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On a Unifying Approach to Decomposition Theorems of Yosida-Hewitt Type (*).

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Abstract. – *In this paper we deal with a very general form of the Yosida-Hewitt theorem on the decomposition of measures into countably additive («normal») and purely finitely additive («antinormal») parts. It expands a previous one by the authors with the aim of joining two different standpoints to the Yosida-Hewitt type theorems. The first goes back to the original publication defining the «antinormal» part as a certain disjoint complement to the «normal» one. The second approach goes deeper and characterizes this disjoint complement intrinsically i.e. as a measure, functional or operator which is equal to zero on a huge set. These two points of view are common for the publications connected, respectively, with measure theory and theory of vector lattices; the second allows important applications. The unification of these approaches gives an opportunity to derive new information in the case of vector measures. We have taken the opportunity of this paper also to furnish a survey of the topic.*

1. – Introduction.

The Yosida-Hewitt theorem [28] (the YH theorem, for short) asserts that each bounded, finitely additive measure ν can be uniquely represented in the form $\nu = \nu_1 + \nu_2$, where ν_1 is a countably additive measure, while ν_2 is a purely finitely additive measure. The latter means that ν_2 is disjoint to any countably additive measure, i.e. if a countably additive measure ϱ is such that $0 \leqslant \varrho \leqslant |\nu_2|$, then $\varrho = 0$ (here $|\cdot|$ stands for the variation of measures). As a consequence, one easily derives a decomposition for any continuous functional on the Banach space L^∞ into an integral («normal») functional (generated by a function from L^1) and a functional which is disjoint to any integral functional (an «antinormal» functional).

The YH theorem has many versions and generalizations in diverse settings, e.g. functionals on vector lattices and spaces of vector-valued functions, measures with values in Banach spaces, topological groups and vector lattices, etc. Our paper has two

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objectives. The first, but not the main, goal is in the presentation of a unified approach which includes all cases mentioned above and shows, moreover, that they can be easily reduced to one theorem—the YH theorem for operators in vector lattices. The second, and the main, goal of this paper is to obtain in the same cases a deeper description of the «antinormal» part which allows many important applications (see references below).

Taking into account the long history of YH theorems and the variety of settings, we have found reasonable to provide the reader with the following detailed Section 2 which covers the principal settings and all preliminaries. The rest of our paper is organized as follows. In Section 3 we give the strong form of the YH-decomposition for order bounded operators from a VL E with the Egorov property to an order complete VL F with the countable sup property (Theorem 1), pointing also out that the latter condition on F is essential. Then we introduce the strict singularity for operators and extend Lozanovskiĭ's results to operators. Section 4 deals with operators between vector lattice normed spaces. In this context we establish the YH Theorem for majorized operators (Theorem 4). Finally, in Section 5 the decomposition of o-weakly compact operators from a VL to a Banach space is obtained (Theorems 5 and 6) and then applied to the case of vector measures.

It is probably worth to be noted that the setting of majorized operators is sufficient to deal with vector measures of bounded variation while results for the more general case of s-bounded measures come from considering o-weakly compact operators. We like to point out the centrality of the role played by Theorem 1 in the proofs of Theorems 4 and 7.

Despite of the great generality of our setting, the results of our work are new for many classical spaces of measurable functions, L^{∞} being the main example. The equivalence between the weak and strong forms of YH-decompositions is clear in the case of spaces where there is an o.d.i. with order continuous norm (e.g. for Orlicz and Marcinkiewicz spaces, more generally, for all rearrangement invariant spaces excluding L^{∞}). But this is not the case, say, for such spaces with mixed norm as $L^1[L^{\infty}]$ and $L^{\infty}[L^1]$ to which our theory also applies.

We are conscious that a good part of the material presented (in particular Theorems 1 and 7) is not dependent on linearity and can be extended to such things like ℓ-groups, topological groups and homomorphisms. However, we decided to remain confined to the concrete case of vector spaces. Our notation mainly follows that of classical treatises like [11], [29] and [1].

2. – Preliminaries.

2.1. *Weak and strong forms of the Yosida-Hewitt decompositions.*

The variety of YH theorems deals with the splitting of a given object into the sum of a «normal» part and its «disjoint» complement («antinormal» part). At the first sight the terms of «disjointness» look rather different in the diverse settings. What is common is that if «something» is «majorized» by such a disjoint complement and is «normal» then this «something» is zero.

The results of this type look somehow tricky because the existence of a linear projector is, usually, considered as a non-trivial fact in the framework of linear analysis—it is well-known how difficult is to construct projections onto infinite-dimensional subspaces of Banach spaces (in non-Hilbert space setting). But there is a wide class of linear spaces where the existence of a lot of special projectors is a triviality—this is the class of order complete vector lattices. In every order complete vector lattice there are special band projections onto every band—and there are a lot of bands. For example, every element generates a principal band. These are undergraduate facts from the theory of vector lattices (cf. [1, 11, 12, 19, 25, 29]) known, mainly, from 1930s.

We will prove that, in fact, all the types of the YH theorem, mentioned above, can be easily reduced to the vector lattice setting where the result is trivial.

Any decomposition into «normal» and «antinormal» parts where the «antinormal» part is described in terms of disjointness to the «normal» one is referred to as *the weak form of the YH-decomposition.* The majority of works on vector measures is exactly devoted to this kind of decompositions but, as far as we know, there are almost no applications of these results outside this theory itself. The point is that here we have too poor information about the «antinormal» part.

We are mostly interested in *the strong forms of the YH-decomposition*, where an intrinsic description of the «antinormal» part is given by characterizing this part as a functional, operator, or measure which vanishes on a «huge» subset of the domain. Such a kind of an antinormal part is referred to as a *singular* one. This strong form was, apparently, known only in the vector lattice setting. We will expand our investigations to all of the settings mentioned in the beginning. Naturally, if a certain weak form of the YH-decomposition is already known, the only thing we should prove, to get the corresponding strong form, is that any «antinormal» object is «singular».

In the case of functionals, the strong form of the YH-decomposition has a variety of different and unexpected applications. The idea is just opposite to the interest to such a kind of pathology as «antinormal» objects. In fact, the description of the antinormal part in many cases gives an opportunity to prove that the object is «normal» using the decomposition and proving that the «antinormal» part is really zero (of course, the starting point is the consideration of the huge set where it vanishes). Impressive examples are given by Khavin's proof of the Mooney–Khavin theorem from complex analysis (the idea of using the YH theorem belongs to G. Ya. LOZANOVSKIĬ), DUBOVITSKIĬ-MILYUTIN approach to control theory in infinite-dimensional setting, applications to subdifferential calculus. Moreover, some generalized forms of the YH theorem are also behind the «theory of optimization without compactness», invented in the paper by BUKHVALOV and LOZANOVSKIĬ [7] (see [6, 7, 11]). In its turn, the cited theory has a wide range of applications to optimization, geometry of Banach spaces and lattices, complex analysis, best approximation, game theory, mathematical economics.

In contrast to the weak form of the YH-decomposition, which is always true, practically without additional assumptions, the strong form of the YH-decomposition needs some assumptions. They are essential from the formal point of view, but they are not restrictive in practice, since the theory works in any space of measurable functions.

2.2. *The Yosida-Hewitt theorem for functionals in vector lattices*

Since we need some basic facts on decompositions in vector lattices, we shall start with a presentation of several YH-type results from this field, giving, at the same time, a historical commentary.

In the sequel the following abbreviations are in use: VL stands for «vector lattice», VLs stands for «vector lattices».

Let E be an arbitrary VL. Then the space $E_n^\sim$ of order continuous functionals, as well as the space $E_{n\sigma}^\sim$ of σ-order continuous functionals, forms a band in the order complete VL $E^\sim$ of all order bounded functionals [25]. Since there is a band projection onto any band in an order complete VL, we derive immediately the following weak forms of the YH-decomposition:

$$\text{(2.1)} \qquad E^\sim = E_n^\sim \oplus (E_n^\sim)^d, \qquad E^\sim = E_{n\sigma}^\sim \oplus (E_{n\sigma}^\sim)^d .$$

Here M^d stands for the disjoint complement of a set M. We will get the case of measures simply considering the VL of bounded measurable functions.

To state the strong form of the YH-decomposition we recall that a functional $\varphi \in E^\sim$ is said to be *singular* (resp. σ-singular) if there exists an order dense (resp. σ-order dense) ideal G in E such that φ vanishes on G. We shall use the notation $E_s^\sim$ (resp. $E_{s\sigma}^\sim$) for the set of singular (resp. σ-singular) functionals. The ideal G plays the role of the «huge» set mentioned above. We also recall that an ideal G in a VL E is said to be an order dense ideal—o.d.i., for short—(resp. σ-order dense ideal—σ-o.d.i., for short) if for any $e \in E_+$ there is a net $\{g_\alpha\} \subset G$ such that $0 \leq g_\alpha \uparrow e$ (resp. there is a sequence $\{g_m\} \subset G$ such that $0 \leq g_m \uparrow e$). In the case of a VL E, which is an ideal in the order complete VL of measurable functions on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$, an order bounded functional φ is singular (in this case it is the same as σ-singular) iff

$$\text{(2.2)} \qquad \forall e \in E \quad \forall A \in \mathcal{F}\colon \mu(A) > 0 \ \exists B \in \mathcal{F} \quad \text{such that } \mu(B) > 0, \ \ B \subseteq A \text{ and } \varphi(e\chi_B) = 0 .$$

It is an old result by A. G. PINSKER [12, 25] that in every order complete VL the set of singular (in the appropriate sense) functionals forms an o.d.i. in the corresponding set of antinormal functionals (this result is also true for operators; it was generalized to arbitrary VLs by A. I. VEKSLER). It was W. A. J. LUXEMBURG who proved in 1965 that, under some mild conditions, any antinormal functional is singular and obtained in this way a powerful intrinsic description (see [29,6] for original references), i.e., under some practically always fulfilled hypothesis the following is true:

(2.3) $$E^{\sim} = E_n^{\sim} \oplus E_s^{\sim}, \qquad E^{\sim} = E_{n\sigma}^{\sim} \oplus E_{s\sigma}^{\sim}.$$

The proof was based on sophisticated techniques dealing with functionals on Banach lattices (cf. [29,11]). It could be hardly carried out to the case of operators in a more or less straightforward manner. Almost at the same time G. Ya. LOZANOVSKIĬ found a much more elementary proof which basically depends only on the diagonal theorem for almost everywhere convergence. The result was presented in the seminar of Prof. B. Z. VULIKH at Leningrad State University, but G. Ya. LOZANOVSKIĬ did not publish his proof because of appearance of the work by W. A. J. LUXEMBURG. Later A. C. ZAANEN has found the same proof independently and has included it as a hint to the solution of Exercise 90.14 in [29]. This proof has been carried out to the case of operators acting in VL in the paper [2]. Exactly this kind of generalization is the starting point of this work.

The strong form of the YH-decomposition for functionals on the spaces $E(X)$ of measurable vector-functions has been derived by V. L. LEVIN [15] and A. V. BUKHVALOV [7]. Numerous applications are given in [7,16]. These were the very first results of this type in non-VL setting.

Investigation of integral representability of subdifferentials (see [16]) needs a more strong property of singularity which was invented by G. Ya. LOZANOVSKIĬ [17] who introduced localizable functionals (we prefer to call these functionals *strictly singular*). The difference is easily seen comparing the formula (2.2) with the following definition of strict singularity of $\varphi \in E^{\sim}$ for a VL E of measurable functions:

(2.4) $$\forall A \in \mathcal{F}\colon \mu(A) > 0 \ \exists B \in \mathcal{F} \text{ such that } \mu(B) > 0, \ B \subseteq A \text{ and } \varphi(e\chi_B) = 0 \ \forall e \in E.$$

This difference consists of the possibility to find B which does not depend on the choice of e. It occurs, for example, that on Orlicz spaces the classes of singular and strictly singular functionals coincide but there are some Marcinkiewich spaces for which this is not true [17,18]. Generalizations of results from [17] to the case of operators are resumed from [2] in section 3.

2.3. *The Yosida-Hewitt theorem for vector measures.*

A rather general YH decomposition for measures, to be intended here as finitely additive functions defined on a field $\mathcal{F}$ of subsets of a set Ω, is due to T. Traynor. He considers in [21,22] the case of group-valued measures giving, with our terminology, a decomposition of weak type since the antinormal part is described in terms of a certain disjointness.

To be less vague, let us consider a vector measure m defined on $\mathcal{F}$ and with values in a Banach Space (BS, for short) Y. We remind that m is said to be *s-bounded* (= *strongly additive* in [9]) if $m(A_n)$ tends to zero whenever $\{A_n\}$ is a disjoint sequence from $\mathcal{F}$. Moreover, the concept of pure finite additivity is extended to vector measures by means of the following definition: m is said to be purely finitely additive if for any countably additive vector measure ν and for any positive ε, there is a set $A \in \mathcal{F}$ such that $\|m(B)\|_Y \leqslant \varepsilon$ whenever $B \subseteq A$ and $\|\nu(C)\|_Y \leqslant \varepsilon$ whenever $C \subseteq \Omega \backslash A$.

Traynor's decomposition states that any s-bounded vector measure m can be uniquely decomposed as $m = m_1 + m_2$ where m_1 and m_2 are s-bounded vector measures, m_1 is countably additive and m_2 is purely finitely additive.

Since we have stated the above theorem only for vector space-valued (rather than group-valued) measures, we must note that in this form it coincides with [9], Theorem 8, p. 30. A formal difference is in the description of pure finite additivity. The component m_2 is characterized in [9] by the fact that, for any $y^* \in Y^*$, the real measure $y^* \circ m_2$ is purely finitely additive. On the other hand, for s-bounded vector measures, the latter property is equivalent to the pure finite additivity.

It follows a formula (see also [20], for the case of a VL Y) for the countably additive component of an s-bounded m:

$$m_1(A) = \lim_{\{A_n\} \in \mathcal{P}_A} \lim_n m(A_n), \tag{2.5}$$

where $\mathcal{P}_A$ denotes the set (directed by $\{A_n\} \geqslant \{B_n\}$ iff $A_n \subseteq B_n$ $\forall n \in \mathbb{N}$) of all increasing sequences $\{A_n\}$ from $\mathcal{F}$ such that $A_n \uparrow A$.

Decompositions for vector measures can be seen as consequences of decompositions of operators defined on VLs. The idea of using some versions of the generalized YH theorem for operators to derive results for vector measures is not pretty new. It was applied in this way to the vector-lattice-valued measures in [20] and to the vector measures in [13]. But, surprisingly, only the weak form of the YH-decomposition was under consideration.

To fix notation for the sequel, let us assume that $m\colon \mathcal{F} \to Y$ is a vector measure and denote by T_m the associated operator acting on the space $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$-simple real functions in the usual way:

$$T_m\left(\sum_i \alpha_i \chi_{A_i}\right) = \sum_i \alpha_i m(A_i). \tag{2.6}$$

Denote $\mathcal{N}(m) := \{A \in \mathcal{F}\colon B \subseteq A,\ B \in \mathcal{F} \Rightarrow m(B) = 0\}$.

Let $\mathcal{L}^\infty(\mathcal{F})$ be the space of all real-valued functions on Ω that are uniform limits of $\mathcal{F}$-simple functions (so for a σ-field $\mathcal{F}$ we have precisely bounded $\mathcal{F}$-measurable functions). As it is well known, formula (2.6) is the germ for a one-to-one linear correspondence between $B(\mathcal{L}^\infty(\mathcal{F}), Y)$, the space of all bounded operators from $\mathcal{L}^\infty(\mathcal{F})$ to Y, and the space of all bounded vector measures from $\mathcal{F}$ to Y. Starting from a σ-finite measure space $(\Omega, \mathcal{F}, \lambda)$, one establishes, more particularly, a one-to-one correspondence between $B(L^\infty(\lambda), Y)$ and the space of all bounded vector measures $m\colon \mathcal{F} \to Y$ that vanish on λ-null sets. It is easy to imagine that decompositions of T_m will correspond to decompositions of m. Naturally, one has to transfer hypothesis concerning m to corresponding hypothesis relative to the operator T_m defined on the suitable VL of functions.

3. - The Yosida-Hewitt theorem for operators in vector lattices.

Throughout this paper all VLs will be assumed Archimedean. If E and F are VLs and F is order complete (Dedekind complete), the symbols $L^{\sim}(E, F)$, $L_n^{\sim}(E, F)$ and $L_{n\sigma}^{\sim}(E, F)$ denote the spaces of operators from E to F that are respectively *regular, order continuous* and *σ-order continuous*. If there is no reason to emphasize the role played by E and F, we simply write $L^{\sim}$, $L_n^{\sim}$ and $L_{n\sigma}^{\sim}$: The spaces $L_n^{\sim}$ and $L_{n\sigma}^{\sim}$ are bands in $L^{\sim}$. The *null ideal* of an operator $T \in L^{\sim}(E, F)$ is $N(T) := \{e \in E: |T|(|e|) = 0\}$. Throughout the paper E and F are VLs, the latter being order complete.

DEFINITION 1. – An operator $T \in L^{\sim}(E, F)$ is called *singular* if it vanishes on a suitable o.d.i. $G \subseteq E$. The set of singular operators is denoted by $L_s^{\sim} = L_s^{\sim}(E, F)$. Analogously if T vanishes on a σ-order dense (= super order dense) ideal, then T is called *σ-singular* and the set of such operators is denoted by $L_{s\sigma}^{\sim} = L_{s\sigma}^{\sim}(E, F)$.

It is obvious that $L_s^{\sim}$ and $L_{s\sigma}^{\sim}$ are ideals in $L^{\sim}$. Our goal is to learn when they are bands.

The weak form of the YH-decomposition for operators is proved without any additional assumptions in the same trivial way as (2.1). As an immediate corollary the YH theorem for VL-valued measures [20] can be derived.

To formulate our result for the strong form of the YH-decomposition we need several definitions more. It is well known that this form is not valid, even in the case of functionals, without certain additional assumptions. For example, in the VL $E = C[0, 1]$ there are no non-trivial order continuous functionals (cf. [25]) but the Rieman integral gives an example of a non singular functional.

We recall that a VL E has the *Egorov property* [19] if for every $0 \leq f \in E$ and for every double sequence $f_{n,k} \in E$, we have that $0 \leq f_{n,k} \uparrow_k f$ (for all $n \in \mathbb{N}$) implies the existence of a sequence $0 \leq g_m \uparrow f$ for which the relation

$$g_m \leq f_{n, j(m, n)}$$

holds (where m, n are arbitrary and $j(m, n)$ is suitable).

We also recall that a VL is said to have the *countable sup property*, if whenever an arbitrary subset D has a supremum, then there exists an at most countable subset C of D with the same supremum.

Both, the Egorov property and the countable sup property are true for the spaces of measurable functions on a σ-finite measure spaces.

To cover also the more general case of semi-finite measure spaces we recall the following definition. A VL E is said to be *weakly σ-distributive* if for every $0 < f \in E$ and for every double sequence $f_{n,k} \in E$, we have that $0 \leq f_{n,k} \uparrow_k f$ (for all $n \in \mathbb{N}$) implies the existence of a sequence $j(n)$ such that there is an element $g > 0$ with $g \leq f_{n, j(n)}$ for every n.

THEOREM 1. – *Let E be a VL with the Egorov property and F an order complete VL with the countable sup property. Then*

$$L^{\sim}(E, F) = L^{\sim}_{n\sigma}(E, F) \oplus L^{\sim}_{s\sigma}(E, F). \tag{1}$$

THEOREM 2. – *Let E be a weakly σ-distributive VL and F an order complete VL with the countable sup property. Then*

$$L^{\sim}(E, F) = L^{\sim}_{n}(E, F) \oplus L^{\sim}_{s}(E, F). \tag{2}$$

PROOF. – See [2]. ∎

REMARK 1. – Let us consider the band projection $P_1 : L^{\sim} \to L^{\sim}_{n\sigma}$ and, by [1], Theorem 4.6, we have for $T \in L^{\sim}_{+}$ and $e \in E_+$,

$$T_1(e) := P_1 T(e) = \inf\{\sup_k T(e_k) :\ 0 \leq e_k \uparrow e\}. \tag{3}$$

The proof of theorems 1 and 2 mainly consist in improving formula (3) showing that in our case the infimum is really attained. Formula (3) needs to be refined to be applicable to derive formulas like (2.5) for the countably additive part of a vector measure. The refinement we are speaking about is well known and consists of taking into account only components of e:

$$T_1(e) = \inf\{\sup_k T(e_k) :\ 0 \leq e_k \uparrow e, (e - e_k) \wedge e_k = 0\ \ \forall k \in \mathbb{N}\}. \tag{3$'$}$$

This formula (as well as its uncountable analogue for the order continuous case) is valid under assumption that E has the principal projection property.

REMARK 2. – If E is an order complete VL with the Egorov property and the continuum hypothesis (CH) is true, then E has the countable sup property [12]. This means that there is no difference between formulas (1) and (2) since $L^{\sim}_{n} = L^{\sim}_{n\sigma}$ and $L^{\sim}_{s} = L^{\sim}_{s\sigma}$. It is interesting to mention that without the order completeness assumption this is not the case. Let us consider the space $E = C(\beta\mathbb{N} \backslash \mathbb{N})$ of continuous functions on the Stone-Čech remainder of the set of natural numbers. It is proved in [23] that this space has the Egorov property, hence both formulas (1) and (2) are true. But they look in quite an opposite manner. Indeed, this Banach lattice has σ-order continuous norm which implies that $L^{\sim}(E, F) = L^{\sim}_{n\sigma}(E, F)$ for any VL F, i.e. (1) is read as

$$L^{\sim} = L^{\sim}_{n\sigma} \oplus \{0\}.$$

On the other hand, there are no non-trivial ideals with order continuous norm in E [23]. This (in combination with σ-order continuity) implies that $L^{\sim}_{n}(E, F) = \{0\}$ at least in the case when F has a total set of order continuous functionals. Hence, (2) is read as

$$L^{\sim} = \{0\} \oplus L_s^{\sim},$$

that is $L_{n\sigma}^{\sim} = L_s^{\sim}$ in this case.

REMARK 3. – The countable sup property of F is essential for the validity of Theorem 1. This is illustrated by the following example. Take $E = L^{\infty}(0, 1)$ and, by the Kreins-Kakutani representation theorem, realize it as a space $C(Q)$ for an extremally disconnected compact Hausdorff space Q. Let T be the identical embedding of $E := C(Q)$ into $F := l^{\infty}(Q)$. We claim that $T \in (L_n^{\sim})^d \backslash L_s^{\sim}$.

It is clear that T does not belong to $L_s^{\sim}$. To show that $T \in (L_n^{\sim})^d$, let T_n be the order continuous component of T and f_0 the constant function 1 on Q. Proving $T_n(f_0) = 0$ we get that $T_n = 0$. Let $\{U_\alpha\}$ be a base of neighbourhoods of an arbitrary point $q \in Q$. Set $F_\alpha = Q \backslash U_\alpha$ and $\alpha_1 \geqslant \alpha_2$ if $U_{\alpha_1} \subseteq U_{\alpha_2}$. By complete regularity, for any α we get $f_\alpha \in C(Q)$ with $f_\alpha(q) = 0$ and $f_\alpha = 1$ over F_α. As a consequence, we can affirm that $f_\alpha \uparrow f_0$ in $C(Q)$ (Q has no isolated points since $C(Q)$ is order isomorphic to $L^{\infty}(0, 1)$) and by [1], Theorem 4.6 $T_n(f_0) \leqslant \sup_\alpha T(f_\alpha)$ (in $l^{\infty}(Q)$), hence $T_n(f_0)(q) = 0$. Since q is arbitrary we have finished.

DEFINITION 2. – An operator $T \in L^{\sim}(E, F)$ is called *strictly singular* if for any non-trivial band B in E, there exists a non-trivial band $K \subseteq B$ such that $T\restriction_K = 0$. The set of all strictly singular operators is denoted by $L_{ss}^{\sim} = L_{ss}^{\sim}(E, F)$.

Obviously $L_{ss}^{\sim} \subseteq L_s^{\sim}$. In [17,18] examples of singular but non strictly singular functionals are given (the case of Marcinkiewicz spaces is considered in detail). It is easy to see that $L_{ss}^{\sim}(L^{\infty}, F) = L_s^{\sim}(L^{\infty}, F)$.

PROPOSITION 1 [2]. – *The ideal $L_{ss}^{\sim}$ is order dense in $L_s^{\sim}$.*

PROPOSITION 2 [2]. – *For a VL E which is decomposable into the direct sum of the bands with both the countable sup and the Egorov property, $L_{ss}^{\sim}(E, F)$ is σ-order closed, namely*

$$T_k \in L_{ss}^{\sim}\ (k \in \mathbb{N});\ 0 \leqslant T_k \uparrow T \in L^{\sim} \Rightarrow T \in L_{ss}^{\sim}.$$

We recall that an element e of a VL is said to be of *countable type* if the principal ideal generated by e has the countable sup property. Now Propositions 1 and 2 give

THEOREM 3. – *Let E be a VL satisfying the conditions of Proposition 2 and F an order complete VL. Then, if $T \in L_s^{\sim}(E, F)$ is of countable type, it follows that $T \in L_{ss}^{\sim}(E, F)$.*

It is shown in [17] that in the space $L^{\infty}[L^1]$ every singular functional is strictly singular despite of the fact that its dual does not possess the countable sup property.

4. – Vector-lattice-normed spaces.

We would like to turn the attention of the reader to the concept of *vector-lattice-normed space* (VLNS, for short) which was introduced by L.V. Kantorovich in 1930s but has been overlooked by many mathematicians. This concept is fruitful when it is necessary to develop some general approach to similar notions in Banach spaces and in VLs.

DEFINITION 3. – A triple (X, p, E) is called a VLNS if X is a vector space, E is a VL (the *norming lattice*) and p is an *abstract (lattice-valued) norm* on X, i.e. a mapping $p: X \to E_+$ such that the following natural conditions are satisfied:

(*a*) $p(x) = 0$ if and only if $x = 0$;

(*b*) $p(x_1 + x_2) \leqslant p(x_1) + p(x_2)$; $x_1, x_2 \in X$;

(*c*) $p(\lambda x) = |\lambda| p(x)$; $\lambda \in \mathbb{R}$, $x \in X$.

Any VL E may be considered as a VLNS with the norming VL equal to E itself and $p(\cdot) = |\cdot|$. Any normed space may be considered as a VLNS with the real line as a norming lattice and $p(\cdot) = \|\cdot\|$.

Typical examples of VLNS are given by spaces of measurable vector-valued functions and by spaces with mixed norm. We recall that the space of measurable vector-valued functions $E(X)$, constructed from a Banach function space E on a measure space (T, Σ, μ) and a Banach space X, is defined as the space of all (strongly) measurable functions f: $T \to X$ such that $\|\!| f |\!\| := \|f(\cdot)\| \in E$. It is provided with the norm $\|f\| = \| \|\!| f |\!\| \|$ of mixed type. Clearly, $E(X)$ is a VLNS with the norming space E and with the abstract norm $\|\!|\cdot|\!\|$. Some other examples are certain spaces of scalarly measurable functions [4] and the spaces of operators with abstract norm [12].

Let (X, p, E) and (Y, q, F) be two VLNS. An operator $T: X \to Y$ is called a *majorized operator* if a positive operator (a *majorant*) $U: E \to F$ exists such that

$$q(Tx) \leqslant U(p(x)) \qquad \forall x \in X.$$

In the case of an order complete F we can consider the infimum (in the space $L^{\sim}(E, F)$) of all majorants for T; it is denoted by $\|\!| T |\!\|$ and still majorizes T, namely $q(Tx) \leqslant \|\!| T |\!\|(p(x))$. In this case the space $M(X, Y)$ of all majorized operators is a VLNS with the abstract norm $\|\!|\cdot|\!\|$ (taking values in $L^{\sim}$). From now on the norming space F is assumed to be order complete.

DEFINITION 4. – An operator $T \in M(X, Y)$ is called *bo-continuous* (we write $T \in M_n(X, Y)$) if $p(x_\alpha) \xrightarrow{(o)} 0$ implies $q(Tx_\alpha) \xrightarrow{(o)} 0$ (here the usual order convergence of nets in the corresponding VLs is assumed).

An operator $T \in M(X, Y)$ is called *singular* ((we write $T \in M_s(X, Y)$) if an o.d.i. $G \subseteq E$ exists such that $p(x) \in G$ implies $T(x) = 0$.

PROPOSITION 3. – *Let $T \in M(X, Y)$. Then T is singular if and only if $\| T \|$ is singular.*

PROOF. – (*of the «only if» part*). Assume G is an o.d.i. in E with $p(x) \in G \Rightarrow T(x) = 0$. Let $0 \leq f \in G$. By [14] we have

$$\| T \|(f) = \sup\left\{ \sum_{i=1}^{n} q(Tx_i) \colon \sum_{i=1}^{n} p(x_i) \leq f \right\}.$$

Evidently, from $\sum_i p(x_i) \leq f$ we have $p(x_i) \in G$ and, therefore, $q(Tx_i) = 0$. This gives $\| T \|(f) = 0$. ■

Two other properties play an important role in the context of VLNS: the decomposability (or Kantorovich's) condition and the *bo*-completeness.

DEFINITION 5. – The lattice norm p of a VLNS (X, p, E) is called *decomposable* if $p(x) \leq e_1 + e_2$ $(x \in X, e_i \in E_+)$ implies the existence of $x_i \in X$ with $x = x_1 + x_2$ and $p(x_i) \leq e_i$. A VLNS (X, p, E) is called *bo-complete* if for any net $\{x_\alpha\}$ in X we have that $p(x_\alpha - x_\beta) \xrightarrow{(o)} 0$ implies the existence of $x \in X$ such that $p(x_\alpha - x) \xrightarrow{(o)} 0$.

It is clear that the decomposability is inspired to the so called Riesz decomposition property to which is equivalent for VLs. All other examples of VLNS that have been presented fulfil both conditions of Definition 5.

From [14] we report the following

THEOREM A. – *Let (X, p, E) be a VLNS with decomposable norm and (Y, q, F) a bo-complete VLNS. Then:*

(1) *the norm $\| \cdot \|$ of the VLNS $M(X, Y)$ is decomposable;*

(2) *for an operator $T \in M(X, Y)$ the equivalence $T \in M_n(X, Y) \Leftrightarrow \| T \| \in L_n^{\sim}$ holds.*

REMARK 4. – WEAK YH DECOMPOSITIONS. From part (1) of the previous Theorem A it is evident that for any given band B of $L^{\sim}$, we have the possibility of representing (in a unique way) a majorized operator $T \in M(X, Y)$ as a sum $T = T_1 + T_2$ of majorized operators such that $\| T \| = \| T_1 \| + \| T_2 \|$, $\| T_1 \| \in B$ and $\| T_2 \| \in B^d$.

Weak YH decompositions then correspond to the choices $B = L_n^{\sim}$, $B = L_{n\sigma}^{\sim}$, while taking into account also part (2) of theorem A, and by means of Theorem 1 and Proposition 3, we get, for example, the following strong YH-decomposition.

THEOREM 4. – *Let (X, p, E) be a VLNS with decomposable norm and E is weakly σ-distributive. Assume further that (Y, q, F) is bo-complete and F is order complete and with the countable sup property. Then, any $T \in M(X, Y)$ can be uniquely decomposed as $T = T_1 + T_2$ where: $T_1 \in M_n(X, Y)$, $T_2 \in M_s(X, Y)$ and $\| T \| = \| T_1 \| + \| T_2 \|$.*

This result generalizes the YH theorem for the functionals on the space $E(X)$ of vector-valued functions [7, 5, 15].

In a similar way one could introduce spaces $M_{n\sigma}$ and $M_{s\sigma}$, the sequential analogs of spaces M_n and M_s, and prove the following strong YH-decomposition formula: $M = M_{n\sigma} \oplus M_{s\sigma}$.

5. - *o*-weakly compact operators.

In this section we shall give decompositions of operators acting from VLs to BSs both for their intrinsic interest and because of our wish of dealing with vector measures. The aim is to derive, in certain cases, new information about the purely finitely additive part as we did in the previous sections for the «antinormal» part of an operator.

Let us assume that $m\colon \mathcal{F} \to Y$ is a vector measure. First, one should note that $T_m \in M(S(\mathcal{F}), Y)$ if and only if m is of bounded variation (in which case the formula $\| T_m \|(f) = \int f d|m|$ holds for $f \in S(\mathcal{F})$). This tells us, evidently, that in the setting of majorized operators we can only obtain decompositions for measures of bounded variation. For example, applying to T_m the decomposition contained in Remark 4 for the case $B = L_{n\sigma}^{\sim}$ we obtain the Yosida-Hewitt decomposition for vector measures of bounded variation as it is stated, say, in the book [9], Theorem 8, p. 30. Of course, this is the usual version of the Yosida-Hewitt theorem where the purely finitely additive component is described in terms of its «disjointness», with respect to any countably additive measure. As we have already remarked, we are interested to go further such a kind of characterization of pure finite additivity.

To derive decompositions for s-bounded vector measures (both weak, like in section 2.3, and strong, like in our final Corollary 3), we must abandon majorized operators to deal with a wider class of operators that corresponds to the class of s-bounded vector measures. The operators we are speaking about are the, so called, *o-weakly compact operators.*

Let E be a VL and Y a BS. We assume that E admits separating order dual $E^{\sim}$. This will not cause any restrictions for our application to vector measures. With this assumption we can identify E with a subset in the second order dual $E^{\sim\sim}$ in the canonical way, and make use of duality theory. We recall from [10] that a linear operator $T\colon E \to Y$ is o-weakly compact if, for any $e \in E_+$, the set $\{Tf\colon |f| \leq e\}$ is relatively weakly compact. o-Weakly compact operators are precisely those that in [3] were called *monotonely Cauchy*. It is relevant for our purposes that

PROPOSITION A [10, Proposition 4.3]. - *A vector measure m is s-bounded if and only if the associated operator T_m is o-weakly compact.*

PROPOSITION 4. - *Any majorized operator $T\colon E \to Y$ from a VL E to a BS Y is o-weakly compact.*

PROOF. – In fact, T can be factorized through L_1 and, therefore, can be written as $T = T' \circ j$ where $j: E \to L_1$ is positive and $T': L_1 \to Y$. Hence, T maps order intervals in E into relatively compact subsets of Y. ∎

For an *o-bounded* operator $T: E \to Y$ (i.e. such that $T[0, e]$ is norm bounded in Y for every $e \in E_+$) we can have the adjoint operator T^* at our disposal. It is defined as usual:

$$T^*: Y^* \to E^\sim, \quad \langle T^* y^*, f\rangle := \langle y^*, Tf\rangle \quad (y^* \in Y^*, f \in E).$$

Moreover, a «natural» monotone (Riesz) seminorm ϱ_T on E can be associated to T, i.e. if Y_1^* denotes the unit ball of Y^*, then

$$\varrho_T(f) := \sup_{y^* \in Y_1^*} \langle |T^* y^*|, |f|\rangle = \sup\{\|Te\|_Y : |e| \leq |f|\}.$$

Denote by I_E the ideal generated by E in $E^{\sim\sim}$.

We can also define the operator

$$T^{**}: I_E \to Y^{**}, \quad \langle T^{**} \varphi, y^*\rangle := \langle \varphi, T^* y^*\rangle \quad (y^* \in Y^*, \varphi \in I_E).$$

PROPOSITION 5. – *Let $T: E \to Y$ be an o-bounded operator. Assume further that B is a band in $E^\sim$. Then T can be uniquely decomposed as a sum of two o-bounded operators $T_i: \to Y^{**}$ $(i = 1, 2)$ such that $y^* \circ T_1 \in B$ and $y^* \circ T_2 \in B^d$ for all $y^* \in Y^*$.*

PROOF. – We consider the two projections $P_1: E^\sim \to B$ and $P_2: E^\sim \to B^d$, and set $V_i := P_i \circ T^*$. Evidently

$$T^{**} = V_1^* + V_2^*.$$

According to the fact that T^{**} coincides with T on E, we define

$$T_1 := V_1^* \restriction_E \quad \text{and } T_2 := V_2^* \restriction_E.$$

Since, evidently, $y^* \circ T_i = P_i T^* y^*$ the assertion is proved. ∎

To give readers a quick reference, we shall report from [10] the next Theorem B that will be useful in the sequel.

THEOREM B [10, Theorem 4.2]. – *For an o-bounded operator $T: E \to Y$ the following statements are equivalent:*

(1) *T is o-weakly compact;*

(2) $T^{**}(I_E) \subseteq Y$;

(3) *each monotone order bounded sequence in E is ϱ_T-Cauchy.*

COROLLARY 1. – *Let T: $E \to Y$ be an o-weakly compact operator. Then the following are equivalent:*

(1) $y^* \circ T \in E_{n\sigma}^{\sim}$ *(resp.* $E_n^{\sim}$*) for all* $y^* \in Y^*$*;*

(2) $\|Tf_n\|_Y \to 0$ *(resp.* $\|Tf_\alpha\|_Y \to 0$*) whenever in* E *we have* $f_n \downarrow 0$ *(resp.* $f_\alpha \downarrow 0$*).*

PROOF. – To prove that (1) implies (2), assume $e_k \downarrow 0$ in E. Certainly (by Theorem B), Te_k is norm converging to some $y \in Y$. On the other hand, $\langle Te_k, y^* \rangle$ tends to zero, for $k \to \infty$, since $y^* \circ T$ belongs to $E_{n\sigma}^{\sim}$. It follows that $y = 0$. Considering that in the assertion (3) of Theorem B one can replace sequences by nets, the above argument works also for nets. ■

In the present context we need to define the concepts corresponding, for example, to those of σ-order continuous operator, singular operator and a concept that corresponds to the disjointness from any σ-order continuous operator.

DEFINITION 6. – An operator T: $E \to Y$ is called *σ-smooth* (resp. *smooth*) if $f_n \downarrow 0$ (resp. $f_\alpha \downarrow 0$) in E implies $\|Tf_n\|_Y \to 0$ (resp. $\|Tf_\alpha\|_Y \to 0$). An operator T is called *σ-singular* (resp. *singular*) if the ideal $N(T) := \{e \in E: 0 \leq f \leq |e| \Rightarrow Tf = 0\}$ is σ-order dense (resp. order dense) in E. Finally, T is called *purely non-σ-smooth* (resp. *purely non-smooth*) if, for every $y^* \in Y^*$, we have that $y^* \circ T \in (E_{n\sigma}^{\sim})^d$ (resp. $\in (E_n^{\sim})^d$).

The following (weak) decomposition corresponds to the weak form of the Yosida-Hewitt theorem for s-bounded measures.

THEOREM 5. – *Any o-weakly compact operator T: $E \to Y$ can be uniquely decomposed as a sum of two o-weakly compact operators T_1, T_2 from E to Y such that T_1 is σ-smooth and T_2 is purely non-σ-smooth.*

PROOF. – Apply Proposition 5 for $B = E_{n\sigma}^{\sim}$. We must prove that the operators T_i are Y-valued. With the notation of the proof of Proposition 5, we claim: $V_i^*(f) \in Y$ whenever $f \in E$ and $i = 1, 2$. To prove this claim we must show, given $f \in E$, the existence of $y_f \in Y$ such that

$$\langle V_i^*(f), y^* \rangle = \langle y_f, y^* \rangle \quad \forall y^* \in Y^* .$$

From the relation $|P_i^* f| \leq |f|$ follows that $P_i^* f$ belongs to the ideal I_E and therefore from Theorem B we derive $y_f := T^{**} P_i^* f \in Y$. Now the computation follows:

$$\langle y_f, y^* \rangle = \langle P_i^* f, T^* y^* \rangle = \langle f, V_i y^* \rangle = \langle V_i^* f, y^* \rangle .$$

So our claim has been proved.

For $0 \leq f \in E$ note that

$$\langle f, |T^* y^*| \rangle \geq \langle P_i^* f, |T^* y^*| \rangle \geq \langle P_i^* f, T^* y^* \rangle = \langle T_i f, y^* \rangle .$$

It follows that

$$\varrho_T(f) = \sup_{y^* \in Y_1^*} \langle f, |T^* y^*| \rangle \geq \sup_{y^* \in Y_1^*} |\langle T_i f, y^* \rangle| = \|T_i f\|_Y$$

and

$$\varrho_T \geq \varrho_{T_i} .$$

By means of Theorem B, this inequality tells us that T_i is o-weakly compact. Corollary 1 gives that T_1 is σ-smooth. ■

REMARK 5. – Evidently, by starting with $B = E_n^{\sim}$, a decomposition into a smooth part and a purely non-smooth part also follows.

In Definition 6 pure non-σ-smoothness has been defined according to the definition of pure finite additivity given in [9]. As for measures, many other definitions can be found in the literature, so it is of some interest to note that for o-weakly compact operators two other alternative descriptions of pure non-σ-smoothness are possible.

PROPOSITION 6. – *For an o-weakly compact operator T: $E \to Y$ the following statements are equivalent:*

(1) *T is purely non-σ-smooth;*

(2) *for any σ-smooth operator U: $E \to Y$, any $\varepsilon > 0$ and any $f \in E_+$ we can find $f_1, f_2 \in E_+$ such that $f = f_1 + f_2$, $\varrho_T(f_1) \leq \varepsilon$ and $\varrho_U(f_2) \leq \varepsilon$;*

(3) *there is no non-trivial σ-order continuous monotone seminorm smaller than ϱ_T.*

REMARK 6. – In [3], being inspired to the literature concerning the so-called Frechet-Nikodym topologies (see [22] and [26]), pure non-σ-smoothness was defined in terms of singularity in the lattice of locally solid topologies on E. Condition (2) in the above proposition is precisely a translation of that definition. With this in mind, two things become clear: first in Proposition 6 we have that (2) implies (3), second a reformulation of (2) is possible by replacing U: $E \to Y$ with U: $E \to Z$ where Z is any BS. On the other hand, when Y is the real line, evidently (1) and (2) coincide.

PROOF OF PROPOSITION 6. – (1) $\Rightarrow$ (2) can be deduced from the results of [3] (see also [27] for the case of measures). Let us fix U, ε, and f. The fact that T is o-weakly compact permits us to find a finite set of functionals $y_1^*, \ldots, y_n^* \in Y^*$ such that

$$[0, f] \cap \bigcap_{i=1}^{n} \{x \in E\colon \varrho_{y_i^* \circ T}(x) \leq \varepsilon\} \subseteq [0, f] \cap \{x \in E\colon \varrho_T(x) \leq \varepsilon\}$$

(for example, apply [3], Theorem 3.2).

Since $y_1^* \circ T \in (E_{n\sigma}^{\sim})^d$ and U is σ-smooth, it is possible to find $a_1, b_1 \in [0, f]$ such that $f = a_1 + b_1$, $\varrho_U(a_1) \leq \varepsilon/n$ and $\varrho_{y_1^*} \circ T(b_1) \leq \varepsilon$. Similarly we decompose b_1 as $b_1 = a_2 + b_2$ where $a_2, b_2 \in [0, b_1]$ and $\varrho_U(a_2) \leq \varepsilon/n$, $\varrho_{y_2^* \circ T}(b_2) \leq \varepsilon$. Proceeding in this way, we will finally get

$$f = a_1 + \ldots + a_n + b_n \text{ where } \varrho_U(a_i) \leqslant \frac{\varepsilon}{n} \text{ and } \varrho_{y_i^* \circ T}(b_n) \leqslant \varepsilon \ \forall i = 1, \ldots n.$$

So $f_1 = b_n$ and $f_2 = a_1 \ldots + a_n$.

(2) $\Rightarrow$ (3): See Remark 6.

(3) $\Rightarrow$ (1): Decomposing T according to Theorem 5, we have $T = T_1 + T_2$. As we proved, $\varrho_{T_1} \leqslant \varrho_T$. Therefore, since T_1 is σ-smooth iff ϱ_{T_1} is σ-order continuous, we have $\varrho_{T_1} = 0$ and $T_1 = 0$. Hence $T = T_2$ is purely non-σ-smooth. ■

Of course, now we recognize Theorem 5 and [3], Theorem 3.4 as the same. In the spirit of the previous sections we shall now furnish a strengthened decomposition theorem for o-weakly compact operators. Namely, we offer, under suitable assumptions, new information on the purely non-σ-smooth part. This decomposition (in its full generality it is stated in Theorems 7 and 8) will be applied to vector measures. First we need the following lemma.

LEMMA 1. – *A countable intersection of σ-order dense ideals in a VL with the Egorov property is also a σ-order dense ideal.*

PROOF. – Let E be our VL and $\{G_k\}$ a countable family of σ-o.d.i. in E. Set $G = \bigcap_k G_k$. Take an arbitrary $f \in E$, $f > 0$. We will construct a sequence $f_m \uparrow f$ such that $f_m \in G$. Since it is evident that any finite intersection of G_k is a σ-o.d.i., for any $n \in \mathbb{N}$ we can choose a sequence $g_{n,k} \uparrow_k f$ such that $g_{n,k} \in \bigcap_{i=1}^{n} G_i$. Using the Egorov property we can find a sequence $0 \leqslant f_m \uparrow f$ such that for all m and n a suitable index $j(m, n)$ exists with the property $f_m \leqslant g_{n, j(m,n)}$. So any $f_m \in G$. ■

COROLLARY 2. – *A countable intersection of order dense ideals in a σ-weakly distributive VL is also an order dense ideal.*

PROOF. – One needs to repeat the proof of Lemma 1 with minor modifications. ■

THEOREM 6. – *Let T: $E \to Y$ be o-weakly compact. Assume further that E has the Egorov property and Y is separable. Then:*

(1) *The operator T is purely non-σ-smooth iff it is σ-singular.*

(2) *If the domain of T has also the countable sup property, the operator is purely non-smooth iff it is singular.*

PROOF. – Let T be purely non-σ-smooth. By Theorem 5, T coincides with T_2, i.e. with the restriction to E of $(P_2 \circ T^*)^*$. Here P_2 is the projection of $E^\sim$ onto $(E^\sim_{n\sigma})^d$ which is the same as $E^\sim_{s\sigma}$ because of the Egorov property. Now we prove that T_2 is null on a σ-o.d.i. in E. For any $y^* \in Y_1^*$ consider the null-ideal $N_{y^*} := N(P_2 T^* y^*) = \{e \in E\colon |P_2 T^* y^*|(|e|) = 0\}$. Since $P_2 T^* y^* \in E_{s\sigma}^{\sim}$, the ideal N_{y^*} is σ-order dense. Once we have proved that $G := \bigcap_{y^* \in Y_1^*} N_{y^*}$ is a σ-o.d.i., then an obvious calculation gives that T_2 is null on G. Because of separability of Y, the ball Y_1^* endowed with the weak*-topology is metrizable and admits a countable (weak*-)dense subset D.

Claim: $G = \bigcap_{y^* \in D} N_{y^*}$. To prove this claim let g be an element of $\bigcap_{y^* \in D} N_{y^*}$. So we have

$$|P_2 T^* y^*|(|g|) = 0 \tag{4}$$

for all $y^* \in D$. To get $g \in G$, the equation (4) must be proved for all $y^* \in Y_1^*$. Therefore let us take an arbitrary y^* and choose a sequence (y_k^*) in D weak*-converging to y^*. Of course, by the definition of modulus, (4) is proved if we show that

$$P_2 T^* y^*(e) = 0 \quad \forall e \in E \quad \text{such that} \quad |e| \leq |g|\,.$$

We have:

$$0 = \langle e, P_2 T^* y_k^* \rangle = \langle P_2^* e, T^* y_k^* \rangle = \langle T^{**} P_2^* e, y_k^* \rangle .$$

Since $P_2^* e \in I_E$, we already observed that $T^{**} P_2^* e$ is really in Y. Because y_k^* weak*-converges to y^*, we get $0 = \langle T^{**} P_2^* e, y^* \rangle = P_2 T^* y^*(e)$. Now an appeal to Lemma 1 will let us to conclude that G is a σ-o.d.i. The proof for the non-trivial implication of the assertion (2) is the same; finally one will invoke Corollary 2. ∎

Corollary 2 gives an analogue of Theorem 6 for the order continuous case.

REMARK 7. – In Theorem 6 the hypothesis of separability of the range space of the operator can be removed. In fact, we have proved Theorem 6 in [2] omitting separability. From this point of view it is clear that formally the proof given here is less powerful. However, we have preferred to present a different approach with respect to [2], because here only handy duality arguments has been used while the proof exposed in [2] is more in the direction of linking known facts.

THEOREM 7 [2]. – *Let E be a VL with the Egorov property, Y a BS and $T\colon E \to Y$ an o-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$ where T_1 is σ-smooth and T_2 is σ-singular.*

THEOREM 8 [2]. – *Let E be a weakly σ-distributive VL, Y a BS and $T\colon E \to Y$ an o-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$ where T_1 is smooth and T_2 is singular.*

To finish let us consider a σ-finite measure space $(\Omega, \mathcal{F}, \lambda)$. If m is an s-bounded vector measure vanishing on $\mathcal{N}(\lambda)$, by means of Proposition 4, the operator $T_m\colon L^\infty(\lambda) \to Y$ is not only continuous but also o-weakly compact. Therefore we have the following

COROLLARY 3 [2]. – *Let* $(\Omega, \mathcal{F}, \lambda)$ *be a* σ*-finite measure space,* Y *a BS and* $m: \mathcal{F} \to Y$ *an s-bounded vector measure with* $m\restriction_{\mathcal{N}(\lambda)} = 0$. *Then* m *can be uniquely decomposed as* $m = m_1 + m_2$, *where* m_1 *and* m_2 *are s-bounded and null on* $\mathcal{N}(\lambda)$, m_1 *is countably additive and* $\mathcal{N}(m_2)$ *is an o.d.i. in* $\mathcal{F}$, *i.e.*

$$\forall A \in \mathcal{F}: \lambda(A) > 0 \ \exists B \in \mathcal{F} \quad \text{such that } \lambda(B) > 0, \ B \subseteq A \text{ and } m(B') = 0 \ \forall B': \ B' \subseteq B.$$

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