Function Spaces $W^{1,p}$, BV , $N^{\lambda,p}$ Characterized by the Motions of $\mathbb{R}^n$ (*).

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Abstract. – We characterize the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ ($p \in]1, +\infty[$), the space $BV(\mathbb{R}^n)$ of functions of bounded variation, the Nikol'skij spaces $N^{\lambda,p}(\mathbb{R}^n) = B_{p,\infty}^{\lambda}(\mathbb{R}^n)$ ($p \in [1, +\infty[$, $\lambda \in]0, 1[$) by means of the group of all motions—i.e., orientation preserving isometries—of $\mathbb{R}^n$ (these spaces are usually defined or characterized by means of differences of functions, involving only the translations). Moreover, we show that the characterizations we present here give rise to norms equivalent to the usual ones. This is done also with the goal of extending the results from $\mathbb{R}^n$ to noncompact homogeneous manifolds, in particular in connection with applications to PDE's.

0. – Introduction.

This paper deals with function spaces defined on $\mathbb{R}^n$, namely with the Sobolev spaces $W^{1,p}$ ($p \in]1, +\infty[$), the space BV of functions of bounded variation, the Nikol'skij spaces $N^{\lambda,p}$ ($p \in [1, +\infty[$, $\lambda \in]0, 1[$), which form a special class of Besov spaces ($N^{\lambda,p} = B_{p,\infty}^{\lambda}$), and with the closed separable subspaces $\dot{N}^{\lambda,p}$ of $N^{\lambda,p}$.

The spaces $W^{1,p}$ and BV are usually defined by means of distributional derivatives; the spaces $N^{\lambda,p}$ were initially defined in terms of differences, which allow the characterization of $W^{1,p}$ and BV as well. Both concepts of derivative and of difference involve the translations of $\mathbb{R}^n$, i.e. all these function spaces can be described by their behaviour with respect to the action of the translation group.

Indeed, if u is a function of $L^p(\mathbb{R}^n)$, then the finiteness of

$$(0.1) \quad \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-\lambda} \|u(x+h) - u(x)\|_{L^p(\mathbb{R}^n)}$$

measures the «smoothness» of the function u , i.e. it determines to which classes of the previous spaces the function u belongs.

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(In this quotient one has to set $\lambda = 1$ for the spaces $W^{1,p}$, $p = \lambda = 1$ for the space BV ; for the spaces $N^{\lambda,p}$ one has to substitute the supremum over h with the condition that the limit for $h \rightarrow 0$ of the quotient must be zero.)

The aim of this paper is to investigate what kind of results we obtain if we substitute the translations with the group of all motions (i.e. orientation preserving isometries) of $\mathbb{R}^n$. More precisely we replace in (0.1) $u(x+h)$ by $u(g \cdot x)$, where g is a motion, and $|h|$ by a properly defined distance of g from the identity transform. In the case of the spaces $W^{1,p}$ and BV , we also replace the distributional derivatives by the vector fields given by the infinitesimal generators of the one-parameter subgroups of the motion group.

Apart from the intrinsic interest of the subject, our investigations are motivated as follows: we would extend, again using differences and isometries, the definition of these function spaces to those Riemannian manifolds which possess—like $\mathbb{R}^n$ —a transitive Lie transformation group of isometries (namely, the so-called homogeneous manifolds, provided with an invariant Riemannian structure). It is clear that we need to know that in $\mathbb{R}^n$ the new definition is equivalent to the usual one.

Moreover, we hope that the planned treatment on manifolds could prove useful also in connection with applications to PDE's on Riemannian homogeneous manifolds, since it permits the exploitation of the invariance of the Laplace-Beltrami operator with respect to isometries.

For instance in [23; Sections 1 and 2] a regularity result of type $N^{\lambda,1}$ was proven for the solution of the two-phase Stefan problem in a bounded domain of $\mathbb{R}^n$; the demonstration makes use, among other things, of the invariance property of the Laplacian with respect to the translations of $\mathbb{R}^n$. For a compact homogeneous manifold M it was the definition of $N^{\lambda,1}(M)$ by means of the isometries which led to an analogous result in [6; Section 3] (see also [22; 5,c]).

This paper is self-contained; however, it can also be seen as a completion and extension of the first part of [5] and of n. 2 of [3]. The note [4] is a preliminary version of some sections of the present paper.

The plan of this paper is as follows. In Section 1 we recall the basic definitions which we will use later on. Section 2 deals with some geometric preliminaries on motions in $\mathbb{R}^n$. Sections 3, 4, 5 and 6 are respectively devoted to the function spaces $W^{1,p}$, BV , $N^{\lambda,p}$, and $N^{\lambda,p}$ (both for non-compactly supported and for compactly supported functions). In Section 7 we collect some counterexamples, especially in order to show that the results for functions with compact support do not hold for all functions. Finally, Section 8 concludes the paper with additional remarks, in particular with some considerations on homogeneous manifolds.

Since this note has two authors, the following remark may be appropriate: Section 2 was supplied by M. P. BERNARDI, while Sections 3, 5, and 6 are due to C. BONDIOLI; finally, Sections 4 (that was the first to be devised), 7 and 8 result from common work.

1. – Notation and definitions.

In the sequel, we will consider mostly functions defined on $\mathbb{R}^n$ and in this case we will omit, for shortness, $\mathbb{R}^n$. Similarly, $\int$ means $\int_{\mathbb{R}^n}$.

We write $|x|$ for the euclidean norm of $x \in \mathbb{R}^n$, and analogously $|A|$ denotes the euclidean norm of the matrix $A = [a_{ij}]$ in $\mathbb{R}^{(n^2)}$.

If r is a positive number, let B_r be the closed ball in $\mathbb{R}^n$, centred at the origin and of radius r .

We denote by C_c the space of all continuous functions on $\mathbb{R}^n$ with compact support, equipped with the sup-norm, and with C'_c the corresponding dual space, equipped with the strong topology (i.e., C'_c is the space of all finite measures on $\mathbb{R}^n$).

Let $\mathcal{O}$ denote the space of all C^∞ functions on $\mathbb{R}^n$ with compact support. Moreover, if r is a positive number, let $\mathcal{O}_r = \{\varphi \in \mathcal{O} \mid \text{supp } \varphi \subseteq B_r\}$. Finally let $\mathcal{O}'$ be the space of all distributions on $\mathbb{R}^n$.

This paper is concerned with the following four classes of function spaces.

(I) The *Sobolev spaces* $W^{1,p}$ with $1 < p \leq +\infty$:

$$W^{1,p} = \{u \in L^p \mid \text{for every } i = 1, \dots, n, D_i u \in L^p\},$$

equipped with the norm

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \sum_{i=1}^n \|D_i u\|_{L^p}^p \right)^{1/p}$$

(obvious modification for $p = \infty$).

(II) The space BV of *functions of bounded variation* (which replaces, in some sense, the case $p = 1$ of (I)):

$$BV = \{u \in L^1 \mid \text{for every } i = 1, \dots, n, D_i u \in C'_c\},$$

equipped with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \sum_{i=1}^n \|D_i u\|_{C'_c}.$$

(III) The *Nikol'skij spaces* $N^{\lambda,p}$ with $p \in [1, +\infty[$, $\lambda \in]0, 1[$:

$$N^{\lambda,p} = \left\{ u \in L^p \mid \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-\lambda} \|\tau_h u - u\|_{L^p} < +\infty \right\}$$

(here $\tau_h u(x) = u(x+h)$ and so $\tau_h u - u$ is the usual first difference $\Delta_h^1 u$ of the function u), equipped with the norm

$$\|u\|_{N^{\lambda,p}} = \|u\|_{L^p} + \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-\lambda} \|\tau_h u - u\|_{L^p}.$$

Note that the spaces $N^{\lambda,p}$ form a special class of Besov (or Lipschitz) spaces, namely, $N^{\lambda,p} = B_{p\infty}^{\lambda}$.

(IV) The *closed subspaces* $\mathring{N}^{\lambda,p}$ of $N^{\lambda,p}$ with $p \in [1, +\infty[$, $\lambda \in]0, 1[$:

$$\mathring{N}^{\lambda,p} = \{u \in L^p \mid \|\tau_h u - u\|_{L^p} = o(|h|^{\lambda}) \text{ for } h \rightarrow 0\},$$

with the norm inherited from $N^{\lambda,p}$.

For references on these function spaces see, for example, [1], [19], [27], [34], [35], [37], for the cases (I) and (III); for the space BV we refer, for instance, to the books [13] and [37]; for the spaces $N^{\lambda,p}$ see, for example, [10; Chapter 3], [33] and [34; n. 2.11].

We need also some algebraic and geometric notation.

We want to make use of the isometries of the euclidean space $\mathbb{R}^n$, which preserve the orientation; we call them *motions* of $\mathbb{R}^n$ and denote by $\mathcal{M}$ the group they constitute. So $\mathcal{M}$ is a Lie group, namely $\mathcal{M} = SO(n) \ltimes \mathbb{R}^n$, where $SO(n)$ denotes the special orthogonal group (i.e., the group of all $n \times n$ orthogonal matrices of determinant 1); $\mathcal{M}$ is the connected component of the identity in the whole group $O(n) \ltimes \mathbb{R}^n$ of all isometries of $\mathbb{R}^n$.

Briefly, we denote the motion $M_{(A,b)}: x \mapsto Ax + b$ with (A, b) and, where useful, we identify $M_{(A,b)}$ with the $(n+1) \times (n+1)$ matrix

$$(1.1) \quad \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.$$

Moreover, for the sake of simplicity, A stands for the motion $(A, 0)$ and b for (I, b) .

We recall that the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ consists of all $n \times n$ skew-symmetric matrices and that the Lie algebra of $\mathcal{M}$ is equal to $\mathfrak{so}(n) \oplus \mathbb{R}^n$; we will denote this by $\mathfrak{m}$ and its elements by (X, a) . Similarly to (1.1), the elements of $\mathfrak{m}$ can also be written in the form

$$(1.2) \quad \begin{pmatrix} X & a \\ 0 & 0 \end{pmatrix},$$

which suggests the definition of the bracket (see, e.g., [24; III.7]).

Finally, let $\exp$ and $\exp_{\mathcal{M}}$ be respectively the exponential mappings of $\mathfrak{so}(n)$ onto $SO(n)$ and of $\mathfrak{m}$ onto $\mathcal{M}$.

2. – Some geometric and algebraic preliminaries.

In order to simplify the exposition, we recall that for every $X \in \mathfrak{so}(n)$ there exists an orthonormal basis of $\mathbb{R}^n$, with respect to which X is represented by a matrix of the

following canonical form

$$(2.1) \quad X = \begin{bmatrix} X_1 & & & \\ & \ddots & & \\ & & X_l & \\ & & & X^* \end{bmatrix},$$

where for every $i = 1, \dots, l$

$$(2.2) \quad X_i = \begin{bmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{bmatrix} \quad \text{with } \theta_i \in \mathbb{R}, \text{ and } \theta_i \neq 2k\pi \quad \forall k \in \mathbb{Z},$$

and

$$(2.3) \quad X^* = \begin{bmatrix} X_1^* & & \\ & \ddots & \\ & & X_{m-l}^* \end{bmatrix} \quad \text{if } n = 2m \text{ is even,}$$

$$X^* = \begin{bmatrix} X_1^* & & \\ & \ddots & \\ & & X_{m-l}^* \\ & & & 0 \end{bmatrix} \quad \text{if } n = 2m + 1 \text{ is odd,}$$

where for every $j = 1, \dots, m - l$

$$(2.4) \quad X_j^* = \begin{bmatrix} 0 & -2k_j\pi \\ 2k_j\pi & 0 \end{bmatrix} \quad \text{with } k_j \in \mathbb{Z}.$$

From now on we assume that for every $X \in \mathfrak{so}(n)$ the coordinate system is chosen so that (2.1) holds.

With respect to the same basis the matrix $A = \exp X \in SO(n)$ is written as

$$(2.5) \quad A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_l \\ & & & I \end{bmatrix},$$

where I stands for the identity matrix and for every $i = 1, \dots, l$

$$(2.6) \quad A_i = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}.$$

Since we are mainly interested in the group $SO(n)$, we may suppose that in (2.2) and in (2.4)

$$(2.7) \quad \theta_i \in]-\pi, \pi] \setminus \{0\} \quad (i = 1, \dots, l) \quad \text{and} \quad k_j = 0 \quad (j = 1, \dots, m-l).$$

We note that in this case

$$(2.8) \quad |X| = \min \{ |Y| \mid Y \in \mathfrak{so}(n) \text{ and } \exp Y = \exp X \}.$$

From the canonical form of X one immediately recognizes that $\mathbb{R}^n$ is an orthogonal direct sum: $\text{Im } X \oplus \text{Ker } X$ (this is a general property of normal matrices). Moreover, under the assumption (2.7), $\text{Ker } X = \{x \in \mathbb{R}^n \mid \exp X \cdot x = x\}$ and the restriction of X to $\text{Im } X$, i.e.

$$\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_l \end{pmatrix}$$

is non singular. So every $a \in \mathbb{R}^n$ can be decomposed uniquely as

$$(2.9) \quad a = Xv + w \quad \text{with } v \in \text{Im } X \text{ and } w \in \text{Ker } X.$$

Let us remark that a and v are orthogonal vectors.

By means of (2.9), one can easily verify the following lemma (which could also be derived from (1.2), using the exponential function of matrices and again (2.9)).

LEMMA 2.1. – For every $(X, a) \in \mathfrak{m}$ the one-parameter subgroup $\exp_{\mathfrak{M}} t(X, a)$ is given by the formula

$$(2.10) \quad \exp_{\mathfrak{M}} t(X, a) = (\exp tX, (\exp tX - I)v + tw) \quad (t \in \mathbb{R}),$$

where v and w are defined as in (2.9).

REMARK 2.1. – For later use, let us explicitly write the following simple consequences of formula (2.10):

$$(2.11) \quad (\exp_{\mathfrak{M}} t(X, a))^{-1} = \exp_{\mathfrak{M}} t(-X, -a) = ((\exp tX)^{-1}, ((\exp tX)^{-1} - I)v - tw);$$

$$(2.12) \quad \text{if } a \in \text{Ker } X, \quad \text{then } \exp t(X, a) = (\exp tX, ta).$$

REMARK 2.2. – Lemma 2.1 allows us, if we know $(A, b) \in \mathfrak{M}$, to find explicitly $(X, a) \in \mathfrak{m}$ such that $\exp_{\mathfrak{M}}(X, a) = (A, b)$. In greater detail, the statement which follows, holds.

Let $(A, b) \in \mathfrak{M}$. We suppose that A is written as in (2.5). Let X be the only matrix in $\mathfrak{so}(n)$ such that $\exp X = A$ and (2.7) holds. With respect to X , we decompose b as $b = b' + b''$, with $b' \in \text{Im } X$ and $b'' \in \text{Ker } X$ (so that $b' \perp b''$). Let v be the only vector in

Im X such that $b' = (\exp X - I)v$. Then

$$(2.13) \quad (A, b) = \exp_{\mathcal{M}}(X, a), \quad \text{where } a = Xv + b''.$$

REMARK 2.3. – Formula (2.10) can be given a geometric interpretation. For simplicity let us consider the case $n = 3$ (the case $n = 2$ is obtained as a by-product).

Let $(X, a) \in \mathfrak{so}(3) \oplus \mathbb{R}^3 = \mathfrak{m}$. We suppose that X is written in the canonical form (2.1) and (2.2) and that $\exp X \neq I$ (so that the axis of the rotation $\exp X$ is chosen as the x_3 -axis of the coordinate system in $\mathbb{R}^3$). We decompose $a = Xv + w$ as in (2.9). Then

$$\exp_{\mathcal{M}}(X, a) = (\exp X, (\exp X - I)v + w) = (\exp X, (\exp X - I)v) \circ (I, w)$$

is the composition in $\mathcal{M}$ (in both orders) of the following two motions:

the translation (I, w) determined by w (therefore in direction of the x_3 -axis);

the rotation $(\exp X, (\exp X - I)v)$ of angle θ_1 , whose axis is the straight line parallel to the x_3 -axis and passing through the point $-v$.

REMARK 2.4. – Usually, to represent a motion of $\mathbb{R}^n$ in canonical form, one chooses suitably both the origin of the coordinate system and its axes (see, e.g., [28; Chapter 10]). Nevertheless we prefer - here and later on - to fix, once and for all, the origin of the coordinate system.

In the next sections we will have to measure the distance between a motion and the identity; so for the rest of this section we make some considerations on this subject. In [5] (and in [3], [6]), in order to value how «far» a motion $(A, b) = \exp_{\mathcal{M}}(X, a)$ is from the identity, we used $|X| + |b|$. More precisely, let us recall the following definition.

DEFINITION 2.1. – Let $A \in SO(n)$ be represented in the form $A = \exp X$, with X satisfying (2.7). We define the distance of the motion $(A, b) = (\exp X, b)$ from the identity map $(I, 0)$ as $d(A, b) = |X| + |b|$.

We emphasize that $|X| + |b|$ does not depend on the orthonormal basis fixed in $\mathbb{R}^n$.

Recalling (2.13), it is quite natural as proven here, that $|b|$ can be replaced by $|a|$. This gives us the following lemma.

LEMMA 2.2. – Let $(\exp X, b) = \exp_{\mathcal{M}}(X, a)$ (where X has the canonical form (2.1) and (2.7) holds). Then

$$(2.14) \quad \frac{2}{\pi} |a| \leq |b| \leq |a|.$$

PROOF. — We expose the proof for $n = 3$; the general case can be treated in the same way. Let

$$X = \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = Xv + w, \quad \text{where } Xv = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \text{ and } w = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}.$$

From (2.10) with $t = 1$, we derive $b = (\exp X - I)v + w = b' + w$. So we have only to compare $|Xv|$ and $|b'|$.

An easy calculation leads to:

$$|Xv| = |\theta| |v|; \quad |b'| = |(\exp X - I)v| = \left(2 \sin \frac{|\theta|}{2} \right) |v| = \frac{2 \sin \frac{|\theta|}{2}}{|\theta|} |Xv|.$$

Since

$$(2.15) \quad 0 < \frac{|\theta|}{2} \leq \frac{\pi}{2} \quad \text{implies} \quad \frac{2}{\pi} \leq \frac{2 \sin \frac{|\theta|}{2}}{|\theta|} < 1,$$

we obtain $2\pi^{-1} |Xv| \leq |b'| \leq |Xv|$ and therefore $2\pi^{-1} |a| \leq |b| \leq |a|$.

REMARK 2.5. — Again, we can explain some of the previous formulas from a geometric point of view.

First, with respect to (2.10), let us remark that a is the tangent vector to the curve $\eta(t) = \exp_{\mathcal{M}} t(X, a) \cdot O$ (where $O = (0, 0)$) at the time $t = 0$. In fact

$$\dot{\eta}(0) = \lim_{t \rightarrow 0} \frac{\exp tX - I}{t} v + w = Xv + w = a.$$

So $|a|$ represents also the arc-length of η between O and $b = (A, b) \cdot O$ (while evidently $|b|$ is the corresponding euclidean distance).

In this way also Lemma 2.2 gains a clearer meaning.

In particular, let $n = 2$ and let us consider a motion $(A, b) = \exp_{\mathcal{M}}(X, a)$, with $A \neq I$ and $b \neq 0$. Then $\eta(t)$ describes a circle through the origin of radius $|v|$ (see Remark 2.3). Also in this case we have to compare the length of an arc of circle with the length of the corresponding chord. So, in relation to (2.15), we notice that the term $2 \sin(|\theta|/2)$ stands for the euclidean distance of any point of a unit circle from its image under the rotation around the centre, with angle $\theta \in]-\pi, \pi]$ (while $|\theta|$ is the corresponding arc-length).

This last remark will be useful also in the proof of the next lemma.

Lemma 2.2 shows that $|X| + |a|$ and $|X| + |b|$ could be good candidates to define the distance between the motion $(\exp X, b)$ and the identity.

However, later on, in order to consider functions with non compact support as well, we will be forced to use a minus direct method to value the difference of a motion from the identity; to this aim, we will constantly employ the inequalities proven in the following lemma.

LEMMA 2.3. – *Let (A, b) be a motion of $\mathbb{R}^n$; suppose $A = \exp X$, where $X \in \mathfrak{so}(n)$ is such that (2.7) holds.*

Then, for every positive number r , the following inequalities hold:

$$(2.16) \quad \max_{B_r} |Ax + b - x| \leq r |X| + |b| \leq \sqrt{\frac{n}{2}} \pi \max_{B_r} |Ax + b - x|.$$

PROOF. – It is not difficult to verify that $\max_{B_r} |Ax + b - x|$ is attained on the boundary ∂B_r of B_r ; so we have only to consider $|x| = r$.

First, we suppose that the motion is a rotation A around the origin, different from the identity. In this case, we choose as before an orthonormal basis of $\mathbb{R}^n$ with respect to which A is represented by a matrix of the canonical form (2.5), where the θ_i 's satisfy (2.7) and moreover $|\theta_1| \geq |\theta_2| \geq \dots \geq |\theta_l| > 0$.

So for every $x \in \mathbb{R}^n$ we have

$$|Ax - x|^2 = \sum_{i=1}^l (x_{2i-1}^2 + x_{2i}^2) \left(2 \sin \frac{|\theta_i|}{2} \right)^2.$$

Using the method of Lagrange's multipliers a simple computation shows:

If $|\theta_1| = |\theta_2| = \dots = |\theta_k| > |\theta_{k+1}| \geq \dots$ ($1 \leq k \leq l$), then the maximum of $|Ax - x|$ is attained at the points x of ∂B_r such that $x_{2k+1} = \dots = x_n = 0$. Let S denote the set of these points; S is a $(2k-1)$ -dimensional sphere, of radius r .

If $x \in S$, then $|Ax - x| = 2r \sin(|\theta_1|/2)$; so, setting $\theta = (\theta_1, \dots, \theta_l)$, we obtain

$$\max_{B_r} |Ax - x| = 2r \sin \frac{\|\theta\|_\infty}{2}$$

(where $\|\theta\|_\infty$ stands for $\max\{|\theta_1|, \dots, |\theta_l|\}$).

Since $\|\theta\|_\infty \leq |\theta| \leq \sqrt{l} \|\theta\|_\infty$, by (2.15) we get

$$(2.17) \quad \max_{B_r} |Ax - x| \leq r |\theta| \leq \sqrt{l} \frac{\pi}{2} \max_{B_r} |Ax - x|.$$

Since $|X| = \sqrt{2} |\theta|$, we can conclude that

$$(2.18) \quad \sqrt{2} \max_{B_r} |Ax - x| \leq r |X| \leq \sqrt{n} \frac{\pi}{2} \max_{B_r} |Ax - x|.$$

Now let us consider the general case of a motion (A, b) ; we claim that

$$(2.19) \quad \max_{B_r} |Ax + b - x|^2 \geq \max_{B_r} |Ax - x|^2 + |b|^2.$$

In fact, $|Ax + b - x|^2 = |Ax - x|^2 + |b|^2 + 2(Ax - x, b)$; moreover, since the set S (see the first part of the proof) is a sphere with centre in the origin, if $x \in S$, also $-x \in S$; so we can always consider x_0 such that

$$|Ax_0 - x_0| = \max_{B_r} |Ax - x| \text{ and } (Ax_0 - x_0, b) \geq 0.$$

From (2.18) and (2.19) we derive

$$(2.20) \quad \max_{B_r} |Ax + b - x|^2 \geq \frac{4}{n\pi^2} r^2 |X|^2 + |b|^2 \geq \frac{2}{n\pi^2} (r|X| + |b|)^2$$

and again from (2.18) we get

$$(2.21) \quad \max_{B_r} |Ax + b - x| \leq r|X| + |b|.$$

The lemma is thus proven.

REMARK 2.6. – We point out that $(n/2)^{1/2}\pi$ on the right-hand side of (2.16) cannot be replaced by a constant c , independent of n . In fact, let us consider the following example: let n be any even integer and set $r = 1$, $A = -I$, $b = 0$. Then $|X| = \sqrt{n}\pi$ and $\max_{B_1} |Ax - x| = 2$.

3. – Characterization of the Sobolev spaces $W^{1,p}$ via motions.

In this section we study the Sobolev spaces $W^{1,p}$, where $1 < p \leq +\infty$ (see Section 1, (I) for their definition). We always use the notation and the assumptions stated in Section 2.

It is well known (see, e.g., [8; Théorèmes VIII.3 and IX.3]) that the following theorem holds.

THEOREM A. – *Let $p \in]1, +\infty]$. For functions in L^p the following conditions are equivalent:*

- (i) $u \in W^{1,p}$;
- (ii) *there exists $C > 0$ such that, for every $i = 1, \dots, n$, for every $\varphi \in \mathcal{O}$,*

$$(3.1) \quad \left| \int u(x) D_i \varphi(x) dx \right| \leq C \|\varphi\|_{L^p} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right);$$
- (iii) $q_p(u) = \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-1} \|\tau_h u - u\|_{L^p} < +\infty$.

Moreover $\|u\|_{L^p} + \inf \{C \mid (3.1) \text{ holds}\}$ and $\|u\|_{L^p} + q_p(u)$ are equivalent to $\|u\|_{W^{1,p}}$.

We now state and prove a characterization of the spaces $W^{1,p}$ by means of the group $\mathcal{M}$, which is analogous to the characterization given by (iii).

DEFINITION 3.1. – Let $u \in L^p$, $p \in]1, +\infty]$. For every positive number r and for every motion $(A, b) = (\exp X, b) \neq (I, 0)$, we define:

$$(3.2) \quad Q_{(A, b), r, p}(u) = \frac{\|\tau_{(A, b)} u - u\|_{L^p(B_r)}}{r|X| + |b|}$$

and

$$(3.3) \quad Q_{(A, b), r, p}(u) = \frac{\|\tau_{(A, b)} u - u\|_{L^p(B_r)}}{\max_{B_r} |Ax + b - x|},$$

where $\tau_{(A, b)} u(x) = u(Ax + b)$.

THEOREM 3.1. – Let $u \in L^p$, $p \in]1, +\infty]$. Then $u \in W^{1,p}$ if and only if

$$(3.4) \quad Q_p(u) = \sup_{r > 0} \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} Q_{(A, b), r, p}(u) < +\infty$$

or equivalently:

$$(3.5) \quad Q_p(u) = \sup_{r > 0} \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} Q_{(A, b), r, p}(u) < +\infty.$$

Moreover $\|u\|_{W^{1,p}}$, $\|u\|_{L^p} + Q_p(u)$ and $\|u\|_{L^p} + Q_p(u)$ are equivalent norms in $W^{1,p}$.

PROOF. – First, let us consider $p \neq +\infty$. We recall that, in this case, $\mathcal{O}$ is dense in $W^{1,p}$.

Let $u \in L^p$, satisfying (3.4). If in (3.4) we choose $A = I$ (and therefore $X = 0$), then by Theorem A it follows immediately that $u \in W^{1,p}$. Moreover, $q_p(u) \leq Q_p(u)$.

Viceversa, let $u \in W^{1,p}$ ($1 < p < +\infty$).

First let $u \in \mathcal{O}$. We want to estimate $\|\tau_{(A, b)} u - u\|_{L^p(B_r)}$. To this aim, let us consider the curve $\alpha(t) = \exp tX \cdot x + tb$, with $t \in [0, 1]$. We have

$$(3.6) \quad |u(\exp X \cdot x + b) - u(x)| = \left| \int_0^1 \nabla u(\exp tX \cdot x + tb) \cdot (\exp tX \cdot X \cdot x + b) dt \right| \leq$$

$$\leq \int_0^1 \sum_{i=1}^n |D_i u(\exp tX \cdot x + tb)| \cdot |\exp tX \cdot X \cdot x + b| dt \leq$$

$$\leq \left[\sum_{i=1}^n \int_0^1 |D_i u(\exp tX \cdot x + tb)| dt \right] \cdot [|X| \cdot |x| + |b|],$$

as $|\exp tX \cdot X \cdot x| = |X \cdot x| \leq |X| \cdot |x|$, since $\exp tX \in SO(n)$.

So, if $|x| < r$ we obtain

$$|u(\exp X \cdot x + b) - u(x)| \leq \left[\sum_{i=1}^n \int_0^1 |D_i u(\exp tX \cdot x + tb)| dt \right] \cdot [r|X| + |b|]$$

and by the Schwarz-Hölder inequality we conclude

$$|u(\exp X \cdot x + b) - u(x)|^p \leq [r|X| + |b|]^p \cdot \left[\sum_{i=1}^n \left(\int_0^1 |D_i u(\exp tX \cdot x + tb)|^p dt \right)^{1/p} \right]^p.$$

If $(a_1, \dots, a_n)$ is an n -tuple of positive real numbers and $p \in [1, +\infty[$, then $(a_1 + \dots + a_n)^p \leq n^{p-1}(a_1^p + \dots + a_n^p)$; so it follows that

$$|u(\exp X \cdot x + b) - u(x)|^p \leq n^{p-1} [r|X| + |b|]^p \cdot \left[\sum_{i=1}^n \int_0^1 |D_i u(\exp tX \cdot x + tb)|^p dt \right].$$

Therefore

$$\begin{aligned} (3.7) \quad \|u(\exp X \cdot x + b) - u(x)\|_{L^p(B_r)}^p &\leq \\ &\leq n^{p-1} [r|X| + |b|]^p \cdot \left[\sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^n} |D_i u(\exp tX \cdot x + tb)|^p dx dt \right] = \\ &= n^{p-1} [r|X| + |b|]^p \cdot \left[\sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^n} |D_i u(x)|^p dx dt \right] = \\ &= n^{p-1} [r|X| + |b|]^p \sum_{i=1}^n \|D_i u\|_{L^p}^p. \end{aligned}$$

So, for functions $u \in \mathcal{O}$, we have proven that (3.4) holds and moreover

$$Q_p(u) \leq n^{1/p'} \left(\sum_{i=1}^n \|D_i u\|_{L^p}^p \right)^{1/p}.$$

Since $\mathcal{O}$ is dense in $W^{1,p}$, we obtain the same result for every $u \in W^{1,p}$.

Finally, by Lemma 2.3, the proof for $p \in]1, +\infty[$ is concluded.

Now let us consider $p = +\infty$.

If $u \in L^\infty$ verifies (3.4), then, as in the previous case, $u \in W^{1,\infty}$ and $q_\infty(u) \leq Q_\infty(u)$.

Conversely, let $u \in W^{1,\infty}$. We recall that $W^{1,\infty}$ consists of Lipschitz-continuous functions:

$$|u(x) - u(y)| \leq \|\nabla u\|_{L^\infty} |x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

where $\|\nabla u\|_{L^\infty} = \max \{\|D_i u\|_{L^\infty} \mid i = 1, \dots, n\}$ (see, e.g., [8; Chapitre IX, Remarque 8]).

Therefore, for every $(A, b) \in \mathcal{H}$, for every $r > 0$, we have

$$\max_{B_r} |u(Ax + b) - u(x)| \leq \|\nabla u\|_{L^\infty} \max_{B_r} |Ax + b - x|.$$

So u verifies (3.5) and $\mathcal{Q}_\infty(u) \leq \|\nabla u\|_{L^\infty}$ and the proof is concluded also for $p = +\infty$.

Theorem 3.1 assumes a simpler form if the support of the function u is compact. Namely, we can prove the following theorem (compare with Definition 2.1):

THEOREM 3.2. — *Let $u \in L^p$, $p \in]1, +\infty]$, with compact support. Then $u \in W^{1,p}$ if and only if*

$$(3.8) \quad \tilde{\mathcal{Q}}_p(u) = \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} (|X| + |b|)^{-1} \|\tau_{(A, b)} u - u\|_{L^p} < +\infty$$

or equivalently:

$$(3.9) \quad \tilde{\mathcal{Q}}_p(u) = \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} \left(\max_{x \in B_1} |Ax + b - x| \right)^{-1} \|\tau_{(A, b)} u - u\|_{L^p} < +\infty.$$

Moreover, for all functions u whose support is contained in a fixed compact set, $\|u\|_{W^{1,p}}$, $\|u\|_{L^p} + \tilde{\mathcal{Q}}_p(u)$ and $\|u\|_{L^p} + \tilde{\mathcal{Q}}_p(u)$ are equivalent norms.

PROOF. — We have to prove the «only if» part, because, again by Theorem A, the «if» part is obvious.

First let us suppose $p \neq +\infty$. Since $\text{supp } u$ is compact, we can choose a sequence (u_j) in $\mathcal{O}$ satisfying: $u_j \rightarrow u$ in $W^{1,p}$ and there exists $R > 1$ such that $\text{supp } u_j \subseteq B_R$ for every j . Then we have:

$$\begin{aligned} \|u_j(Ax + b) - u_j(x)\|_{L^p} &\leq \|u_j(Ax + b) - u_j(Ax)\|_{L^p} + \|u_j(Ax) - u_j(x)\|_{L^p} \\ &\leq q_p(u_j) |b| + \|u_j(Ax) - u_j(x)\|_{L^p}. \end{aligned}$$

So we only have to estimate

$$\|u_j(Ax) - u_j(x)\|_{L^p} = \|u_j(\exp X \cdot x) - u_j(x)\|_{L^p}.$$

Since $\|u_j(\exp X \cdot x) - u_j(x)\|_{L^p} = \|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_R)}$, proceeding as in (3.7) in the proof of Theorem 3.1, we obtain

$$\|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_R)}^p \leq n^{p-1} (R|X| + |b|)^p \sum_{i=1}^n \|D_i u_j\|_{L^p}^p.$$

So we conclude

$$\begin{aligned} (3.10) \quad \|u_j(Ax + b) - u_j(x)\|_{L^p} &\leq q_p(u_j) |b| + n^{1/p'} (R|X| + |b|) \left(\sum_{i=1}^n \|D_i u_j\|_{L^p}^p \right)^{1/p} \\ &\leq CR \|u_j\|_{W^{1,p}} (|X| + |b|). \end{aligned}$$

By density we obtain the desired result.

Now let $p = \infty$ and let $\text{supp } u \subseteq B_R$. Once again we have

$$\begin{aligned} \|u(Ax + b) - u(x)\|_{L^\infty} &\leq \|u(Ax + b) - u(Ax)\|_{L^\infty} + \|u(Ax) - u(x)\|_{L^\infty} \leq \\ &\leq \|\nabla u\|_{L^\infty} |b| + \|u(Ax) - u(x)\|_{L^\infty(B_R)} \leq \|\nabla u\|_{L^\infty} (|b| + \max_{B_R} |Ax - x|) \leq \\ &\leq \|\nabla u\|_{L^\infty} R(|X| + |b|) \quad (\text{by Lemma 2.3}). \end{aligned}$$

So also in this case the proof is concluded.

As a second step, we give a characterization of the spaces $W^{1,p}$ (again by means of the group $\mathcal{M}$), analogous to the condition (ii) of Theorem A that involves the distributional derivatives of the functions.

Let us start with the term $\int u(x) D_i \varphi(x) dx$ and let us reinterpret D_i as the infinitesimal generator of the one-parameter group of translations $T_{te_i} : x \mapsto x + te_i$ (where the vectors e_i form the canonical basis of $\mathbb{R}^n$), i.e.:

$$D_i \varphi(x) = \lim_{t \rightarrow 0} \frac{\varphi(T_{te_i}(x)) - \varphi(x)}{t}.$$

Now we want to replace D_i with the infinitesimal generators of all the one-parameter subgroups of $\mathcal{M}$. Therefore we consider the Lie algebra $\mathfrak{m}$ of the group $\mathcal{M}$; for any vector $(X, a) \in \mathfrak{m}$, let

$$(X, a)^* \varphi(x) = \lim_{t \rightarrow 0} \frac{\varphi(\exp_{\mathcal{M}} t(X, a) \cdot x) - \varphi(x)}{t} \quad (\varphi \in \mathcal{O}).$$

We have to pay attention that the vectors e_i corresponding to the derivatives D_i are of norm 1, which may not be the case for (X, a) ; therefore we have to look for a normalizing factor.

Now we can state the following theorem.

THEOREM 3.3. – *Let $u \in L^p$, $p \in]1, +\infty]$. The two following conditions are equivalent (also in the sense of equivalent norms):*

(i) $u \in W^{1,p}$;

(ii') *there exists $c > 0$ such that, for every $(X, a) \in \mathfrak{m}$, for every $r > 0$, and for every $\varphi \in \mathcal{O}_r$,*

$$(3.11) \quad \left| \int u(x) (X, a)^* \varphi(x) dx \right| \leq c \|\varphi\|_{L^{p'}} (r|X| + |a|).$$

PROOF. – First we suppose that $u \in L^p$ satisfies (3.11); if we put $X = 0$ and $a = e_i$ in (3.11), we obtain

$$\left| \int u(x) D_i \varphi(x) dx \right| \leq c \|\varphi\|_{L^{p'}}, \quad \forall \varphi \in \mathcal{O},$$

that is $u \in W^{1,p}$ and $C \leq c$ (see Theorem A (ii)).

Conversely, let $u \in W^{1,p}$. Let (X, a) be in $\mathfrak{m}$ and φ be in $\mathcal{O}_r$. Since for every $t \neq 0$

$$\begin{aligned} (3.12) \quad \frac{1}{t} \int u(x) [\varphi(\exp_{\mathfrak{M}} t(X, a)x) - \varphi(x)] dx &= \\ &= \frac{1}{t} \int [u((\exp_{\mathfrak{M}} t(X, a))^{-1}x) - u(x)] \varphi(x) dx = \\ &= \frac{1}{t} \int_{B_r} [u((\exp_{\mathfrak{M}} t(X, a))^{-1}x) - u(x)] \varphi(x) dx, \end{aligned}$$

we obtain, again by the Schwarz-Hölder inequality

$$\left| \int u(x) \frac{\varphi(\exp_{\mathfrak{M}} t(X, a)x) - \varphi(x)}{t} dx \right| \leq \frac{1}{|t|} \|\tau_{(\exp_{\mathfrak{M}} t(X, a))^{-1}} u - u\|_{L^p(B_r)} \cdot \|\varphi\|_{L^{p'}}.$$

Recalling (2.11), we have

$$(\exp_{\mathfrak{M}} t(X, a))^{-1} = (\exp(-tX), [\exp(-tX) - I]v - tw)$$

and therefore, by Theorem 3.1,

$$\begin{aligned} \left| \int u(x) \frac{\varphi(\exp_{\mathfrak{M}} t(X, a)x) - \varphi(x)}{t} dx \right| &\leq \\ &\leq \frac{Q_p(u)}{|t|} [r|tX| + |(\exp(-tX) - I)v - tw|] \cdot \|\varphi\|_{L^{p'}} = \\ &= Q_p(u) \left[r|X| + \left| \frac{\exp(-tX) - I}{t} v - w \right| \right] \cdot \|\varphi\|_{L^{p'}}. \end{aligned}$$

Letting $t \rightarrow 0$, by the Lebesgue dominated convergence theorem and by (2.9), we conclude

$$\begin{aligned} \left| \int u(x) (X, a)^* \varphi(x) dx \right| &\leq Q_p(u) [r|X| + |-Xv - w|] \|\varphi\|_{L^{p'}} = \\ &= Q_p(u) (r|X| + |a|) \|\varphi\|_{L^{p'}}. \end{aligned}$$

So u verifies (3.11) and the greatest lower bound of the constants c such that (3.11) holds, does not exceed $Q_p(u)$. By Lemma 2.2 the proof is complete.

REMARK 3.1. – Evidently we can reformulate also Theorem 3.3 using Lemmas 2.2 and 2.3, i.e., in (3.11) we can replace $r|X| + |a|$ by $\max_{B_r} |Ax + b - x|$.

REMARK 3.2. – As (ii) of Theorem A, also (ii') of Theorem 3.3 can be given a distributional interpretation. Namely, if $u \in C^\infty$ and $\varphi \in \mathcal{O}$, then by (3.12) we obtain

$$\int u(x)(X, a)^* \varphi(x) dx = \int (-X, -a)^* u(x) \varphi(x) dx.$$

So it seems natural to define:

if $u \in \mathcal{O}'$, then for every $(X, a) \in \mathfrak{m}$, $(X, a)^* u$ is the distribution such that

$$\langle (X, a)^* u, \varphi \rangle = \langle u, (-X, -a)^* \varphi \rangle \quad \text{for every } \varphi \in \mathcal{O}.$$

COROLLARY 3.1. – Let $u \in W^{1,p}$, $p \in]1, +\infty]$. Then $(X, a)^* u$ (in the distributional sense) is represented by a function $f \in L^p_{\text{loc}}$. Moreover, there exists $k \in \mathbb{N}$ such that $(1 + |x|)^{-k} f \in L^p$ (and therefore f is a tempered distribution).

PROOF. – In condition (3.11), $(X, a)^* u$ can be interpreted as a linear continuous functional on $L^{p'}(B_r)$, for every $r > 0$. Therefore, by the Riesz representation theorem for the L^q spaces, we can conclude that $(X, a)^* u \in L^p_{\text{loc}}$ (otherwise one can reason as follows: $(X, a)^*$ is a first order differential operator with C^∞ coefficients; since $u \in W^{1,p}$, $(X, a)^* u \in L^p_{\text{loc}}$). Let us denote $(X, a)^* u = f$.

Now we prove that there exists $k \in \mathbb{N}$ such that $(1 + |x|)^{-k} f \in L^p$. By Theorem 3.3, we have

$$\|f\|_{L^p(B_r)} = \sup_{\varphi \in \mathcal{O}_r, \|\varphi\|_{L^{p'}} \leq 1} \left| \int_{B_r} f(x) \varphi(x) dx \right| \leq c(r|X| + |a|).$$

Let us consider the sequence of annuli $B_{j+1} \setminus B_j$ ($j = 1, 2, \dots$). We have, for $k \in \mathbb{N}$,

$$\begin{aligned} \int_{B_{j+1} \setminus B_j} \frac{|f(x)|^p}{(1 + |x|)^{kp}} dx &\leq \int_{B_{j+1} \setminus B_j} \frac{|f(x)|^p}{j^{kp}} dx \leq \frac{1}{j^{kp}} \int_{B_{j+1}} |f(x)|^p dx \leq \\ &\leq \frac{1}{j^{kp}} \cdot c^p ((j+1)|X| + |a|)^p \leq c' \frac{(j+1)^p}{j^{kp}}. \end{aligned}$$

If we choose $k \in \mathbb{N}$ such that $k > (p+1)/p$, we obtain

$$\left\| \frac{f}{(1 + |x|)^k} \right\|_{L^p}^p = \int_{B_1} \frac{|f(x)|^p}{(1 + |x|)^{kp}} dx + \sum_{j=1}^{+\infty} \int_{B_{j+1} \setminus B_j} \frac{|f(x)|^p}{(1 + |x|)^{kp}} dx < +\infty,$$

as we had to show.

Let us conclude this section with a statement for compactly supported functions. The proof can easily be deduced from Theorem 3.2 and from the proof of Theorem 3.3.

THEOREM 3.4. — *Let $u \in L^p$, $p \in]1, +\infty]$, with compact support. The two following conditions are equivalent:*

(i) $u \in W^{1,p}$;

(ii'') *there exists $c > 0$ such that, for every $(X, a) \in \mathfrak{m}$ and for every $\varphi \in \mathcal{D}$,*

$$(3.13) \quad \left| \int u(x)(X, a)^* \varphi(x) dx \right| \leq c \|\varphi\|_{L^{p'}} (|X| + |a|).$$

Moreover, for all functions u whose support is contained in a fixed compact set, (3.13) gives rise to a norm equivalent to the classical one.

REMARK 3.3. — Let $u \in W^{1,p}$ with compact support. The formula (3.13) of Theorem 3.4 (or Corollary 3.1) shows that $(X, a)^* u$ is represented by a function $f \in L^p$ with

$$\|f\|_{L^p} = \sup_{\varphi \in \mathcal{D}, \varphi \neq 0} \|\varphi\|_{L^{p'}}^{-1} \left| \int u(x)(X, a)^* \varphi(x) dx \right|.$$

We notice that for a fixed $u \in W^{1,p}$ with compact support the map $\mathcal{L}: \mathfrak{m} \rightarrow L^p$ defined by $\mathcal{L}(X, a) = (X, a)^* u$ is linear (cf. [25; Chapter IV, §17]) and also continuous, since $\mathfrak{m}$ is finite-dimensional. The seminorm to which (3.13) gives rise is precisely $\|\mathcal{L}\|$ (as linear continuous operator).

4. — Characterization of the space BV via motions.

In this section we study the space BV of functions with bounded variation (see Section 1, (II) for their definition). We follow the scheme of Section 3.

It is well known (see, e.g., [8; Théorème IX.3 and Remarque 6]); [10; Theorem 2.1.7]) that the following theorem holds.

THEOREM B. — *For functions in L^1 the following conditions are equivalent:*

(i) $u \in BV$;

(ii) *there exists $C > 0$ such that, for every $i = 1, \dots, n$, for every $\varphi \in \mathcal{D}$,*

$$(4.1) \quad \left| \int u(x) D_i \varphi(x) dx \right| \leq C \|\varphi\|_{L^\infty};$$

(iii) $q_1(u) = \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-1} \|\tau_h u - u\|_{L^1} < +\infty$.

Moreover $\|u\|_{L^1} + \inf \{C | (4.1) \text{ holds} \}$ and $\|u\|_{L^1} + q_1(u)$ are equivalent to $\|u\|_{BV}$.

In order to characterize the space BV in terms of the motions of $\mathbb{R}^n$, we need the following approximation theorem (see [2] and [18]).

THEOREM C. - *Let $u \in BV$. Then there exists a sequence (u_j) of functions in $C^\infty \cap W^{1,1}$ such that:*

$$u_j \rightarrow u \quad \text{in } L^1; \quad q_1(u_j) \rightarrow q_1(u) \quad \text{in } \mathbb{R}.$$

Now let us prove a characterization of the space BV by means of the group $\mathfrak{M}$.

DEFINITION 4.1. - *Let $u \in L^1$. For every positive number r and for every motion $(A, b) = (\exp X, b) \neq (I, 0)$, we define:*

$$(4.2) \quad Q_{(A, b), r, 1}(u) = \frac{\|\tau_{(A, b)} u - u\|_{L^1(B_r)}}{r|X| + |b|}$$

and

$$(4.3) \quad Q_{(A, b), r, 1}(u) = \frac{\|\tau_{(A, b)} u - u\|_{L^1(B_r)}}{\max_{B_r} |Ax + b - x|},$$

where $\tau_{(A, b)} u(x) = u(Ax + b)$.

THEOREM 4.1. - *Let $u \in L^1$. Then $u \in BV$ if and only if*

$$(4.4) \quad Q_1(u) = \sup_{r > 0} \sup_{(A, b) \in \mathfrak{M}, (A, b) \neq (I, 0)} Q_{(A, b), r, 1}(u) < +\infty$$

or equivalently:

$$(4.5) \quad Q_1(u) = \sup_{r > 0} \sup_{(A, b) \in \mathfrak{M}, (A, b) \neq (I, 0)} Q_{(A, b), r, 1}(u) < +\infty.$$

Moreover $\|u\|_{BV}$, $\|u\|_{L^1} + Q_1(u)$ and $\|u\|_{L^1} + Q_1(u)$ are equivalent norms in BV .

PROOF. - First, suppose $Q_1(u) < +\infty$. If we put $A = I$ in (4.4), we get

$$\sup_{r > 0} \sup_{b \neq 0} |b|^{-1} \cdot \int_{B_r} |u(x+b) - u(x)| dx \leq Q_1(u).$$

By Theorem B, this shows that $u \in BV$ and that $q_1(u) \leq Q_1(u)$.

Conversely, let $u \in BV$. Since

$$(4.6) \quad \begin{aligned} \int_{B_r} |u(\exp X \cdot x + b) - u(x)| dx &\leq \\ &\leq \int_{B_r} |u(\exp X \cdot x + b) - u(\exp X \cdot x)| dx + \int_{B_r} |u(\exp X \cdot x) - u(x)| dx \leq \\ &\leq q_1(u)|b| + \int_{B_r} |u(\exp X \cdot x) - u(x)| dx, \end{aligned}$$

we have to estimate $\int_{B_r} |u(\exp X \cdot x) - u(x)| dx$.

To this aim, let $(u_j)_{j \in \mathbb{N}}$ be as in Theorem C. We have

$$\begin{aligned}
 (4.7) \quad & \int_{B_r} |u_j(\exp X \cdot x) - u_j(x)| dx \leq \int_0^1 dt \int_{B_r} |\nabla u_j(\exp tX \cdot x) \cdot X \cdot \exp tX \cdot x| dx \leq \\
 & \leq \int_0^1 dt \int_{B_r} |\nabla u_j(\exp tX \cdot x)| \cdot |X \cdot \exp tX \cdot x| dx \leq \\
 & \leq \int_0^1 dt \int_{B_r} \left(\sum_{i=1}^n |D_i u_j(\exp tX \cdot x)| \right) |X| |\exp tX \cdot x| dx \leq r |X| \left(\sum_{i=1}^n \|D_i u_j\|_{L^1} \right).
 \end{aligned}$$

Since, for $i = 1, \dots, n$, $\|D_i u_j\|_{L^1} \leq q_1(u_j)$, we obtain

$$\int_{B_r} |u_j(\exp X \cdot x) - u_j(x)| dx \leq n q_1(u_j) r |X|.$$

Letting $j \rightarrow +\infty$ and recalling Theorem C, we have: for every $r > 0$

$$\int_{B_r} |u(\exp X \cdot x) - u(x)| dx \leq n q_1(u) r |X|$$

and therefore

$$\int_{B_r} |u(\exp X \cdot x + b) - u(x)| dx \leq n q_1(u) (r |X| + |b|).$$

So we conclude: (4.4) holds, $Q_1(u) \leq n q_1(u)$, and the proof is complete.

Also in this case, for functions with compact support, the characterization becomes simpler. In fact we can choose all the approximating functions u_j with supports contained in a fixed ball B_R , whose radius R depends on u but not on j . Now we can proceed as in Theorem 3.2; so Theorem 4.1 takes the following form:

THEOREM 4.2. - *Let $u \in L^1$ with compact support. Then $u \in BV$ if and only if*

$$(4.8) \quad \tilde{Q}_1(u) = \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} (|X| + |b|)^{-1} \|\tau_{(A, b)} u - u\|_{L^1} < +\infty$$

or equivalently:

$$(4.9) \quad \tilde{Q}_1(u) = \sup_{(A, b) \in \mathcal{H}, (A, b) \neq (I, 0)} \left(\max_{x \in B_1} |Ax + b - x| \right)^{-1} \|\tau_{(A, b)} u - u\|_{L^1} < +\infty.$$

Moreover, for all functions u whose support is contained in a fixed compact set, $\|u\|_{BV}$, $\|u\|_{L^1} + \tilde{Q}_1(u)$ and $\|u\|_{L^1} + \tilde{Q}_1(u)$ are equivalent norms.

The next theorem furnishes a characterization of the space BV , which generalizes the condition (ii) of Theorem B, as in Section 3 Theorem 3.2 extends the condition (ii) of Theorem A.

THEOREM 4.3. – *Let $u \in L^1$. The two following conditions are equivalent (also in the sense of equivalent norms):*

(i) $u \in BV$;

(ii') *there exists $c > 0$ such that, for every $(X, a) \in \mathfrak{m}$, for every $r > 0$, and for every $\varphi \in \mathcal{O}_r$,*

$$(4.10) \quad \left| \int u(x)(X, a)^* \varphi(x) dx \right| \leq c \|\varphi\|_{L^\infty} (r|X| + |a|).$$

PROOF. – First suppose that the inequality above holds. If in (4.10) we choose $X = 0$ and, for $i = 1, \dots, n$, $a = e_i$, we obtain

$$\left| \int u(x) D_i \varphi(x) dx \right| \leq c \|\varphi\|_{L^\infty}, \quad \forall \varphi \in \mathcal{O},$$

that is $u \in BV$ and $C \leq c$.

Conversely, let $u \in BV$. If $\varphi \in \mathcal{O}_r$ and $(X, a) \in \mathfrak{m}$, as in Theorem 3.3 we have

$$\begin{aligned} \frac{1}{t} \int u(x) (\varphi(\exp_{\mathfrak{M}} t(X, a) \cdot x) - \varphi(x)) dx &= \\ &= \frac{1}{t} \int_{B_r} (u(\exp(-tX) \cdot x + (\exp(-tX) - I)v - tw) - u(x)) \varphi(x) dx \end{aligned}$$

and therefore, by Theorem 4.1,

$$\begin{aligned} \left| \int u(x) \frac{\varphi(\exp_{\mathfrak{M}} t(X, a) \cdot x) - \varphi(x)}{t} dx \right| &\leq \\ &\leq Q_1(u) \left[r|X| + \left| \frac{\exp(-tX) - I}{t} v - w \right| \right] \cdot \|\varphi\|_{L^\infty}. \end{aligned}$$

Letting $t \rightarrow 0$, we conclude

$$\left| \int u(x)(X, a)^* \varphi(x) dx \right| \leq Q_1(u)(r|X| + |a|) \|\varphi\|_{L^\infty},$$

So u verifies (4.10) and the greatest lower bound of the constants c , such that (4.10) holds, does not exceed $Q_1(u)$. By Lemma 2.2 the proof is complete.

REMARK 4.1. – The same remark as (3.1) can be made for the space BV .

COROLLARY 4.1. – *Let $u \in BV$. Then $(X, a)^* u$ (in the distributional sense) is a regular Borel measure μ . Moreover, there exists $k \in \mathbb{N}$ such that $\int (1 + |x|)^{-k} d|\mu|(x) < +\infty$, i.e., μ is a tempered measure.*

PROOF. – By Theorem 4.3, $(X, a)^* u$ is a linear continuous functional on the space of all continuous functions with support contained in B_r , equipped with the L^∞ -norm. Therefore, by the Riesz representation theorem, there is a regular Borel measure μ , which represents $(X, a)^* u$ (see [30; Chapter 2]) (otherwise one can follow the alternative method of reasoning mentioned in the proof of Corollary 3.1).

Moreover, by (4.10), the total variation of μ restricted to B_r , i.e. $\int_{B_r} d|\mu|(x)$, does not exceed $c(r|X| + |a|)$.

Since

$$\int (1 + |x|)^{-k} d|\mu|(x) = \int_{B_1} (1 + |x|)^{-k} d|\mu|(x) + \sum_{j=1}^{+\infty} \int_{B_{j+1} \setminus B_j} (1 + |x|)^{-k} d|\mu|(x),$$

reasoning as in Corollary 3.1, we obtain that for $k > 2$ the series is convergent, and therefore μ is a tempered measure.

Finally, let us note that also for the space BV a theorem analogous to Theorem 3.4 holds, namely we have:

THEOREM 4.4. – *Let $u \in L^1$ with compact support. The two following conditions are equivalent:*

(i) $u \in BV$;

(ii'') *there exists $c > 0$ such that, for every $(X, a) \in \mathfrak{m}$ and for every $\varphi \in \mathcal{O}$,*

$$(4.11) \quad \left| \int u(x)(X, a)^* \varphi(x) dx \right| \leq c \|\varphi\|_{L^\infty} (|X| + |a|).$$

Moreover, for all functions u whose support is contained in a fixed compact set, (4.11) gives rise to a norm equivalent to the classical one.

REMARK 4.2. – Let $u \in BV$ with compact support. Then it is clear that $(X, a)^* u$ is a finite measure (cf. Remarks 3.2 and 3.3) whose total variation is equal to

$$\sup_{\varphi \in \mathcal{O}, \varphi \neq 0} \|\varphi\|_{L^\infty}^{-1} \left| \int u(x)(X, a)^* \varphi(x) dx \right|.$$

Now for a fixed $u \in BV$ with compact support we can consider the map $\mathcal{L}: \mathfrak{m} \rightarrow C'_c$ defined by $\mathcal{L}(X, a) = (X, a)^* u$ and we conclude as in Remark 3.3.

5. - Characterization of $N^{\lambda,p}$ via motions.

In this section we study the Nikol'skij spaces $N^{\lambda,p}$ with $p \in [1, +\infty[$ and $\lambda \in]0, 1[$ (see Section 1, (III) for their definition). Let us denote

$$(5.1) \quad q_{\lambda,p}(u) = \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-\lambda} \|u(x+h) - u(x)\|_{L^p}.$$

As in Section 4, we need the following approximation theorem (see, e.g., [26; II, 9] and, for the proof, [5; Theorem 1] and [7; Theorem 10]).

THEOREM D. - *If $u \in N^{\lambda,p}$, there exists a sequence $(u_j)_{j=0,1,\dots}$ of functions in $C^\infty \cap W^{1,p}$ satisfying the conditions:*

$$u = \sum_{j=0}^{+\infty} u_j \quad \text{in } L^p;$$

there exists $c_1 > 0$, independent of u , such that for every j

$$\|u_j\|_{L^p} \leq c_1 q_{\lambda,p}(u) 2^{-\lambda j},$$

$$\|\nabla u_j\|_{L^p} \leq c_1 q_{\lambda,p}(u) 2^{(1-\lambda)j}.$$

Now we proceed as in Sections 3 and 4; we use the same notation and we state, for the spaces $N^{\lambda,p}$, the proposition analogous to Theorems 3.1 and 4.1. The last part of the proof we give here is different from the ones of Theorems 3.1 and 4.1; indeed, it is close to the proof of Theorem 2 in [5], where, however, only functions with compact support are considered (see also the proof of Lemma 2.2 in [29]).

DEFINITION 5.1. - *Let $u \in L^p$, $p \in [1, +\infty[$ and let $\lambda \in]0, 1[$. For every positive number r and for every motion $(A, b) = (\exp X, b) \neq (I, 0)$, we define:*

$$(5.2) \quad Q_{(A,b),r,\lambda,p}(u) = \frac{\|\tau_{(A,b)}u - u\|_{L^p(B_r)}}{(r|X| + |b|)^\lambda}$$

and

$$(5.3) \quad Q_{(A,b),r,\lambda,p}(u) = \frac{\|\tau_{(A,b)}u - u\|_{L^p(B_r)}}{(\max_{B_r} |Ax + b - x|)^\lambda},$$

where $\tau_{(A,b)}u(x) = u(Ax + b)$.

THEOREM 5.1. - *Let $u \in L^p$, p and λ as above. Then $u \in N^{\lambda,p}$ if and only if*

$$(5.4) \quad Q_{\lambda,p}(u) = \sup_{r>0} \sup_{(A,b) \in \mathcal{M}, (A,b) \neq (I,0)} Q_{(A,b),r,\lambda,p}(u) < +\infty$$

or equivalently:

$$(5.5) \quad \mathcal{Q}_{\lambda,p}(u) = \sup_{r>0} \sup_{(A,b) \in \mathcal{K}, (A,b) \neq (I,0)} \mathcal{Q}_{(A,b),r,\lambda,p}(u) < +\infty.$$

Moreover $\|u\|_{N^{\lambda,p}}$, $\|u\|_{L^p} + \mathcal{Q}_{\lambda,p}(u)$ and $\|u\|_{L^p} + \mathcal{Q}_{\lambda,p}(u)$ are equivalent norms in $N^{\lambda,p}$.

PROOF. – First, suppose $\mathcal{Q}_{\lambda,p}(u) < +\infty$; reasoning as in Theorem 4.1, we get $u \in N^{\lambda,p}$ and $q_{\lambda,p}(u) \leq \mathcal{Q}_{\lambda,p}(u)$.

Conversely, let $u \in N^{\lambda,p}$. Again, as in (4.6), we have

$$(5.6) \quad \|\tau_{(A,b)} u - u\|_{L^p(B_r)} \leq q_{\lambda,p}(u) |b|^\lambda + \|\tau_A u - u\|_{L^p(B_r)}.$$

So we have to estimate $\|\tau_A u - u\|_{L^p(B_r)}^p = \int_{B_r} |u(\exp X \cdot x) - u(x)|^p dx$.

To this aim, let (u_j) be as in Theorem D. Now we adapt the first part of the proofs of Theorem 3.1 and of Theorem 4.1. We obtain

$$(5.7) \quad \int_{B_r} |u_j(\exp X \cdot x) - u_j(x)|^p dx \leq n^{p-1} r^p |X|^p \left(\sum_{i=1}^n \|D_i u_j\|_{L^p}^p \right)$$

and therefore it follows that

$$\|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_r)} \leq n^{1/p'} r |X| c_1 q_{\lambda,p}(u) 2^{(1-\lambda)j} \leq nr |X| c_1 q_{\lambda,p}(u) 2^{(1-\lambda)j}.$$

So we have, for every j , both the following estimates at our disposal:

$$(5.8) \quad \|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_r)} \leq \begin{cases} 2c_1 q_{\lambda,p}(u) 2^{-\lambda j}, \\ nr |X| c_1 q_{\lambda,p}(u) 2^{(1-\lambda)j}. \end{cases}$$

Let us choose, on each occasion, the better of the two estimates; therefore, let $j_0 \in \mathbb{N}$ be such that

$$2^{j_0} \leq 2(nr|X|)^{-1} < 2^{j_0+1}$$

(thus in (5.8) $2c_1 q_{\lambda,p}(u) 2^{-\lambda j} \geq nr |X| c_1 q_{\lambda,p}(u) 2^{(1-\lambda)j}$ if and only if $j \leq j_0$; clearly, if $2(nr|X|)^{-1} < 1$, we choose the first estimate for every $j \in \mathbb{N}$).

Consequently we obtain

$$(5.9) \quad \|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)} \leq \sum_{j=0}^{j_0} \|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_r)} + \sum_{j=j_0+1}^{+\infty} \|u_j(\exp X \cdot x) - u_j(x)\|_{L^p(B_r)} \leq \\ \leq nr |X| c_1 q_{\lambda,p}(u) \frac{2^{(1-\lambda)(j_0+1)}}{2^{1-\lambda} - 1} + 2c_1 q_{\lambda,p}(u) 2^{-\lambda(j_0+1)} \sum_{j=0}^{+\infty} 2^{-\lambda j} \leq$$

$$\begin{aligned}
&\leq \frac{2^{1-\lambda}}{2^{1-\lambda}-1} c_1 nr |X| q_{\lambda,p}(u) 2^{j_0(1-\lambda)} + \frac{2}{1-2^{-\lambda}} c_1 q_{\lambda,p}(u) 2^{(j_0+1)(-\lambda)} \leq \\
&\leq c_2 q_{\lambda,p}(u) (nr |X| (2^{j_0})^{(1-\lambda)} + (2^{j_0+1})^{-\lambda}) \leq \\
&\leq c_2 q_{\lambda,p}(u) \left(nr |X| \left(\frac{2}{nr |X|} \right)^{1-\lambda} + \left(\frac{nr |X|}{2} \right)^\lambda \right) = \\
&= c_3 q_{\lambda,p}(u) (nr |X|)^\lambda = c_4 q_{\lambda,p}(u) (r |X|)^\lambda.
\end{aligned}$$

Recalling (5.6) we get

$$\|\tau_{(A,b)} u - u\|_{L^p(B_r)} = \|u(\exp X \cdot x + b) - u(x)\|_{L^p(B_r)} \leq c_5 q_{\lambda,p}(u) (|b|^\lambda + r^\lambda |X|^\lambda).$$

Since $\lambda \in]0, 1[$, for every $a, b > 0$, $a^\lambda + b^\lambda < 2^{1-\lambda}(a+b)^\lambda$ and therefore we can conclude that there exists $c > 0$ such that

$$(5.10) \quad \|u(\exp X \cdot x + b) - u(x)\|_{L^p(B_r)} \leq c q_{\lambda,p}(u) (r |X| + |b|)^\lambda.$$

So we have proven: (5.4) holds, $Q_{\lambda,p}(u) \leq c q_{\lambda,p}(u)$, and the proof is complete.

REMARK 5.1. – Note that the constant c_2 , which appears in the estimation (5.9), tends to infinity for $\lambda \rightarrow 0$ and for $\lambda \rightarrow 1$.

Once again, if a function $u \in L^p$ has compact support, there exist $R > 0$ and a sequence (u_j) as in Theorem D, such that $\text{supp } u_j \subseteq B_R$ for every j . So Theorem 5.1 can be reformulated as follows (see [5; Theorem 2]) and [7; Theorem 10]).

THEOREM 5.2. – *Let $u \in L^p$ with compact support; let p and λ be as above. Then $u \in N^{\lambda,p}$ if and only if*

$$(5.11) \quad \tilde{Q}_{\lambda,p}(u) = \sup_{(A,b) \in \mathcal{M}, (A,b) \neq (I,0)} (|X| + |b|)^{-\lambda} \|\tau_{(A,b)} u - u\|_{L^p} < +\infty$$

or equivalently:

$$(5.12) \quad \tilde{Q}_{\lambda,p}(u) = \sup_{(A,b) \in \mathcal{M}, (A,b) \neq (I,0)} \left(\max_{x \in B_1} |Ax + b - x| \right)^{-\lambda} \|\tau_{(A,b)} u - u\|_{L^p} < +\infty.$$

Moreover, for all functions u whose support is contained in a fixed compact set, $\|u\|_{N^{\lambda,p}}$, $\|u\|_{L^p} + \tilde{Q}_{\lambda,p}(u)$ and $\|u\|_{L^p} + \tilde{Q}_{\lambda,p}(u)$ are equivalent norms.

6. – Characterization of $\mathring{N}^{\lambda,p}$ via motions.

In this section we study the spaces $\mathring{N}^{\lambda,p}$ (see Section 1,(IV) for their definition). We go on using the notation of the previous sections.

$\mathring{N}^{\lambda,p}$ is a closed subspace of $N^{\lambda,p}$; unlike $N^{\lambda,p}$ it is separable, for it is the closure in $N^{\lambda,p}$ of the Sobolev space $W^{1,p}$ (see [10; Chapter 3, Proposition 3.1.6] and [34; p. 180]).

Also in this case we characterize the function space $\mathring{N}^{\lambda,p}$ by means of the motion group. To this aim, we state in advance the following two lemmas. The first one is a simple consequence of Theorems 3.1, 4.1 and 5.1 with $b = 0$.

LEMMA 6.1. – *Let $\mu \in]0, 1]$ (for $\mu = 1$, we mean: $N^{1,p} = BV$ if $p = 1$, $N^{1,p} = W^{1,p}$ if $p > 1$). If u is an element of $N^{\mu,p}$, then for all λ with $0 < \lambda < \mu$ u satisfies*

$$\lim_{r|X| \rightarrow 0} \frac{\|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)}}{(r|X|)^\lambda} = 0.$$

LEMMA 6.2. – *If $u \in \mathring{N}^{\lambda,p}$, then*

$$\lim_{r|X| \rightarrow 0} \frac{\|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)}}{(r|X|)^\lambda} = 0.$$

PROOF. – Fix $\varepsilon > 0$. Since the Sobolev space $W^{1,p}$ is dense in $\mathring{N}^{\lambda,p}$, we choose $v_\varepsilon \in W^{1,p}$ such that $Q_{\lambda,p}(u - v_\varepsilon) < \varepsilon/2$. So we obtain

$$\begin{aligned} \frac{\|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)}}{(r|X|)^\lambda} &\leq \frac{\|(u - v_\varepsilon)(\exp X \cdot x) - (u - v_\varepsilon)(x)\|_{L^p(B_r)}}{(r|X|)^\lambda} + \\ &+ \frac{\|v_\varepsilon(\exp X \cdot x) - v_\varepsilon(x)\|_{L^p(B_r)}}{(r|X|)^\lambda} \leq \frac{\varepsilon}{2} + \frac{\|v_\varepsilon(\exp X \cdot x) - v_\varepsilon(x)\|_{L^p(B_r)}}{(r|X|)^\lambda}. \end{aligned}$$

Now we apply Lemma 6.1 and we conclude the proof.

The next theorem characterizes $\mathring{N}^{\lambda,p}$ by means of the motions of $\mathbb{R}^n$.

THEOREM 6.1. – *Let $u \in L^p$, $p \in [1, +\infty[$; let $\lambda \in]0, 1[$. Then $u \in \mathring{N}^{\lambda,p}$ if and only if*

$$(6.1) \quad \lim_{(r|X| + |b|) \rightarrow 0} \frac{\|u(\exp X \cdot x + b) - u(x)\|_{L^p(B_r)}}{(r|X| + |b|)^\lambda} = 0.$$

PROOF. – First suppose $u \in L^p$ be such that (6.1) holds. We fix $\varepsilon > 0$ and determine the corresponding $\delta > 0$. So, if $X = 0$, for every $r > 0$ and for every $b \in \mathbb{R}^n$ with $0 < |b| < \delta$ we have

$$\frac{\|u(x + b) - u(x)\|_{L^p(B_r)}}{|b|^\lambda} < \varepsilon.$$

Letting $r \rightarrow +\infty$, we obtain (for $0 < |b| < \delta$)

$$\frac{\|u(x+b) - u(x)\|_{L^p}}{|b|^\lambda} \leq \varepsilon,$$

that is $u \in \dot{N}^{\lambda,p}$.

Conversely, let $u \in \dot{N}^{\lambda,p}$. Fix $\varepsilon > 0$; let δ_1 be such that for $|b| < \delta_1$

$$\|u(x+b) - u(x)\|_{L^p(B_r)} < \frac{\varepsilon}{2} |b|^\lambda;$$

by Lemma 6.2 there exists δ_2 such that for $r|X| < \delta_2$

$$\|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)} < \frac{\varepsilon}{2} (r|X|)^\lambda.$$

Let $\delta = \min(\delta_1, \delta_2)$. If $r|X| + |b| < \delta$, we have

$$\begin{aligned} \|u(\exp X \cdot x + b) - u(x)\|_{L^p(B_r)} &\leq \|u(x+b) - u(x)\|_{L^p(B_r)} + \|u(\exp X \cdot x) - u(x)\|_{L^p(B_r)} < \\ &< \frac{\varepsilon}{2} |b|^\lambda + \frac{\varepsilon}{2} (r|X|)^\lambda < \varepsilon (r|X| + |b|)^\lambda, \end{aligned}$$

that is u satisfies (6.1).

As in the previous sections, Theorem 6.1 can be formulated in a slightly different way, using Lemma 2.3.

COROLLARY 6.1. – *Let $u \in L^p$; let p and λ be as above. Then $u \in \dot{N}^{\lambda,p}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $r > 0$, for every $(A, b) \in \mathcal{N}$, $(A, b) \neq (I, 0)$*

$$\max_{x \in B_r} |Ax + b - x| < \delta \quad \text{implies} \quad \frac{\|u(Ax + b) - u(x)\|_{L^p(B_r)}}{(\max_{x \in B_r} |Ax + b - x|)^\lambda} < \varepsilon.$$

In the case of functions with compact support, Theorem 6.1 reads as follows.

THEOREM 6.2. – *Let $u \in L^p$ with compact support; let p and λ be as above. Then $u \in \dot{N}^{\lambda,p}$ if and only if*

$$(6.2) \quad \lim_{|X| + |b| \rightarrow 0} \frac{\|u(\exp X \cdot x + b) - u(x)\|_{L^p}}{(|X| + |b|)^\lambda} = 0.$$

PROOF. – The «if» part is immediate, so we prove the converse.

Let $u \in \dot{N}^{\lambda,p}$ with $\text{supp } u \subseteq B_R$. We fix $\varepsilon > 0$; let δ_1 be such that for $|b| < \delta_1$

$$\|u(x+b) - u(x)\|_{L^p} < \frac{\varepsilon}{2} |b|^\lambda;$$

by Lemma 6.2 there exists δ_2 such that for every X with $R|X| < \delta_2$, i.e. for every X

with $|X| < R^{-1}\delta_2$,

$$\|u(\exp X \cdot x) - u(x)\|_{L^p} = \|u(\exp X \cdot x) - u(x)\|_{L^p(B_R)} < \frac{\varepsilon}{2R^\lambda} (R|X|)^\lambda = \frac{\varepsilon}{2} |X|^\lambda.$$

Let $\delta = \min(\delta_1, R^{-1}\delta_2)$. If $|X| + |b| < \delta$, we obtain

$$\|u(\exp X \cdot x + b) - u(x)\|_{L^p} < \frac{\varepsilon}{2} |b|^\lambda + \frac{\varepsilon}{2} (|X|)^\lambda < \varepsilon (|X| + |b|)^\lambda.$$

Once again, Theorem 6.2 could be reformulated in different ways; we choose the following one, which makes use of the Lie algebra $\mathfrak{m}$.

COROLLARY 6.2. – *Let $u \in L^p$ with compact support; let p and λ be as above. Then $u \in \dot{N}^{\lambda,p}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every vector $(X, a) \in \mathfrak{m}$ with $0 < |(X, a)| < \delta$,*

$$|(X, a)|^{-\lambda} \|u(\exp_{\mathfrak{M}}(X, a) \cdot x) - u(x)\|_{L^p} < \varepsilon.$$

REMARK 6.1. – Let us note that Section 4 of [16] is in some way connected with our investigations on the spaces $W^{1,2}$ and $\dot{N}^{\lambda,2}$. In [16] the author considers functions $u \in C_0^\infty(K)$, where K is a fixed compact set. For every one-parameter local group of transformations defined by a first order differential operator Y , he introduces the seminorm $|u|_{Y,\lambda}$ ($u \in C_0^\infty(K)$; $0 < \lambda \leq 1$). We notice that in the special case of a one-parameter (global) group of motions (and with our notation) $|u|_{Y,\lambda}$ takes the form $\sup_{t \neq 0} |t|^{-\lambda} \|u(\exp_{\mathfrak{M}} t(X, a) \cdot x) - u(x)\|_{L^2}$.

In Lemma 4.2 of [16] (see also Theorem 4.3) an estimate of $|u|_{Y,\lambda}$ by means of the seminorm $q_{\lambda,2}(u)$ is proven, which can be compared with our formulas (3.10) for $\lambda = 1$ and (5.10) for $\lambda < 1$.

7. – Counterexamples.

In this section we set down some counterexamples, chiefly in order to show that the statements about functions with compact support (i.e., Theorems 3.2, 4.2, 5.2, 6.2 and 3.4, 4.4) are no longer true without the hypothesis of the compactness of the supports. Finally we add a remark about Corollaries 3.1 and 4.1, in order to show that better results cannot be expected. For the sake of simplicity, we expose all the examples for $n = 2$.

1) The first examples, regarding Theorems 3.2, 4.2, and 5.2, are already listed in [7; n.11] and can easily be proven as in [3; n.2, ex.1] for $N^{\lambda,1}$. However we repeat them here both for reader's convenience and because they are necessary for the sequel (for instance they immediately provide a counterexample to Theorem 6.2).

Let us fix an angle $\theta_0 \in]0, \pi/2[$ and let $k \in \mathbb{N}$ with $k > 2$ (we will choose k at the end of the example, because k depends on the space we are dealing with). Now we

consider the sequence of angles

$$(7.1) \quad \theta_0, \theta_1 = k^{-1}\theta_0, \dots, \theta_l = k^{-l}\theta_0, \dots \quad (l \in \mathbb{N})$$

and the corresponding rotations $\exp X_l$, where

$$X_l = \begin{pmatrix} 0 & -\theta_l \\ \theta_l & 0 \end{pmatrix}.$$

Note that $|X_l| = \sqrt{2}\theta_l = \sqrt{2}k^{-l}\theta_0$. On the x_1 -axis let (P_l) be the sequence of points $(\theta_l^{-1}, 0)$ and for every l let T_l be the translation which maps P_0 into P_l . We denote with S_0 the square with vertices $(\theta_0^{-1}, 0), (\theta_0^{-1} + 1, 0), (\theta_0^{-1} + 1, 1), (\theta_0^{-1}, 1)$ and with S_l the square $T_l(S_0)$ (we underline that the squares S_l are mutually disjoint).

We fix $\psi_0 \in \mathcal{D}$ with $\text{supp } \psi_0 \subseteq S_0$ and $\psi_0 \geq 0$; we define $\psi_l = 2^{-l}\psi_0 \circ T_l^{-1}$ (so $\text{supp } \psi_l \subseteq S_l$); finally let u be the non-compactly supported function defined by

$$(7.2) \quad u = \sum_{l=0}^{+\infty} \psi_l.$$

For every k we have:

$$(i) \quad u \in L^p, \quad \forall p \in [1, +\infty]$$

$$\text{and } \|u\|_{L^p} = \left(\frac{2^p}{2^p - 1} \right)^{1/p} \|\psi_0\|_{L^p}, \quad \text{if } p < +\infty; \quad \|u\|_{L^\infty} = \|\psi_0\|_{L^\infty};$$

$$(ii) \quad u \in W^{1,p}, \quad \forall p \in [1, +\infty]$$

$$(\text{therefore } u \in BV \text{ and } u \in N^{\lambda,p}, \quad \forall \lambda \in]0, 1[\text{ and } \forall p \in [1, +\infty[),$$

$$\text{because } \frac{\|\tau_h u - u\|_{L^p}}{|h|} \leq \sum \frac{\|\tau_h \psi_l - \psi_l\|_{L^p}}{|h|} = 2 \frac{\|\tau_h \psi_0 - \psi_0\|_{L^p}}{|h|};$$

$$(iii) \quad \|\tau_{\exp X_l} u - u\|_{L^p} \geq 2^{1/p} \|\psi_l\|_{L^p} = 2^{1/p} \cdot 2^{-l} \|\psi_0\|_{L^p} \text{ if } p < +\infty;$$

$$\|\tau_{\exp X_l} u - u\|_{L^\infty} \geq 2^{-l} \|\psi_0\|_{L^\infty};$$

$$(iv) \quad \frac{\|\tau_{\exp X_l} u - u\|_{L^p}}{|X_l|^\lambda} \geq c \cdot \left(\frac{k^\lambda}{2} \right)^l \quad (c \text{ constant}).$$

So we conclude (recalling that $k \in \mathbb{N}$, $k > 2$):

(I) $u \in W^{1,p}$ ($p \in [1, +\infty]$), but u does not satisfy (3.8) for any k ;

(II) $u \in BV$, but u does not satisfy (4.8) for any k ;

(III) $u \in N^{\lambda,p}$, but u does not satisfy (5.11) for any k such that $k^\lambda > 2$;

(IV) $u \in \dot{N}^{\lambda,p}$, but u does not satisfy (6.2) for any k such that $k^\lambda > 2$.

Moreover, we can use the same functions ψ_l in order to show that, if we do not consider functions with support contained in a fixed compact set, $\tilde{Q}_p$, $\tilde{Q}_1$, $\tilde{Q}_{\lambda,p}$ (respectively for (I), (II), (III)) do not lead to norms equivalent to the classical ones. We illustrate the case $\tilde{Q}_p$ ($p \in]1, +\infty[$) since the others can be treated in the same way.

Let u be as in (7.2) and let u_N be the compactly supported functions defined by

$$u_N = \sum_{l=0}^N \psi_l, \quad N = 0, 1, 2, \dots$$

For every N we have:

$$(v) \quad \|u_N\|_{W^{1,p}} \leq \|u\|_{W^{1,p}};$$

$$(vi) \quad \tilde{Q}_p(u_N) \geq \frac{\|\tau_{\exp X_N} u_N - u_N\|_{L^p}}{|X_N|} \geq c \cdot \left(\frac{k}{2}\right)^N$$

(see (iv) above).

Therefore

$$\lim_{N \rightarrow +\infty} \tilde{Q}_p(u_N) = +\infty$$

and we have obtained the desired result.

This remark could be repeated at the end of example 2. which follows; we will leave it out for the sake of brevity.

2) Now we expose the counterexamples concerning Theorems 3.4 and 4.4.

First, let us recall that in the two dimensional case, if $\varphi \in \mathcal{O}$ and

$$X = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \quad \text{with } \theta \in]-\pi, \pi] \text{ and } \theta \neq 0,$$

then

$$(7.3) \quad (X, 0)^* \varphi(x_1, x_2) = \theta(-x_2 D_1 \varphi(x_1, x_2) + x_1 D_2 \varphi(x_1, x_2)).$$

Let u be again the function defined in (7.2) (the integer k which appears in (7.1) can be any integer > 2 , as above in the cases (I) and (II)). We claim that there exists a sequence (φ_m) of functions in $\mathcal{O}$, with

$$(7.4) \quad \|\varphi_m\|_{L^p} = \|\varphi_0\|_{L^p} \quad \forall m \in \mathbb{N}, \quad \forall p \in [1, +\infty],$$

such that for every $X \in \mathfrak{so}(2)$, $\exp X \neq I$,

$$(7.5) \quad \lim_{m \rightarrow +\infty} \left| \int u(x) (X, 0)^* \varphi_m(x) dx \right| = +\infty.$$

In order to define the functions φ_m we proceed as follows. Let us recall again that the closed squares S_i are mutually disjoint; therefore there exists $\varepsilon > 0$ such that, if we define $\tilde{S}_0$ the square which is homothetic to S_0 with respect to their common centre and has sides of length $1 + 2\varepsilon$, then $\tilde{S}_0$ does not intersect S_1 . Let η be a C^∞ function satisfying: $\eta(x) = 1 \forall x \in S_0$; $\eta(x) = 0 \forall x \notin \tilde{S}_0$. Finally, let $\varphi_0(x_1, x_2) = x_2 \eta(x_1, x_2)$ and

for every $m \in \mathbb{N}$ $\varphi_m = \varphi_0 \circ T_m^{-1}$ (T_m is the translation defined in 1), hence the condition (7.4) is fulfilled).

So by (7.3) we obtain

$$\begin{aligned} \left| \int u(x)(X, 0)^* \varphi_m(x) dx \right| &= \left| \int_{S_m} u(x)(X, 0)^* \varphi_m(x) dx \right| = \left| \int_{S_m} \psi_m(x_1, x_2) \theta x_1 dx_1 dx_2 \right| = \\ &= |\theta| \int_{S_m} \psi_m(x_1, x_2) x_1 dx_1 dx_2 \geq |\theta| \theta_m^{-1} \int_{S_m} \psi_m(x) dx = \left(\frac{k}{2} \right)^m \frac{|\theta|}{\theta_0} \|\psi_0\|_{L^1}. \end{aligned}$$

The condition (7.5) is thus proven. So we conclude:

(I) $u \in W^{1,p}$ ($p \in [1, +\infty]$), but u does not satisfy (ii'') of Theorem (3.4);

(II) $u \in BV$, but u does not satisfy (ii'') of Theorem (4.4).

Notice that we have in fact proven stronger statements. Namely we have demonstrated that for every $X \in \mathfrak{so}(2)$ (with $\theta \in]-\pi, \pi[\setminus \{0\}$) and for every $c > 0$ there exists $\varphi \in \mathcal{O}$ such that

$$(7.6) \quad \left| \int u(x)(X, 0)^* \varphi(x) dx \right| \geq c |X| \|\varphi\|_{L^p} \quad p \in [1, +\infty].$$

This consideration leads to the third counterexample concerning Corollaries 3.1 and 4.1.

3) The formula (7.6) shows that for every X as above

(I) the distribution $(X, 0)^* u$ cannot be represented by a function in L^p ($p \in [1, +\infty]$), but only by a function in L_{loc}^p ;

(II) the distribution $(X, 0)^* u$ cannot be represented by a function in L^1 , but only by a function in L_{loc}^1 , and therefore it is not a finite measure.

8. – Some remarks on homogeneous manifolds.

This section contains remarks, some of which might perhaps prove useful for the definition of function spaces on homogeneous manifolds.

We begin with two short remarks on topologies. In formula (2.16) of Lemma 2.3 we have introduced the expression $\max_{B_r} |Ax + b - x|$ and we have often used it in the sequel. It suggests the following definition: for every $g_1, g_2 \in \mathcal{M}$ and $r > 0$, let

$$d_r(g_1, g_2) = \max_{x \in B_r} |g_1(x) - g_2(x)|.$$

Then each d_r is a left invariant metric (which is not right invariant) on the group $\mathcal{M}$. As a consequence of general properties, we have: the family of the d_r 's induces on $\mathcal{M}$ the compact-open topology, i.e. the balls of all d_r 's form a subbase of the open sets in

$\mathcal{M}$ (see [12; Section XII.8, last remark]; clearly, it is sufficient to take r in the set of positive integers); moreover, the compact-open topology is the natural topology for the transformation groups (see [15; Chapter IV §2]), and in the present case it coincides with the euclidean topology of $\mathcal{M}$, considered as a subset of $\mathbb{R}^{(n^2)} \times \mathbb{R}^n$ (see [17; Chapter 7, Theorem 15]).

However, in this case, we must notice that all metrics d_r 's are equivalent; so it is not for topological reasons that we had to use all of them in the preceding pages.

Correspondingly in the Lie algebra $\mathfrak{m}$ of $\mathcal{M}$ we could consider the norms $\|(X, a)\|_r = r|X| + |a|$ (needless to say, all these norms are equivalent, too). At this point Lemma 2.3 could in fact give us a property of the exponential map $\exp_{\mathcal{M}}$ with respect to the norms $\|(X, a)\|_r$ and to the metrics d_r . It also seems possible to extend in part Remarks 3.3 and 4.2 to non-compactly supported functions.

Let us pass to the second remark.

The metrics $d_r(g, id)$ make sense also for the isometries g of $\mathbb{R}^n$ which are not motions, whereas evidently in this case the expression $r|X| + |a|$ has no meaning. In particular, if g is the reflexion with respect to an affine subspace $\mathcal{A}$ (so that $g \in \mathcal{M}$ if and only if $\text{codim } \mathcal{A}$ is even) and $\text{dist}(O, \mathcal{A}) = \delta$, then $d_r(g, id) = 2(r + \delta)$, for every $r > 0$.

Let us recall that the family of all reflections with respect to hyperplanes generates the group $O(n) \ltimes \mathbb{R}^n$. So one might perhaps conjecture that if a function $u \in L^p$ satisfies, e.g., (3.5) for all reflections g with respect to hyperplanes, then u satisfies (3.5) also for all motions and hence $u \in W^{1,p}$. This is untrue: let $u \in L^p$, with $\text{supp } u$ out of a ball B_ϱ , with $\varrho > 0$. Then we claim that u satisfies (3.5) for all reflections.

In fact, let g be a reflection with respect to a hyperplane, which is $\delta \geq 0$ distant from the origin. We consider

$$\mathcal{Q}_{g,r,p}(u) = \frac{\|u(g \cdot x) - u(x)\|_{L^p(B_r)}}{\max_{x \in B_r} |g \cdot x - x|} = \frac{\|u(g \cdot x) - u(x)\|_{L^p(B_r)}}{2(r + \delta)}.$$

If $2\delta + r \leq \varrho$, then $\mathcal{Q}_{g,r,p}(u) = 0$; if $2\delta + r > \varrho$, then also $2(\delta + r) > \varrho$ and therefore $\mathcal{Q}_{g,r,p}(u) \leq 2\|u\|_{L^p} \varrho^{-1}$. Since $2\|u\|_{L^p} \varrho^{-1}$ is independent of g and r , the assertion is proven.

The same remark (without leaving the group $\mathcal{M}$) can be made about the family of all reflections with respect to affine subspaces of codimension 2; as is known, if $n \geq 3$, this family generates $\mathcal{M}$ (see [11; Annexe II, n.11]).

All in all, we could expect what we have just said. For the function spaces we consider, those which are important are the small increments. Now, it is not possible to get small increments if one uses only reflexions. Moreover, all this confirms that it is convenient to use only the motions and not all the isometries.

After these preliminary remarks, let us briefly discuss homogeneous manifolds.

By homogeneous manifold we mean a manifold M which possesses, like $\mathbb{R}^n$, a transitive Lie transformation group G , provided with a G -invariant metric d and with a G -invariant measure (so that each transformation g of G is an isometry). Owing to the preceding remark, we will suppose that the group G is connected (as is consequently the manifold M). As done for $\mathbb{R}^n$, let us fix a point $p_0 \in M$; for every $r > 0$ let B_r denote the closed ball centred in p_0 and of radius r ; next let us introduce in G the metrics d_r , i.e. for every $r > 0$ and $g_1, g_2 \in G$ we define

$$d_r(g_1, g_2) = \max_{x \in B_r} d(g_1(x), g_2(x)).$$

Now analogously to the seminorms $\mathcal{Q}$ of the previous sections we define for functions $u \in L^p(M)$

$$\mathcal{Q}_{g,r,\lambda,p}(u) = \frac{\|\tau_g u - u\|_{L^p(B_r)}}{(d_r(g, id))^\lambda}.$$

This should be an intrinsic way to introduce function spaces of type (I), (II), (III), (IV) on M (see [7; Sections 10 and 11]).

Evidently it would be desirable (but it does not always seem possible) to have at our disposal a transitive group which is smaller than the whole group of isometries and can therefore in some sense play the role of the translation group of $\mathbb{R}^n$.

This is surely possible if M itself is a Lie group; see Section 7.6 of [35], where this approach is developed.

This is still possible if M is a Riemannian symmetric manifold (see [15] and [21]). Namely, in [21; Chapter II, §1] the translation (or displacement) group of M is defined as the group of isometries generated by the compositions of two geodesic symmetries. However, it may happen that this group coincides with the whole isometry group G of M ; for instance, it happens if G is a simple group. In [31] and in some previous papers the author used this translation group successfully in order to define function spaces of the Sobolev type on symmetric manifolds; in fact, if $\dim M = m$, he used m (suitably chosen) one-parameter subgroups of it and their infinitesimal generators.

We add a remark, again to show how difficult it can be to try to get a convenient small subgroup of G . If we put

$$G_b = \{g \in G \mid \sup_{x \in M} d(g \cdot x, x) < +\infty\},$$

G_b is a subgroup of G . If $M = \mathbb{R}^n$ and $G = \mathcal{M}$, then G_b is the translation group, which is sharply 1-transitive on $\mathbb{R}^n$.

This result does not hold for any manifold: G_b can be transitive on M , but not sharply 1-transitive or it is even possible that G_b is no longer transitive on M .

In fact, if M is compact, $G_b = G$. In order to justify the second statement let us consider the Klein model of the hyperbolic plane. Namely (see [9; Sections 28 and 29]), let Γ be a non degenerate conic in the real projective plane $P_2(\mathbb{R})$, M the inner region of Γ , d the Hilbert metric in M , G the isometry group of M , i.e. the group of those projec-

tive transformations which leave Γ (and therefore M) fixed. Then we have: G acts transitively on M ; $G_b = \{id\}$. In fact, if $\sup_{x \in M} d(g \cdot x, x) < +\infty$, then g must fix Γ pointwise, therefore g is the identity transform.

This last remark extends to all symmetric manifolds of non-compact type (see [15; Chapter VI, ex. C.1, p. 278-9]).

We will also recall that Nikol'skij and Besov spaces on homogeneous manifolds (in particular with regard to the approximation theory) are investigated by several Russian mathematicians; we refer for instance to the survey article [20] and to the references therein.

Finally let us mention that in [14] Sobolev spaces are defined and studied on an arbitrary metric space.

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