On Smooth Subcanonical Varieties of Codimension 2 in P^n, $n \geq 4$ (*) (**).

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Summary. – *We study subcanonical codimension 2 subvarieties of P^n, $n \geq 4$, using as our main tool the rank 2 vector bundle canonically associated to them. With this method we prove first that every smooth canonical surface in P^4 is a complete intersection. Next we study smooth varieties of codimension 2 in P^n, $n \geq 6$; it is well known that all of them are subcanonical and R. Hartshorne conjectured that they are always complete intersections, if $n \geq 7$. We prove this conjecture in the particular case of a variety X for which the integer e such that $\omega_X = \mathcal{O}_X(e)$ is 0 or negative. This result, togheter with a strong result by Z. Ran, provides a quadratic bound for the degree of a non-complete intersection variety of codimension 2 in P^n, $n \geq 6$.*

Introduction.

This paper is concerned about smooth subvarieties X of the complex projective space P^n, $n \geq 4$, whose canonical divisor is a multiple of an hyperplane section: such subvarieties are called « subcanonical »; this class contains all smooth « canonical » varieties, i.e. varieties embedded in P^n by a sublinear system of the canonical system.

The main examples of subcanonical varieties are the complete intersections; indeed if $X = H_1 \cap H_2 \cap \ldots \cap H_n$, H_i hypersurface of degree d_i in P^n, then $\mathcal{O}_X(\sum d_i - n - 1) = \omega_X$.

We only consider the case codim $(X, P^n) = 2$. In this situation, by a standard construction, the normal bundle of X can be lifted to a rank 2 vector bundle E on P^n and X can be viewed as the zero locus of a global section of E. Many properties of X are strictly connected with properties of E: E has Chern classes $c_2(E) = \deg X$ and $c_1(E) =$ the integer such that $\omega_X = \mathcal{O}_X(c_1(E) - n - 1)$; moreover X is a complete intersection of hypersurfaces of degree a and b if and only if $E = \mathcal{O}_{P^n}(a) \oplus \mathcal{O}_{P^n}(b)$, i.e. E splits into a sum of line bundles.

This correspondence between codimension 2 subcanonical varieties and rank 2 vector bundles is the main tool of our investigation.

In section 1 we look at the case $n = 4$. In [C], it is raised the question of the existence of a smooth, non complete intersection, canonical surface S on P^4; using

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Severi's formula for the number of nodal points, it is showed that S must have degree 12 and arithmetic genus 4. We are able to prove that such surface cannot exist. Indeed the vector bundle associated to S, twisted by -3, would give a rank 2 bundle E on $\boldsymbol{P}^4$ with Chern classes $c_1(E) = 0$ and $c_2(E) = 3$. In [B-E], BARTH and ELENCWAJG claimed the non-existence of such a bundle, but their proof only works in the case « E stable » while for the case « E non stable » they refer to a theorem of GRAUERT and SCHNEIDER ([G-S], 3.1) whose proof is incomplete ([Zentralblatt], 412-32014; [Math. Reviews], 58 n. 1279; [S], p. 92). For our problem the existence of the surface S would imply that the bundle E is *semi*-stable, so we limit our exmination to such bundles. We have two ways to get the result: in the first we use BARTH-ELENCWAJG's construction, together with some supplementary information on the cohomology of E, obtained by the cohomology of S (which is partially known by Kodaira vanishing, Riemann-Roch theorem and the fact that S must be linearly normal ([Se])) and we get a contradiction. In the second way we prove the non-existence of a semi-stable but not stable rank 2 vector bundle E on $\boldsymbol{P}^4$ with Chern classes $c_1(E) = 0$ and $c_2(E) = 3$: first we show that E would have a section whose zero locus X is a non-reduced, locally complete intersection, multiplicity 3 structure on a plane, then we prove that such an X must be degenerate, and this is unconsistent with $c_1(E) = 0$; this, together with the correct part of Barth-Elencwajg's theorem, gives the non-existence of S.

While there are a lot of examples of smooth, non complete intersection, subcanonical curves in $\boldsymbol{P}^3$, when the dimension raises the situation seems to be much poorer; HORROCKS and MUMFORD showed in [H-M] that there are abelian surfaces embedded in $\boldsymbol{P}^4$ and they are examples of non complete intersection, subcanonical varieties; all of them are related, up to projective transformations, to the same rank 2 vector bundle E, moreover, as far as we know, E is the unique known example of a vector bundle of rank 2 in $\boldsymbol{P}^4$ which is not the sum of line bundles (up to twisting by $\mathcal{O}_{\boldsymbol{P}^n}(m)$).

For $n > 4$ we know no examples of smooth subcanonical varieties, of codimension 2 in $\boldsymbol{P}^n$, except complete intersections.

If $n \geqslant 6$, then every smooth codimension 2 subvariety X of $\boldsymbol{P}^n$ is subcanonical, indeed BARTH and LARSEN showed that the PICARD group of X is $\boldsymbol{Z}$, generated by the class of an hyperplane section, hence in particular $\omega_X = \mathcal{O}_X(e)$ for some integer e.

In 1974 HARTSHORNE conjectured that all smooth codimension 2 subvarieties of $\boldsymbol{P}^n$, $n \geqslant 7$, are complete intersections. In section 2 we give a short survey on the progresses made in this direction from 1974 till now; we point our attention on a recent preprint of Z. RAN, which seems to provide some useful tool for the study of codimension 2 subvarieties. Using Barth-Larsen's theorem, Ran's results and the Riemann-Roch formula for a vector bundle, we are able to prove a very particular case of the conjecture: we prove that if X is smooth of codimension 2 in $\boldsymbol{P}^n$, $n \geqslant 6$ and $\omega_X = \mathcal{O}_X(e)$, $e \leqslant 0$, then X is complete intersection.

In the last section we briefly study smooth subcanonical threefolds X in $\boldsymbol{P}^5$. By Riemann-Roch formula for threefolds, we see that fixed the integer e such that

$\omega_X = \mathcal{O}_X(e)$, in general only finitely many values are allowed for $d = \deg X$; then reducing ourselves to a general hyperplane section of X and using some result of the theory of surfaces in $\boldsymbol{P}^4$, we prove that X is complete intersection if $e \leqslant 2$.

We end giving, as a corollary of Ran's theorems and of our theorem of section 2, the following lower bound for the degree of a non C.I. smooth codim 2 subvariety of $\boldsymbol{P}^n$, $n \geqslant 6$: if $d = \deg X$ and X is not C.I., then $d > (n+2)^2/4$. This bound is much better than the linear one given in [B-V] and it is, as far as we know, the first quadratic bound on the degree of a non C.I. codimension 2 subvariety of $\boldsymbol{P}^n$.

We wish to thank Z. RAN for some useful conversations on this subject.

0. – Preliminaries.

Once forever, $\boldsymbol{P}^n$ means the projective n-space over the *complex field*.

Sometimes we shall abbreviate in the text, « complete intersection » with « C.I. ».

A subscheme $X \subseteq \boldsymbol{P}^n$ is said to be « degenerate » if it is contained in some hyperplane.

i) If X is a smooth subcanonical variety of codimension 2 in $\boldsymbol{P}^n$, $n \geqslant 3$, I_X is its ideal sheaf, then there is a unique non-trivial extension of sheaves on $\boldsymbol{P}^n$, $0 \to \mathcal{O}_{\boldsymbol{P}^n} \to E \to I_X(a) \to 0$ such that E is a rank 2 vector bundle with Chern classes $c_1(E) = a$ and $c_2(E) = \deg X$: E is the extension to $\boldsymbol{P}^n$ of the normal bundle of X. Moreover $\omega_X = \mathcal{O}_X(a-n-1)$ and X is the zero locus of a global section of E (see [H3], § 1). More generally this construction holds when X is any locally C.I. scheme of codimension 2 in $\boldsymbol{P}^n$, such that, for some integer e, $\mathcal{O}_X(e)$ is a dualizing sheaf for X.

ii) We shall use the following definition of stability for rank 2 vector bundles on $\boldsymbol{P}^n$, which is equivalent to the one given in [H3] or in [M] (see [H3], Prop. 3.1). Let c_1 be the first Chern class of E. Then:

a) E is « stable » if $H^0(E(a)) = 0$, $\forall a \leqslant c_1/2$;

b) E is « semistable » if $H^0(E(a)) = 0$, $\forall a < c_1/2$.

iii) Let X be a subcanonical, locally C.I., reduced codimension 2 subscheme of $\boldsymbol{P}^3$, zero locus of a section of the vector bundle E.

If X is contained in a smooth quadric surface, then X is C.I. or it is a disjoint union of lines ([H1], p. 231).

If X is contained in a quadric cone Q, then X is always C.I.: indeed if $\deg X$ is even, then X is a Cartier divisor of Q, but $\operatorname{Pic} Q = \boldsymbol{Z}$, generated by the class of an hyperplane section; if $\deg X$ is odd, then one can see that for a generic line $r \in Q$, $X \cup r$ is a Cartier divisor on Q, hence $X \cup r$ is C.I., i.e. X is linked to r; it follows that X is arithmetically COHEN-MACAULAY ([Ra], 2.3) so that E is a sum of line bundles.

§ 1. – This section is mainly devoted to the proof of the following

THEOREM 1. – *Every smooth canonical surface S in $\boldsymbol{P}^4$ is complete intersection.*

Of course we may assume that S is non-degenerate, otherwise the claim is obvious. In this case, using Riemann-Roch theorem as in [H1], App. A, p. 434, we get the formula:

$$d^2 - 10d - 5(H\cdot K) - 2K^2 + 12 + 12p_a = 0 \tag{1}$$

($d = \deg S$, p_a = arithmetic genus of S, H = hyperplane section of S, K = canonical divisor of S) which in our situation gives:

$$0 = d(d-17) + 12p_a + 12 \qquad \text{(see also [C], corol. 6.6)}\,.$$

As it is pointed out in [C], § 6, this relation tells us that only the following cases may occur:

a) $d = 8{,}9$; $p_a = 5$.

In this case S must be C.I. (see [C], § 6)

b) $d = 12$; $p_a = 4$.

In this case the irregularity $q(S)$ of S is 1, hence S cannot be C.I.

From now on, let S indicate a canonical surface of degree 12 and arithmetic genus 4 in $\boldsymbol{P}^4$; let I_S be its ideal sheaf. All we need to show is that such S cannot exist.

The proof uses Barth-Elencwajg's theory of the spectrum of a rank 2 vector bundle (see [B-E]).

Let E_0 be the vector bundle associated to S $(0\cdot i)$, then E_0 has Chern classes $c_1(E_0) = 6$ and $c_2(E_0) = 12$. Put $E = E_0(-3)$; it is easy to see that $c_1(E) = 0$ and $c_2(E) = 3$.

In [B-E], § 4, BARTH and ELENCWAJG claim that there are no rank 2 vector bundles E on $\boldsymbol{P}^4$ with $c_1(E) = 0$ and $c_2(E) = 3$; this would imply immediately the theorem. But indeed they only prove the case « E stable », while for the case « E non stable » they refer to a theorem of GRAUERT and SCHNEIDER ([G-S], 3.1) whose proof is unfortunately incomplete ([Zentralblatt] 412-32014, [Math.Reviews] 58, n. 1279; see also [S], p. 92).

In our case, however, we can get some more informations about the cohomology of E looking at the cohomology groups $H^i(I_S)$ and $H^i(\theta_S)$; this permits to handle the case in which E is not stable. In fact we have not to use Barth-Elencwajg's result, but rather to repeat their construction, making the computations in a different way, using the surface S.

Now we give a short account of Barth-Elencwajg's construction.

Let E be a rank 2 vector bundle on $\boldsymbol{P}^n$, $n \geq 3$; let $L \subseteq \boldsymbol{P}^n$ be a line such that $E_{|L} = 2\mathcal{O}_L$; such a line exists, for instance, if E is semistable. The idea is to study E looking at its restrictions to the planes passing through L.

Let $p\colon \tilde{P} \to \boldsymbol{P}^n$ be the blowing up of $\boldsymbol{P}^n$ along L; if $P_L = \boldsymbol{P}^{n-2}$ is the projective space which parametrizes the planes π of $\boldsymbol{P}^n$ containing L, then $\tilde{P}$ can be viewed as the subset of $\boldsymbol{P}^n \times P_L$ defined by $\tilde{P} = \{(x, \pi)\colon x \in \pi\}$, so we have a canonic projection $q\colon \tilde{P} \to P_L$. Geometrically this map can be constructed as follows: fix a $(n-2)$-linear subspace $P_L \subseteq \boldsymbol{P}^n$, disjoint from L, then send every point x of $\boldsymbol{P}^n - L$ to $\overline{(x, L)} \cap P_L$ and send every plane π of $\boldsymbol{P}^n$ containing L (i.e. every point of the exceptional divisor of $\tilde{P}$) to $\pi \cap P_L$.

Define $\varkappa_1 = R^1 q_* p^*(E(-1))$; by the theory of change of basis it follows that $\varkappa_1$ is a vector bundle on $\boldsymbol{P}^{n-2} = P_L$ of rank equal to $c_2(E)$.

We shall use only the following properties of $\varkappa_1$, which are proved in [B-E], Prop. 2.2.1:

1) $\varkappa_1 = \check{\varkappa}_1$;

2) $h^0(\varkappa_1) = h^1(E(-1))$;

3) $h^0(\varkappa_1(-1)) = h^1(E(-2))$;

4) $h^1(\varkappa_1(-1)) = h^2(E(-2))$.

Moreover we shall use the following crucial fact: if $n = 3$, then $\varkappa_1$ is a vector bundle on $\boldsymbol{P}^1$, so it splits into a direct sum of line bundles. Now let $n > 3$; for a general linear 3-space H, $L \subseteq H \subseteq \boldsymbol{P}^n$, we may repeat the construction for $E_{|H}$ so we get a vector bundle on $\boldsymbol{P}^1$; let we call it $\varkappa_1^H$. Let m be the line of P_L corresponding to H, then $\varkappa_{1|m}$ also splits into a sum of line bundles. We have:

5) $\varkappa_1^H \simeq \varkappa_{1|m}$ as bundles on $\boldsymbol{P}^1$ ([B-E], § 2).

PROOF OF TH. 1. We shall use the existence of S to evaluate the dimension of some cohomology group of E and to get a contradiction.

By construction we have the following exact sequences:

$$0 \to \mathcal{O}_{\boldsymbol{P}^4} \to E(3) \to I_S(6) \to 0 \tag{2}$$

$$0 \to I_S \to \mathcal{O}_{\boldsymbol{P}^4} \to \mathcal{O}_S \to 0 . \tag{3}$$

Since S is not a C.I., by [C], Prop. 5.9 it follows that S cannot be contained in any quadric hypersurface, i.e. $h^0(I_S(2)) = 0$. By the cohomology sequence of (2), twisted by -4, this implies $h^0(E(-1)) = 0$, which means, by definition, that E is *semi*-stable. It follows that for a generic line $L \subseteq \boldsymbol{P}^4$, $E_{|L} = 2\mathcal{O}_L$, so we may apply Barth-Elencwajg's construction.

Fix such a line L; then we get a bundle $\varkappa_1$ on a projective plane $P_L = \boldsymbol{P}^2$ which parametrizes all the planes of $\boldsymbol{P}^4$ passing through L. We have rank $(\varkappa_1) = c_2(E) = 3$;

then for every line $m \subseteq P_L$, $\varkappa_{1|m}$ splits into a sum of 3 line bundles and since $\check{\varkappa}_{|1m} \simeq \varkappa_{1|m}$ by property 1) above, then we must have:

$$\varkappa_{1|m} \simeq \theta_m(-k) \oplus \theta_m \oplus \theta_m(+k) \tag{4}$$

k a non-negative integer, eventually depending on m.

Take any 3-dimensional linear subspace $H \supseteq L$ and put $F = E_{|H}$. F is still semi-stable, indeed $F_{|L} = 2\theta_L$ and $L \subseteq H$ so that, by semicontinuity, $F_{|m} = 2\theta_m$ for a generic line m in H, hence any section of $F(-1)$ must vanish identically on a generic line, hence it must vanish everywhere.

We have an exact sequence

$$0 \to E(-3) \to E(-2) \to F(-2) \to 0$$

from which we get

$$H^1(E(-2)) \to H^1(F(-2)) \to H^2(E(-3)).$$

But, by (2), $h^2(E(-3)) = h^2(I_S)$ and by (3) $h^2(I_S) = h^1(\theta_S) = q(S) = 1$. On the other hand, by (2), $h^1(E(-2)) = h^1(I_S(1))$ which is 0 since, by a theorem of SEVERI (see [Se]), S must be linearly normal. It follows $h^1(F(-2)) \leqslant 1$.

Let m be the line of P_L corresponding to H, then by property 5) we know that $\varkappa_{1|m} \simeq \varkappa_1^H$ and the bundle on the right hand side is obtained by Barth-Elencwajg's construction applied to F, hence by property 3), $h^0(\varkappa_1^H(-1)) = h^0(F(-2)) \leqslant 1$.

It follows by (4) that we have only 2 possibilities for $\varkappa_{1|m}$, namely:

a) $\varkappa_{1|m} \simeq 3\theta_m$ if $h^0(\varkappa_{1|m}(-1)) = 0$;

b) $\varkappa_{1|m} \simeq \theta_m(-1) \oplus \theta_m \oplus \theta_m(1)$ if $h^0(\varkappa_{1|m}(-1)) = 1$.

Case *a)* can be excluded, as in [B-E], p. 18, looking at the Atiyah-Rees invariant of F, $\alpha(F) = h^0(F(-2)) + h^2(F(-2))$. In this case $h^0(F(-2)) = 0$ since F is semi-stable and, by property 4), $h^2(F(-2)) = h^1(\varkappa_1^H(-1)) = h^1(\varkappa_{1|m}(-1)) = 0$ so that $\alpha(F) = 0$; on the other hand, since F is the restriction to $\boldsymbol{P}^3$ of a vector bundle in $\boldsymbol{P}^4$, by [A-R], Prop. 7.2 we must have $\alpha(F) = \Delta(\Delta - 1)/12$ (mod 2), where $\Delta = (c_1^2 - 4c_2)/4$, that is $\alpha(F) = 1$, absurd.

So, varying H among the hyperplanes of $\boldsymbol{P}^4$ through L, we see that for every line m in P_L, $\varkappa_{1|m} \simeq \theta_m(-1) \oplus \theta_m \oplus \theta_m(1)$ so that, by definition, $\varkappa_1$ is uniform. Uniform rank 3 vector bundles on $\boldsymbol{P}^2$ were classified by ELENCWAJG (see [E]). In our situation this classification implies that $\varkappa_1$ is one of the following:

i) $\theta_{\boldsymbol{P}^2}(-1) \oplus \theta_{\boldsymbol{P}^2} \oplus \theta_{\boldsymbol{P}^2}(1)$;

ii) $T\boldsymbol{P}^2(-2) \oplus \theta_{\boldsymbol{P}^2}(1)$ ($T\boldsymbol{P}^2$ = tangent bundle);

iii) $TP^2(-1) \oplus \theta_{P^2}(-1)$;

iv) $S^2 TP^2 \otimes \theta_{P^2}(-3)$; ($S^2 = 2^{nd}$ symmetric power).

By property 3), $h^0(\varkappa_1(-1)) = h^1(E(-2))$ and this is 0, again by (2) and by the fact that S is linearly normal. This excludes case i) and case ii).

In case iii) we have $h^1(\varkappa_1(-1)) = h^1(TP^2(-2)) = 0$, but, on the other hand, by property 4), $h^1(\varkappa_1(-1)) = h^2(E(-2))$ and by (2) $h^2(E(-2)) = h^2(I_S(1))$; by (3) $h^2(I_S(1)) = h^2(\theta_S(1))$ and by duality $h^2(\theta_S(1)) = h^0(\theta_S) = q(S)$ which is 1, a contradiction.

It remains case iv). By property 2), $h^0(\varkappa_1) = h^1(E(-1))$ which, by (2), is equal to $h^1(I_S(2))$. By (3), since S is not contained in any quadric, we have $h^1(I_S(2)) = h^0(\theta_S(2)) - h^0(\theta_{P^4}(2))$; we may use Riemann-Roch theorem on S to compute $h^0(\theta_S(2))$, in fact by duality $h^2(\theta_S(2)) = h^0(\theta_S(-1)) = 0$ and $h^1(\theta_S(2))$ is 0 by Kodaira vanishing; it follows by Riemann-Roch $h^0(\theta_S(2)) = 17$ hence $h^0(\varkappa_1) = h^1(I_S(2)) = 2$. But the bundle $S^2 TP^2 \otimes \theta_{P^2}(-3)$ has no global sections, since $S^2 TP^2 \otimes \theta_{P^2}(-3) \oplus \theta_{P^2}$ is isomorphic to End (TP^2) and TP^2, being stable, has only constants as endomorphisms (for more details, see [B-E], p. 22).

This excludes case iv) and proves the theorem.

REMARK 2. – As far as we know, the following can be said about the classification of smooth subcanonical surfaces in P^4.

Put $\omega_S = \theta_S(e)$.

a) $e < 0$. It is well known by the theory of surfaces that there are only complete intersections. They all are degenerate if $e < -1$, while for $e = -1$ there are the degenerate cubic surface and the Del Pezzo surface in P^4, which is complete intersection of two quadrics.

b) $e = 0$. This very interesting case was completely solved by HORROCKS and MUMFORD. Indeed if $\omega_S = \theta_S$ and S is not C.I., then it follows from the classification of surfaces ([Be], Th. VIII.2) that S must be an abelian variety of degree 10, and $q(S) = 2$. HORROCKS and MUMFORD proved in [H-M] that there are abelian surfaces with an embedding in P^4 of degree 10; the corresponding vector bundle E is unique up to projective transformations. Up to shifting, E is the only known example of indecomposable rank 2 vector bundle on P^n, $n > 3$ (in characteristic 0).

c) $e = 1$. By Th. 1, there are only complete intersections.

d) $e = 2k$, $k \geq 1$. If E is the HORROCKS-MUMFORD's bundle, then by its cohomology (calculated in [H-M], p. 74) it follows that a general section of $E(k)$ has a codimension 2 zero locus which is a smooth surface S with $\omega_S = \theta_S(2k)$ and $q(S) = 0$.

We do not know in P^4 examples of smooth, non C.I., subcanonical surfaces with odd e, or examples of smooth subcanonical surfaces of general type with $q(S) \neq 0$.

We have an alternative way for proving Th. 1. In the previous argument we used the fact that the semistable bundle E was related to the surface S, in order to compute part of its cohomology and prove its non-existence. Using a different method, we are able to say something more about rank 2 vector bundles on $\boldsymbol{P}^4$ with $c_1 = 0$ and $c_2 = 3$; namely we can state the following:

PROPOSITION 3. – *There are no semistable vector bundles on* $\boldsymbol{P}^4$ *with Chern classes* $c_1 = 0$ *and* $c_2 = 3$.

PROOF. – Let E be such a bundle. By BARTH-ELENCWAJG's theorem ([B-E], § 4) E cannot be stable, so $h^0(E) \neq 0$ and $h^0(E(-1)) = 0$. It follows that E has a global section whose zero locus X has codimension 2 or is empty.

If $X = \emptyset$, then E would split into a sum of line bundles ([H3], 1.0.1) $E = \mathcal{O}_{\boldsymbol{P}^4}(a) \oplus \oplus \mathcal{O}_{\boldsymbol{P}^4}(b)$, with $ab = 3$, $a + b = 0$, absurd.

It follows codim $(X, \boldsymbol{P}^4) = 2$ and $\mathcal{O}_X(-5)$ is a dualizing sheaf for X, moreover X must be locally C.I. of degree 3.

Let Y be a general hyperplane section of X; Y is subcanonical and $\mathcal{O}_Y(-4)$ is a dualizing sheaf, indeed Y is the zero locus of a global section of the bundle $E_{|H}$. We show that Y is a triple line examinating subcanonical subschemes of degree 3 in $\boldsymbol{P}^3$, locally C.I. (for a similar argument, see [H3], 9.1).

Since deg $Y = 3$ and $\omega_Y = \mathcal{O}_Y(-4)$, by reasons of genus Y cannot be reduced. Y must be connected, otherwise a connected component should be a line L, but $\omega_L = \mathcal{O}_L(-2)$.

The case Y formed by a double line Y' and a line L intersecting in a point can be excluded; in fact the inclusion $f: L \hookrightarrow Y$ implies, by [H1], III, Ex. 7.2 and 6.10, the isomorphism $\omega_L \simeq f^! \omega_Y$, which gives, by definition of $f^!$, a non zero map $f_* \mathcal{O}_L \to \mathcal{O}_Y(-2)$. The image of the section 1_L of $f_* \mathcal{O}_L$ must vanish on $L - (Y' \cap L)$ since every section of $\mathcal{O}_Y(-2)$ has support on Y'. But by [H4], Prop. III.6, the formation of $f^!$ commutes with flat pullback and on the open set $U = L - (Y' \cap L)$ f « is » the identity, so the image of 1_L cannot vanish on U, a contradiction.

It follows that Y must be a triple line; then every hyperplane section of X has support on a line, hence the support of X must be a plane π of $\boldsymbol{P}^4$, hence X is a non-reduced structure on a plane and X is locally C.I., so it has no embedded components moreover deg $X = 3$. We show that such an X must be degenerate.

Choose coordinates in $\boldsymbol{P}^4$, x, y, z, w, t, such that π is defined by $x = y = 0$; put I = homogeneous ideal of X = ideal spanned by the homogeneous polynomials which vanish on X.

STEP 1. – If $F \in \boldsymbol{C}[x, y, z, w, t]$ is a homogeneous polynomial which vanishes on X at a closed point $P \in \pi$ (i.e. the image of F in $\mathcal{O}_{\boldsymbol{P}^4,P}$ belongs to the ideal of X in $\mathcal{O}_{\boldsymbol{P}^4,P}$) then F vanishes on X in an open subset $U \subseteq X$ and U is dense on X since X has support on π and no embedded components. Thus F vanishes on the whole X, i.e. $F \in I$. A fortiori the same is true if F vanishes on X at a non closed point of π, i.e. at the generic point of a closed subscheme of π.

STEP 2. - Put $\bar{x} = x/t$ and define similarly $\bar{y}, \bar{z}, \bar{w}$. The ring $\boldsymbol{C}(\bar{z}, \bar{w})[\bar{x}, \bar{y}]_{(\bar{x},\bar{y})} = A_\xi$ can be canonically identified with the local ring of the generic point of π in $\boldsymbol{P}^4$. Let us continue to indicate, by abuse, the images of $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ in A by the same letters. Let $I_\xi =$ ideal of X in A. Since X has support on π, A_ξ/I_ξ is artinian and since X has multiplicity 3 at all points of π, then length $A_\xi/I_\xi = 3$. But this is possible only if $\bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3$ all belong to I_ξ. By step 1 this imply $x^3, x^2y, xy^2, y^3 \in I$.

STEP 3. - For every closed point $P \in \pi$, I must contain a homogeneous element of the form $\phi = \alpha(z, w, t)x + \beta(z, w, t)y +$ terms of higher degree in x, y, with $\alpha(P) \neq 0$ or $\beta(P) \neq 0$. Indeed X is locally C.I., so let $\varphi_1, \varphi_2 \in \boldsymbol{C}[x, y, z, w, t]$ be homogeneous elements which define hypersurfaces H_1, H_2 whose intersection locally at P is X. Put, for $i = 1, 2$, $\phi_i = \alpha_i(z, w, t)x + \beta_i(z, w, t)y +$ terms of higher degree in x, y; if $\alpha_1, \alpha_2, \beta_1, \beta_2$ all vanish at P then H_1, H_2 both have multiplicity at least 2 at P, so X has multiplicity at least 4 at P, absurd. Then one of the ϕ_i's is of the required form and it belongs to I by step 1.

STEP 4. - It follows by step 2 and step 3 that I contains an element which can be written in the form $\phi = Q(Ax + By) + Cx^2 + Dxy + Ey^2$, with $Q, A, B, C, D, E \in \boldsymbol{C}[z, w, t]$, A, B non both identically 0, $Q \neq 0$ and A, B without common factors.

Let $\phi' = A'x + B'y +$ terms of higher degree in x, y, be another element of I; then we claim that there exists $Q' \in \boldsymbol{C}[z, w, t]$ with $A'x + B'y = Q'(Ax + By)$. This is obvious if $A'x + B'y \equiv 0$, so assume this is not the case.

Let $\varphi = q(a\bar{x} + b\bar{y}) + c\bar{x}^2 + d\bar{x}\bar{y} + e\bar{y}^2 \in \boldsymbol{C}[\bar{x}, \bar{y}, \bar{z}, \bar{w}]$ be a dehomogeneization of ϕ, where $q = Q/t^{\deg Q}$ and a, b, c, d, e are defined in the same way. Similarly take $\varphi' = a'\bar{x} + b'\bar{y} +$ terms of higher degree in $\bar{x}, \bar{y}$, as a dehomogeneization of ϕ'. By abuse we shall also consider φ and φ' as elements of $\boldsymbol{C}(z, w)[x, y]$ or A_ξ.

Since $qa \neq 0$ or $qb \neq 0$, then in $A_\xi/(\varphi)$, $\bar{x} \in (\bar{y})$ or $\bar{y} \in (\bar{x})$ so that length $A_\xi/I_\xi =$ $=$ length $A_\xi/(\varphi, \bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3) = 3$ hence $(\varphi, \bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3)$, which is contained in I_ξ, must be equal to I_ξ.

Since $\varphi' \in I_\xi$ then there exists $\sigma \in \boldsymbol{C}(\bar{z}, \bar{w})[\bar{x}, \bar{y}]$, $\sigma \notin (\bar{x}, \bar{y})$ such that $\sigma\varphi' \in (\varphi, \bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3)\boldsymbol{C}(\bar{z}, \bar{w})[\bar{x}, \bar{y}]$ so that $\sigma\varphi' = \varrho\varphi +$ terms of degree at least 3 in $\bar{x}, \bar{y}$; since σ must have non-vanishing constant term, then there are elements $\sigma_0, \varrho_0 \in \boldsymbol{C}(z, w)$, $\sigma_0, \varrho_0 \neq 0$ such that $\sigma_0(a'\bar{x} + b'\bar{y}) = \varrho_0 q(a\bar{x} + b\bar{y})$. Taking away denomitators, we may assume $\sigma_0, \varrho_0 \in \boldsymbol{C}[\bar{z}, \bar{w}]$ i.e. the previous relation holds in $\boldsymbol{C}[\bar{x}, \bar{y}, \bar{z}, \bar{w}]$. Dividing by the common factors of σ_0 and $\varrho_0 q$, we find relatively prime elements $\sigma_1, \varrho_1 \in$ $\in \boldsymbol{C}[\bar{z}, \bar{w}]$ with $\sigma_1(a'\bar{x} + b'\bar{y}) = \varrho_1(a\bar{x} + b\bar{y})$, i.e. $\sigma_1 a' = \varrho_1 a$ and $\sigma_1 b' = \varrho_1 b$. Since σ_1 does not divide ϱ_1, it must divide both a and b, but by assumption A and B are relatively prime in $\boldsymbol{C}[z, w, t]$, so their dehomogeneization are too.

This implies $\sigma_1 \in \boldsymbol{C}$ so $a' = a/\sigma_1$ and $b' = b/\sigma_1$ and homogeneizing suitably, we find what we claimed.

STEP 5. – A and B are homogeneous of the same degree. If they are not constant there is a point $P \in \pi$ where they both vanish. By the previous step, this means that no element of I can be written as $\alpha(z, w, t)x + \beta(z, w, t) +$ terms of higher degree in x, y, with $\alpha(P) \neq 0$ or $\beta(P) \neq 0$; this contradicts step 3.

Thus A and B are constant. After a suitable change of the coordinates x and y, assume $A = 1$, $B = 0$; so every element of I is of the form: $Q'x +$ terms of higher degree in x, y.

STEP 6. – Now $\phi \in \boldsymbol{C}[x, y, z, w]$ is $qx + cx^2 + dxy + ey^2$. It is clear that in A_ξ, $\varphi_0 = qx + ey^2$ also belong to $I_\xi = (\varphi, \bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3)$, so that the element $\varphi_0 = qx + ey^2$ also belong to $I_\xi = (\varphi, \bar{x}^3, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{y}^3)$, so that the element $\phi_0 = Qx + Ey^2$, which is a homogeneization of φ_0, also belong to I by step 1; again if Q, E have a common factor $F \in \boldsymbol{C}[z, w, t]$ and $Q = Q_0F$, $E = E_0F$, then $Q_0x + E_0y^2$ also vanishes on X at the generic point ξ so by step 1 $Q_0x + E_0y^2 \in I$; it follows that we may assume Q and E relatively prime in $\boldsymbol{C}[z, w, t]$.

STEP 7. – If $Q \in \boldsymbol{C}$, by reason of degree $E = 0$ and X is degenerate. So it remains to show that if Q is not a costant we get a contradiction.

In fact in this case there exists a point $P \in \pi$ such that $Q(P) = 0$, $E(P) \neq 0$ since Q, E have no common factor. Moreover by step 5 and step 3 there is an element $\phi' = A'x +$ terms of higher degree in x, y I such that $A'(P) \neq 0$. By changing the coordinates z, w, t, we may assume $P = (0, 0, 0, 0, 1)$; let I_P be the ideal of X in $\theta_{\boldsymbol{P}^4,P} = \boldsymbol{C}[\bar{x}, \bar{y}, \bar{z}, \bar{w}]_{(\bar{x},\bar{y},\bar{z},\bar{w})}$. Let π' be the plane defined by $z = w = 0$; then $\pi' \cap X$ has support in P and degree 3, so we must have length $\theta_{\boldsymbol{P}^4,P}/I_P + (\bar{z}, \bar{w}) = 3$; but I_P contains elements $\varphi = q(\bar{z}, \bar{w})\bar{x} + e(\bar{z}, \bar{w})\bar{y}^2$ with $q(0, 0) = 0$, $e(0, 0) \neq 0$ and $\varphi' = a'(\bar{z}, \bar{w})\bar{x} +$ terms of higher degree in x, y, where $a'(0, 0) \neq 0$. It follows that $\theta_{\boldsymbol{P}^4,P}/I_P + (\bar{z}, \bar{w}) = \boldsymbol{C}[\bar{x}, \bar{y}]/(\bar{x}, \bar{y}^2)$ so it has length 2, absurd.

To finish the proof of the proposition we only need to note that X would be the C.I. of a hyperplane and a cubic hypersurface; hence $E = \theta_{\boldsymbol{P}^4}(3) \oplus \theta_{\boldsymbol{P}^4}(1)$, absurd since $c_1(E) = 0$.

COROLLARY 4. – *In $\boldsymbol{P}^4$ there are no irreducible, reduced canonical surfaces S of degree 12 with locally complete intersection singularities.*

PROOF. – S would be the zero locus of a section of a rank 2 vector bundle E' with Chern classes $c_1(E') = 6$ and $c_2(E') = 12$. If S is not contained in any quadric hypersurface, then by $0 \to \theta_{\boldsymbol{P}^4} \to E' \to I_S(6) \to 0$ ($I_S =$ ideal sheaf of S) it follows that E' would be semistable, hence $E = E'(-3)$ would be a semistable rank 2 vector bundle on $\boldsymbol{P}^4$ with Chern classes $c_1(E) = 0$ and $c_2(E) = 3$, absurd by the previous Proposition.

Suppose S is contained in a quadric: then for a general hyperplane H, $C = S \cap H$ is a reduced irreducible curve in $\boldsymbol{P}^3$ which is subcanonical, since it is the zero locus of a section of $E'|_H$; since C is contained in a quadric surface, it must be C.I. by 0.iii), hence $E'|_H \simeq \theta_H(2) \oplus \theta_H(q)$ and q is a positive integer such that $q+2 = c_1(E'|_H) = 6$ and $2q = c_2(E'|_H) = 12$, absurd.

REMARK 5. – By the discussion at the beginning of this section, it follows that the previous Corollary implies Th. 1.

The multiple points allowed for S in Corol. 4 are different from the ones allowed in [C]. Indeed in [C] general isolated singularities were allowed, hence S might be not locally C.I.

We shall use Prop. 3 also in section 3.

§ 2. – In this section we are going to study subvarieties of codimension 2 in $\boldsymbol{P}^n$, $n \geqslant 6$.

Our interest moves from the following conjecture, stated by R. HARTSHORNE in 1974 (see [H2])

CONJECTURE. – *Let X be a smooth subvariety of dimension r in $\boldsymbol{P}^n$. If $r > (2/3)n$, then X is a complete intersection.*

In particular, if codim $X = 2$, then the conjecture implies that X is C.I. if $n \geqslant 7$; in the same paper Hartshorne also posed the question about the existence of a non C.I., smooth subvariety of codimension 2 in $\boldsymbol{P}^6$.

The conjecture arose from a theorem of Barth and Larsen, which we shall use in the following form:

THEOREM (BARTH-LARSEN) (see [H2], th. 2.2). – *Let X be a nonsingular variety of dimension r in $\boldsymbol{P}^n$, then:*

a) the restriction map $H^i(\boldsymbol{P}^n, \boldsymbol{C}) \to H^i(X, \boldsymbol{C})$ is an isomorphism for $i \leqslant 2r - n$;

b) if $r \geqslant (n+2)/2$, then Pic $(X) = \boldsymbol{Z}$, *generateed by the class of an hyperplane section.*

Note that for complete intersections, *a*) and *b*) above are consequences of Lefschetz's theorem.

Part *b*) implies that if X has «small» codimension in $\boldsymbol{P}^n$, then it is subcanonical, since we must have $\omega_X = \theta_X(e)$ for some integer e; in particular this holds for every codimension 2 smooth subvariety of $\boldsymbol{P}^n$, $n \geqslant 6$.

Another consequence of Barth-Larsen's theorem that we need to point out is the following: if codim $X = 2$ we have $h^i(\theta_X) = 0$ for $0 < i \leqslant r-2$. Indeed in this case, by *a*), $h^i(X, \boldsymbol{C}) = \begin{pmatrix} 1 & i \text{ odd} \\ 0 & i \text{ even} \end{pmatrix}$; on the other hand, since $i > 0$, by Hodge decomposition we have $h^i(X, \boldsymbol{C}) = \sum_{p+q=i} h^{p,q}(X, \boldsymbol{C}) \geqslant h^{0,i}(X, \boldsymbol{C}) + h^{i,0}(X, \boldsymbol{C}) = 2h^i(\theta_X)$.

Hartshorne's paper also contains a wide survey on this subject up to 1974. After 1974 few progresses seems to be made in proving or confuting the conjecture; there is the following Zak's extension of a Severi's theorem:

THEOREM (Zak) (see [F-L]). – *If* $r \geqslant (2/3)n$, *then any smooth subvariety of* $\boldsymbol{P}^n$ *of dimension* r *is linearly normal.*

Further progresses were made by Z. RAN in a recent preprint (see [R]). We give an account of the results of Ran that we are going to use.

Let X be a smooth, non-degenerate codimension 2 subvariety of $\boldsymbol{P}^n$, which is subcanonical (this last condition holds automatically, by Barth-Larsen's theorem, if $n \geqslant 6$). Let E be the rank 2 vector bundle associated to X; put $c_1 = c_1(E)$, $c_2 = c_2(E) = \deg X$ and for every t define $f(t) = c_2(E(t - c_1)) = c_2 - tc_1 + t^2$.

Ran proves that for every $k \leqslant n-2$ and for a generic point $P \in \boldsymbol{P}^n - X$ the cone of $(k+1)$-secants to X passing through P has degree $f(0) \dots f(k)/k!$ ([R], p. 3). It follows that if $f(0) \dots f(k) \neq 0$ then X cannot be contained in any surface W of degree k, since any $(k+1)$-secant to W must be contained in W ([R], Th. 2).

Using this result, Ran proves that X is C.I. if either:

a) $c_1 \geqslant (c_2/m) + m$ for some $m \in (0, n-2]$;

b) $2\sqrt{c_2} \leqslant c_1 \leqslant 2n - 4$.

In particular, if there are integers a and b such that $c_1 = a + b$ and $c_2 = ab$ (i.e. E has the same Chern classes of $\mathcal{O}_{\boldsymbol{P}^n}(a) \oplus \mathcal{O}_{\boldsymbol{P}^n}(b)$) and one of them lies in $(0, n-2]$ (this holds automatically if $c_1 \leqslant 2n-4$, $a, b > 0$) then by *a*), X is C.I.

Note that if X is contained in a surface of degree $k \leqslant n-2$, then we must have $f(0) \dots f(k) = 0$, hence $0 = f(i) = c_2 - ic_1 + i^2$ for some $i \leqslant n-2$, hence $c_2 = i(c_1 - i)$, $c_1 = i + (c_1 - i)$ and $i \in (0, n-2]$ since $c_2 = \deg X > 0$; it follows that X is C.I.

Using the previous discussion and Riemann-Roch formula for vector bundles we are able to prove the following particular case of the conjecture:

THEOREM 6. – Let $X \subseteq \boldsymbol{P}^n$, $n \geqslant 6$ be a smooth subvariety of codimension 2; then by Barth-Larsen's theorem $\omega_X = \mathcal{O}_X(e)$ for some integer e. If $e \leqslant 0$ then X is complete intersection.

PROOF. – The isomorphism $\omega_X \simeq \mathcal{O}_X(e)$ induces on $\boldsymbol{P}^n$ an extension

$$0 \to \mathcal{O}_{\boldsymbol{P}^n} \to E \to I_X(e+n+1) \to 0 \tag{1}$$

where I_X is the ideal sheaf of X and E is the rank 2 vector bundle associated to X, with Chern classes $c_1(E) = e + n + 1$ and $c_2(E) = \deg X$. X is C.I. if and only if E splits into a sum of line bundles.

By [E-F], Corol. 1.7, E splits if and only if its restriction $E_{|H}$ to a general hyperplane H does. $E_{|H}$ is the vector bundle associated to the subscheme $X \cap H$ of $\boldsymbol{P}^{n-1}$; for H general, $X \cap H$ is still nonsingular and $\omega_{X \cap H} = \mathcal{O}_{X \cap H}(e+1)$ by adjonction formula; hence cutting with hyperplanes and making induction, we may reduce ourselves to prove the statement only for $e \leqslant 0$, $n = 6$ and $e = 0$, all $n \geqslant 6$.

Suppose $n = 6$, $e \leqslant -1$. Choose hyperplanes H_1 and H_2 such that $V = X \cap H_1$ and $S = V \cap H_2$ are smooth of dimension 3 and 2 respectively. Hence $\omega_V = \theta_V(e+1)$ and $\omega_S = \theta_S(e+2)$. We claim that S is C.I. This follows from Remark 2*a*) for $e \leqslant -3$ and from theorem 1 if $e = -1$; if $e = -2$ the claim follows from Remark 2*b*) since the irregularity $h^1(\theta_S)$ of S is 0; indeed by the exact sequence

$$0 \to \theta_V(-1) \to \theta_V \to \theta_S \to 0$$

we get

$$H^1(\theta_V) \to H^1(\theta_S) \to H^2(\theta_V(-1))$$

and by duality $h^1(\theta_V) = h^2(\theta_V(-1))$ which is 0 by Kodaira vanishing.

Thus we may assume $e = 0$, $n \geqslant 6$. We distinguish two cases.

CASE 1. - n odd, $n = 2k+1$.

Since $\dim X = n-2$ and, by Kodaira vanishing, $h^i(\theta_X(1)) = 0$ for $0 < i \leqslant n-2$, then $\chi(\theta_X(1)) = h^0(\theta_X(1))$. By Zak's theorem X is linearly normal hence, if it is *non-degenerate*, $h^0(\theta_X(1)) = n+1$. It follows from (1) twisted by $-n$ that

$$\chi(E(-n)) = \chi(\theta_{\boldsymbol{P}^n}(-n)) \tag{2}$$

for every vector bundle E on $\boldsymbol{P}^n$ with $c_1(E) = n+1$, which has a section whose zero locus is a smooth non-degenerate variety of codimension 2.

Riemann-Roch theorem assures us that $\chi(E(-n))$ can be expressed as a polynomial, with total degree $\leqslant n$, in the Chern classes $c_1 = c_1(E(-n)) = -n+1 = -2k$ and $c_2 = c_2(E(-n)) = d - n^2$ where $d = \deg X$.

Now fix n; $\chi(E(-n))$ becomes a polynomial T in c_2, which must satisfy (2), hence we see that only a finite number of values are allowed for $d = \deg X$, if X is non-degenerate, provided that T is non constant. We prove that T is not constant by computing its leading term.

A quick way to do it, following [H1], App. A, § 3, is to carry the computation on a sum of line bundles: put $F = \theta_{\boldsymbol{P}^n}(a) \oplus \theta_{\boldsymbol{P}^n}(b)$, then $c_1(F) = a+b$, $c_2(F) = ab$ and

$$\chi(F) = \binom{a+n}{n} + \binom{b+n}{n};$$

the left hand side can be uniquely expressed as a polynomial in $c_1(F)$ and $c_2(F)$; this polynomial gives the Riemann-Roch formula for a generic rank 2 vector bundle.

We need a technical

LEMMA. – *Let a, b, k be integers, $k \geq 0$ and put $a + b = c_1$, $ab = c_2$. Then*

i) $a^{2k} + b^{2k} = (-1)^k 2c_2^k +$ *terms of lower degree in* c_2;

ii) $a^{2k+1} + b^{2k+1} = (-1)^k(2k+1)c_1 c_2^k +$ *terms of lower degree in* c_2.

PROOF. – The proof is done by induction on k. Let us write « $\equiv$ » to mean « equal up to terms of lower degree in c_2 ». Both formulas are obvious for $k = 0$, so let us suppose $k > 0$.

i) $a^{2k} + b^{2k} = (a^k + b^k)^2 - 2c_2^k$; if k is even, by induction $a^k + b^k \equiv (-1)^{k/2} 2c_2^{k/2}$ so that $a^{2k} + b^{2k} \equiv 2c_2^k$; if k is odd, by induction again, $(a^k + b^k)$ cannot contain terms of degree k in c_2, so i) is proved.

ii) $a^{2k+1} + b^{2k+1} = (a^{2k} + b^{2k})(a + b) - ab(a^{2k-1} + b^{2k-1})$; by i) we have $(a^{2k} + b^{2k})(a + b) \equiv 2c_1 c_2^k(-1)^k$ and by induction $ab(a^{2k-1} + b^{2k-1}) \equiv c_2^k(-1)^{k-1}(2k-1)c_1$; adding we find ii).

Now we return to the proof of the theorem.

We have:

$$(3) \qquad \chi(F) = \binom{a+n}{n} + \binom{b+n}{n} =$$
$$= \big((a+n)(a+n-1)\dots(a+1) + (b+n)(b+n-1)\dots(b+1)\big)/n! =$$
$$= \big((a^n + b^n) + n(n+1)(a^{n-1} + b^{n-1})/2 + (\text{some coefficient})\,(a^{n-2} + b^{n-2}) + \dots\big)/n!$$

hence, by the Lemma, replacing $n = 2k + 1$, $ab = c_2$, $a + b = c_1 = -2k$, we have:

$$\chi\big(E(-n)\big) = \frac{1}{(2k+1)!}\big((-1)^k(2k+1)(-2k)c_2^k + (-1)^k 2c_2^k(2k+1)(2k+2)/2 + \\ + \text{terms of lower degree in } c_2\big).$$

Thus the leading coefficient of T is $\big(2(-1)^k(2k+1)\big)/(2k+1)!$ and T has degree k in c_2.

It follows that, for fixed n, equation (2) has at most k roots in c_2. We know yet some of these roots: they are the numbers $d - n^2$ where d is the degree of a non-degenerate C.I. of two hypersurfaces whose degrees have sum $n + 1 = 2k + 2$. This gives exactly k distinct values for d, namely $d_1 = 2(2k)$, $d_2 = 3(2k-1)$, ..., $d_k = (k+1)^2$, hence the corresponding values for $d - n^2$ exaust all the roots of (2).

It follows that if X is non-degenerate, its degree must be one of the d_i's, hence its associated vector bundle has the same Chern classes of a sum of line bundles of positive degree. Since by assumptions $c_1(E) = n + 1 \leq 2n - 4$, it follows by Ran's theorems that X is C.I.

If X is degenerate, then it is obviously C.I., so the case « n odd » is proved.

CASE 2. – n even, $n = 2k$.

In this case we construct a formula similar to (2) which is valid also for degenerate X. We look at $\chi(\theta_X)$; by Barth-Larsen's theorem if $\dim X = r$ then $h^i(\theta_X) = 0$, $0 < i \leqslant r - 2$, moreover since $r > 3$, this implies also $h^{r-1}(\theta_X) = 0$ by duality; finally $h^r(\theta_X) = h^0(\theta_X) = 1$ hence we have $\chi(\theta_X) = 2$.

From (1) twisted by $-n-1$, it follows:

$$\chi\big(E(-n-1)\big) = 1 + \chi\big(\theta_{P^n}(-n-1)\big) \tag{4}$$

and this holds for every rank 2 vector bundle E with $c_1(E) = n + 1$, which has a section whose zero locus is a smooth subvariety of codimension 2.

But again $\chi\big(E(-1-n)\big)$ can be expressed as a polynomial in $c_1 = c_1\big(E(-1-n)\big) = -n-1 = -2k-1$ and $c_2 = c_2\big(E(-n-1)\big) = \deg X = d$, hence fixing n, $\chi\big(E(-n-1)\big)$ becomes a polynomial T' in c_2 which must satisfy (4).

For the same computations as before, replacing in (3) $n = 2k$, $ab = c_2$, $a + b = c_1 = -2k-1$, by the Lemma

$$\chi\big(E(-n-1)\big) = \frac{1}{(2k)!}\big((-1)^k 2c_2^k\big) + \text{terms of lower degree in } c_2$$

hence T' has leading term $(-1)^k 2/(2k)!$ and degree k. It follows that for fixed n, equation (4) has k roots in $c_2 = d$. But we know yet these roots: they are the degrees of complete intersections of two hypersurfaces whose degrees have sum $n+1$, namely they are the (distinct) numbers $\delta_1 = 1(2k)$, $\delta_2 = 2(2k-1), \dots, \delta_k = k(k+1)$. Once again it follows that the degree of X must be one of the δ_i's, hence E has the same Chern classes of a sum of line bundles of positive degree. Since $c_1(E) \leqslant 2n-4$, by Ran's theorems this implies that X is C.I.

This completes the proof of the theorem.

REMARK 7. – We cannot use equation (4) to prove the case «n odd» because it becomes an identity: indeed it has degree $\leqslant k$ in $c_2 = d$ while complete intersections give $k+1$ distinct values for d, namely $1(2k+2), \dots, (k+1)(k+1)$.

We cannot use equation (2) to prove the case «n even» because it has degree k in $c_2 = d - n^2$ and non degenerate complete intersections give only $k-1$ values for $d - n^2$, so they do not exaust all the roots of (2) but possibly there is a missing value.

§ 3. – Let us examinate more closely the case of smooth subcanonical threefolds X in $\boldsymbol{P}^5$.

REMARK 8. – Put $\omega_X = \theta_X(e)$ and let S be a general smooth hyperplane section of X; then $\omega_S = \theta_S(e+1)$, moreover the irregularity of S is 0, indeed $h^1(\theta_X) = 0$

by Barth-Larsen's theorem and we have the exact sequence $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_S) \to \to H^2(\mathcal{O}_X(-1))$ and, by Kodaira vanishing, $h^2(\mathcal{O}_X(-1)) = 0$.

If $e \leqslant 0$ it follows from Th. 1 and Remark 2 that S is C.I., hence also X is C.I.

PROPOSITION 9. – *If $e = 1$, then X is a complete intersection in $\boldsymbol{P}^5$.*

PROOF. – With the previous terminology, put $d = \deg X = \deg S$; S is a smooth surface in $\boldsymbol{P}^4$ and $\omega_S = \mathcal{O}_S(2)$ hence the formula (1) of § 1 ([H1], p. 434) gives strong restrictions on the possible values for d: indeed S is a surface of general type hence we must have $\chi(\mathcal{O}_S) > 0$ ([Be], Th. X.4) hence the only possible values for d are 4, 6, 10, 12, 16, 18, 22, 24.

There are no smooth surfaces in $\boldsymbol{P}^4$ with $\omega_S = \mathcal{O}_S(2)$ and $d = 4$, while every such surface of degree 6, 10, 12 is C.I. In fact a general hyperplane section C of S is a smooth, connected subcanonical curve with $\omega_C = \mathcal{O}_C(3)$, hence with genus $g = (3/2)d + 1$; for $d = 4$ no such curve exists; for $d = 6$ C is a plane curve and for $d = 10$ C is contained in a quadric, by CASTELNUOVO's bound; in both cases C must be C.I. (0.iii); for $d = 12$, by RIEMANN-ROCH, C is contained in a cubic and, if it does not lie on a quadric, it must lie on a irreducible quartic, thus C and S are C.I.

S cannot have degree 18 or 22, indeed by formula (1) of § 1, we obtain respectively $\chi(\mathcal{O}_S) = 15$ and $\chi(\mathcal{O}_S) = 18$; in both cases, since $q(S) = 0$, the geometric genus $p_g(S)$ is less than $15 = h^0(\mathcal{O}_{\boldsymbol{P}^4}(2))$, thus S is contained in a quadric, hence C must be the complete intersection in $\boldsymbol{P}^3$ of a quadric and another surface; this is impossible for reasons of degree, since $\omega_C = \mathcal{O}_C(3)$.

S cannot have degree 24 since we should have, by § 1, (1), $\chi(\mathcal{O}_S) = 8$ and this contradicts the celebrated Yau's inequality $K_S \cdot K_S \leqslant 9\chi(\mathcal{O}_S)$ for a surface of general type ($K_S =$ canonical divisor of S).

In the case $d = 16$, we have $\chi(\mathcal{O}_S) = 16$ and, by RIEMANN-ROCH, $\chi(\mathcal{O}_S(1)) = 8$ and $p_g(S) = h^0(\mathcal{O}_S(2)) = 15$ since $q(S) = 0$. From the exact sequence $0 \to \mathcal{O}_X \to \to \mathcal{O}_X(1) \to \mathcal{O}_S(1) \to 0$ we obtain: $\chi(\mathcal{O}_S(1)) = \chi(\mathcal{O}_X(1)) - \chi(\mathcal{O}_X)$ which is $2 \cdot \chi(\mathcal{O}_X(1))$ by duality, i.e. $\chi(\mathcal{O}_X(1)) = 4$; by Zak's theorem $h^0(\mathcal{O}_X(1)) = 6$; by duality $h^3(\mathcal{O}_X(1)) = 1$ and $h^2(\mathcal{O}_X(1)) = h^1(\mathcal{O}_X)$ which is 0 by Barth-Larsen's theorem; furthemore $h^1(\mathcal{O}_X(2)) = 0$ by Kodaira vanishing; for reasons of degree, X cannot be C.I., hence it cannot lie on a quadric, thus $h^0(\mathcal{O}_X(2)) \geqslant 21$. Putting all these numbers together, the exact sequence:

$$0 \to H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2)) \to H^0(\mathcal{O}_S(2)) \to H^1(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_X(2)) \to 0$$

gives the contradiction.

PROPOSITION 10. – *If $\omega_X = \mathcal{O}_X(2)$ then X is a complete intersection in $\boldsymbol{P}^5$.*

PROOF. – Put $d = \deg X$. By Ran's theorems (§ 2) if X is contained in any cubic hypersurface, it is C.I. Thus assume X not contained in any cubic. The vector bundle E associated to X has Chern classes $c_1(E) = 8$ and $c_2(E) = d$; by the sequence $0 \to \theta_{P^5} \to E \to I_X(8) \to 0$ (I_X = ideal sheaf of X), since X is not contained in a cubic, it follows that $h^0(E(-5)) = 0$, i.e. E is semistable.

Suppose $d \leqslant 16$; then by [B], § 3, E is not stable; it follows that $E(-4)$ has a global section whose zero locus Y, by the semistability of E, is empty or has codimension 2. Since $\deg Y = c_2(E(-4)) = d - 16$, this implies that $d = 16$ and $Y = \emptyset$, so that, by [H3], Th. 1.0.1, E is a sum of line bundles, i.e. X is C.I.

By Riemann-Roch formula on X (see [Mu], p. 40), if $D \subseteq X$ is a divisor $\chi(\theta_X(D)) = D^3/6 - (K_X D^2)/4 + (K_X^2 + c_2(TX))/12 - K_X(c_2(TX))/24$ (TX = tangent bundle, K_X = canonical divisor). By a straightforward computation, if $\omega_X = \theta_X(e)$ then $c_2(TX) = h^2(15 + e(6 + e) - d)$ where h is the class of an hyperplane section in the Chow ring of X; thus in our case $\chi(\theta_X(3)) = d(37 - d)/6$. By Kodaira vanishing, $\chi(\theta_X(3)) = h^0(\theta_X(3))$; if $d \geqslant 24$ this is less than $56 = h^0(\theta_{P^5}(3))$, hence X belongs to a cubic, absurd for reasons of degree.

It remains to examinate what happens if the degree varies in the range $16 < d < 24$. Reducing ourselves to a general hyperplane section and using formula (1) of § 1 as before, we see that the only possible value for d is 19. In this case, for a general hyperplane H_1 $E(-4)|_H$ would be semistable by [M], Th. 3.1 and would have Chern classes $c_1 = 0$ and $c_2 = 3$; this is impossible for a rank 2 vector bundle on P^4, by Prop. 3.

REMARK 11. - Let X be a smooth codimension 2 subvariety of P^6, with $\omega_X = \theta_X(e)$. The previous discussion shows that X is C.I. if $e \leqslant 1$. The same technique allows us to state the same result for $e = 2$.

Namely, formula (1) of § 1 for a generic intersection of X with a linear 4-space, implies that a priori $d = \deg X$ can only have the values $d \leqslant 20$ or $d \geqslant 24$. In the first case X is contained in a quartic hypersurface, hence, by Ran's theorems, it is C.I. In the second case, if V is a general smooth hyperplane section of X, by RIEMANN-ROCH, $\chi(\theta_V) = 1 + d(42 - d)/8$ so that, since by Kodaira vanishing and by Barth-Larsen's theorem $\chi(\theta_V) = h^0(\theta_V) - h^3(\theta_V) = 1 - h^0(\theta_V(3))$, so we have $h^0(\theta_V(3)) \leqslant 56 = h^0(\theta_{P^5}(3))$; it follows that V is contained in a cubic hypersurface of P^5, hence by Ran's theorems V and X are C.I.

Other cases can be handled in this way, but they seem to be too particular to be interesting.

We note that Ran's results *a*), *b*) and our theorem 6 give at once the following interesting result.

THEOREM 12. - *Let X be a non-singular codimension 2 subvariety of P^n, $n \geqslant 6$; put $c_2 := d := \deg X$. If $d \leqslant (n+2)^2/4$, then X is a complete intersection.*

Indeed look at Ran's inequalities *a*), *b*). Put $\omega_X = \theta_X(e)$. If $c_1 \leqslant n + 1$, then X is a complete intersection by theorem 6. If $n + 2 \leqslant c_1 \leqslant 2n - 4$, by Ran's inequality *b*), we have $c_2 \geqslant c_1^2/4 \geqslant (n+2)^2/4$. If $c_1 \geqslant 2n - 3$, by Ran's inequality *b*), we have $c_2 > (n-2)(c_1 - n + 2) \geqslant (n-2)(n-1)$.

Note that theorem 12 gives the first known quadratic bound for this problem. BARTH and VAN DE VEN proved the existence of a bound and then in [B-V] gave a linear bound.

REMARK 13. – We wish to point out the following extension, due to GRIFFITH and EVANS, of a classical theorem of GHERARDELLI for subcanonical curves (see [G]):

if X is a smooth subcanonical variety of codimension 2 in $\boldsymbol{P}^n$, $n \geqslant 3$, then X is C.I. if and only if $\forall m$ the map

$$H^0(\theta_{\boldsymbol{P}^n}(m)) \to H^0(\theta_X(m))$$

is surjective (see [E-G], Th. 2.4).

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