi, \pi])$ is a linear and continuous map, hence there exists a constant $C > 0$ such that $\|\Xi h\| \leq C\|h\|_1$ for all $h \in \mathcal{Y}$. Therefore:

$$\|\Xi S_g \eta - \Xi S_g \mu\| \leq C\|S_g \eta - S_g \mu\|_1 \leq Cg(1 + 6g\pi)\|\eta - \mu\|, \qquad \eta, \mu \in \mathcal{M}_1$$

and then the map ΞS_g: $\mathcal{M}_1 \to \mathcal{M}_1$ is a contraction mapping (if g is small enough). This completes the proof. ∎

REMARK 6.4. – Recalling Theorem 4.8, Theorem 6.3 states that, if $g \geq 0$ is small enough, then there exists one and only one physical solution Ω defined by Problem A. Moreover, by the part iii) of Theorem 4.8, since $\varphi \in \mathcal{M}_1 \subset \mathcal{M}_0$, Ω is a convex subset of $\mathbb{C}$. ∎

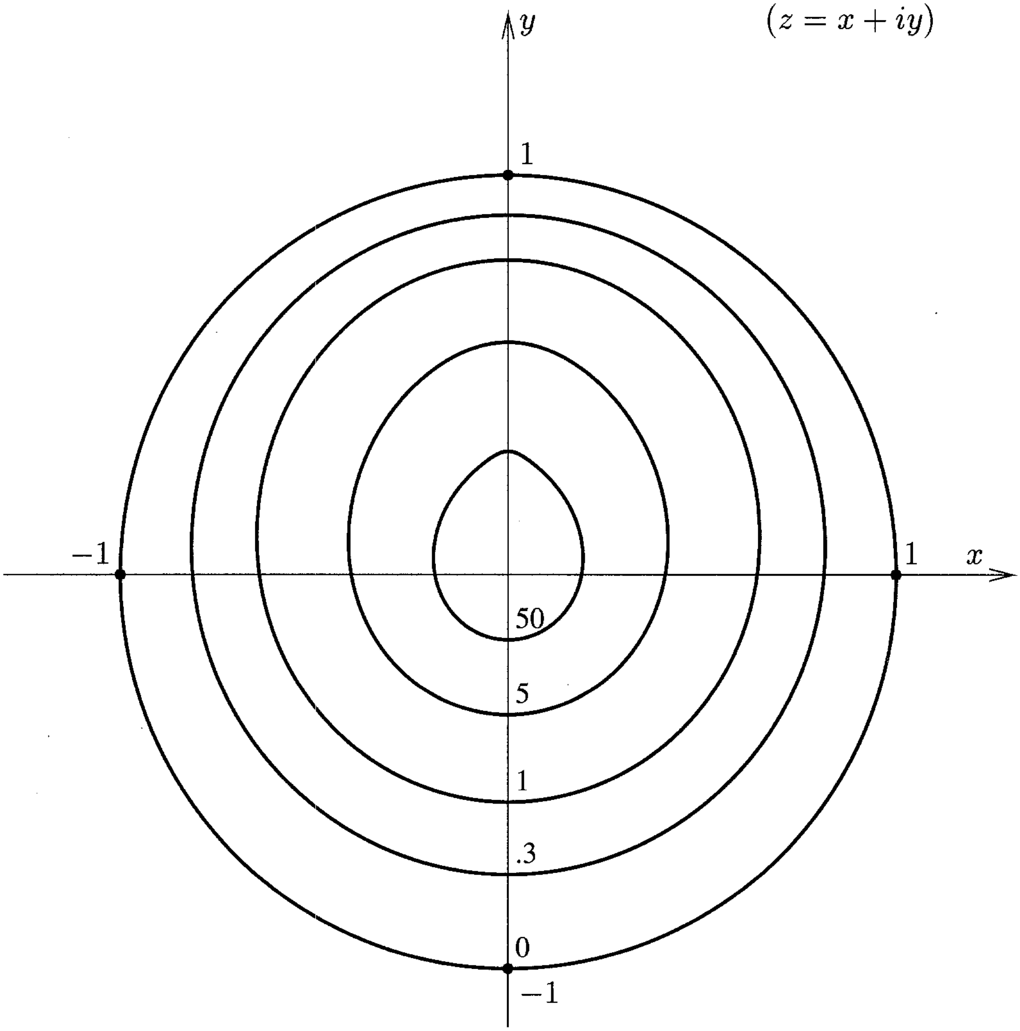

Fig. 2. – Monotone behaviour of the set Ω as a function of g.

7. – Some numerical experiment.

The weak formulation of the problem can be used to obtain an heuristic approximation of the problem. Discretizing the integral which appears in the definition (4.19) of operator S_g, we can introduce an approximate operator S_g^h. In a similar way we can define an operator Ξ_h starting from the definition (4.7) of operator Ξ. Setting now $\varphi_0 \equiv 0$ and computing (by the program MATLAB) $\varphi_{n+1} = \Xi_h S_g^h \varphi_n$, we can experimentally obtain, after 7 – 8 iterations, that $|\varphi_{n+1} - \varphi_n| < 10^{-5}$. Choosing the values of $g = 0$, .3, 1, 5, 50, we can determine the corresponding shape of Ω. The results, printed using the software POSTSCRIPT and described in Figure 2, show us a monotone behaviour of Ω as a function of g: I am not able to verify this property from the theoretical point of view.

Acknowledgments. Paper supported by I.A.N.-C.N.R. (Pavia-Italy) and by the 40% M.U.R.S.T. Project «Problemi non lineari dell'Analisi e delle Applicazioni fisiche chimiche e biologiche...» (Italy).

REFERENCES

[1] E. ALESSANDRINI - D. BOFFI - A. TORELLI, *Study of an obstacle vortex free boundary problem*, to appear on Boll. U.M.I.
[2] P. L. DUREN, *Theory of H^p Spaces*, Academic Press, New York (1970).
[3] P. L. DUREN, *Univalent Functions*, Springer-Verlag (1983).
[4] J. B. GARNETT, *Bounded Analytic Functions*, Academic Press, New York (1981).
[5] H. HELSON, *Harmonic Analysis*, Addison Wesley, London (1983).
[6] A. LEZZI-A. TORELLI, *On the use of the complex method in the study of a vortex free boundary problem*, J. Math. Anal. and Appl., **186**, n. 3, (1994), pp. 649-673.
[7] N. G. LLOYD, *Degree Theory*, Cambridge University Press (1978).
[8] CH. POMMERENKE, *Boundary Behaviour of Conformal Maps*, Springer-Verlag (1992).
[9] M. H. PROTTER - H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, INC., Englewood Cliffs, New Jersey (1967).
[10] A. TORELLI, *Study of a vortex free boundary problem by the complex method*, on *Boundary Value Problems for Partial Differential Equations and Applications*, Masson, Paris (1993).
[11] A. TORELLI, *A plane vortex free boundary problem: existence and uniqueness results using the complex method*, in *Proceedings of the Meeting: Non Linear Boundary Problems (Donetsk-Laspi, Ukraina, 24-29 May 1993)*.
[12] A. TORELLI, *A plane steady vortex free boundary problem: existence and uniqueness results using elementary methods of harmonic analysis*, in *Proceedings of the Meeting: Elliptic and Parabolic P.D.E.'s and Applications (Capri, Italy, 19-23 September 1994)*.