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A New Variational Characterization of Jacobi Fields along Geodesics (*)

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Abstract. – *A classical result of Riemannian geometry states that Jacobi fields along geodesics of a Riemannian manifold (Q, g) can be obtained as geodesics of the so-called «complete lift» of the metric g itself to the tangent bundle TQ. We show that this classical result is in fact a very simple consequence of a completely general theorem of Calculus of Variations.*

0. – Introduction.

It is well known that the second variation of any action functional $\mathcal{A} = \int \mathcal{L}$ governs the behaviour of the action itself in the neighborhood of critical sections and that the Hessian of the Lagrangian $\mathcal{L}$ defines a quadratic form which allows to distinguish between minima, maxima and degenerate critical sections [1]. In the case of geodesics of a Riemannian manifold (Q, g), which are critical sections for the energy functional, those fields which govern the transition from geodesics to geodesics, (*i.e.* those vectorfields which make the second variation to vanish identically modulo boundary terms) are called *Jacobi fields* and they are solutions of a second-order differential equation known as *Jacobi equation of geodesics* [2]. From the theory of «complete lifts» to a tangent bundle TQ [3], [4] it is also known that the geodesics of a «completely lifted connection» project over the geodesics defined by the given connection in the base space Q and define Jacobi fields along them.

Suitably *generalized Jacobi equations* along critical sections are in fact an outcome of the second variation of any action functional $\mathcal{A} = \int \mathcal{L}$ and have been considered in the literature from a «structural viewpoint» (see, *e.g.*, [5], [6], [7], [8]); in particular, it was shown in [9] that the second variation $\delta^2 \mathcal{A}$ and the ensuing generalised

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Jacobi equations define a suitable notion of «curvature» for any given (first-order) variational principle, which takes a particularly significant form in the case of generalised harmonic Lagrangians.

The Lagrangian characterisation of these equations (for first-order Lagrangians) has been recently reconsidered in [10], where it was shown how to recast the system formed by the original Euler-Lagrange equation together with its corresponding Jacobi equation as a single variational equation generated by the *first-order deformed Lagrangian.* In this paper we shall illustrate the power of this simple result by discussing a straightforward application to Riemannian Geometry: we shall in fact show that the aforementioned characterisation of Jacobi fields as geodesics of a suitably lifted metric is in fact a particular case of the theorem.

1. – Complete lifts and geodesics.

In this Section we shall recall some preliminaries and notation and shortly discuss those parts of the theory of tangent bundle lifts which are relevant to our purposes.

Let Q be a n-dimensional manifold and (TQ, Q, τ_Q) its tangent bundle. Let Γ be a linear symmetric (i.e., torsionless) connection in Q, having local components Γ^i_{jk} in any local chart $(U; q^i)$ of Q; we denote by ∇_Γ the covariant derivative operator associated to Γ. If g is a (pseudo)-Riemannian metric in Q, having local components g_{ij}, we denote by Γ_g its *Levi-Civita connection,* having local components $\Gamma^i_{jk}(g) = 1/2 g^{im}(\partial_k g_{jm} + \partial_j g_{mk} - \partial_m g_{jk})$; the Riemannian covariant derivative w.r. to Γ_g will be simply denoted by ∇_g. The Levi-Civita connection is characterised as the only torsionless connection such that g is parallel:

$$\nabla_g(g) = 0 \tag{1.1}$$

i.e., locally:

$$\partial_i g_{jk} = g_{jm}\Gamma^m_{ki} + g_{im}\Gamma^m_{jk}. \tag{1.2}$$

The *geodesics* of a linear connection Γ are those curves $\gamma: \boldsymbol{R} \to Q$ whose tangent vector $\dot\gamma$ is parallel along γ, i.e., it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$; in local components:

$$\ddot q^i + \Gamma^i_{jk}\dot q^j \dot q^k = 0\,. \tag{1.3}$$

The *Jacobi fields* of Γ are those vectorfields $\eta = \eta^i\partial_i$ defined along geodesics γ by the differential equation:

$$\nabla^2_{\dot\gamma}\eta + \mathrm{Riem}(\eta, \dot\gamma, \dot\gamma) = 0\,, \tag{1.4}$$

where $\nabla^2_{\dot\gamma}$ denotes the second-order covariant derivative along the curve γ and Riem is the tri-linear mapping defining the Riemannian curvature of Γ. It is well known (see, e.g., [2] and [11]) that Jacobi fields generically define infinitesimal deformations of geodesics into families of nearby geodesics.

If $\Gamma = \Gamma_g$ is the Levi-Civita connection of a (pseudo)-Riemannian metric g in Q, then the geodesics γ can be characterised as those curves which make stationary the *energy functional* of (Q, g), i.e., the integral along γ of the norm $g(\dot{\gamma}, \dot{\gamma})$ of the tangent vector $\dot{\gamma}$. Therefore, the metric geodesic equation (1.2) with $\Gamma = \Gamma_g$ is the Euler-Lagrange equation of the variational principle based on the energy functional; in this case, the Jacobi equation (of geodesic deviation) is generated by a suitable manipulation on the second functional derivative of the same energy functional (see [2] for details).

Let us now recall that there exist several «lifts» of geometric objects from a manifold Q to its tangent bundle TQ, as thoroughly discussed in [3]. Among them a particularly important case is given by so-called «complete lifts», which shall be hereafter described in local components in any natural chart $(TU;\, q^i, u^i)$ of TQ associated to any local chart $(U;\, q^i)$ of Q. Let us first suppose that $g = g_{ij}\, dq^i\, dq^j$ is a (pseudo)-Riemannian metric on Q; then its complete lift is the (pseudo)-Riemannian metric g^C defined in TQ by the local expression

$$g^C = 2g_{ij}\, \delta u^i\, dq^j, \tag{1.5}$$

where δu^i denotes the following:

$$\delta u^i = du^i + \Gamma^i_{mk}\, u^m\, dq^k. \tag{1.6}$$

For any function $f\colon Q \to \boldsymbol{R}$ we define a function $\partial f\colon TQ \to \boldsymbol{R}$ by setting locally:

$$(\partial f)(q^i, u^i) \equiv (\partial_j f)\, u^j \tag{1.7}$$

(in the jet-bundles terminology, ∂f is the «formal derivative» of the given function f; see, e.g., [12]). With this notation and using eqn. (1.2) it is easily seen that g^C can be locally expressed by:

$$g^C = (\partial g_{ij})\, du^i\, du^j + 2g_{ij}\, du^i\, dq^j \tag{1.8}$$

i.e., the $(2n \times 2n)$ matrix of g^C is

$$g^C = \begin{pmatrix} \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \tag{1.9}$$

Let now Γ be a linear connection on Q, with local components Γ^k_{ij} in any local chart $(U;\, q^i)$. The complete lift of Γ (see [3], pp. 38-39) is the linear connection Γ^C defined in TQ as follows. We first introduce a convenient notation: denote also by $(TU;\, q^k, q^{\bar{k}})$ the natural tangent chart $(TU;\, q^k, u^k)$ (i.e., we use «barred indices» $\bar{k}, \bar{m} = 1, 2, \dots, n$ in the fibres of TQ as corresponding to the «unbarred indices» $k, m = 1, 2, \dots, n$ in the base Q); we denote by $\tilde{\Gamma}^C_{AB}$ the components of Γ^C in this tangent chart, with indices

$A, B, \ldots$ either «barred» or «unbarred». Then Γ^C has local components:

$$\tilde{\Gamma}^k_{mn} = \Gamma^k_{mn} \tag{1.10a}$$

$$\tilde{\Gamma}^k_{m\bar{n}} = \tilde{\Gamma}^k_{\bar{n}m} = 0\,, \tag{1.10b}$$

$$\tilde{\Gamma}^k_{\bar{m}\bar{n}} = 0\,, \tag{1.10c}$$

$$\tilde{\Gamma}^{\bar{k}}_{mn} = (\partial_i \Gamma^k_{mn})\, u^i, \tag{1.10d}$$

$$\tilde{\Gamma}^{\bar{k}}_{m\bar{n}} = \tilde{\Gamma}^{\bar{k}}_{\bar{n}m} = \Gamma^k_{mn}, \tag{1.10e}$$

$$\tilde{\Gamma}^{\bar{k}}_{\bar{m}\bar{n}} = 0\,. \tag{1.10f}$$

The following results are well known.

THEOREM 1 ([3], 6.6 page 45). – *Let (Q, g) be a Riemannian manifold. The complete lift $(\nabla_g)^C$ of the Levi-Civita connection ∇_g of g is the Levi-Civita connection of the complete lift g^C of the metric g; i.e.:*

$$(\nabla_g)^C = \nabla_{g^C}. \tag{1.11}$$

THEOREM 2 ([3], Th. 9.1 page 58). – *Let Γ be any linear connection in Q. Whenever $\hat{\gamma}$: $\boldsymbol{R} \to TQ$ is a geodesic of the complete lift Γ^C then its projection $\gamma \equiv \tau_Q \circ \hat{\gamma}$: $\boldsymbol{R} \to Q$ is a geodesic of Γ and the vectorfield defined by $\hat{\gamma}$ along γ is a Jacobi field for Γ.*

The classical proof of Thm. 1 above relies on the calculation of the Christoffel symbols of (1.9). The classical proof of Thm. 2 relies instead on the explicit calculation of the geodesic equation in TQ for the connection locally expressed by (1.10); one finds that the $2n$ components of the geodesic equation in TQ split into n equations in the base Q, corresponding to the «unbarred» indices, and n equations in the fibres, corresponding to the «barred» indices; the first set coincides with the geodesic equation of Γ in Q; in virtue of (1.10*d*) the connection components of Γ^C involve first derivatives of the connection components of Γ and, accordingly, the second set can be manipulated to the form (1.4) which involves the Riemann curvature of Γ.

An immediate consequence of Thm.s 1 and 2 above is the following:

THEOREM 3. – *Let (Q, g) be a Riemannian manifold. Then the system formed by the geodesic equation of g in Q and the Jacobi equation associated to g in TQ is the geodesic equation in TQ of the complete lift metric g^C. Therefore this system follows from a variational principle on TQ based on the energy functional defined by the lifted metric g^C.*

2. – The Lagrangian characterization of generalized Jacobi equations.

We shall now briefly recall some fundamental notion from the Calculus of Variations on fibered manifolds (see [12] for notation). Let (B, M, π) be a fibered manifold and $\mathcal{L}\colon J^1\pi \to \Lambda^n(T^*M)$ be a first-order Lagrangian (density) on B. Here $J^1\pi$ denotes the first-jet prolongation of π and $\Lambda^n(T^*M)$ is the bundle of n-forms of M, n being the dimension of the base manifold M. Locally:

$$\mathcal{L} = L(x^\lambda, y^i, y^i_\lambda)\,\boldsymbol{ds}\,, \tag{2.1}$$

where $(J^1U; x^\lambda, y^i, y^i_\lambda)$ is any natural chart in $J^1\pi$ and $\boldsymbol{ds}$ is the local volume of $(U; x^\lambda)$.

The action of $\mathcal{L}$ is defined by

$$\mathcal{A} = \int_\Omega (j^1\sigma)^*\mathcal{L}\,, \tag{2.2}$$

where Ω is any compact domain in M with regular boundary $\partial\Omega$ and $\sigma \in \Gamma(\pi)$ is any local section (defined in an open subset containing Ω).

One defines then the «first variation» $\delta\mathcal{A}$ of $\mathcal{A}$ by considering homotopic variations $\eta \equiv \delta\sigma \in \chi_V(\pi)$ with fixed values at the boundary $\partial\Omega$ (see, e.g., [12]), $\chi_V(\pi)$ being the space of vertical vectorfields of π. The result is given by the well known (local) expression:

$$\delta\mathcal{A} = \int_\Omega \delta\mathcal{L} \equiv \int_\Omega \left[\frac{\partial\mathcal{L}}{\partial y^i}\eta^i + \frac{\partial\mathcal{L}}{\partial y^i_\mu}\eta^i_\mu\right]\boldsymbol{ds}\,, \tag{2.3}$$

which integrates by parts to give:

$$\delta\mathcal{A} = \int_\Omega e_i(\mathcal{L})\,\eta^i\,\boldsymbol{ds} + \int_{\partial\Omega} p^\mu_i(\mathcal{L})\,\eta^i\,\boldsymbol{ds}_\mu\,, \tag{2.4}$$

where $\boldsymbol{ds}_\mu$ is the surface element of $\partial\Omega$, defined so that $\boldsymbol{ds}_\mu \wedge dx^\mu = (-1)^\mu\boldsymbol{ds}$, $p^\mu_i(\mathcal{L})$ are the *canonical momenta* defined by

$$p^\mu_i(\mathcal{L}) = \frac{\partial\mathcal{L}}{\partial y^i_\mu} \tag{2.5}$$

and $e_i(\mathcal{L})$ is defined by:

$$e_i(\mathcal{L}) = \partial_i\mathcal{L} - \frac{d}{dx^\mu}(p^\mu_i(\mathcal{L}))\,. \tag{2.6}$$

Defining thence *critical sections* as those sections along which $\delta\mathcal{A}$ vanishes for any η with fixed values at the boundary, from (2.4) we see that they are characterised by the equation:

$$(j^2\sigma)^*[e(\mathcal{L})] = 0\,, \tag{2.7}$$

which is called the *Euler-Lagrange equation.* Here $[e(\mathcal{L})]$ is a global bundle morphism

$e(\mathcal{L}): J^2\pi \to \Lambda^m(T^*M) \otimes V^*\pi$, where $V^*\pi$ is the dual bundle of the vertical vector bundle $V\pi$, called the *Euler-Lagrange morphism* and locally defined by:

$$e(\mathcal{L}) = e_i(\mathcal{L})\, \boldsymbol{ds} \otimes dy^i. \tag{2.8}$$

The local expressions (2.4) have a global meaning; the first variation of $\mathcal{L}$ is in fact globally defined through a further global bundle morphism $f(\mathcal{L}): J^1\pi \to \to \Lambda^{m-1}(T^*M) \otimes V^*\pi$, locally expressed by:

$$(j^1\sigma)^* f(\mathcal{L}) = (p_i^\mu(\mathcal{L}) \circ j^1\sigma)\, \boldsymbol{ds}_\mu \otimes dy^i \tag{2.9}$$

and the following holds for $T\mathcal{L}$:

$$(j^1\sigma)^*[T\mathcal{L}(\eta)] = (j^2\sigma)^* \langle e(\mathcal{L}) | \eta \rangle + (j^1\sigma)^* d\langle f(\mathcal{L}) | \eta \rangle \tag{2.10}$$

for any local section σ and any vertical vectorfield η which projects onto σ. Here and in the sequel $\langle \,|\, \rangle$ denotes standard duality between forms and vectorfields. Equation (2.10) is called the *global first variation formula* of $\mathcal{L}$. The global counterpart of eqn. (2.4) is thence the following:

$$\delta \mathcal{A} = \int_\Omega \langle e(\mathcal{L}) | \eta \rangle + \int_{\partial\Omega} \langle f(\mathcal{L}) | \eta \rangle. \tag{2.11}$$

In order to study the stability properties of critical sections, i.e. of the solutions of the Euler-Lagrange equation (2.3), one considers next the variation of the action under second-order deformations of σ. The local expression for the second variation of $\mathcal{A}$ is then given by (see, e.g., [7]):

$$\delta^2 \mathcal{A} = \frac{1}{2}\left[\int_\Omega \left[\frac{\partial \mathcal{L}}{\partial y^i} \varrho^i + \frac{\partial \mathcal{L}}{\partial y^i_\mu} \varrho^i_\mu \right] \boldsymbol{ds} + \right.$$
$$\left. + \int_\Omega \left[\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \eta^i \eta^j + 2 \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j_\mu} \eta^i \eta^j_\mu + \frac{\partial^2 \mathcal{L}}{\partial y^i_\mu y^j_\nu} \eta^i_\mu \eta^j_\nu \right] \boldsymbol{ds} \right], \tag{2.12}$$

where $\varrho = \delta^2 \sigma$ denotes the second variation of σ. Equation (2.12) is the local counterpart of the global expression

$$\delta^2 \mathcal{A} = \frac{1}{2}\left[\int_\Omega \langle e(\mathcal{L}) | \varrho \rangle + \int_\Omega Hess_{\mathcal{L}}(j^1\eta) + \int_{\partial\Omega} \langle f(\mathcal{L}) | \varrho \rangle \right], \tag{2.13}$$

where the n-form $Hess_{\mathcal{L}}(j^1\eta)$, called the *Hessian* of $\mathcal{L}$, is locally given by the quadratic expression corresponding to the second term of (2.12).

A second order equation for η along critical sections, called the *(generalized) Jacobi equation*, can be obtained from (2.13) by suitable integrations by parts on *Hess*. In fact, as discussed e.g. in [7], eq. (2.13) can be conveniently rewritten as follows

$$\delta^2 \mathfrak{A} = \frac{1}{2}\left[\int_{\Omega} \langle e(\mathcal{L}) | \varrho \rangle + \int_{\Omega} \langle Jac_{\mathcal{L}}(j^2\eta) | \eta \rangle + \int_{\partial\Omega} \hat{f}(\mathcal{L})(\eta, \varrho) \right], \tag{2.14}$$

where $\hat{f}(\mathcal{L})(\eta, \varrho)$ is a new boundary term depending both on η and ϱ and $Jac_{\mathcal{L}}(j^2\eta)$ is locally given by

$$Jac_{\mathcal{L}}(j^2\eta) = J_i(j^2\eta) dy^i$$

with

$$J_i(j^2\eta) = \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}\eta^j + \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j_\mu}\eta^j_\mu - \frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial y^i_\mu \partial y^j}\eta^j + \frac{\partial^2 \mathcal{L}}{\partial y^i_\mu \partial y^j_\nu}\eta^j_\nu \right). \tag{2.15}$$

Equating $J_i(j^2\eta)$ to zero, i.e. setting

$$Jac_{\mathcal{L}}(j^2\eta) = 0 \tag{2.16}$$

gives rise to the standard form of the *(generalized) Jacobi equation.*

Let us now remark that eqn. (2.3) defines in fact a new Lagrangian density in the bundle $V\pi$ as follows. Equation (2.3) contains the first variation $\delta\mathcal{L}$: $V(J^1\pi) \to \Lambda^n(T^*M)$, locally defined by:

$$\delta\mathcal{L} \equiv \frac{\partial\mathcal{L}}{\partial y^i}\eta^i + \frac{\partial\mathcal{L}}{\partial y^i_\mu}\eta^i_\mu. \tag{2.17}$$

Recall that there is a natural bundle isomorphism $\mathfrak{J}$: $V(J^1\pi) \to J^1(V\pi)$, locally defined by:

$$(x^\lambda, y^i, y^i_\mu; \eta^i, \eta^i_\mu) \mapsto (x^\lambda, y^i, \eta^i; y^i_\mu, \eta^i_\mu). \tag{2.18}$$

Then a new Lagrangian density $\mathcal{L}_1$: $J^1(V\pi) \to \Lambda^n(T^*M)$ is defined by:

$$\mathcal{L}_1 = \delta L \circ (\mathfrak{J})^{-1} \tag{2.19}$$

and it is called the *first-order deformed Lagrangian.*

The following has been proved in [10].

THEOREM 4. – *Let (B, M, π) be any fibered manifold and $\mathcal{L}$ be any first order Lagrangian density on π. Then the system formed by the Euler-Lagrange equation* (2.3) *and the Jacobi equation* (2.16) *of $\mathcal{L}$ is equivalent to the Euler-Lagrange equations of $\mathcal{L}_1$.*

The above results hold in particular for Lagrangians over curves, i.e., for variational principles based on the tangent bundle. Since this case will be the relevant one for our purposes, we shall shortly report here how the results above specialise to Classical Mechanics. Let then Q be a smooth manifold and $B = \boldsymbol{R} \times Q$; then $J^1\pi = \boldsymbol{R} \times TQ$ and a Lagrangian density is a mapping $\mathcal{L}: \boldsymbol{R} \times TQ \to \Lambda^1(T^*\boldsymbol{R})$; locally:

$$\mathcal{L} = L(t, q^i, u^i)\, dt, \tag{2.20}$$

where $(\boldsymbol{R} \times TU; t, q^i, u^i)$ is any natural chart in $\boldsymbol{R} \times TQ$. The *canonical momenta* are defined by

$$p_i(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial u^i} \tag{2.21}$$

and $e_i(\mathcal{L})$ is defined by:

$$e_i(\mathcal{L}) = \partial_i \mathcal{L} - \frac{d}{dt}(p_i(\mathcal{L}))\,. \tag{2.22}$$

A local section of B is naturally identified with a curve $\gamma: \boldsymbol{R} \to Q$ and Euler-Lagrange equations are second order equations specifying «critical curves» in Q. The vertical bundle $V\pi$ identifies with $\boldsymbol{R} \times TQ$ and a vertical vectorfield $\eta = \eta^i \partial_i$ is nothing but a vectorfield $\eta \in \chi(Q)$. The second variation of $\mathcal{A}$ is then given by:

$$\delta^2 \mathcal{A} = \frac{1}{2}\left[\int_\Omega \left[\frac{\partial \mathcal{L}}{\partial q^i}\varrho^i + \frac{\partial \mathcal{L}}{\partial u^i}\dot{\varrho}^i\right] dt + \right.$$
$$\left. + \int_\Omega \left[\frac{\partial^2 \mathcal{L}}{\partial q^i \partial q^j}\eta^i \eta^j + 2\frac{\partial^2 \mathcal{L}}{\partial q^i \partial u^j}\eta^i \dot{\eta}^j + \frac{\partial^2 \mathcal{L}}{\partial u^i u^j}\dot{\eta}^i \dot{\eta}^j\right] dt\right]. \tag{2.23}$$

The Jacobi equation $Jac_{\mathcal{L}}(j^2\eta)$ is then locally given by

$$J_i(j^2\eta) = \frac{\partial^2 \mathcal{L}}{\partial q^i \partial q^j}\eta^j + \frac{\partial^2 \mathcal{L}}{\partial q^i \partial u^j}\dot{\eta}^j - \frac{d}{dt}\left(\frac{\partial^2 \mathcal{L}}{\partial u^i \partial q^j}\eta^j + \frac{\partial^2 \mathcal{L}}{\partial u^i \partial u^j}\dot{\eta}^j\right). \tag{2.24}$$

The first variation $\delta\mathcal{L}$ is locally defined by:

$$\delta\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial q^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i}\dot{\eta}^i \tag{2.25}$$

and the natural bundle isomorphism $\mathfrak{J}: \boldsymbol{R} \times TTQ \to \boldsymbol{R} \times TTQ$ is locally defined by:

$$(t, q^i, u^i; \eta^i, \dot{\eta}^i) \mapsto (t, q^i, \eta^i; u^i, \dot{\eta}^i); \tag{2.26}$$

then the new Lagrangian density $\mathcal{L}_1$ is a mapping $\mathcal{L}_1: \boldsymbol{R} \times TTQ \to \Lambda^1(T^*\boldsymbol{R})$.

3. – The main result revisited.

We are now ready to show that Theorem 3 is a simple consequence of the Theorem 4 recalled in the previous Section, thus completing our claim.

Let us then consider a Riemannian metric $g = g_{ij}\, dq^i\, dq^j$ in Q. The energy functional of g is based on the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{ij} u^i u^j . \tag{3.1}$$

According to eqn. (2.24) and using the symmetry of g the associated first-order deformation Lagrangian is thence given by:

$$\mathcal{L}_1 = \frac{1}{2} (\partial_k g_{ij}) u^i u^j \eta^k + g_{ij} u^j \dot{\eta}^i , \tag{3.2}$$

which, using eqn. (1.2), becomes immediately:

$$\mathcal{L}_1 = g_{ij} [\dot{\eta}^i + \Gamma^i_{mk} u^m \eta^k] u^j . \tag{3.3}$$

Then, from eqn. (1.6) we see that $\mathcal{L}_1$ is in fact the energy Lagrangian of the lifted metric $g^C = 2 g_{ij}\, \delta u^i\, dq^j$. Accordingly, Theorem 4 applied to $\mathcal{L}$ entails that the geodesic equation of g^C in TQ is equivalent to the system formed by the geodesic equation of g in Q together with the Jacobi equation of g, which is nothing but Theorem 3.

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