Oscillations of First-Order Differential Inequalities with Deviating Arguments (*).

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Summary. – *This paper contains some new results on the oscillatory behaviours of differential inequalities* (N_z) *and* (L_z) *caused by retarded, advanced and general deviating arguments* $g_i(t)$ $(i = 1, \ldots, n)$.

1. – Introduction.

In this paper we consider the oscillatory behaviour of first-order functional-differential inequalities of the forms

$$(N_z) \qquad (-1)^z y'(t) \operatorname{sgn} y(t) \geqslant p(t) \prod_{i=1}^{n} |y(g_i(t))|^{r_i}$$

and

$$(L_z) \qquad (-1)^z y'(t) \operatorname{sgn} y(t) \geqslant \sum_{i=1}^{n} p_i(t) |y(g_i(t))| ,$$

where $z = 1, 2$, r_i $(i = 1, \ldots, n)$ are nonnegative numbers with $r_1 + \ldots + r_n = 1$, the functions $g_i\colon R_+ \to R_+ = [0, \infty)$ and $p, p_i\colon R_+ \to (0, \infty)$ $(i = 1, \ldots, n)$ are continuous and $\lim_{t\to\infty} g_i(t) = \infty$. We consider only solutions of (N_z) or (L_z) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a solution of (N_z) or (L_z) is called oscillatory if it has no last zero, otherwise it is called nonoscillatory.

In recent years, the oscillations of the solutions of first-order functional-differential equations and inequalities caused by retarded or advanced arguments, has been studied in the papers [1]-[17]. A characteristic feature of these papers is the fact that the results obtained there are not valid for corresponding ordinary differential equations and inequalities. In this paper we give sufficient conditions under which all solutions of (N_z) or (L_z) oscillate. Our results generalize and improve some results of the papers [2] and [4]-[15]. Some specific comparisons to known results will be made in the text of the paper.

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2. – Retarded differential inequalities.

In this section we will assume without further mention that $g_i(t) \leqslant t$ $(i = 1, \dots, n)$ on R_+ and let $h_i(t) = \sup_{0 < s \leqslant t} g_i(s)$.

THEOREM 1. – *Assume that*

$$\sum_{k=1}^{n} \liminf_{t\to\infty} r_k \cdot \int_{g_k(t)}^{t} p(s)\, ds > \frac{1}{e}, \tag{1}$$

then all solutions of (N_1) *oscillate.*

PROOF. – Suppose that there exists a nonoscillatory solution $y(t)$ of (N_1). Without loss of generality we may suppose that $y(t)$ is positive for $t \geqslant t_0$. There is a $t_1 \geqslant t_0$ such that $y(g_i(t)) > 0$ $(i = 1, \dots, n)$ for $t \geqslant t_1$ From (N_1) it follows, that $y(t)$ is decreasing for $t \geqslant t_1$. Since $g_i(t) \leqslant h_i(t)$ $(i = 1, \dots, n)$ for $t \geqslant t_1$, then from (N_1) we get for $t \geqslant t_1$

$$-y'(t) \geqslant p(t) \prod_{i=1}^{n} [y(h_i(t))]^{r_i}. \tag{2}$$

We follow similar arguments as in [6]-[8]. Dividing (2) by $y(t)$ and next integrating from $h_k(t)$ to t we obtain for $t \geqslant t_1$

$$\int_{h_k(t)}^{t} w(s) p(s)\, ds \leqslant \ln \frac{y(h_k(t))}{y(t)}, \qquad (k = 1, \dots, n),$$

where

$$w(t) = \prod_{i=1}^{n} \left[\frac{y(h_i(t))}{y(t)}\right]^{r_i} \geqslant 1 \qquad \text{for } t \geqslant t_1.$$

Multiplying both sides of the above inequalities by r_k and next adding these inequalities we derive

$$\sum_{k=1}^{n} r_k \int_{h_k(t)}^{t} w(s) p(s)\, ds \leqslant \sum_{k=1}^{n} r_k \ln \frac{y(h_k(t))}{y(t)} = \ln w(t).$$

Let $\gamma = \liminf_{t\to\infty} w(t)$. Then $\gamma \geqslant 1$ and is finite or infinite.

(*a*) Case γ is finite. Then taking limit inferiors on both sides of the last inequality, we obtain

$$\gamma \sum_{k=1}^{n} \liminf_{t\to\infty} r_k \int_{h_k(t)}^{t} p(s)\, ds \leqslant \ln \gamma$$

that is

$$\sum_{k=1}^{n} \liminf_{t\to\infty} r_k \int_{h_k(t)}^{t} p(s)\, ds \leqslant \frac{\ln \gamma}{\gamma} \leqslant \frac{1}{e},$$

which gives a contradiction, since (1) is equivalent to the following condition

$$\sum_{k=1}^{n} \liminf_{t\to\infty} r_k \int_{h_k(t)}^{t} p(s)\, ds > \frac{1}{e}. \tag{3}$$

(*b*) Case γ is infinite. Then

$$\lim_{t\to\infty} w(t) = \infty. \tag{4}$$

From (3) it follows, that there exists a $k \in \{1, \ldots, n\}$ such that

$$\liminf_{t\to\infty} r_k \int_{h_k(t)}^{t} p(s)\, ds \geqslant C > 0, \tag{5}$$

where C is some constant. Then (cf. [4]) for any $t \geqslant t_2 \geqslant t_1$ there exists a $t^* > t$ such that

$$r_k \int_{h_k(t^*)}^{t} p(s)\, ds \geqslant \frac{C}{2} \quad \text{and} \quad r_k \int_{t}^{t^*} p(s)\, ds \geqslant \frac{C}{2}. \tag{6}$$

Integrating now (2) from $h_k(t)$ to t we derive

$$-y(t) + y(h_k(t)) \geqslant \int_{h_k(t)}^{t} p(s) \prod_{i=1}^{n} [y(h_i(s))]^{r_i}\, ds$$

which imply, by monotonicy of $h_i(t)$ and $y(t)$,

$$\frac{y(h_k(t))}{y(t)} \geqslant w(t) \int_{h_k(t)}^{t} p(s)\, ds.$$

In view of (4) and (5), from the last inequality we get

$$\lim_{t\to\infty} \frac{y(h_k(t))}{y(t)} = \infty. \tag{7}$$

Now from (2), by the fact that $y(h_i(t)) \geqslant y(t)$ $(i = 1, \ldots, n)$ for $t \geqslant t_1$, we obtain

$$-y'(t)[y(t)]^{r_k - 1} \geqslant p(t)[y(h_k(t))]^{r_k}. \tag{8}$$

Integrating both sides of this inequality from $h_k(t^*)$ to t (cf. [4]), by (6), we find for $t \geq t_2$

$$[y(h_k(t^*))]^{r_k} - [y(t)]^{r_k} \geq r_k \int_{h_k(t^*)}^{t} p(s)[y(h_k(s))]^{r_k} ds \geq r_k [y(h_k(t))]^{r_k} \cdot \int_{h_k(t^*)}^{t} p(s)\, ds \geq \frac{C}{2} [y(h_k(t))]^{r_k}.$$

Now, integrating (8) from t to t^*, similarly as above, we obtain

$$[y(t)]^{r_k} - [y(t^*)]^{r_k} \geq r_k \int_{t}^{t^*} p(s)[y(h_k(s))]^{r_k}\, ds \geq \frac{C}{2} [y(h_k(t^*))]^{r_k}. \tag{10}$$

Therefore from (9) and (10) we have

$$[y(t)]^{r_k} \geq \left(\frac{C}{2}\right)^2 [y(h_k(t))]^{r_k}, \qquad t \geq t_2,$$

which contradicts (7). Thus the proof is complete.

REMARK 1. – Similar result as in Theorem 1 has been obtained recently by LADAS [6] in the case $n = 1$ and $g_1(t) = t - \tau_1$, τ_1 is positive constant, by KOPLATADZE and CHANTURIA [4] in the case $n = 1$.

COROLLARY 1. – *Consider the retarded differential inequality*

$$y(t)\left[y'(t) + q(t)y(t) + p(t) \prod_{i=1}^{n} [y(g_i(t))]^{r_i}\right] \leq 0, \tag{11}$$

where r_i $(i = 1, \dots, n)$ are the ratio of odd natural numbers with $r_1 + \dots + r_n = 1$, $q \colon R_+ \to R$ is continuous function, $p(t)$ and $g_i(t) \leq t$ are the same as in (N_z). Let

$$\sum_{k=1}^{n} \liminf_{t\to\infty} r_k \int_{g_k(t)}^{t} p(s) \exp\left(\sum_{j=1}^{n} r_j \int_{g_j(s)}^{s} q(v)\, dv\right) ds > \frac{1}{e}. \tag{12}$$

Then all solutions of (11) *oscillate.*

PROOF. – Putting in (11) $y(t) = x(t) \exp\left(-\int_0^t q(v)\, dv\right)$ we obtain

$$x(t)\left[x'(t) + p(t) \exp\left(\sum_{j=1}^{n} r_j \int_{g_j(t)}^{t} q(v)\, dv\right) \cdot \prod_{i=1}^{n} [x(g_i(t))]^{r}\right] \leq 0.$$

By Theorem 1 $x(t)$ oscilate. Therefore $y(t)$ also oscillate.

REMARK 2. – In the case $g_i(t) = t - \tau_i$, τ_i is a positive constant, and $q(t) \geqslant 0$ for $t \in R_+$, the analogous problem as in Corollary 1 has been considered by LADAS and STAVROULAKIS [7] for $n = 1$, and by STAVROULAKIS [15] for $n \geqslant 1$. According to Theorems 1 and 2 of [15] all solutions of (11) with $g_i(t) = t - \tau_i$ are oscillatory if

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds > \frac{1}{e} \exp\left(-\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s)\, ds\right), \qquad \tau = \min(\tau_1, \ldots, \tau_n)\,. \tag{13}$$

Notice that if the condition (13) is satisfied, then also the condition (12) holds.

THEOREM 2. – *Each one of the following conditions*

$$\liminf_{t\to\infty} \int_{g_k(t)}^{t} p_k(s) \exp\left(\int_{g_k(s)}^{s} \sum_{\substack{i=1\\ i\neq k}}^{n} p_i(v)\, dv\right) ds > \frac{1}{e} \tag{14}$$

for some $k \in \{1, \ldots, n\}$,

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds > \frac{1}{e}, \qquad g(t) = \max\big(g_1(t), \ldots, g_n(t)\big)\,, \tag{15}$$

implies that every solution of (L_1) *oscillates.*

PROOF. – Let $y(t)$ be a nonoscillatory solution of (L_1) and let $y(t) > 0$ and $y(g_i(t)) > 0$ $(i = 1, \ldots, n)$ for $t \geqslant t_1$. Since $y(g_i(t)) \geqslant y(g(t)) \geqslant y(t)$ for $t \geqslant t_1$, therefore from (L_1) we get respectively

$$y'(t) + y(t) \sum_{\substack{i=1\\ i\neq k}}^{n} p_i(t) + p_k(t) y(g_k(t)) \leqslant 0$$

and

$$y'(t) + y(g(t)) \sum_{i=1}^{n} p_i(t) \leqslant 0\,.$$

By Corollary 1, in view of (14) and (15) respectively, $y(t)$ oscillate. But this contradicts our assumption that $y(t) > 0$. Thus the proof is complete.

REMARK 3. – The analogous results as in the case (15) of Theorem 2 has been obtained by LADDE [9] and by LADAS and STAVROULAKIS [8] for $g_i(t) = t - \tau_i$.

3. – Advanced differential inequalities.

In this section we will assume without further mention that $g_i(t) \geqslant t$ $(i = 1, \ldots, n)$ on R_+. By a similar argument as in the proofs of Theorems 1 and 2 we can obtain the following dual results about advanced differential inequalities (N_2) and (L_2).

THEOREM 3. – *Assume that*

$$\sum_{k=1}^{n} \liminf_{t\to\infty} r_k \int^{g_k(t)} p(s)\, ds > \frac{1}{e}, \tag{16}$$

then all solutions of (N_2) *oscillate.*

THEOREM 4. – *Each one of the following conditions*

$$\liminf_{t\to\infty} \int_t^{g_k(t)} p_k(s) \exp\left(\int_s^{g_k(s)} \sum_{\substack{i=1\\ i\neq k}}^{n} p_i(v)\, dv \right) ds > \frac{1}{e} \tag{17}$$

for some $k \in \{1, \dots, n\}$,

$$\liminf_{t\to\infty} \int_t^{g(t)} \sum_{i=1}^{n} p_i(s)\, ds > \frac{1}{e}, \qquad g(t) = \min\left(g_1(t), \dots, g_n(t)\right), \tag{18}$$

implies that every solution of (L_2) *oscillates.*

REMARK 4. – From Theorem 4, in the case $g_i(t) = t + \tau_i$, $\tau_i > 0$ $(i = 1, \dots, n)$, we obtain some results of LADAS and STAVROULAKIS [8] if $n \geqslant 1$ and of KUSANO [5] if $n = 1$. We notice that the condition (17) is better than the analogous condition of [8] which has the form

$$\liminf_{t\to\infty} \int_t^{t+\tau_k} p_k(s)\, ds > \frac{1}{e}.$$

4. – Inequalities with general deviating arguments.

In this section we consider the differential inequalities (N_z) and (L_z) with general deviating arguments $g_i(t)$, not necessarily retarded $(g_i(t) \leqslant t)$ or advanced $(g_i(t) \geqslant t)$ arguments.

We denote

$$D = \{t \in R_+ \colon g_i(t) \leqslant t \ (i = 1, \dots, n)\},$$
$$A = \{t \in R_+ \colon g_i(t) \geqslant t \ (i = 1, \dots, n)\}.$$

Let $a_i, d_i \colon R_+ \to R_+$ $(i = 1, \dots, n)$ be nondecreasing continuous functions such that

$$\begin{cases} d_i(t) \leqslant t \leqslant a_i(t) \quad \text{for } t \in R_+, \\ g_i(t) \leqslant d_i(t) \quad \text{for } t \in D \quad \text{and} \quad a_i(t) \leqslant g_i(t) \quad \text{for } t \in A. \end{cases} \tag{19}$$

Set

$$D_i(t) = D \cap [d_i(t), t] , \quad A_i(t) = A \cap [t, a_i(t)] .$$

THEOREM 5. – *If*

$$\limsup_{t\to\infty} \prod_{k=1}^{n} \Big[\int_{D_k(t)} p(s)\, ds \Big]^{r_k} > 1 , \tag{20}$$

then all solutions of (N_1) *oscillate.*

PROOF. – Let $y(t)$ be a nonoscillatory solution of (N_1) and let $y(t) > 0$ and $y(g_i(t)) > 0$ $(i = 1, \dots, n)$ for $t \geqslant t_1$. Integrating (N_1) from $d_k(t)$ to t we obtain

$$y(d_k(t)) - y(t) \geqslant \int_{d_k(t)}^{t} p(s) \prod_{i=1}^{n} [y(g_i(s))]^{r_i} ds \geqslant \int_{D_k(t)} p(s) \prod_{i=1}^{n} [y(g_i(s))]^{r_i} ds , \quad (k = 1, \dots, n)$$

for $t \geqslant t_2 \geqslant t_1$. Since $y(t)$ is decreasing, in view of (19), we have for $s \in D_k(t)$ $(k = 1, \dots, n)$ and $t \geqslant t_2$

$$y(g_i(s)) \geqslant y(d_i(s)) \geqslant y(d_i(t)) , \quad (i = 1, \dots, n) .$$

Thus, we derive for $t \geqslant t_2$

$$y(d_k(t)) \geqslant \prod_{i=1}^{n} [y(d_i(t))]^{r_i} \int_{D_k(t)} p(s)\, ds , \quad (k = 1, \dots, n) .$$

Raising both sides of the above inequality to r_k and next multypling these inequalities we obtain

$$\prod_{k=1}^{n} [y(d_k(t))]^{r_k} \geqslant \prod_{k=1}^{n} \Big[\prod_{i=1}^{n} [y(d_i(t))]^{r_i} \Big]^{r_k} \cdot \prod_{k=1}^{n} \Big[\int_{D_k(t)} p(s)\, ds \Big]^{r_k} .$$

Since $r_1 + \dots + r_n = 1$, then the last inequality gives

$$1 \geqslant \prod_{k=1}^{n} \Big[\int_{D_k(t)} p(s)\, ds \Big]^{r_k} ,$$

which contradicts the condition (20).

In exactly the same way we can prove the following theorem.

THEOREM 6. – *If*

$$\limsup_{t\to\infty} \prod_{k=1}^{n} \Big[\int_{A_k(t)} p(s)\, ds \Big]^{r_k} > 1 , \tag{21}$$

then all solutions of (N_2) *oscillate.*

REMARK 5. – Similar results as in Theorems 5 and 6, in the case $n = 1$, have been obtained in the paper [2]. The analogous problem for $r_1 + \ldots + r_n > 1$ (< 1) has been considered in the papers [2, 3] and [16, 17].

COROLLARY 2. – *Consider the retarded differential inequality* (11), *where* p, q, g_i *and* r_i *are the same as in Corollary* 1. *If*

$$\limsup_{t\to\infty} \prod_{k=1}^{n} \left[\int_{d_k(t)}^{t} p(s) \exp\left(\sum_{j=1}^{n} r_j \int_{g_j(s)}^{s} q(v)\, dv \right) ds \right]^{r_i} > 1 , \tag{22}$$

then all solutions of (11) *oscillate.*

REMARK 6. – In the case of retarded differential equations and inequalities oscillation criteria of similar nature as in Corollary 2 have been obtained by NAITO [11], SFICAS and STAIKOS [12], STAIKOS and STAVROULAKIS [14]. According to Th. 2 of [11] or Th. 4 of [12] or Th. 2 of [14] all solutions of (11) with $q(t) = 0$, are oscillatory if

$$\limsup_{t\to\infty} \int_{d(t)}^{t} p(s)\, ds > 1 , \qquad d(t) = \max\big(d_1(t), \ldots, d_n(t)\big) . \tag{23}$$

We notice that the condition (23) implies (22).

THEOREM 7. – *If*

$$\limsup_{t\to\infty} \left\{ \sum_{k=1}^{n} \int_{D_k(t)} p_k(s)\, ds + 2 \sum_{\substack{k<i \\ k,i=1}}^{n} \sqrt{\int_{D_k(t)} p_i(s)\, ds \cdot \int_{D_i(t)} p_k(s)\, ds} \right\} > n , \tag{24}$$

then all solutions of (L_1) *oscillate.*

PROOF. – Suppose, that (L_1) has a nonoscillatory solution $y(t) > 0$ for $t \geqslant t_1$. Therefore $y(g_i(t)) > 0$ for $t \geqslant t_2 \geqslant t_1$. Integrating (L_1) from $d_k(t)$ to t we get for $t \geqslant t_2$

$$y\big(d_k(t)\big) - y(t) \geqslant \int_{d_k(t)}^{t} \sum_{i=1}^{n} p_i(s) y\big(g_i(s)\big)\, ds , \qquad (k = 1, \ldots, n) ,$$

which gives, by (19) and monotonicity of $y(t)$,

$$y\big(d_k(t)\big) \geqslant \sum_{i=1}^{n} \int_{D_k(t)} p_i(s) y\big(g_i(s)\big)\, ds \geqslant \sum_{i=1}^{n} y\big(d_i(t)\big) \cdot \int_{D_k(t)} p_i(s)\, ds , \qquad (k = 1, \ldots, n) .$$

Dividing now the both sides of the above inequality by $y(d_k(t))$ and next adding these inequalities we derive

$$n \geqslant \sum_{k=1}^{n} \int\limits_{D_k(t)} p_k(s)\, ds + \sum_{\substack{k<i\\ k,i=1}}^{n} \left[\frac{y(d_i(t))}{y(d_k(t))} \cdot \int\limits_{D_k(t)} p_i(s)\, ds + \frac{y(d_k(t))}{y(d_i(t))} \cdot \int\limits_{D_i(t)} p_k(s)\, ds\right].$$

Using the fact that

$$\frac{y(d_i(t))}{y(d_k(t))} \cdot \int\limits_{D_k(t)} p_i(s)\, ds + \frac{y(d_k(t))}{y(d_i(t))} \cdot \int\limits_{D_i(t)} p_k(s)\, ds \geqslant 2 \sqrt{\int\limits_{D_k(t)} p_i(s)\, ds \cdot \int\limits_{D_i(t)} p_k(s)\, ds}\,,$$

the last inequality implies

$$n \geqslant \sum_{k=1}^{n} \int\limits_{D_k(t)} p_k(s)\, ds + 2 \sum_{\substack{k<i\\ k,i=1}}^{n} \sqrt{\int\limits_{D_k(t)} p_i(s)\, ds \cdot \int\limits_{D_i(t)} p_k(s)\, ds}\,,$$

which contradicts (24).

In the case $y(t) < 0$ the proof is analogous. Thus, the proof of Theorem is complete.

In exactly the same way we can prove the following theorem.

THEOREM 8. – *If*

$$\limsup_{t\to\infty} \left\{\sum_{k=1}^{n} \int\limits_{A_k(t)} p_k(s)\, ds + 2 \sum_{\substack{k<i\\ k,i=1}}^{n} \sqrt{\int\limits_{A_k(t)} p_i(s)\, ds \cdot \int\limits_{A_i(t)} p_k(s)\, ds}\right\} > n\,, \tag{25}$$

then all solutions of (L_2) *oscillate.*

REFERENCES

[1] C. H. ANDERSON, *Asymptotic oscillation results for solutions to first-order nonlinear differential-difference equations of advanced type*, J. Math. Anal. Appl., **24** (1968), pp. 430-439.

[2] A. F. IVANOV - V. N. SHEVELO, *On oscillation and asymptotic behaviour of solutions for differential-functional first-order equations*, Ukrain. Mat. Z., **33** (1981), pp. 745-751 (Russian).

[3] Y. KITAMURA - T. KUSANO, *Oscillations of first-order nonlinear differential equations with deviating arguments*, Proc. Amer. Math. Soc., **78** (1980), pp. 64-68.

[4] R. G. KOPLATADZE - T. A. CHANTURIA, *On oscillatory and monotonous solutions of first-order differential equations with deviating argument*, Differencial'nye Uravnienija, **18** (1982), pp. 1463-1465 (Russian).

[5] T. KUSANO, *On even order functional differential equations with advanced and retarded arguments*, J. Differential Equations, **45** (1982), pp. 75-84.

[6] G. LADAS, *Sharp conditions for oscillations caused by delays*, Applicable Analysis, **9** (1979), pp. 93-98.

[7] G. LADAS - I. P. STAVROULAKIS, *On delay differential inequalities of first order*, Funkcial. Ekvac., **25** (1982), pp. 105-113.

[8] G. LADAS - I. P. STAVROULAKIS, *Oscillations caused by several retarded and advanced arguments*, J. Differential Equations, **44** (1982), pp. 134-152.

[9] G. S. LADDE, *Oscillations caused by retarded perturbations of first order linear ordinary differential equations*, Atti Acad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., **63** (1977), pp. 351-359.

[10] A. D. MYSHKIS, *On the solutions of linear homogeneous differential equations of the first order and stable type with retarded argument*, Mat. Sb., **28** (1951), pp. 641-658 (Russian).

[11] M. NAITO, *Oscillations of differential inequalities with retarded arguments*, Hiroshima Math. J., **5** (1975), pp. 187-192.

[12] Y. G. SFICAS - V. A. STAIKOS, *Oscillations of differential equations with deviating arguments*, Funkcial. Ekvac., **19** (1976), pp. 35-43.

[13] W. E. SHREVE, *Oscillation in first order nonlinear retarded argument differential equations*, Proc. Amer. Math. Soc., **41** (1973), pp. 565-568.

[14] V. A. STAIKOS - I. P. STAVROULAKIS, *Bounded oscillations under the effect of retardations for differential equations of arbitrary order*, Proc. Roy. Soc. Edinburgh, **77** A (1977), pp. 129-136.

[15] I. P. STAVROULAKIS, *Non-linear delay differential inequalities*, Nonlinear Analysis, Theory, Meth. Appl., **6** (1982), pp. 389-396.

[16] J. WERBOWSKI, *On oscillation behavior of solutions of differential equations generated by deviating arguments*, in « Differential Equations and Applications », Proceedings of the Second Conference, pp. 135-138, Rousse, 1982.

[17] J. WERBOWSKI, *Oscillations of nonlinear differential equations caused by deviating arguments*, Funkcial. Ekvac., **25** (1982), pp. 295-301.