Asymptotic Nonuniform Nonresonance Conditions in the Periodic-Dirichlet Problem for Semi-Linear Wave Equations (*).

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Summary. – *We consider the existence of weak solutions of the periodic-Dirichlet problem on* $]0, 2\pi[\times]0, \pi[$ *for the semi-linear wave equation*

$$u_{tt} - u_{xx} - f(t, x, u) = 0$$

when

$$\alpha(t, x) \leqslant \liminf_{|u|\to\infty} u^{-1} f(t, x, u) \leqslant \limsup_{|u|\to\infty} u^{-1} f(t, x, u) \leqslant \beta(t, x)$$

and α *and* β *satisfy some nonresonance conditions of non uniform type withr espect to two consecutive nonzero eigenvalues of the associated linear problem. The proof is based upon one generalized continuation theorem for some perturbations of mappings which are not of Fredholm type.*

0. – Introduction.

Let $J =]0, 2\pi[\times]0, \pi[$ and let $f\colon J\times \boldsymbol{R} \to \boldsymbol{R}$ be a function such that $f(\cdot, \cdot, u)$ is measurable on J for each $u \in \boldsymbol{R}$, $f(t, x, \cdot)$ is continuous on $\boldsymbol{R}$ for a.e. $(t, x) \in J$. Assume moreover that, for each $r > 0$, there exists $h_r \in L^2(J)$ such that

$$|f(t, x, u)| \leqslant h_r(t, x) \tag{0.1}$$

when $(t, x) \in J$ and $|u| \leqslant r$, with $L^2(J)$ the space of measurable Lebesgue square integrable real functions on J. We shall then say that f satisfies the *Caratheodory conditions for* $L^2(J)$.

Consider the semi-linear wave equation

$$u_{tt} - u_{xx} - f(t, x, u) = 0\ . \tag{0.2}$$

A *weak solution of the periodic-Dirichlet problem on* J *for* (0.2) will be a $u \in L^2(J)$ such that

$$\int_J u(t, x)[v_{tt}(t, x) - v_{xx}(t, x)]\, dt\, dx = \int_J f\big(t, x, u(t, x)\big)\, v(t, x)\, dt\, dx$$

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for every $v \in C^2(\bar{J})$ satisfying the boundary conditions

$$v(t, 0) = v(t, \pi) = 0 \qquad (t \in [0, 2\pi]) ,$$
$$v(0, x) - v(2\pi, x) = v_t(0, x) - v_t(2\pi, x) = 0 \qquad (x \in [0, \pi]) .$$

In particular, the periodic-Dirichlet problem on J for the nonhomogeneous linear equation

$$u_{tt} - u_{xx} - \lambda u = h(t, x)$$

is uniquely solvable for every $h \in L^2(J)$ if and only if

(0.4) $$\lambda \notin \{n^2 - m^2 \colon m \in Z, n \in N^*\} = \{\dots \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots\}$$

(see e.g. [1] or [2]). In [3], it has been proved that the periodic-Dirichlet problem on J for (0.2) has ar least a weak solution if there exists real numbers p, q, r such that

$$\lambda_N < p \leqslant u^{-1} f(t, x, u) \leqslant q < \lambda_{N+1}$$

for some $N \in Z$, a.e. $(t, x) \in J$ and all $u \in \boldsymbol{R}$ with $|u| \geqslant r$, and if moreover the function sign$p f(t, x, \cdot)$ is nondecreasing for a.e. $(t, x) \in J$. In this paper, we generalize this result for $N \notin \{-1, 0\}$ by proving the following

THEOREM 1. – *Assume that the inequalities*

(0.5) $$\alpha(t, x) \leqslant \liminf_{|u| \to \infty} u^{-1} f(t, x, u) \leqslant \limsup_{|u| \to \infty} u^{-1} f(t, x, u) \leqslant \beta(t, x)$$

hold uniformly a.e. in $(t, x) \in J$ *where* α *and* β *are functions in* $L^\infty(J)$ *such that for some* $N \neq 0$ *and* -1 *one has*

(0.6) $$\lambda_N \leqslant \alpha(t, x) \leqslant \beta(t, x) \leqslant \lambda_{N+1}$$

a.e. with

$$\lambda_N < \alpha(t, x)$$

on a set of positive measure and

$$\beta(t, x) < \lambda_{N+1}$$

on a set of positive measure.

Then if the function $\operatorname{sign} \lambda_N \cdot f(t, x, \cdot)$ *is nondecreasing for a.e.* $(t, x) \in J$, *the periodic-Dirichlet problem on* J *for equation* (0.2) *has at least one weak solution.*

Conditions of the form (0.5) and (0.8) were first introduced in elliptic Dirichlet problems by the authors in [6] and then used in [7] for periodic solutions of ordinary differential equations. The approach used in those papers has to be substantially modified here because the abstract realization in $L^2(J)$ of the wave operator $\partial^2/\partial t^2 - \partial^2/\partial x^2$ with the periodic-Dirichlet boundary conditions on J has a noncompact resolvant. This is due to the fact that the spectrum $\sigma(L)$ of L, which is in fact given by the right hand member in (0.4), contains an eigenvalue of infinite multiplicity (namely $\lambda_0 = 0$), the other ones having however finite multiplicity. It is also the infinite multiplicity of λ_0 which implies the exclusion of the couples (λ_1, λ_0) and (λ_0, λ_1) in the extension of the uniform nonresonance conditions of [3] to the non-uniform situation (with respect to (t, x)) given by the theorem above. Of course, our tech niques would give results for $N = 0$ by assuming α constant and for $N = -1$ by assuming β constant, which still is better than the result of [3]. The reader can easily check the details.

The proof of the theorem is based on two lemmas (for linear problems) which are given in Section 1 and whose assertions are reminiscent of the preliminary lemmas of [6] and [7], but whose proofs are different for the reason explained above. Also the lack of compactness prevents the use of a usual Leray-Schauder's degree argument in the proof of the theorem and we have to make use of a generalized continuation theorem of one of the authors [4] (see also [5]) based on combination of compactness and monotonicity methods. This is where the monotonicity assumption on $f(t, x, \cdot)$ is effectively used. We state here, for the reader's convenience, the special case of the theorem 2 and Remark 1 of [4] which will be used in this paper. This requires some preliminary definitions.

Let H be a real Hilbert space, with inner product $(\cdot, \cdot)$ and corresponding norm $|\cdot|$. Let us denote by Λ the class of operators $L: D(L) \subset H \to H$ which are linear, closed, have domain $D(L)$ dense in H and are such that their kernel $N(L)$ and range $R(L)$ satisfy the condition

$$R(L) = (N(L))^{\perp} .$$

An example of such a L is a self-adjoint operator with closed range. Denote by K the right inverse of L defined by

$$K = [L|_{D(L) \cap R(L)}]^{-1}: R(L) \to R(L) ,$$

and by Q the orthogonal projector onto $R(L)$.

Recall that if $F: H \to H$ is a (possibly) nonlinear operator, then F is said to be *monotone* (resp. *strongly monotone*) on H if, for all u, v in H one has

$$(Fu - Fv,\ u - v) \geqslant 0$$

$$\left(\text{resp. } (Fu - Fv,\ u - v) \geqslant c|u - v|^2,\ c > 0\right),$$

and *demi continuous* on H if

$$u_k \to u \Rightarrow Fu_k \rightharpoonup Fu ,$$

where $\rightharpoonup$ denotes the weak convergence in H.

We can now state the special case of Theorem 2 of [4] used here and refer to it as the continuation lemma.

CONTINUATION LEMMA. – *Let $L \in \mathcal{A}$ (with right inverse K) and let $F: H \to H$ be a monotone demi continuous operator.*

Assume that there exists a linear, strongly monotone operator $A: H \to H$ and a number $\varrho > 0$ such that the following conditions are satisfied:

a) KQF and KQA are compact on the closed ball $\bar{B}(\varrho)$ of center 0 and radius ϱ in H.

b) $F(\bar{B}(\varrho))$ is bounded.

c) $(\forall \lambda \in [0, 1[)$ $(\forall u \in D(L) \cap \partial B(\varrho))$:

$$Lu - (1 - \lambda) Au - \lambda Fu \neq 0 .$$

Then the equation

$$Lu - Fu = 0$$

has at least one solution $u \in D(L) \cap \bar{B}(\varrho)$.

1. – Preliminary lemmas on linear problems.

Let $H = L^2(J)$ with the usual inner product $(\cdot, \cdot)$ and corresponding norm $|\cdot|$. Let

$$v_{mn}(t, x) = \pi^{-1} \exp(imt) \sin(nx)$$

for $m \in Z$ and $n \in N^*$. Each $u \in H$ can be written as the Fourier series

$$u = \sum_{\substack{m\in Z \\ n \in N^*}} u_{mn} v_{mn}$$

where $u_{mn} = (u, v_{mn})$. Note that, since u is real, $\bar{u}_{m,n} = u_{-m,n}$.

We define the abstract realization L in H of the above operator with the periodic-Dirichlet conditions on J as follows. Let

$$D(L) = \left\{u \in H: \sum_{\substack{m\in Z\\ n\in N^*}} (n^2 - m^2)^2 |u_{mn}|^2 < \infty\right\},$$

$$L: D(L) \to H, \qquad u \mapsto Lu = \sum_{\substack{m\in Z\\ n\in N^*}} (n^2 - m^2)\, u_{mn} v_{mn}.$$

L is a self-adjoint operator in H with pure point spectrum consisting of its eigenvalues

$$\sigma(L) = \{n^2 - m^2:\ m \in Z,\ n \in N^*\}.$$

One can see that 0 is an eigenvalue of infinite multiplicity and that the others have finite multiplicity. Moreover, one can show that if $h \in H$, and $u \in D(L)$, then

$$Lu = h$$

if and only if u is a weak solution of the periodic-Dirichlet problem on J for the equation

$$u_{tt} - u_{xx} = h.$$

We refer to [1] or [2] for the corresponding details.

Let us number the eigenvalues of L consecutively, counting from $\lambda_0 = 0$, so that the eigenvalues are

$$\dots < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots,$$

and let λ_N, λ_{N+1} be a pair of consecutive eigenvalues of L, neither of which is zero. Thus, either

$$0 < \lambda_N < \lambda_{N+1}$$

or

$$\lambda_N < \lambda_{N+1} < 0.$$

Let $c = (\frac{1}{2})(\lambda_N + \lambda_{N+1})$ and let $\{E_\lambda: \lambda \in \boldsymbol{R}\}$ be the spectral resolution of L, so that $L = \int_{\boldsymbol{R}} \lambda\, dE_\lambda$. Let P_1, P_2 be the orthogonal projections given by

$$P_1 = \int_{-\infty}^{c} dE_\lambda, \qquad P_2 = \int_{c}^{\infty} dE_\lambda,$$

and let $H_1 = P_1(H)$, $H_2 = P_2(H)$. Then H_1 is spanned by the eigenfunctions of L associated with the eigenvalues λ_i for $i \leqslant N$ and H_2 is spanned by the eigenfunctions associated with λ_i for $i \geqslant N+1$.

Moreover

$$P_1 u = \sum_{n^2 - m^2 \leqslant N} u_{mn} v_{mn}\,, \qquad P_2 u = \sum_{n^2 - m^2 \geqslant N+1} u_{mn} v_{mn}\,.$$

Let us prove the first preliminary result.

LEMMA 1. – *Let* α *(resp.* β*) be functions in* $L^\infty(J)$ *such that*

$$\lambda_N \leqslant \alpha(t, x)$$
$$\big(\textit{resp. } \beta(t, x) \leqslant \lambda_{N+1}\big)$$

a.e. with

$$\lambda_N < \alpha(t, x)$$
$$\big(\textit{resp. } \beta(t, x) < \lambda_{N+1}\big)$$

on a set of positive measure. Then there is a positive number $\delta_1 > 0$ *(resp.* $\delta_2 > 0$*) such that for any* $p \in L^\infty(J)$ *satisfying*

$$\alpha(t, x) \leqslant p(t, x)$$
$$\big(\textit{resp. } p(t, x) \leqslant \beta(t, x)\big)$$

and all $u_1 \in D(L) \cap H_1$ *(resp.* $u_2 \in D(L) \cap H_2$*), one has*

$$(Lu_1 - pu_1, u_1) \leqslant -\delta_1 |u_1|^2 \tag{1.1}$$

$$\big(\textit{resp. } (Lu_2 - pu_2, u_2) \geqslant \delta_2 |u_2|^2\big)\,. \tag{1.2}$$

PROOF. – The arguments for (1.1) and (1.2) are virtually identical, so we will present the details only for (1.1). First of all, because $p(t, x) \geqslant \lambda_N$ for *a.e.* $(t, x) \in J$, it is easy to show that $(Lu_1 - pu_1, u_1) \leqslant 0$. Moreover, $\alpha(t, x) \leqslant p(t, x)$ *a.e.* on J and hence

$$(Lu_1 - pu_1, u_1) \leqslant (Lu_1 - \alpha u_1, u_1)\,. \tag{1.3}$$

Thus if there is a $\delta_1 > 0$ such that

$$(Lu_1 - \alpha u_1, u_1) \leqslant -\delta_1 |u_1|^2 \tag{1.4}$$

for all $u_1 \in D(L) \cap H_1$, this will also establish (1.1). Suppose there is no such $\delta_1 > 0$. Then there is a sequence $\{u_{1k}\}$, which we will denote simply by $\{u_k\}$, in $D(L) \cap H_1$ with $|u_k| = 1$ and

$$-k^{-1} \leqslant (Lu_k - \alpha u_k, u_k), \qquad k = 1, 2, \dots . \tag{1.5}$$

Now let $\lambda_{N-1} < a < \lambda_N < b < \lambda_{N+1}$ and define projections $\tilde{P}$, $\overline{P}$ by

$$\tilde{P} = \int_{-\infty}^{a} dE_\lambda \quad \text{and} \quad \overline{P} = \int_{a}^{b} dE_\lambda ,$$

with $\tilde{H} = \tilde{P}(H)$ and $\overline{H} = \overline{P}(H)$, so that $H_1 = \tilde{H} \oplus \overline{H}$, $\tilde{H}$ is spanned by the eigenfunctions associated with the λ_i for $i \leqslant N-1$ and $\overline{H}$ is the finite-dimensional space spanned by the eigenfunctions associated with λ_N. We will write $u_k = \tilde{u}_k + \overline{u}_k$ where $\tilde{u} = \tilde{P}u$, $\overline{u} = \overline{P}u$. Now, since $\alpha(t, x) \geqslant \lambda_N$ for a.e. $(t, x) \in J$, we have from (1.5) that

$$-k^{-1} \leqslant (Lu_k - \lambda_N u_k, u_k)$$

which reduces to

$$-k^{-1} \leqslant (L\tilde{u}_k - \lambda_N \tilde{u}_k, \tilde{u}_k) = \sum_{n^2 - m^2 \leqslant \lambda_{N-1}} (n^2 - m^2)|u_{knm}|^2 - \lambda_N |\tilde{u}_k|^2 \leqslant (\lambda_{N-1} - \lambda_N)|\tilde{u}_k|^2 ,$$

so that $|\tilde{u}_k| \to 0$ as $k \to \infty$:

Now $\overline{H}$ is finite-dimensional and since $1 = |u_k|^2 = |\tilde{u}_k|^2 + |\overline{u}_k|^2$, we have that a subsequence of $\{u_k\}$, which we may relabel as $\{u_k\}$, converges strongly to some $\overline{u} \in \overline{H}$, with $|\overline{u}| = 1$. Consequently, we must have $\overline{u}(t, x) \neq$ a.e. on J and

$$\begin{aligned} -k^{-1} \leqslant (Lu_k - \alpha u_k, u_k) &= (L\tilde{u}_k - \alpha\tilde{u}_k, \tilde{u}_k) - 2\int_J \alpha(t, x)\,\tilde{u}_k(t, x)\,\overline{u}_k(t, x)\,dt\,dx \\ &+ (L\overline{u}_k - \alpha\overline{u}_k, \overline{u}_k) \leqslant -(\lambda_N - \lambda_{N-1})|\tilde{u}_k|^2 - 2\int_J \alpha(t, x)\,\tilde{u}_k(t, x)\,\overline{u}_k(t, x)\,dx \\ &+ \int_J \big(\lambda_N - \alpha(t, x)\big)|\overline{u}_k(t, x)|^2\,dt\,dx . \end{aligned}$$

Using $\tilde{u}_k \to 0$ and $\overline{u}_k \to \overline{u}$ as $k \to \infty$, we obtain

$$0 \leqslant \int_J \big(\lambda_N - \alpha(t, x)\big)|\overline{u}(t, x)|^2\,dt\,dx ,$$

and since $\lambda_N \leqslant \alpha(t, x)$ for a.e. $(t, x) \in J$, we have

$$\int_J \big(\lambda_N - \alpha(t, x)\big)|\overline{u}(t, x)|^2\,dt\,dx = 0 . \tag{1.6}$$

However, (1.6) contradicts $\bar{u}(t, x) \neq 0$ a.e. on J since by hypothesis $\lambda_N - \alpha(t, x) < 0$ on some subset of J of positive measure. This contradiction proves (1.4) and hence (1.1) and the proof of lemma 1 is complete.

LEMMA 2. – *Let α and β be functions in $L^\infty(J)$ such that*

$$\lambda_N \leqslant \alpha(t, x) \leqslant \beta(t, x) \leqslant \lambda_{N+1}$$

a.e. with

$$\lambda_N < \alpha(t, x)$$

on a set of positive measure and

$$\beta(t, x) < \lambda_{N+1}$$

on a set of positive measure. Then there are numbers $\delta > 0$ and $\varepsilon > 0$ such that for any $p \in L^\infty(J)$ satisfying

$$\alpha(t, x) - \varepsilon \leqslant p(t, x) \leqslant \beta(t, x) + \varepsilon$$

a.e. on J, one has

$$|Lu - pu| \geqslant \delta |u|$$

for all $u \in D(L)$.

PROOF. – Suppose the conclusion of the lemma is false. Then there exists a sequence $\{u_k\}$ in $D(L)$ with $|u_k| = 1$ and a sequence $\{p_k\}$ in $L^\infty(J)$ with

$$\alpha(t, x) - \frac{1}{k} \leqslant p_k(t, x) \leqslant \beta(t, x) + \frac{1}{k} \tag{1.7}$$

a.e. for each $k = 1, 2, \ldots$, and

$$|Lu_k - p_k u_k| \leqslant k^{-1},$$

$k = 1, 2, \ldots$. That is

$$Lu_k - p_k u_k = f_k \tag{1.8}$$

with $|f_k| \leqslant k^{-1}$ and $|u_k| = 1$ $(k = 1, 2, \ldots)$.

Writing $u_k = u_{1k} + u_{2k}$ with $u_{1k} = P_1 u_k$, $u_{2k} = P_2 u_k$, we have $u_{1k} \in D(L) \cap H_1$ and $u_{2k} \in D(L) \cap H_2$ for $k = 1, 2, \ldots$, and, taking inner products with (1.8), we have

$$(Lu_k - p_k u_k, u_{2k} - u_{1k}) = (f_k, u_{2k} - u_{1k}),$$

which reduces upon expansion to

$$(1.9) \qquad (Lu_{2k} - p_k u_{2k}, u_{2k}) - (Lu_{1k} - p_k u_{1k}, u_{1k}) = (f_k, u_{2k} - u_{1k}) .$$

Now, by (1.7),

$$\alpha(t, x) \leqslant p_k(t, x) + \frac{1}{k}$$

and

$$p_k(t, x) - \frac{1}{k} \leqslant \beta(t, x)$$

a.e. for each $k = 1, 2, \ldots$. So, using Lemma 1 we obtain the existence of $\delta_1 > 0$ and $\delta_2 > 0$ such that, for all $k = 1, 2, \ldots$, one has

$$\left(Lu_{1k} - \left(p_k + \frac{1}{k}\right) u_{1k}, u_{1k}\right) \leqslant - \delta_1 |u_{1k}|^2$$

and

$$\left(Lu_{2k} - \left(p_k - \frac{1}{k}\right) u_{2k}, u_{2k}\right) \geqslant \delta_2 |u_{2k}|^2 .$$

Combining with (1.9) and using Schwarz inequality, this gives

$$\delta_2 |u_{2k}|^2 - \frac{1}{k} |u_{2k}|^2 + \delta_1 |u_{1k}|^2 - \frac{1}{k} |u_{1k}|^2 \leqslant |f_k| \, |u_{2k} - u_{1k}| ,$$

and hence

$$\delta_2 |u_{2k}|^2 + \delta_1 |u_{1k}|^2 \leqslant \frac{4}{k}$$

which implies that $u_k = u_{1k} + u_{2k}$ converges strongly to zero. This contradicts $|u_k| = 1$ and thus proves the lemma

2. – The proof of theorem 1.

We now return to the periodic-Dirichlet problem on J for the semi-linear wave equation (0.2) and proceed to the proof of Theorem 1 stated in the introduction.

Let $\delta > 0$ and $\varepsilon > 0$ be given by Lemma 2. By (0.5) we can find $r > 0$ such that, for a.e. (t, x) in J and all u with $|u| \geqslant r$, we have

$$\alpha(t, x) - \varepsilon \leqslant u^{-1} f(t, x, u) \leqslant \beta(t, x) + \varepsilon ,$$

This implies, by (0.1) that

$$|f(t, x, u)| \leqslant (C + \varepsilon)|u| + h_r(t, x)$$

for a.e. $(t, x) \in J$ and all $u \in \boldsymbol{R}$, with $C = \lambda_{N+1}$ if $\lambda_N > 0$ and $|\lambda_N|$ if $\lambda_{N+1} < 0$. Consequently, the mapping F defined on H by

$$(Fu)(t, x) = f\big(t, x, u(t, x)\big)$$

will map H continuously into itself and take bounded sets into bounded sets. Moreover, the weak solutions of the periodic-Dirichlet problem on J for (0.2) will be the solutions in $D(L)$ of the abstract equation in H

$$Lu - Fu = 0 . \tag{2.1}$$

Without loss of generality, we can assume from now on that $\lambda_N > 0$ because, if $\lambda_N < 0$, then $0 > N = -\tilde{N}$ and we can consider the equivalent problem

$$\tilde{L}u - \tilde{F}u = 0$$

where $\tilde{L} = -L$ and $\tilde{F} = -F$. Defining

$$0 < \tilde{\lambda}_N = -\lambda_{-N}, \quad \tilde{\alpha} = -\beta, \quad \tilde{\beta} = -\alpha, \quad \tilde{f} = -f,$$

we see that

$$\sigma(\tilde{L}) = \{\ldots < \tilde{\lambda}_{-2} < \tilde{\lambda}_{-1} < 0 < \lambda_1 < \tilde{\lambda}_2 < \ldots\},$$

that (0.5) and (0.6) hold with $\alpha, \beta, f, \lambda_N, \lambda_{N+1}$ respectively replaced by $\tilde{\alpha}, \tilde{\beta}, \tilde{f}, \tilde{\lambda}_N, \tilde{\lambda}_{N+1}$ and that the function $\operatorname{sign} \tilde{\lambda}_N \cdot \tilde{f}(t, x, \cdot) = \tilde{f}(t, x, \cdot) = \operatorname{sign} \lambda_{-N} f(t, x, \cdot)$ is non decreasing. We are therefore reduced to the case of $\lambda_N > 0$. It implies by our assumptions on $f(t, x, \cdot)$ that F is monotone on H. As the right inverse K of L is clearly compact, we see that KQF is compact on bounded sets of H. Define the linear operator $A: H \to H$ by

$$(Au)(t, x) = \alpha(t, x)\, u(t, x)$$

so that A is continuous and strongly monotone on H, as $\alpha(t, x) \geqslant \lambda_N > 0$ for a.e. $(t, x) \in J$.

According to the continuation lemma stated in the introduction, equation (2.1) will have a solution if the set of possible solutions of the family of equations

$$Lu - (1 - \lambda) Au - \lambda Fu = 0 , \qquad \lambda \in [0, 1] , \tag{2.2}$$

is a priori bounded independently of λ. Define g on $J \times \boldsymbol{R}$ by

$$g(t,x,u) = \begin{cases} u^{-1} f(t,x,u) & \text{if } |u| \geqslant r \\ r^{-1} f(t,x,r)\dfrac{u}{r} + \left(1 - \dfrac{u}{r}\right)\alpha(t,x) & \text{if } 0 \leqslant u < r \\ r^{-1} f(t,x,-r)\dfrac{u}{r} + \left(1 + \dfrac{u}{r}\right)\alpha(t,x) & \text{if } -r \leqslant u \leqslant 0 , \end{cases}$$

and b on $J \times \boldsymbol{R}$ by $b(t,x,u) = f(t,x,u) - g(t,x,u)u$. It is easy to check that

$$\alpha(t,x) - \varepsilon \leqslant g(t,x,u) \leqslant \beta(t,x) + \varepsilon$$

for a.e. $(t,x) \in J$ and all $u \in \boldsymbol{R}$, that h satisfies the Caratheodory conditions for $L^2(J)$ and that

$$|b(t,x,u)| \leqslant 2h_r(t,x) \tag{2.3}$$

for a.e. $(t,x) \in J$ and all $u \in \boldsymbol{R}$. If we define, for each $u \in H$, the linear mapping $G(u): H \to H$ by

$$[G(u)v](t,x) = g\big(t,x,u(t,x)\big)\, v(t,x)$$

and if we define $B: H \to H$ by

$$(Bu)(t,x) = b\big(t,x,u(t,x)\big) ,$$

then, for each $u \in H$, we have

$$Fu = G(u)u + Bu .$$

Thus, if $u \in D(L)$ is a solution of (2.2) for some $\lambda \in [0,1]$, it also satisfies the equation

$$Lu - [(1-\lambda)A - \lambda G(u)]u = \lambda Bu . \tag{2.4}$$

But, by construction, we have, for a.e. $(t,x) \in J$ and every $\lambda \in [0,1]$,

$$\alpha(t,x) - \varepsilon \leqslant (1-\lambda)(Au)(t,x) + \lambda[G(u)u](t,x) \leqslant \beta(t,x) + \varepsilon ,$$

and hence, using Lemma 2, (2.3) and (2.4), we obtain

$$2|h_r| \geqslant |\lambda Bu| = |Lu - [(1-\lambda)A - \lambda G(u)]u| \geqslant \delta |u| ,$$

i.e.

$$|u| \leqslant 2\delta^{-1}|h_r| .$$

The conditions of the Continuation lemma are therefore satisfied for any $\varrho > 2\delta^{-1}|h_r|$, and the proof is complete.

COROLLARY 1. – *Let $f\colon J\times \boldsymbol{R} \to \boldsymbol{R}$ satisfy the Caratheodory conditions for $L^2(J)$ and be such that*

$$\alpha(t, x) \leqslant \frac{f(t, x, u) - f(t, x, v)}{u - v} \leqslant \beta(t, x) \tag{2.5}$$

for a.e. $(t, x) \in J$ and all $u \neq v \in \boldsymbol{R}$, with α and β like in Theorem 1. Then the periodic-Dirichlet problem for equation (0.2) has a unique weak solution.

PROOF. – It follows from (2.5) that condition (0.5) holds and that $\operatorname{sign}\lambda_N \cdot f(t, x, \cdot)$ is non decreasing for a.e. $(t, x) \in J$. Thus, the existence follows from Theorem 1. If now u and v are solutions, then, letting $w = u - v$, w will be a weak solution of the periodic-Dirichlet problem for equation

$$w_{tt} - w_{xx} - [f(t, x, v + w) - f(t, x, v)] = 0 . \tag{2.6}$$

Setting

$$g(t, x, w) = \begin{cases} w^{-1}[f(t, x, v + w) - f(t, x, v)] , & \text{if } w \neq 0 , \\ \alpha(t, x) , & \text{if } w = 0 , \end{cases}$$

we see that (2.6) can be written

$$w_{tt} - w_{xx} - g(t, x, w)\, w = 0 \tag{2.7}$$

with

$$\alpha(t, x) \leqslant g(t, x, w) \leqslant \beta(t, x)$$

for a.e. $(t, x) \in J$ and all $w \in \boldsymbol{R}$. Consequently, by Lemma 2, we easily see from (2.7) that $w = 0$, i.e. $u = v$, and the proof is complete.

REMARK 1. – Condition (2.5) is in particular satisfied if the partial derivative $f'_u(t, x, u)$ exists and satisfies the condition

$$\alpha(t, x) \leqslant f'_u(t, x, u) \leqslant \beta(t, x)$$

for a.e. $(t, x) \in J$ and all $u \in \boldsymbol{R}$, with α and β like in Corollary 1.

REMARK 2. – The above results shows that if J is partitioned into two measurables subsets J_1 and J_2 of positive measure and if p is defined on J by $p(t, x) = \lambda_N$ for $(t, x) \in J_1$ and $p(t, x) = \lambda_{N+1}$ for $(t, x) \in J_2$, then the periodic-Dirichlet problem on J for the equation

$$u_{tt} - u_{xx} - p(t, x)\, u = h(t, x)$$

has a unique weak solution for every $h \in L^2(J)$.

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