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Homogenization of Attractors for Semilinear Parabolic Equations on Manifolds with Complicated Microstructure (*).

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Abstract. – *An approach to a homogenized description of solutions of the Cauchy problem for parabolic equations on Riemannian manifolds with complicated microstructure is presented. This approach covers both linear and non-linear cases and makes it possible to establish a connection between global attractors of the initial problem of the homogenized one.*

1. – Introduction.

We consider on an n-dimensional ($n \geqslant 2$) Riemannian manifold M_ε of complicated microstructure depending on $\varepsilon > 0$ the following initial-boundary problem

$$\frac{\partial u^\varepsilon}{\partial t} - \Delta_\varepsilon u^\varepsilon + f(u^\varepsilon) = h^\varepsilon(x), \qquad x \in M_\varepsilon, \quad t > 0, \tag{1}$$

$$\frac{\partial u^\varepsilon}{\partial \nu_\varepsilon} = 0, \qquad x \in \partial M_\varepsilon, \quad t > 0, \tag{2}$$

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x). \tag{3}$$

Here Δ_ε is the Laplace operator on M_ε, $\partial/\partial\nu_\varepsilon$ is the outer normal derivative on the boundary ∂M_ε of $M_\varepsilon, f(u)$ is a smooth real function on $\boldsymbol{R}^1$ and $h^\varepsilon(x)$, u_0^ε: $M_\varepsilon \to \boldsymbol{R}^1$ are given functions. We suppose that the local structure of the manifold M_ε becomes more and more complicated, when ε tends to zero.

This paper deal with the study of the asymptotic behaviour of the solution $u^\varepsilon(x, t)$

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and of the global attractor $\mathcal{A}_\varepsilon$ of problem (1)-(3) when $\varepsilon \to 0$. One of the main goals here is to learn how the transition to homogenized ($\varepsilon \to 0$) description reflects on the long-time ($t \to +\infty$) dynamics.

Under certain conditions on the manifold M_ε and non-linear term $f(u)$ we first prove that for any finite time interval the limit behaviour of $u^\varepsilon(x, t)$ is described by a solution of the Cauchy problem for a system of two coupled equations. After that we study the long-time dynamics of this homogenized system and show that it possesses a finite-dimensional global attractor $\mathcal{A}$ (for definitions and basic facts on attractors see, e.g. [1, 4, 7, 17]). We investigate the structure of $\mathcal{A}$ and prove that global attractors $\mathcal{A}_\varepsilon$ tend to A in a suitable sense.

In the linear case ($f(u) \equiv 0$) a similar homogenization problem has been studied in [2]. It has been proved that the asymptotic of $u^\varepsilon(x, t)$ is described by a linear diffusion equation with a term non-local in time. This term can be interpretated as memory of the medium (on the memory phenomena for linear homogenized models see also [11-14]). The method developed in [2] essentially relies on the linearity of the problem. The main ingredients there are the Laplace transformation in time and the study of the corresponding stationary problem by variational methods. Unlike [2] the approach presented here can be applied both to linear and non-linear cases. For the linear case the homogenized coupled system can be reduced to a single diffusion equation with memory term of the same form as in [2].

We also note that the dependence of attractors on parameters for various singularly perturbated systems has been studied by many authors (see, e.g. [1, 3, 5, 7, 8, 10, 16] and the references therein). In this paper we rely on some ideas presented in [3, 5, 7, 8].

The paper is organized as follows. In Section 2 we describe the structure of the manifold M_ε introduce some notations and give preliminary results concerning the properties of solutions of the problem (1)-(3), when $\varepsilon > 0$ is fixed. In Section 3 we formulate our main results. The rest of the paper is devoted to the proofs of the Theorems of Section 3. Section 4 contains the proof of the estimates which guarantee the compactness of the family $\{u_\varepsilon \colon \varepsilon \to 0\}$. In Section 5 we make the limit transition in the weak form of problem (1)-(3). The main point here is to choose the testing function. In Section 6 we study properties of the homogenized and prove the upper semicontinuity of global attractor $\mathcal{A}_\varepsilon$ of the problem (1)-(3), when $\varepsilon \to 0$.

2. - Preliminary consideration.

Now we describe the structure of the manifold M_ε. Let Ω be a smooth bounded domain in $\boldsymbol{R}^n$ ($n \geq 2$) and let

$$F_\varepsilon = \bigcup_{j \in N_\varepsilon} F(x^i, a_\varepsilon)$$

be a union of balls $F(x^i, a_\varepsilon)$ of radius $a_\varepsilon \ll \varepsilon$ ($\lim_{\varepsilon \to 0} a_\varepsilon \varepsilon^{-1} = 0$) with centers in $x^j = j\varepsilon$ ($j \in \boldsymbol{Z}^n$) such that $F(x^i, a_\varepsilon) \in \Omega$. Here N_ε stands for the corresponding set of multiindexes $j \in \boldsymbol{Z}^n$. In $\boldsymbol{R}^{n+1}$ we consider the surfaces (below $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^1$, $(x, y) \in \boldsymbol{R}^{n+1}$):

$$\Omega_\varepsilon = \{(x; 0) \in \boldsymbol{R}^{n+1}:\ x \in \Omega \backslash F_\varepsilon\}$$

and

$$B_\varepsilon^j = (j\varepsilon; 0) + B_\varepsilon\,, \qquad j \in N_i \subset \boldsymbol{Z}^n\,,$$

where

$$B_\varepsilon = \{(x, y) \in \boldsymbol{R}^{n+1}:\ |x|^2 + (y - \sqrt{b^2\varepsilon^2 - a_\varepsilon^2})^2 = b^2\varepsilon^2,\ y \leq 0\}\,.$$

Here b is a parameter such that $a_\varepsilon \varepsilon^{-1} < b < 1$. We assume that

$$M_\varepsilon = \Omega_\varepsilon \cup \left(\bigcup_{j \in N_\varepsilon} B_\varepsilon^j \right),$$

i.e. M_ε consists of a piece of flat submanifold in $\boldsymbol{R}^{n+1}$ with bubbles B_ε^j. We define a Riemannian structure on M_ε by a C^∞ metric tensor

$$g^\varepsilon(x) = \{g_{\alpha\beta}^\varepsilon(x);\ \alpha, \beta = 1, 2, \ldots, n\}\,, \qquad x \in M_\varepsilon\,,$$

and assume the following:

(i) the metric coincides with the euclidean metric of $\boldsymbol{R}^{n+1}$ on Ω_ε;

(ii) the metric is the same for all bubbles B_ε^j, $j \in N_\varepsilon$;

(iii) there exist positive constants C_1 and C_2 such that

$$C_1 \varepsilon^n |\xi|^2 \leq \sum_{\alpha\beta} g_{\alpha\beta}^\varepsilon(x)\, \xi_\alpha \xi_\beta \leq C_2 \varepsilon^n |\xi|^2, \qquad \varepsilon > 0\,, \tag{2.1}$$

for all $x \in B_\varepsilon^j$, $j \in N_\varepsilon$ andfor all $\xi \in \boldsymbol{R}^n$.

The main object of this paper is the problem (1)-(3) on the Riemannian manifold $(M_\varepsilon, g^\varepsilon)$, which can be treated as a model of diffusion in a medium with traps. The corresponding Laplace operator Δ_ε is of the form

$$\Delta_\varepsilon = \frac{1}{\sqrt{|g^\varepsilon|}} \sum_{\alpha, \beta} \frac{\partial}{\partial x_\alpha} \left(\sqrt{|g^\varepsilon|}\, g_\varepsilon^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right),$$

where $|g^\varepsilon| = \det g^\varepsilon$ and $g_\varepsilon^{\alpha\beta}$ are the components of the inverse of the tensor g^ε. We also assume that the function $f(u) \in C^2(\boldsymbol{R}^1)$ possesses the property:

$$\sup\{|f'(u)|:\ u \in \boldsymbol{R}^1\} < \infty \tag{2.2}$$

and there exists a constant $\eta > 0$ such that

$$uf(u) \geqslant \eta u^2 - C_1 , \tag{2.3}$$

$$\mathcal{F}(u) \equiv \int_0^u f(\xi)\,d\xi \geqslant \eta u^2 - C_2 . \tag{2.4}$$

Below dx represents the surface measure on M_ε. In local coordinates $\{x_1, \ldots, x_n\}$ we have $dx = \sqrt{|g^\varepsilon|}dx_1 \ldots dx_n$. We also denote $H^l(V_\varepsilon)$ the Sobolev space of order l on a submanifold $V_\varepsilon \subseteq M_\varepsilon$ and $H_0^l(V_\varepsilon)$ for closure of $C_0^\infty(V_\varepsilon)$ in $H^l(V_\varepsilon)$. We denote by $\|\cdot\|_{l,\varepsilon}$ the norm $H^l(M_\varepsilon)$ and by $\|\cdot\|_\varepsilon$ and $(\cdot, \cdot)_\varepsilon$ the norm and inner product in $L^2(M_\varepsilon)$. In certain obvious cases the index ε in norms and inner products will be omitted.

By standard way (see, e.g. [9, 15]) we can prove the following existence and uniqueness theorem.

THEOREM 2.1. – *Let u_0^ε and h^ε belong to $L^2(M_\varepsilon)$. Then for any interval $[0, T]$ problem* (1)-(3) *has a unique solution $u^\varepsilon(t) = u^\varepsilon(x, t)$ such that*

$$u^\varepsilon(t) \in C(0. T; L^2(M_\varepsilon)) \cap L^2(0, T; H^1(M_\varepsilon)) \tag{2.5}$$

$$\|u^\varepsilon(t)\|_\varepsilon^2 \int_0^t (\|\nabla_\varepsilon u^\varepsilon\|_\varepsilon^2 + \eta\|u^\varepsilon\|_\varepsilon^2)\,d\tau \leqslant \|u_0^\varepsilon\|_\varepsilon^2 + C_1(1 + \|h^\varepsilon\|_\varepsilon^2) \tag{2.6}$$

and

$$\|u^\varepsilon(t)\|_\varepsilon^2 \leqslant \|u_0^\varepsilon\|_\varepsilon^2 e^{-\eta t} + C_2(1 + \|h^\varepsilon\|_\varepsilon^2)(1 - e^{-\eta t}), \tag{2.7}$$

where C_1 and C_2 are independent of ε. The solution $u^\varepsilon(t)$ has the following properties:

i) *if $u_0^\varepsilon \in H^1(M_\varepsilon)$, then*

$$u^\varepsilon(t) \in C(0, T; H^1(M_\varepsilon)) \cap L^2(0, T; H^2(M_\varepsilon))$$

and

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2(0, T; L^2(M_\varepsilon));$$

ii) *if*

$$u_0^\varepsilon \in \left\{v \in H^2(M_\varepsilon):\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial M_\varepsilon\right\} \equiv H_N^2(M_\varepsilon)$$

then

$$u^\varepsilon(t) \in C(0, T; H_N^2(M_\varepsilon))$$

and

$$\frac{\partial u^\varepsilon}{\partial t} \in C(0, T; L^2(M_\varepsilon)) \cap L^2(0, T; H^1(M_\varepsilon)).$$

To obtain additional estimates for the solutions $u^\varepsilon(t)$ we introduce on $H^1(M_\varepsilon)$ the Lyapunov function

$$V_\varepsilon(u) = \frac{1}{2}\|\nabla_\varepsilon u\|_\varepsilon^2 + \int_{M_\varepsilon} \mathcal{F}(u(x))\,dx - (h^\varepsilon, u)_\varepsilon . \tag{2.8}$$

It is clear that V_ε is continuous on $H^1(M_\varepsilon)$ and there exist positive constants α_j and β_j independent of ε such that

$$\alpha_1\|u\|_{1,\varepsilon}^2 - \beta_1 \leq V_\varepsilon(u) \leq \alpha_2\|u\|_{1,\varepsilon}^2 + \beta_2 . \tag{2.9}$$

Here we assume that $\|h^\varepsilon\|_\varepsilon \leq C$ for all $0 < \varepsilon \leq \varepsilon_0$.

One can easily prove (see, e.g. [1, 7, 17]) that the solution $u^\varepsilon(t)$ of problem (1)-(3) with $u_0^\varepsilon \in H^1(M_\varepsilon)$ satisfies

$$V_\varepsilon(u^\varepsilon(t)) + \int_0^t \|\partial_t u^\varepsilon(\tau)\|_\varepsilon^2 d\tau = V_\varepsilon(u_0^\varepsilon). \tag{2.10}$$

LEMMA 2.1. – *Let $u_0^\varepsilon \in H_N^2(M_\varepsilon)$. Then*

$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_\varepsilon^2 + 2\int_0^t \left\| \nabla_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|_\varepsilon^2 d\tau \leq C_1 + C_2 V(u_0^\varepsilon) + \|u_1^\varepsilon\|_\varepsilon^2 , \tag{2.11}$$

where $u_1^\varepsilon = \Delta_\varepsilon u_0^\varepsilon - f(u_0^\varepsilon) + h^\varepsilon$ and $C_{1,2}$ are independent of ε.

PROOF. – Theorem 2.1 implies that $w^\varepsilon(t) = \partial u^\varepsilon/\partial t$ is a solution of the following problem:

$$\frac{\partial \omega^\varepsilon}{\partial t} - \Delta_\varepsilon w^\varepsilon + f'(u^\varepsilon(t))w^\varepsilon = 0, \qquad \frac{\partial \omega^\varepsilon}{\partial n} = 0 \quad \text{on} \quad \partial M_\varepsilon,\ w^\varepsilon(x, 0) = u_1^\varepsilon(x). \tag{2.12}$$

Since $|f'(u)| \leq C$ it is clear that

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon(t)\|_\varepsilon^2 + \|\nabla_\varepsilon w^\varepsilon(t)\|_\varepsilon^2 \leq C\|w^\varepsilon(t)\|_\varepsilon^2 . \tag{2.13}$$

Therefore (2.11) follows from (2.10) and (2.13).

REMARK 2.1. – From (2.10) and (2.13) it also follows that

$$t\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_\varepsilon^2 \leq e^{C_1 t}\{V(u_0^\varepsilon) + C_2\}. \tag{2.14}$$

Therefore using (1), (2.7), (2.9), (2.10) we have

$$t\|\Delta_\varepsilon u^\varepsilon(t)\|_\varepsilon^2 \leq C_1 e^{Ct}(1+\|u^\varepsilon\|_{1,\varepsilon}^2), \tag{2.15}$$

if we assume that $\|h_\varepsilon\|_\varepsilon \leq C$ for all $0<\varepsilon\leq\varepsilon_0$.

Theorem 2.1 makes it possible to define an evolution operator S_t^ε on the space $H^1(M_\varepsilon)$ by the formula $S_t^\varepsilon u_0^\varepsilon = u^\varepsilon(t)$, where $u^\varepsilon(t)$ is the solution of the problem (1)-(3). It is not difficult to show that S_t^ε is a C^1-smooth nonlinear semigroup in the space $H^1(M_\varepsilon)$ and to prove (see, e.g. [1, 17]) the following

THEOREM 2.2. – *The dynamical system $(S_t^\varepsilon, H^1(M_\varepsilon))$ for every $\varepsilon>0$ has compact global attractor, i.e. there is a compact set $\mathcal{A}_\varepsilon$ in $H^1(M_\varepsilon)$ such that $S_t^\varepsilon \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon$ for $t\geq 0$ and*

$$\lim_{t\to+\infty} \sup\{\mathrm{dist}_{H^1(M_\varepsilon)}(S_t^\varepsilon v, \mathcal{A}_\varepsilon):\ v\in B\} = 0$$

for any bounded set B in $H^1(M_\varepsilon)$. This attractor $\mathcal{A}_\varepsilon$ has finite Hausdorff dimension.

REMARK 2.2. – Using (2.7), (2.11), (2.15) and the formula

$$u^\varepsilon(t) = e^{-L_{\varepsilon,\gamma}t}u_0^\varepsilon + \int_0^t e^{-L_{\varepsilon,\gamma}(t-\tau)}(\gamma u^\varepsilon(\tau) - f(u^\varepsilon(\tau)) + h^\varepsilon)\,d\tau,$$

where $L_{\varepsilon,\gamma} = -\Delta_\varepsilon + \gamma$ with the Neumann boundary condition on ∂M_ε, $\gamma>0$, it is easy to show that for any trajectory $u^\varepsilon(t)$ lying in the attractor $\mathcal{A}_\varepsilon$ we have the estimates:

$$\left\|\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_\varepsilon^2 + \|\Delta_\varepsilon u^\varepsilon(t)\|_\varepsilon^2 + C\|\nabla_\varepsilon u^\varepsilon(t)\|_\varepsilon^2 + \|u^\varepsilon(t)\|_\varepsilon^2 < C_1 \tag{2.16}$$

and

$$\int_{-\infty}^{\infty}\left(\left\|\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_\varepsilon^2 + \left\|\nabla_\varepsilon\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_\varepsilon^2\right)dt \leq C_2, \tag{2.17}$$

where C_1 and C_2 are independent of ε, $0<\varepsilon\leq\varepsilon_0$.

3. – Formulation of main results.

We introduce a parameter to describe the asymptotic behaviour of manifolds. For simplicity we will suppose $0\in\Omega$, and denote

$$G_\varepsilon = \left\{(x;0)\in \boldsymbol{R}^{n+1}:\ a_\varepsilon \leq |x| < \frac{\varepsilon}{2}\right\}, \qquad D_\varepsilon = B_\varepsilon \cup G_\varepsilon,$$

We set

$$\lambda_\varepsilon = \inf\left\{ \frac{\|\nabla_\varepsilon v\|^2_{L^2(D_\varepsilon)}}{\|v\|^2_{L^2(D_\varepsilon)}} : v \in H_0^1(D_\varepsilon)\right\}. \tag{3.1}$$

λ_ε is the first eigenvalue of the Dirichlet problem

$$\Delta_\varepsilon + \lambda_\varepsilon v = 0, \quad x \in D_\varepsilon; \qquad v = 0, \quad x \in \partial D_\varepsilon. \tag{3.2}$$

Our main assumption concerning to behaviour of the bubbles B_ε^j (and manifold M_ε) is the existence of the limits

$$\lambda = \lim_{\varepsilon \to 0} \lambda_\varepsilon \quad \text{and} \quad \mu = \lim_{\varepsilon \to 0} \varepsilon^{-n} m_\varepsilon > 0, \tag{3.3}$$

where

$$m_\varepsilon \doteq Vol(B_\varepsilon) = \int_{B_\varepsilon} \sqrt{|g^\varepsilon|}\, dx_1 \dots dx_n. \tag{3.3}$$

REMARK 3.1. – It is easy to see that

$$0 < \lambda_\varepsilon \leqslant C \begin{cases} a_\varepsilon^{n-2}\varepsilon^{-n}, & n > 2, \\ |\mathrm{In}\, a_\varepsilon|^{-1}\varepsilon^{-2}, & n = 2. \end{cases}$$

Moreover, if the metric on M_ε coincides with the metric induced from $\boldsymbol{R}^{n+1}$ outside of small neighbourhoods of the boundaries ∂B_ε^j, one can prove that the condition

$$a_\varepsilon = \begin{cases} a\varepsilon^{n/(n-2)}, & n > 2, \\ \exp(-1/\varepsilon^2), & n = 2, \end{cases}$$

implies that limits (3.3) exist and $\lambda = (1/2)a^{n-2}b^{-n}$ and $\mu = \omega_n$, where ω_n is the volume of the unit sphere in $\boldsymbol{R}^{n+1}$ (see [2] for a closely related assertion). From this observation and (2.1) it also follows that for existence of limits (3.3) it is necessary that

$$C_1\varepsilon^{n/(n-2)} \leqslant a_\varepsilon \leqslant C_2\varepsilon^{n/(n-2)} \quad \text{for } n \geqslant 3$$

and

$$C_1 \exp\left(-1/\varepsilon^2\right) \leqslant a_\varepsilon \leqslant C_2 \exp\left(-1/\varepsilon^2\right) \quad \text{for } n = 2.$$

Let P_ε be a bounded operator from $L^2(M_\varepsilon)$ into $L^2(\Omega)$ defined by the formula

$$(P_\varepsilon u)(x) = \begin{cases} u(x), & x \in \Omega_\varepsilon, \\ 0 & x \in \Omega \backslash \Omega_\varepsilon, \end{cases}$$

and let Q_ε be the operator which maps a function $u \in L^2(M_\varepsilon)$ into poly-linear spline

$Q_\varepsilon u$ associated with a net $\{x^j = j\varepsilon,\ j \in N_\varepsilon\}$ such that

$$(Q_\varepsilon u)(x^j) = \frac{1}{m_\varepsilon} \int_{B_\varepsilon^j} u(x)\, dx\,, \qquad j \in N_\varepsilon\,.$$

It is clear that Q_ε is a linear bounded operator from $L^2(M_\varepsilon)$ into $H^1(\Omega_{N_\varepsilon})$, where Ω_{N_ε} is the union of elementary cubes corresponding to the net $\{j\varepsilon\colon j \in N_\varepsilon\}$. If we set $Q_\varepsilon u(x) = 0$ for $x \in \Omega \backslash \Omega_{N_\varepsilon}$, we can also consider Q_ε as a bounded operator from $L^2(M_\varepsilon)$ into $L^2(\Omega)$.

The first main result of the paper is the following

THEOREM 3.1. – *Let $u^\varepsilon(t)$ be the solution of the problem* (1)-(3). *Assume that*

i) *for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|u_0^\varepsilon\|_{1,\varepsilon} + \|\nabla Q_\varepsilon u_0^\varepsilon\|_{L^2(\Omega_{N_\varepsilon})} \leq C$$

and

$$\|h^\varepsilon\|_{1,\varepsilon} + \|\nabla Q_\varepsilon u^\varepsilon\|_{L^2(\Omega_{N_\varepsilon})} \leq C\,,$$

where the constant C is independent of ε;

ii) *there exist functions u_0, v_0, h_1, h_2 from $L^2(\Omega)$ such that $P_\varepsilon u_0^\varepsilon \to u_0$, $Q_\varepsilon u_0^\varepsilon \to v_0$, $P_\varepsilon h^\varepsilon \to h_1$, $Q_\varepsilon h^\varepsilon \to h_2$ strongly in $L^2(\Omega)$;*

iii) *there exist limits* (3.3).

Then for any interval $[0, T]$ we have that

$$\lim_{\varepsilon \to 0} \{\max_{[0,T]} \|P_\varepsilon u^\varepsilon(t) - u(t)\|^2_{L^2(\Omega)} + \max_{[0,T]} \|Q_\varepsilon u^\varepsilon(t) - v(t)\|^2_{L^2(\Omega)}\} = 0\,, \tag{3.4}$$

where the pair of functions $u(t) = u(x, t)$ and $v(t) = v(x, t)$ is the solution of the problem:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda\mu(u - v) + f(u) = h_1(x)\,, \qquad x \in \Omega,\ \ t > 0\,, \tag{3.5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\,, \qquad u|_{t=0} = u_0(x)\,, \tag{3.6}$$

$$\frac{\partial u}{\partial t} + \lambda(v - u) + f(v) = h_2(x)\,, \qquad x \in \Omega,\ \ t > 0\,, \tag{3.7}$$

$$v|_{t=0} = v_0(x)\,, \tag{3.8}$$

The proof of this theorem consists of two parts. The main point of the first one is to obtain a uniform estimate

$$\int_0^T \|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} dt < C\,. \tag{3.9}$$

In the second part we make a limit transition in the equation (1) on testing functions of special structure. In order to prove the uniqueness of limits we also use the following

THEOREM 3.2. – *Assume that* (2.2)-(2.4) *are satisfied and* $U_0 = (u_0, v_0) \in \mathcal{F}_0 = L^2(\Omega) \times L^2(\Omega)$. *Then the problem* (3.5)-(3.8) *has a unique generalized solution* $U(t) = (u(t), v(t))$ *belonging to the space* $C(\boldsymbol{R}_+, \mathcal{F}_0)$. *Moreover, if* $U_0 \in \mathcal{F}_1 = H^1(\Omega) \times L^2(\Omega)$ *then*

$$U(t) \in C(\boldsymbol{R}_+, \mathcal{F}_1) \quad \text{and} \quad \frac{d}{dt} U(t) \in L^2(\boldsymbol{R}_+, \mathcal{F}_0) \tag{3.10}$$

if $U_0 \in \mathcal{F}_2 = H^1(\Omega) \times L^2(\Omega)$ *and* $h_2 \in H^1(\Omega)$ *then*

$$U(t) \in C(\boldsymbol{R}_+, \mathcal{F}_2) \quad \text{and} \quad \frac{d}{dt} U(T) \in L^2(\boldsymbol{R}_+, L^2(\Omega) \times H^1(\Omega)) . \tag{3.11}$$

The proof of this theorem is of standard character and relies on the methods presented in [9].

Theorem 3.2 allows us to define the evolutionary semigroup S_t in each of the spaces $\mathcal{F}_i$ by the formula $S_t U_0 = U(t)$, where $U(t)$ is the solution of the problem (3.5)-(3.8). If we consider this semigroup in $\mathcal{F}_2$, then we can prove the following assertion on the existence of a global attractor.

THEOREM 3.3. – *Assume that* (2.2)-(2.4) *are satisfied and*

$$\lambda + \inf\{f'(u) \colon u \in \boldsymbol{R}^1\} > 0 , \qquad h_2(x) \in H^1(\Omega) . \tag{3.12}$$

Then the dynamical system $(S_t, \mathcal{F}_2)$ *has a weak global attractor* $\mathcal{F}$. *This attractor has finite Hausdorff dimension as a compact set in* $\mathcal{F}_0$.

In order to prove this theorem we rely on certain results from [6, 17]. Recall (see [1, 4, 17]) that a weak global attractor $\mathcal{A}$ is a bounded weakly closed set in $\mathcal{F}_2$ such that (i) $S_t \mathcal{A} = \mathcal{A}$ for any $t > 0$ and (ii) for any weak neighbourhood $\mathcal{O}$ of A and for any bounded set $B \subset \mathcal{F}_2$ we have $S_t B \subset \mathcal{O}$, when $t \geq t_0(B, \mathcal{O})$.

At last using Theorem 3.1 and estimates (2.16) and (2.17) we prove the second main result of the paper.

THEOREM 3.4. – *Assume that* (2.2)-(2.4), (3.12) *and the assumptions of Theorem* 3.1 *are satisfied. Then we have*

$$\lim_{\varepsilon \to 0} \sup_{u_\varepsilon \in \mathcal{A}_\varepsilon} \left\{ \inf_{(u, v) \in \mathcal{A}} \|P_\varepsilon u^\varepsilon - u\|^2_{L^2(\Omega)} + \|Q_\varepsilon u^\varepsilon - v\|^2_{L^2(\Omega)}) \right\} = 0 .$$

This theorem means that the global attractor $\mathcal{A}_\varepsilon$ of problem (1)-(3) tends to a weak global attractor $\mathcal{A}$ of the homogenized system (3.5)-(3.8).

4. – Uniform estimates.

Now we begin the proof of Theorem 3.1. In this section we establish our main Lemma 4.1 on uniform boundness of the norms $\|Q_\varepsilon u^\varepsilon\|_{H^1(\Omega_{N_\varepsilon}\times(0,T))}$. This lemma and estimates for $P_\varepsilon u^\varepsilon$ which directly follow from (2.6) and (2.10) make it possible to extract from $\{P_\varepsilon u^\varepsilon\}$ and $\{Q_\varepsilon u^\varepsilon\}$ subsequences strongly convergent in $L^2(\Omega\times(0,T))$. Below we consider the case $n\geq 3$ only. For the case $n=2$ the consideration should be repeated word by word with slight modifications in the estimates. We assume that the conditions (i)-(iii) of Theorem 3.1 are satisfied.

At first we note that (2.7) and (2.10) imply that the solution $u^\varepsilon(x,t)$ satisfies the estimate

$$\|u^\varepsilon(t)\|_\varepsilon^2+\|\nabla_\varepsilon u^\varepsilon(t)\|_\varepsilon^2+\int_0^t\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_\varepsilon^2 d\tau\leq C_T \tag{4.0}$$

for any $t\in[0,T]$. Since the metric g^ε coincides with the euclidean one on Ω_ε, we have

$$\|P_\varepsilon u^\varepsilon(t)\|_{L^2(\Omega)}^2+\|\nabla_\varepsilon P_\varepsilon u^\varepsilon(t)\|_{L^2(\Omega)}^2+\int_0^t\left\|P_\varepsilon\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^2(\Omega)}^2 d\tau\leq C_T\,. \tag{4.1}$$

The remaining part of this section is devoted to the proof of a similar estimate for $Q_\varepsilon u^\varepsilon(t)$.

Let us introduce the following notation:

$$u_k^\varepsilon(x,t)=u^\varepsilon(x^k+x,t),\qquad x^k=l\varepsilon,\qquad k\in N_\varepsilon,\ \ x\in D_\varepsilon;$$

$$u_k^{\varepsilon,in}(t)=\frac{1}{m_\varepsilon}\int_{B_\varepsilon}u_k^\varepsilon(x,t)\,dx;$$

$$u_k^{\varepsilon,ex}(t)=\frac{1}{m'_\varepsilon}\int_{G_\varepsilon}u_k^\varepsilon(x,t)\,dx;$$

where $u^\varepsilon(x,t)$ is the solution of problem (1)-(3), the sets B_ε, G_ε and D_ε are defined in Sections 2 and 3, $m_\varepsilon=Vol(B_\varepsilon)$ and $m'_\varepsilon=Vol(G_\varepsilon)$. We also use the notation

$$w^\varepsilon\equiv w_{kl}^\varepsilon(x,t)=u_k^\varepsilon(x,t)-u_l^\varepsilon(x,t),\qquad x\in D_\varepsilon,\ \ k,l\in N_\varepsilon$$

and

$$w^\#=w_{kl}^\#(t)=u_k^{\varepsilon,\#}(t)-u_l^{\varepsilon,\#}(t),\qquad k,l\in N_\varepsilon$$

where $\#$ is either «*in*» or «*ex*».

It is clear from (2.7) and (2.10) that for any $t \geq 0$

$$\|Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \left\| Q_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega)} d\tau \leq C_T . \tag{4.2}$$

The main result of this section is

LEMMA 4.1. – *For any $T > 0$ we have*

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} \leq C_T , \qquad t \in [0, T],$$

where C_T is a constant independent of ε.

In order to prove this Lemma it is sufficient to obtain appropriate estimates for $w_{kl}^{in}(t)$. We will use the following preliminary assertions.

LEMMA 4.2. – *Let $a_\varepsilon \leq a\varepsilon^{n/(n-2)}$ $(n > 2)$ and let $v^\varepsilon(x) \in H_0^1(D_\varepsilon)$ be the solution of the problem* (3.2) *such that*

$$\int_{B_\varepsilon} v^\varepsilon(x)\, dx = m_\varepsilon . \tag{4.3}$$

Then we have the following estimates:

(i) $|D^\alpha v^\varepsilon(x)| \leq \dfrac{C\varepsilon^n}{|x|^{n-2+|\alpha|}}$ for $x \in \tilde{G}_\varepsilon$ and $|x| \geq \varepsilon/4$;

(ii) $\displaystyle\int_{G_\varepsilon} |v^\varepsilon(x)|^2 dx \leq C\varepsilon^{n+2}$;

(iii) $\displaystyle\int_{D_\varepsilon} |v^\varepsilon(x)|^2 dx = \int_{B_\varepsilon} |v^\varepsilon(x)|^2 dx + O(\varepsilon^{n+2}) = m_\varepsilon + O(\varepsilon^{n+2})$;

(iv) $\displaystyle\int_{D_\varepsilon} |\nabla_\varepsilon v^\varepsilon(x)|^2 dx = \lambda_\varepsilon m_\varepsilon + O(\varepsilon^{n+2})$;

(v) $\displaystyle\int_{\Gamma(a_\varepsilon)} \frac{\partial v^\varepsilon}{\partial n} d\sigma = \lambda_\varepsilon m_\varepsilon$ and $\displaystyle\int_{\Gamma(\varepsilon/2)} \frac{\partial v^\varepsilon}{\partial n} d\sigma = \lambda_\varepsilon m_\varepsilon + O(\varepsilon^{n+1})$;

where $\Gamma(a_\varepsilon)$ and $\Gamma(\varepsilon/2)$ are the inner and outer boundaries of the ring G_ε, and the normal vector n is directed towards the center of the ring G_ε.

PROOF. – It is easy to see that for $v^\varepsilon(x)$ we have the following inequalities of

Poincaré and Friedrichs type:

$$\int_{B_\varepsilon} (v^\varepsilon - 1)^2 dx \le C\varepsilon^2 \int_{B_\varepsilon} |\nabla_\varepsilon v^\varepsilon|^2 dx; \tag{4.4}$$

$$\int_{G_\varepsilon} |v^\varepsilon|^2 dx \le C\varepsilon^2 \int_{G_\varepsilon} |\nabla v_\varepsilon v^\varepsilon|^2 dx. \tag{4.5}$$

Since

$$\int_{D_\varepsilon} |\nabla_\varepsilon v^\varepsilon|^2 dx = \lambda_\varepsilon \left\{ m_\varepsilon +; \int_{B_\varepsilon} (v^\varepsilon - 1)^2 dx + \int_{G_\varepsilon} |v^\varepsilon|^2 dx \right\},$$

from (3.3), (4.4) and (4.5) we have

$$\int_{D_\varepsilon} |\nabla_\varepsilon v^\varepsilon|^2 dx \le C\varepsilon^n \tag{4.6}$$

and the property (iv) follows. Now (ii) and (iii) follow from (4.5) and (4.6). Using Green's formula we get

$$\int_{\Gamma(\varepsilon/2)} \frac{\partial v^\varepsilon}{\partial n} d\sigma = \int_{D_\varepsilon} |\nabla_\varepsilon v^\varepsilon|^2 dx - \int_{D_\varepsilon} \Delta v^\varepsilon (1 - v^\varepsilon) dx.$$

Therefore using (3.2)and (4.3) we obtain

$$\int_{\Gamma(\varepsilon/2)} \frac{\partial v^\varepsilon}{\partial n} d\sigma = \lambda_\varepsilon m_\varepsilon + \lambda_\varepsilon \int_{G_\varepsilon} v^\varepsilon dx.$$

Hence (v) follows from (iii) and from the obvious formula:

$$\int_{\Gamma(a_\varepsilon)} \frac{\partial v^\varepsilon}{\partial n} d\sigma = \int_{\Gamma(\varepsilon/2)} \frac{\partial v^\varepsilon}{\partial n} d\sigma - \lambda_\varepsilon \int_{G_\varepsilon} v^\varepsilon dx.$$

We now prove (i). Let $\Gamma(x, y)$ be the generalized solution of the problem:

$$\Delta\Gamma(x, y) + \lambda_\varepsilon \Gamma(x, y) = -\delta(x - y) \text{ for } x, y \in K_\varepsilon, \qquad \Gamma_\varepsilon(x, y)|_{x \in \partial K_\varepsilon} = 0,$$

where $K_\varepsilon = \{x \in \boldsymbol{R}^n : |x| < \varepsilon/2\}$. It is well known that

$$|D_x^\alpha D_y^\beta \Gamma(x, y)| \le C|x - y|^{-n+2-|\alpha|-|\beta|}. \tag{4.7}$$

For $y \in G_\varepsilon$, $|y| \ge 2a_\varepsilon$, and $x \in D_\varepsilon$ we define the function

$$R_\varepsilon(x, y) = \begin{cases} \Gamma(0, y), & x \in B_\varepsilon, \\ \Gamma(x, y) + (\Gamma(0, y) - \Gamma(x, y))\, \varphi\left(\dfrac{x}{a_\varepsilon}\right), & x \in G_\varepsilon, \end{cases}$$

where $\varphi(x) \in C_0^\infty(\boldsymbol{R}^n)$ satisfies:

$$\varphi(x) = 1 \quad \text{for } |x| \leqslant 1 \qquad \varphi(x) = 0 \quad \text{for } |x| \geqslant \frac{3}{2}.$$

It is clear that $R_\varepsilon(x, y) = 0$, when $x \in \partial D_\varepsilon$ and

$$(\Delta_{\varepsilon, x} + \lambda_\varepsilon) R_\varepsilon(x, y) = \begin{cases} \lambda_\varepsilon \Gamma(0, y), & x \in B_\varepsilon, \\ -\delta(x - y) + \theta_\varepsilon(x, y), & x \in G_\varepsilon, \end{cases}$$

where

$$\theta_\varepsilon(x, y) = a_\varepsilon^{-2} \Delta\varphi(\Gamma(0, y) - \Gamma(x, y)) - 2a_\varepsilon^{-1} \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i} \Gamma(x, y) + \lambda_\varepsilon \Gamma(0, y)\varphi.$$

Using the Green formula we have

$$\int_{D_\varepsilon} v^\varepsilon(x)(\Delta_{\varepsilon, x} + \lambda_\varepsilon) R_\varepsilon(x, y)\, dx = 0.$$

Consequently,

$$v^\varepsilon(y) = \lambda_\varepsilon m_\varepsilon \Gamma(0, y) + \int_{G+_\varepsilon} \theta_\varepsilon(x, y) v^\varepsilon(x)\, dx \tag{4.8}$$

for $y \in G_\varepsilon$ and $|y| \geqslant 2a_\varepsilon$, where $G_\varepsilon^* = \{x \in G_\varepsilon;\ |x| \leqslant (3/2)a_\varepsilon\}$. The property (i) then follows from (ii), (4.7) and (4.8).

LEMMA 4.3. – *Let $v^\varepsilon(x)$ be as in Lemma 4.2. Then we have*

$$\frac{d}{dt} \int_{D_\varepsilon} w^\varepsilon(t) v^\varepsilon dx + (\lambda_\varepsilon + I_{kl}^\varepsilon(t)) \int_{D_\varepsilon} w^\varepsilon(t) v^\varepsilon dx = R_{kl}^\varepsilon(t), \tag{4.9}$$

where

$$I_{kl}^\varepsilon(t) = \frac{1}{m_\varepsilon} \int_0^1 d\tau \int_{B_\varepsilon} f'(u_k^\varepsilon(t) + \tau(u_l^\varepsilon(t) - u_k^\varepsilon(t)))\, v^\varepsilon dx$$

and the quantity $R_{kl}^\varepsilon(t)$ admits the estimate

$$|R_{kl}^\varepsilon(t)| \leqslant C_1 m_\varepsilon(|w^{ex}(t)| + |h^{in}|) +$$

$$+ C_2 \varepsilon^{n/2+1} \{\|w^\varepsilon(t)\|_{L^2(G_\varepsilon)} + \|\nabla_\varepsilon w^\varepsilon(t)\|_{L^2(D_\varepsilon)} + \|h_k\|_{L^2(G_\varepsilon)} + \|\nabla_\varepsilon h_{kl}\|_{L^2(G_\varepsilon)}\}.$$

Here $h_{kl}(x) = h(x_k + x) - h(x_l + x)$ for $x \in D_\varepsilon$, $k, l \in N_\varepsilon$ and h^{in} is defined in the same manner as $u^{\varepsilon, in}$.

PROOF. – We use the equation

$$\frac{d}{dt}\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx - \int_{D_\varepsilon} \Delta w^\varepsilon v^\varepsilon\,dx + \int_{D_\varepsilon} (f(u_k^\varepsilon) - f(u_l^\varepsilon))\,v^\varepsilon\,dx = \int_{D_\varepsilon} h_{kl} v^\varepsilon\,dx\,, \tag{4.10}$$

which follows from (1) and we use the following Lemmas.

LEMMA 4.4

$$\left|\int_{D_\varepsilon} \Delta w^\varepsilon v^\varepsilon\,dx + \lambda_\varepsilon \int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx\right| \leqslant C(\varepsilon^n\,|w^{ex}| + \varepsilon^{n/2+1}\|\nabla w^\varepsilon(t)\|_{L^2(G_\varepsilon)})\,. \tag{4.11}$$

PROOF. – Using Green formula we get

$$\int_{D_\varepsilon} \Delta w^\varepsilon v^\varepsilon\,dx + \lambda_\varepsilon \int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx = -w^{ex}\int_{\Gamma(\varepsilon/2)} \frac{\partial v^\varepsilon}{\partial n}\,d\sigma + \int_{\Gamma(\varepsilon/2)} (w^{ex} - w^\varepsilon)\frac{\partial v^\varepsilon}{\partial n}\,d\sigma\,.$$

Lemma 4.2 (i) implies that

$$\left|\int_{\Gamma(\varepsilon/2)} (w^{ex} - w^\varepsilon)\frac{\partial v^\varepsilon}{\partial n}\,d\sigma\right| \leqslant C\varepsilon \int_{\Gamma(\varepsilon/2)} |w^{ex} - w^\varepsilon|\,d\sigma\,.$$

Therefore, using the trace theorem and the Poincaré inequality in G_ε we obtain (4.11).

LEMMA 4.5.

$$\left|\int_{D_\varepsilon} (f(u_k^\varepsilon) - f(u_l^\varepsilon))\,v^\varepsilon\,dx - I_{kl}^\varepsilon(t)\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx\right| \leqslant \tag{4.12}$$

$$\leqslant C\varepsilon^{n/2+1}\{\|\nabla_\varepsilon w^\varepsilon\|_{L^2(B_\varepsilon)} + \|w^\varepsilon\|_{L^2(G_\varepsilon)}\}\,.$$

PROOF. – It is clear from (2.2) and Lemma 4.2 (ii) that

$$\left|\int_{D_\varepsilon} (f(u_k^\varepsilon) - f(u_l^\varepsilon))\,v^\varepsilon\,dx\right| \leqslant C\|w\|_{L^2(G_\varepsilon)}\|v\|_{L^2(G_\varepsilon)} \leqslant C\varepsilon^{n/2+1}\|w\|_{L^2(G_\varepsilon)}\,.$$

Using Lemma 4.2 and the Hölder and Poincaré inequalities we also get

$$\left|\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx - w^{in} m_\varepsilon\right| \leqslant \left|\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\,dx\right| + \left|\int_{B_\varepsilon} (w^\varepsilon - w^{in})\,v^\varepsilon\,dx\right| \leqslant \tag{4.13}$$

$$\leqslant C\varepsilon^{n/2+1}(\|\nabla_\varepsilon w^\varepsilon\|_{L^2(B_\varepsilon)} + \|w^\varepsilon\|_{L^2(G_\varepsilon)})\,.$$

Obviously

$$\left| \int_{B_\varepsilon} (f(u_k^\varepsilon) - f(u_l^\varepsilon))\, v^\varepsilon\, dx - w^{in} m_\varepsilon I_{kl}^\varepsilon(t) \right| \leqslant C \int_{B_\varepsilon} |w^\varepsilon - w^{in}| |v^\varepsilon|\, dx \leqslant$$

$$\leqslant C\varepsilon^{n/2+1} \|\nabla_\varepsilon w^\varepsilon\|_{L^2(B_\varepsilon)} .$$

Now in order to obtain (4.12) it is sufficient to note that

$$\inf_{u \in \boldsymbol{R}} f'(u) \leqslant I_{kl}^\varepsilon(t) \leqslant \sup_{u \in \boldsymbol{R}} f'(u). \tag{4.14}$$

This follows from the fact that v^ε, first eigenfunction of the Dirichlet problem is positive in D_ε.

As in (4.13) we have

$$\left| \int_{D_\varepsilon} h_{kl} v^\varepsilon\, dx \right| \leqslant |h^{in}| m_\varepsilon + C\varepsilon^{n/2+1} (\|\nabla_\varepsilon h_{kl}\|_{L^2(B_\varepsilon)} + \|h_{kl}\|_{L^2(G_\varepsilon)}) .$$

Therefore Lemma 4.4 and 4.5 give equality (4.9), and this proves Lemma 4.3.

Using (4.9) and (4.14) we get

$$\frac{1}{2} \frac{d}{dt} \left(\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\, dx \right)^2 + (\lambda_\varepsilon + L - \delta) \left(\int_{D_\varepsilon} w^\varepsilon v^\varepsilon\, dx \right)^2 \leqslant \frac{1}{4\delta} [R_{kl}^\varepsilon]^2 \tag{4.15}$$

for any positive δ, where $L = \inf_{u \in \boldsymbol{R}} f'(u)$. Then it follows from Gronwall's lemma that we have

$$\left(\int_{D_\varepsilon} w^\varepsilon(t)\, v^\varepsilon\, dx \right)^2 \leqslant e^{-\alpha_\varepsilon t} \left(\int_{D_\varepsilon} w^\varepsilon(0)\, v^\varepsilon\, dx \right)^2 + \frac{1}{2\delta} \int_0^t e^{-\alpha_\varepsilon (t-\tau)} [R_{kl}^\varepsilon(\tau)]^2\, d\tau \tag{4.16}$$

for any $\delta > 0$, with $\alpha_\varepsilon = 2(\lambda_\varepsilon + L - \delta)$.

It is clear that

$$[R_{kl}^\varepsilon(t)]^2 \leqslant C_1 m_\varepsilon^2 (|w^{ex}|^2 + |h^{in}|^2) + C_2 \varepsilon^{n+2} (Y_k^\varepsilon(t) + Y_l^\varepsilon(t)) ,$$

where

$$Y_l^\varepsilon(t) = \|u^\varepsilon(t)\|_{H^1(D_\varepsilon^j)}^2 + \|h\|_{H^1(D_\varepsilon^j)}^2 .$$

Here $D_\varepsilon^j = (j\varepsilon;\, 0) + D_\varepsilon$ for $j \in N_\varepsilon$. Therefore

$$\sum_{k \in N_\varepsilon} \sum_{l \in \sigma(k)} \frac{1}{m_\varepsilon} [R_{kl}^\varepsilon(t)]^2 \leqslant C_1 \varepsilon^2 (\|u^\varepsilon(t)\|_{H^1(M_\varepsilon)}^2 + \|h\|_{H^1(M_\varepsilon)}^2) +$$

$$+ C_2 \sum_{k \in N_\varepsilon} \sum_{l \in \sigma(k)} \varepsilon^n (|u_k^{ex} - u_l^{ex}|^2 + |h_k^{in} - h_l^{in}|^2) ,$$

where $\sigma(k) = \bar{\sigma}(k) \cap N_\varepsilon$ and $\bar{\sigma}(k)$ is the set of the nearest neighbours of k in $\boldsymbol{Z}^n$. Since any function $\varphi \in H^1(\Omega_\varepsilon)$ can be extended to $\tilde{\varphi} \in H^1(\Omega)$ such that

$$\|\tilde{\varphi}\|_{H_1(\Omega)} \leq C\|\varphi\|_{H_1(\Omega_\varepsilon)}$$

with constant C independent of ε, one can easily verify that

$$\sum_{k \in N_\varepsilon} \sum_{l \in \sigma(k)} \varepsilon^n |u_k^{ex} - u_l^{ex}|^2 \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u(t)|^2 dx .$$

Therefore using inequality

$$C_1 \varepsilon^2 \|\nabla Q_\varepsilon h\|^2_{L^2(\Omega_{N_\varepsilon})} \leq \sum_{k \in N_\varepsilon} \sum_{l \in \sigma(k)} \varepsilon^n |h_k^{in} - h_l^{in}|^2 \leq C_2 \varepsilon^2 \|\nabla Q_\varepsilon h\|^2_{L^2(\Omega_{N_\varepsilon})}, \tag{4.17}$$

we obtain

$$\sum_{k \in N_\varepsilon} \sum_{l \in \sigma(k)} \frac{1}{m_\varepsilon} [R_{kl}^\varepsilon(t)]^2 \leq C\varepsilon^2 (\|\nabla Q_\varepsilon h\|^2_{L^2(\Omega_{N_\varepsilon})} + \|u^\varepsilon(t)\|^2_{1,\varepsilon} + \|h\|^2_{1,\varepsilon}). \tag{4.18}$$

Now using (4.13), (4.16) and (4.17) with $u^\varepsilon(t)$ instead of h we conclude

$$\|\nabla Q_\varepsilon u(t)\|^2_{L^2(\Omega_{N_\varepsilon})} \leq C_1 \|u^\varepsilon(t)\|^2_{1,\varepsilon} + C_2 (\|\nabla Q_\varepsilon u_0\|^2_{L^2(\Omega_{N_\varepsilon})} + \|u_0\|^2_{1,\varepsilon}) e^{-\alpha_\varepsilon t} + \tag{4.19}$$

$$+ C_3 \delta^{-1} \int_0^t e^{-\alpha_\varepsilon (t-\tau)} \{\|u^\varepsilon(\tau)\|^2_{1,\varepsilon} + \|h\|^2_{1,\varepsilon} + \|\nabla Q_\varepsilon h\|^2_{L^2(\Omega_{N_\varepsilon})}\} d\tau.$$

Therefore the assertion of Lemma 4.1 follows from (4.0) and condition (i) of Theorem 3.1.

REMARK 4.1. – Assume that

$$\lambda + \inf_{u \in \boldsymbol{R}} f'(u) > 0 .$$

Then we have $\alpha_\varepsilon > 0$ for ε and δ small enough. Therefore for any trajectory $u^\varepsilon(t)$ lying in the attractor $\mathcal{A}_\varepsilon$ from (4.19) and Remark 2.2 it is easy to see that

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} \leq C_1 + C_2 (1 + \|\nabla Q_\varepsilon u^\varepsilon(s)\|^2_{L^2(\Omega_{N_\varepsilon})}) e^{-\alpha_\varepsilon (t-s)} \tag{4.20}$$

for all $t \geq s$. Since

$$\|\nabla Q_\varepsilon u^\varepsilon(s)\|^2_{L^2(\Omega_{N_\varepsilon})} < C_\varepsilon , \qquad -\infty < s < \infty ,$$

then letting $s \to -\infty$ in (4.20) we conclude that

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} \leq C_1 \tag{4.21}$$

for any $u^\varepsilon(t) \in \mathcal{A}_\varepsilon$, where ε is small enough.

5. - Limit transition.

Let R_1^ε be a linear continuation operator from Ω_ε to Ω possesses the properties:

i) R_1^ε: $H^l(\Omega_\varepsilon) \to H^l(\Omega)$ for $l = 0, 1$ such that

$$\|R_1^\varepsilon \phi\|_{H^1(\Omega)} \leq C\|\phi\|_{H^1(\Omega_\varepsilon)}, \qquad l = 0, 1,$$

where C is a constant, independent of ε;

ii) $R_1^\varepsilon \phi = \phi$ on Ω_ε for all $\phi \in L^2(\Omega_\varepsilon)$.

We also denote by R_2^ε a continuation operator from Ω_{N_ε} to Ω with similar properties. The existence of such operators is easily proved, in view of the structure of the domains Ω_ε and Ω_{N_ε}.

Let $\bar{u}^\varepsilon(t) = R_1^\varepsilon P_\varepsilon u^\varepsilon(t)$ and $\bar{v}^\varepsilon(t) = R_2^\varepsilon Q_\varepsilon u^\varepsilon(t)$, where $u^\varepsilon(t)$ is the solution of problem (1)-(3). Then it follows from (4.1)-(4.3) that the family $\{(\bar{u}^\varepsilon(t); \bar{v}^\varepsilon(t))\}$ is a precompact set in the space $C(0, T; L^2(\Omega) \times L^2(\Omega))$, when ε goes to zero.

In this section we prove that any limiting point $(u(t); v(t) \in C(0, T; L^2(\Omega) \times \times L^2(\Omega))$ of the family $\{(\bar{u}_\varepsilon(t); \bar{v}_\varepsilon(t)): \varepsilon \to 0\}$ is a weak solution of problem (3.5)-(3.8). Below we assume that the conditions (i)-(iii) of Theorem 3.1 are satisfied.

We first rewrite problem (1)-(3) in a weak form. Let $u^\varepsilon(t)$ be the solution of (1)-(3). We denote

$$J_\varepsilon(u^\varepsilon; \psi; V_\varepsilon) = -(u_0^\varepsilon, \psi(0))_{L^2(V_\varepsilon)} - \int_0^T (u^\varepsilon(t), \partial_t \psi(t))_{L^2(V_\varepsilon)} dt - \tag{5.1}$$

$$- \int_0^T (u^\varepsilon(t), \Delta_\varepsilon \psi(t))_{L^2(V_\varepsilon)} dt + \int_0^T (f(u^\varepsilon(t)), \psi(t))_{L^2(V_\varepsilon)} dt - \int_0^T (h, \psi(t))_{L^2(V_\varepsilon)} dt,$$

where $\psi(x, t)$ belongs to the class

$$\mathcal{L}_T = \{\psi(x, t) \in L^2(0, T; H^2(M_\varepsilon)): \partial_t \psi(x, t) \in L^2(0, T; L^2(M_\varepsilon)), \psi(T) = 0\}$$

and V_ε is a submanifold of M_ε. It is clear that

$$J_\varepsilon(u^\varepsilon; \psi; M_\varepsilon) = 0 \quad \text{for all } \psi \in \mathcal{L}_T. \tag{5.2}$$

Now we suppose $\psi(t, x) = \beta(t)\xi_\varepsilon(x)$, where $\beta(t) \in C^1(0, T)$, $\beta(T) = 0$, and the function $\xi_\varepsilon(x)$ is constructed as follows. If $x \in \Omega_\varepsilon$, we set

$$\xi_\varepsilon(x) = \zeta(x) + \sum_{i \in N_\varepsilon} (\zeta(i\varepsilon) - \zeta(x))\,\varphi\left(\frac{x - i\varepsilon}{4a_\varepsilon}\right) + \sum_{i \in N_\varepsilon} (\eta(i\varepsilon) - \zeta(i\varepsilon))\, v^\varepsilon(x - i\varepsilon)\,\varphi\left(\frac{x - i\varepsilon}{\varepsilon}\right)$$

and for $x \in B_\varepsilon^i$ we suppose

$$\xi_\varepsilon(x) = \eta(i\varepsilon) + (\zeta(i\varepsilon) - \eta(i\varepsilon))(1 - v^\varepsilon(x - i\varepsilon)),$$

where $\zeta(x)$ and $\eta(x)$ are smooth functions on Ω and $\varphi(x) = \bar{\varphi}(|x|) \in C_0^\infty(\boldsymbol{R}^n)$ possesses the properties $0 \leq \varphi \leq 1$; $\bar{\varphi}(r) = 1$ for $r \leq 1/4$; $\bar{\varphi}(r) = 0$, if $r \geq 1/3$. It is clear that $\xi_\varepsilon(x)$ is a smooth function on M_ε.

LEMMA 5.1. – *The function $\xi_\varepsilon(x)$ has the following properties:*

$$P_\varepsilon \xi_\varepsilon \to \zeta(x) \text{ strongly in } L^2(\Omega), \tag{5.3}$$

$$P_\varepsilon \Delta \xi_\varepsilon \to \Delta\zeta + \lambda\mu(\eta(x) - \zeta(x)) \quad \text{weakly in } L^2(\Omega), \tag{5.4}$$

when $\varepsilon \to 0$.

PROOF. – Using Lemma 4.2 (ii) we have

$$\|P_\varepsilon \xi_\varepsilon - \zeta\|_{L_2(\Omega_\varepsilon)} \leq C(a_\varepsilon^{n+1} + \varepsilon^{n+1})|N_\varepsilon| \to 0$$

when $\varepsilon \to 0$. Here $|N_\varepsilon|$ is a number of elements of N_ε. Since $Vol(\Omega \backslash \Omega_\varepsilon) \to 0$, we obtain (5.3).

In order to prove (5.4) we first note that Remark 3.1 implies that

$$P_\varepsilon \Delta \left(\sum_{i \in N_\varepsilon} (\zeta(i\varepsilon) - \zeta(x))\, \varphi\left(\frac{x - i\varepsilon}{4a_\varepsilon} \right) \right) \to 0, \qquad \varepsilon \to 0 \tag{5.5}$$

strongly in $L^2(\Omega)$. Therefore it is sufficient to consider the term

$$\chi_\varepsilon(x) = \sum_{i \in N_\varepsilon} (\eta(i\varepsilon) - \zeta(i\varepsilon))\, v^\varepsilon(x - i\varepsilon)\, \varphi\left(\frac{x - i\varepsilon}{\varepsilon} \right).$$

It follows from Lemma 4.2 (i), (ii) that

$$\int_{F_\varepsilon} \left| \Delta \left(v^\varepsilon(x)\, \varphi\left(\frac{x}{\varepsilon} \right) \right) \right|^2 dx \leq C\varepsilon^n,$$

so the family $\{P_\varepsilon(\Delta\chi_\varepsilon)\}$ is bounded in $L^2(\Omega)$. Consequently on this family, weak convergence is the same as weak convergence on smooth functions $\theta(x) \in C_0^\infty(\Omega)$.

It is clear from Lemma 4.2 (i), (ii) that we have

$$\int_{G_\varepsilon} \Delta \left[v^\varepsilon(x)\, \varphi\left(\frac{x}{\varepsilon} \right) \right] \theta(x)\, dx = \theta(0) \int_{G_\varepsilon} \Delta \left[v^\varepsilon(x)\, \varphi\left(\frac{x}{\varepsilon} \right) \right] dx + O(\varepsilon^{n+1}). \tag{5.6}$$

Using Green's formula and Lemma 4.2 (v) we get

$$\int_{G_\varepsilon} \Delta \left(v^\varepsilon(x)\, \varphi\left(\frac{x}{\varepsilon} \right) \right) dx = \int_{\Gamma(a_\varepsilon)} \frac{\partial v^\varepsilon}{\partial n}\, d\sigma = \lambda_\varepsilon m_\varepsilon.$$

Therefore (5.6) gives

$$\int_{\Omega_\varepsilon} P_\varepsilon \chi_\varepsilon \theta(x)\, dx = \sum_{i \in N_\varepsilon} \lambda_\varepsilon m_\varepsilon (\eta(i\varepsilon) - \zeta(i\varepsilon))\, \theta(i\varepsilon) + O(\varepsilon).$$

From this and (5.5) follows (5.4). This proves Lemma 5.1.

If $(u; v)$ is a limit point in $C(0, T; L^2(\Omega) \times L^2(\Omega))$ of the family $\{(\tilde{u}_\varepsilon(t); \tilde{v}_\varepsilon(t)) : \varepsilon \to 0\}$, there exists a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, such that

$$\max_{[0,T]} \|u(t) - \tilde{u}_{\varepsilon_k}(t)\|_{L^2(\Omega)} + \max_{[0,T]} \|v(t) - \tilde{v}_{\varepsilon_k}(t)\|_{L^2(\Omega)} \to 0, \tag{5.7}$$

when $k \to \infty$. Therefore Lemma 5.1 implies that

$$\lim_{k \to \infty} J_{\varepsilon_k}(u^{\varepsilon_k}; \beta \xi_{\varepsilon_k}, \Omega_{\varepsilon_k}) = J_1(u; \eta, \zeta), \tag{5.8}$$

where

$$J_1(u; \eta, \zeta) = -(w_0, \zeta)_{L^2(\Omega)} \beta(0) - \int_0^T (u(t), \zeta)_{L^2(\Omega)} \beta'(t)\, dt - \tag{5.9}$$

$$- \int_0^T (u(t), \Delta\zeta + \lambda\mu(\eta - \zeta))_{L^2(\Omega)} \beta(t)\, dt + \int_0^T (f(u(t)) - h_1, \zeta)_{L^2(\Omega)} \beta\, dt.$$

We now study the asymptotic behaviour of $J_{\varepsilon_k}(u^{\varepsilon_k}; \beta\xi_{\varepsilon_k}, M_{\varepsilon_k} \backslash \Omega_{\varepsilon_k})$.

LEMMA 5.2. – *Let*

$$\gamma_\varepsilon \equiv \sum_{j \in N_\varepsilon} \int_{B_\varepsilon^j} f(u^\varepsilon(x, t))\, \xi_\varepsilon(x)\, dx$$

Then for any interval $[0, T]$ *we have*

$$\left| \gamma_\varepsilon - \sum_{j \in N_\varepsilon} f(u_j^{\varepsilon, in}(t))\, \eta(j\varepsilon)\, m_\varepsilon \right| \leq C_T \varepsilon, \qquad t \in [0, T]. \tag{5.10}$$

PROOF. – The Poincaré inequality and the structure of $\xi_\varepsilon$on B_ε^j give

$$\left| \int_{B_\varepsilon^j} f(u^\varepsilon)\, \xi_\varepsilon(x)\, dx - u_j^{\varepsilon, in}(t) \int_{B_\varepsilon^j} \xi_\varepsilon(x)\, dx \right| \leq \varepsilon \|\nabla_\varepsilon u^\varepsilon(t)\|_{L^2(B_\varepsilon^j)} (\|v^\varepsilon\|_{L^2(B_\varepsilon^j)} + \varepsilon \|\nabla_\varepsilon v^\varepsilon\|_{L^2(B_\varepsilon^j)}).$$

Therefore it follows from Lemma 4.24(iii), (iv) and (4.10) that we have

$$\left| \gamma_\varepsilon - \sum_{j \in N_\varepsilon} f(u_j^{\varepsilon, in}(t)) \int_{B_\varepsilon^j} \xi_\varepsilon(x)\, dx \right| \leq C_T \varepsilon, \qquad t \in [0, T]. \tag{5.11}$$

It is also clear from (2.2), the Poincaré inequality and Lemma 4.2 (iv) that

$$\left| \int_{B_\varepsilon^j} f(u_j^{\varepsilon,\, in})(\zeta(j\varepsilon) - \eta(j\varepsilon))(1 - v^\varepsilon(x - j\varepsilon))\, dx \right| \leqslant$$

$$\leqslant C\varepsilon^{n+1}(1 + |u_j^{\varepsilon,\, in}(t)|) \leqslant C\varepsilon^{n/2+1}(\varepsilon^{n/2} + \|u^\varepsilon(t)\|_{L^2(B_\varepsilon^j)}).$$

So (5.10) follows from (5.11).

As above we can conclude that

$$\left| \sum_{j \in N_\varepsilon} \left\{ \int_{B_\varepsilon^j} u^\varepsilon(t)\, \xi_\varepsilon\, dx - u_j^{\varepsilon,\, in}(t)\, \eta(j\varepsilon)\, m_\varepsilon \right\} \right| \leqslant C_1 \varepsilon \tag{5.12}$$

for all $t \in [0, T]$ and

$$\left| \sum_{j \in N_\varepsilon} \left\{ \int_{B_\varepsilon^j} h\xi_\varepsilon\, dx - h(j\varepsilon)\, \eta(j\varepsilon)\, m_\varepsilon \right\} \right| \leqslant C_2 \varepsilon . \tag{5.13}$$

Using the equality

$$\Delta_\varepsilon \xi_\varepsilon(x) = \lambda_\varepsilon(\zeta(j\varepsilon) - \eta(j\varepsilon))\, v^\varepsilon(x - j\varepsilon), \qquad x \in B_\varepsilon^j ,$$

and Lemma 4.2 it is easy to see that

$$\sum_{j \in N_\varepsilon} \int_{B_\varepsilon^j} u^\varepsilon(t, x)\, \Delta_\varepsilon \xi_\varepsilon(x)\, dx = \lambda_\varepsilon m_\varepsilon \sum_{j \in N_\varepsilon} u_j^{\varepsilon,\, in}(t)(\zeta(j\varepsilon) - \eta(j\varepsilon)) + O(\varepsilon). \tag{5.14}$$

Thus, it follows from (5.7), (5.10), (5.12)-(5.14) that

$$\lim_{k \to \infty} J_{\varepsilon_k}(u^{\varepsilon_k}; \beta\xi_{\varepsilon_k} M_{\varepsilon_k} \backslash \Omega_{\varepsilon_k}) = \mu J_2(v; \eta, \zeta), \tag{5.15}$$

where

$$J_2(v; \eta, \zeta) = -(v_0, \eta)_{L^2(\Omega)}\beta(0) - \int_0^T (v(t), \eta)_{L^2(\Omega)}\beta'(t)\, dt - \tag{5.16}$$

$$-\lambda \int_0^T (v(t), \xi - \eta)_{L^2(\Omega)}\beta'(t)\, dt + \int_0^T (f(v(t)) - h_2, \eta)_{L^2(\Omega)}\beta(t)\, dt .$$

Equations (5.2), (5.8) and (5.15) imply that

$$J_1(u; \eta, \zeta) + \mu J_2(v; \eta, \zeta) = 0 \tag{5.17}$$

for any limiting point $(u(t); v(t)) \in C(0, T; L^2(\Omega) \times L^2(\Omega))$ of the family $\{(\bar{u}_\varepsilon(t); \bar{v}_\varepsilon(t))\colon \varepsilon \to 0\}$. Here $J_1(u; \eta, \zeta)$ and $J_2(v; \eta, \zeta)$ are defined by (5.9) and (5.16), where $\beta(t) \in C^1(0, T)$, $\beta(T) = 0$ and $\eta(x)$, $\zeta(x)$ are any smooth functions on Ω. Therefore it follows easily from (5.17) that $(u(t); v(t))$ is a weak solution of problem (3.5)-(3.8).

REMARK 5.1. – Existence theorem for solutions from $C(0, T; L^2(\Omega) \times L^2(\Omega))$ of the problem (3.5)-(3.8) also follow from the considerations above, under certain conditions concerning the functions u_0, v_0, h_1 and h_2 (see the assumption (ii) of Theorem 3.1).

In order to complete the proof of Theorem 3.1 we only need to prove the uniqueness theorem for the system (3.5)-(3.8). We will do this in the following section.

6. – Properties of the homogenized system.

In this section we prove Theorems 2.2-2.4. We rewrite equations (3.5)-(3.8) as the following system of first order evolution equations in the space $\mathcal{H}_0 = L^2(\Omega) \times \times L^2(\Omega)$:

$$\frac{d}{dt} U + AU = B(U), \qquad U|_{t=0} = U_0 , \tag{6.1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} -\Delta + \lambda\mu & 0 \\ 0 & \lambda \end{pmatrix}, \qquad B(U) = \begin{pmatrix} \lambda\mu v - f(u) + h_1 \\ \lambda u - f(v) + h_2 \end{pmatrix}.$$

It is easy to see that A is a positive self-adjoint operator in $\mathcal{H}_0$ such that

$$(AU, U)_{\mathcal{H}_0} \geq \|\nabla u\|_{L^2(\Omega)} + \gamma\lambda \|U\|^2_{\mathcal{H}_0}, \qquad U \in \mathcal{D}(A^{1/2}), \tag{6.2}$$

where $\gamma = \min(1, \mu)$, and

$$\|B(U)\|_{\mathcal{H}_0} \leq M_1(1 + \|U\|)_{\mathcal{H}_0}, \qquad \|B(U_1) - B(U_2)\| \leq M_2 \|U_1 - U_2\|_{\mathcal{H}_0} . \tag{6.3}$$

If we consider de equation (6.1) in the integral form

$$U(t) = e^{-At} U_0 + \int_0^t e^{-A(t-\tau)} B(U(\tau))\, d\tau , \tag{6.4}$$

then using the fixed point method in the space $C(0, T; \mathcal{H}_0)$ we can easily prove the existence and uniqueness of solutions for $T < T_0$, when T_0 is small enough. It is clear that the function $U(t)$ gives a generalized solution of the system (3.5)-(3.8) on the interval $[0, T]$, $T < T_0$. Using standard methods (see, e.g. [9, 15]) and the properties (2.2)-(2.4) of the function $f(u)$ we see that

$$\|U(t)\|_{\mathcal{H}_0} \leq C_1 \|U_0\|_{\mathcal{H}_0} e^{-\omega t} + C_2(1 - e^{-\omega t}), \tag{6.5}$$

where ω, C_1 and C_2 are positive constants. This estimate allows us to extend the solution $U(t)$ on the whole of $\boldsymbol{R}_+$. The proofs of the properties (3.10) and (3.11) are also of standard character (for similar consideration see, e.g. [9, 15]). This proves Theorem 3.2.

Let S_t be the evolutionary semigroup defined by the formula $S_t U_0 = U(t)$, where $U(t)$ is the solution of problem (6.1). Since

$$\|A^\beta e^{-tA}\| \leq Ct^{-\beta} e^{-\lambda\gamma t}, \qquad t > 0, \quad 0 < \beta < 1,$$

and $\mathcal{D}(A^{1/2}) = \mathcal{F}_1 = H^1(\Omega) \times L^2(\Omega)$, (6.4) and (6.5) imply that S_t has the following the following dissipativity property: there exists a constant $R > 0$ such that for any bounded set B in $\mathcal{F}_0$ we have

$$\|S_t U_0\|_{\mathcal{F}_1} \leq R \quad \text{for all } U_0 \in B \text{ and } t \geq t_0(B). \tag{6.6}$$

LEMMA 6.1. – *Assume that* (3.12) *is satisfied. Then S_t is a dissipative semigroup in the space $\mathcal{F}_2 = H^1(\Omega) \times H^1(\Omega)$, i.e. there exists a constant $R^* > 0$ such that for any bounded set B in $\mathcal{F}_2$ we have*

$$\|S_t U_0\|_{\mathcal{F}_2} \leq R^* \quad \textit{for all } U_0 \in B \textit{ and } t \geq t_0(B). \tag{6.7}$$

PROOF. – Using (3.7) and (3.8) we see that the function $w_k(x, t) = \partial_{x_k} v(x, t)$ satisfies the equation

$$\frac{d}{dt} w_k(t) + (\lambda + f'(v(t)))\, w_k = \lambda \partial_{x_k} u + \partial_{x_k} h_2 .$$

Therefore from (3.12) we get

$$\frac{1}{2}\frac{d}{dt}\|w_k(t)\|^2_{L^2(\Omega)} + \delta\|w_k(t)\|^2_{L^2(\Omega)} \leq C(\|u\|^2_{H^1(\Omega)} + \|h_2\|^2_{H^2(\Omega)}),$$

where $\delta > 0$. Using (6.6) we obtain

$$\|\partial_{x_k} v(t)\|^2_{L^2(\Omega)} \leq \|v(s)\|^2_{H^1(\Omega)} e^{-2\delta(t-s)} + C_R, \qquad t \geq s \geq t_0(B).$$

This estimate and (6.6) imply (6.7).

LEMMA 6.2. – *The semigroup S_t is weakly closed in $\mathcal{F}_2$, i.e. for any $t > 0$, the conditions: $U_n \to U$ and $S_t U_n \to V$ weakly in $\mathcal{F}_2$ for $n \to \infty$ impliy $V = S_t U$.*

PROOF. – The lemma follows from equation (6.4) and from the compactness of the imbedding $\mathcal{F}_2 \to \mathcal{F}_0$.

Lemma 6.1 and 6.2 make it possible to use the results from [1] and to guarantee the existence of weak global attractor $\mathcal{A}$ for the dynamical system $(S_t, \mathcal{F}_2)$. This attractor is a bounded weakly closed set in $\mathcal{F}_2 = H^1(\Omega) \times H^1(\Omega)$. It is also easy to see that the initial data from $\mathcal{F}_2$ is C^1 with respect to the semigroup S_t. Therefore in order to prove the finiteness of the Hausdorff dimension of $\mathcal{A}$ we can use the approach presented in [17].

Let us consider the first variation equation corresponding to (6.1):

$$\frac{d}{dt}W + AW = B'(U(t))W$$

for a trajectory $U(t)$ lying on the attractor $\mathcal{A}$. As in [17] (see also [4]) it is necessary to estimate the quantity

$$\sigma_N(t) = tr\{(A - B(U(t)))Q_N\}$$

for any N dimensional orthoprojector Q_N in the space $\mathcal{F}_0$ such that $Q_N\mathcal{F}_0 \subset \mathcal{F}_1$. It is clear that for $W = (w_1; w_2) \in \mathcal{F}_1$ we have

$$([A - B'(U(t))]W, W)_{\mathcal{F}_0} \geq \|\nabla w_1\|^2 + \alpha_\delta \|w_1\|^2 + \beta_\delta \|\nabla w_2\|^2,$$

where

$$\alpha_\delta = \lambda_\mu - \frac{\lambda^2(1+\mu^2)}{4\delta} + \inf_u f'(u)$$

and

$$\beta_\delta = \lambda + \inf_u f'(u) - \delta$$

for any $\delta > 0$ such that $\beta_\delta > 0$. Let $\{W^k = (w_1^k; w_2^k)\}_{k=1}^N$ be an orthonormal basis of $Q_N\mathcal{F}_0$. Using the equality

$$\sum_{k=1}^N \|w_2^k\|^2 = N - \sum_{k=1}^N \|w_1^k\|^2$$

we get

$$\sigma_N(t) = \beta_\delta N + \sum_{k=1}^N \|w_1^k\|^2 + (\alpha_\delta - \beta_\delta)\sum_{k=1}^N \|w_2^k\|^2 .$$

Now we use the following version of the Sobolev-Lieb-Thirring inequality

$$k_1 \sum_{k=1}^N \|w_1^k\|^2 + \frac{k_2}{[d(\Omega)]^2}\int_\Omega \varrho(x)\,dx \geq \int_\Omega \varrho(x)^{1+2/n}\,dx$$

which follows from [6, Theorem 2.1]. Here $\varrho(x) = \sum_{k=1}^N [w_1^k]^2$, k_1 and k_2 are constants depending on n and on the shape of Ω, $d(\Omega)$ is the diameter of Ω. We obtain

$$\sigma_N(t) = \beta_\delta N + \int_\Omega \left\{\frac{1}{k_1}\varrho(x)^{1+2/n} - \gamma\varrho(x)\right\} dx$$

where

$$\gamma = \frac{k_2}{k_1}[d(\Omega)]^{-2} + \lambda + \frac{\lambda^2(1+\mu^2)}{4\delta} .$$

Since

$$z^{1+2/n} - \gamma k_1 z \geq -\frac{2}{n}\left(\frac{\gamma k_1 n}{n+2}\right)^{1+n/2}$$

for any $z > 0$, we have

$$\sigma_N(t) \geq \beta_\delta N - \frac{|\Omega|}{k_1}\frac{2}{n}\left(\frac{\gamma k_1 n}{n+2}\right)^{1+n/2}.$$

Therefore (see, e.g. [17]) an estimate for the Hausdorff dimension of $\mathcal{A}$ as a compact set in $\mathcal{F}_0$ can be found from the condition

$$N > \frac{2}{n}\frac{|\Omega|}{k_1\beta_\delta}\left(\frac{\gamma k_1 n}{n+2}\right)^{1+n/2}.$$

This proves Theorem 3.3.

REMARK 6.1. - It is easy to see that the function

$$V(u, v) = \frac{1}{2}(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\mu\|u - v\|^2_{L^2(\Omega)}) + \\ + \int_\Omega (F(u) + \mu F(v))\,dx - (h_1, u)^2_{L^2(\Omega)} - \mu(h_2, v)^2_{L^2(\Omega)}$$

is continuous on $\mathcal{F}_2$ and has the following properties

$$V(u(t), v(t)) + \int_0^t \left(\left\|\frac{\partial u}{\partial t}(\tau)\right\|^2_{L^2(\Omega)} + \mu\left\|\frac{\partial u}{\partial t}(\tau)\right\|^2_{L^2(\Omega)}\right)dt = V(u_0, v_0),$$

where $(u(t), v(t))$ is the solution of problem (3.5)-(3.8). This property implies (see, e.g. [1, 4, 7, 17]) $\mathcal{A} = \mathcal{M}_+(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary solutions of the system (3.5)-(3.8) and $\mathcal{M}_+(\mathcal{N})$ is the unstable manifold of $\mathcal{N}$. In particular this means that any trajectory of $S_t U_0$ goes to $\mathcal{N}$, when $t \to +\infty$.

REMARK 6.2. - The assumption (3.12) is of prime importance in the proof of Theorem 3.3. Indeed, for any $\delta > 0$ it is easy to find a function $f(u)$ satisfying (2.2)-(2.4) such that

$$f(v_0) = 0, \qquad \lambda + f'(v_0) = -\delta$$

for some $v_0 \in \boldsymbol{R}$. In this case the pair $(v_0; v_0)$ gives a stationary solution of (3.5)-(3.8). The linearization of (3.5) and (3.7) near $(v_0; v_0)$ has the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} + A_{v_0}\begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad \text{where } A_{v_0} = \begin{pmatrix} -\Delta + \lambda\mu + f'(v_0) & -\lambda\mu \\ -\lambda & \lambda + f'(v_0) \end{pmatrix}.$$

Simple considerations show that for $\mu = 1$ the operator A_{v_0} has an infinite dimensional spectral subspace corresponding to eigenvalues belonging to $\{\lambda\colon \operatorname{Re}\lambda < 0\}$. Therefore the instable manifold $\mathcal{M}_+(v_0; v_0)$ has infinite dimension. Thus the asymptotic behaviour of the system (3.5)-(3.8) without assumption (3.12) cannot be described by a finite-dimensional global attractor.

Now we prove Theorem 3.4 on the upper semicontinuity of the family $\{\mathcal{a}_\varepsilon\colon \varepsilon > 0\}$ of attractors for the problem (1)-(3), when $\varepsilon \to 0$.

It follows from Remarks 2.2 and 4.1 that for any trajectory $\{u^\varepsilon(t) - \infty < t < \infty\}$, belonging to the attractor $\mathcal{a}_\varepsilon$ we have the uniform estimates:

$$\left\| P_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla P_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \|P_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \leq C_1 , \tag{6.8}$$

and

$$\left\| Q_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega_{N_\varepsilon})} + \|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} + \|Q_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Omega_{N_\varepsilon})} \leq C_2 , \tag{6.9}$$

for all $t \in (-\infty, \infty)$ and ε small enough.

Let $u_0^\varepsilon \in \mathcal{a}_\varepsilon$. Then there exist a trajectory $\{u^\varepsilon(t)\colon -\infty < t < \infty\} \subset \mathcal{a}_\varepsilon$ such that $u^\varepsilon(0) = u_0^\varepsilon$ and (6.8) and (6.9) are satisfied. Therefore, as in Section 5 we can find a solution $(u(t); v(t))$ of (3.5)-(3.8) belonging to $C(a, b; \mathcal{F}_0)$ for any a, b such that $-\infty < a < a < +\infty$,

$$\max_{[a,\,b]} \|P_{\varepsilon_k} u^{\varepsilon_k}(t) - u(t)\|_{L^2(\Omega)} + \max_{[a,\,b]} \|Q_{\varepsilon_k} u^{\varepsilon_k}(t) - v(t)\|_{L^2(\Omega)} \to 0 \tag{6.10}$$

for some subsequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$. From (6.8) and (6.9) we also get

$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + \|u\|^2_{L^2(\Omega_\varepsilon)} \leq C_1 ,$$

and

$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq C_2 ,$$

for all t. Consequently $U(t) = (u(t); v(t))$ belongs to a weak global attractor $\mathcal{a}$. Therefore from (6.10) it is easy to extract the assertion of Theorem 3.4.

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