D_π-Property and Normal Subgroups (*).

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Summary. – *D_π-property (π = set of primes) in finite groups is not in general inherited by subgroups. In this paper, as evidence in favor of the following conjecture (F. Gross):*

(○) *If a finite group G satisfies D_π then its normal subgroups satisfy D_π as well.*

the Author shows that if the D_π and the $D_{\pi'}$-properties (π' = set of the primes not in π) hold together in a finite group G, then both are inherited by the normal subgroups of G. As a corollary, the characterization of the groups satisfying both the properties D_π and $D_{\pi'}$ is given in terms of the composition factors.

0. – Introduction.

Let π be a set of primes and let G be a finite group,

Recall that G satisfies D_π if G has a Hall π-subgroup H and each π-subgroup of G is contained in a conjugate of H.

This definition, together with those of the properties C_π and E_π, were set by P. HALL in his well known article « Theorems like Sylow's » [1].

It is straigthforward to see that if a finite group G satisfies E_π (i.e. there exists in G a Hall π-subgroup) then its normal subgroups satisfy E_π as well.

Examples of finite groups satisfying the C_π-property (i.e. having a unique conjugacy class of Hall π-subgroups) but in which there are normal subgroups not satisfying C_π, have been provided by F. GROSS in [2 see pg. 11].

In the same paper, F. GROSS asks the question if the D_π-property is inherited by normal subgroups, or if this holds only for a special class of groups.

Counterexamples to the conjecture:

(○) If a finite group G satisfies D_π then its normal subgroups satisfy D_π

do not seem to exist.

As evidence in favor of this conjecture in this paper, it is shown that if a finite group G satisfies both the D_π and $D_{\pi'}$ properties (π' indicates the set of primes not in π) then normal subgroups have the same properties (see Theorem 1.4).

(*) Entrata in Redazione il 10 marzo 1986.

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The proof of this theorem uses results by Arad-Fisman on factorizable simple groups [3] and so it holds modulo the finite simple groups classification.

Further, in the second section, it is shown that if $PSL(2, q)$ satisfies the D_π-property then its automorphism group satisfies the same property.

This result with the already cited Theorem 1.4 is then applied to give a characterization of all finite groups satisfying both the D_π and $D_{\pi'}$-properties, generalizing the results of the joint paper of the Author with L. SERENA [4].

1. – All groups considered in this paper are finite and notation not explicitely introduced is standard (cfr. [5]).

Let $\boldsymbol{P}$ the set of all primes. Let us denote by π a subset of $\boldsymbol{P}$ and by π' the complementary set of π in $\boldsymbol{P}$.

If n is a natural number, let us denote by $\pi(n)$ the set of primes dividing n.

If G is a finite group indicate by $\pi(G) = \pi(|G|)$.

Further, denote by D_π (resp. C_π, E_π) the class of finite groups satisfying the D_π-property (resp. C_π, E_π).

Finally, $D_{\pi,\pi'}$ (resp. $C_{\pi,\pi'}$) is the class of finite groups belonging to both D_π and $D_{\pi'}$ (resp. C_π and $C_{\pi'}$, E_π and $E_{\pi'}$).

Let us list, in the following proposition, two preliminary observations:

PROPOSITION. – *Let G be a group belonging to D_π and let N be a normal subgroup of G.*

i) *N has Hall π-subgroups and they have the form $H^x \cap N$, where H is a Hall π-subgroup of G and $x \in G$. So they are conjugate in G, in particular, isomorphic.*

ii) *If $N \in C_\pi$, then $N \in D_\pi$.*

PROOF. – i) is straigthforward; ii) is immediate by i) see also [2 Corollary 4.3].

LEMMA 1.1. – *Let G be a group and M be a normal subgroup of G, if G/M is a π-group and if $G \in D_\pi$, then $M \in D_\pi$.*

PROOF. – We have $G = MH$ where H is a Hall π-subgroup of G. By the previous Proposition i), the Hall π-subgroups of M are of the form $H^x \cap M$, $x \in G$. But $x = hm$ for some $h \in H$, $m \in M$ and so $H^x \cap M = H^m \cap M = (H \cap M)^m$.

It follows $M \in C_\pi$, so by Proposition ii), $M \in D_\pi$ as claimed.

LEMMA 1.2. – *Let $2 \in \pi$ and let G be a group in D_π. Let M be a normal subgroup of G. Assume that G has a Hall π'-subgroup K, then $MK \in D_{\pi,\pi'}$.*

PROOF. – It is easily seen that $MK \in E_{\pi,\pi'}$.

By results in [3] $MK \in D_{\pi'}$.

The Hall π-subgroups of MK coincide with those of M and so, by Proposition i) are of the form $H^g \cap M$, $g \in G$ (H indicates a Hall π-subgroup of G).

Since $G = HK$, there exist elements $h \in H$ and $k \in K$ such that $g = hk$. So $H^g \cap M = H^k \cap M = (H \cap M)^k$.

Thus the Hall π-subgroups of MK are conjugate by elements of K, so, in particular, they are conjugate in MK.

It follows $MK \in C_\pi$. But, since each π-subgroup of MK is necessarily contained in M, by Proposition ii) $MK \in D_\pi$. So $MK \in D_{\pi,\pi'}$.

LEMMA 1.3. – *Let M be a simple group and let G be a group such that M is normal in G and $C_G(M) = 1$. Suppose $G \in D_{\pi,\pi'}$, then $M \in D_{\pi,\pi'}$.*

PROOF. – Identifying M with Inn(M) we can write $M \leqslant G \leqslant \mathrm{Aut}(M)$.

Let $2 \in \pi$, so π' is a set of primes, all odd. Since $M \in E_{\pi,\pi'}$, by results in [3], $M \in D_{\pi'}$. We need only to show that $M \in D_\pi$, and, by Proposition ii), it will be enough to show that $M \in C_\pi$.

For our analysis, we can restrict ourselves to the list of factorizable simple groups see [3 Theorem 1.1].

By Lemma 1.1 we can eliminate all the following possibilities for M: A_r; M_{11}; M_{23}; $PSL(2, q)$ (q a prime); $PSL(5, 2)$.

By Lemma 1.2, we can assume G/M to be a π'-group. Further, by Proposition i), we can also assume that the maximal π-subgroups of M are isomorphic, since, by the hypothesis on G, they are necessarily Hall π-subgroups of M. This enables us to eliminate the cases: $M \simeq PSL(2, 2^k)$ and $M \simeq PSL(2, q)$ (with $2, 3 \in \pi$) (for the second case see also next Lemma 2.1).

It only remains to analyze the following two cases:

I) $M = PSL(r, q)$ where r is an odd prime and $(r, q-1) = 1$ $q = p^n$, and A is a maximal parabolic subgroup such that $PSL(r-1, q)$ is involved in A (A indicates here a Hall π-subgroup of M).

II) $M = PSL(2, q)$, where $q = p^n$, $3 < q \not\equiv 1(4)$ and π is such that

$$\pi \frac{(q(q-1))}{2} \subseteq \pi' \quad \text{and} \quad \pi(q+1) \subseteq \pi .$$

In the case I) $PSL(r, q) \simeq SL(r, q)$ and since $(r, q-1) = 1$ and G/M is a π'-group, $|G/M|$ divides n.

It follows then that G/M is isomorphic to a subgroup of the cyclic group of the automorphisms of $GF(p^n)$.

As in [4], by considering the Hall π-subgroups of M,

$$A_1 = \left\{ \begin{array}{c} \\ 1 \\ r-1 \end{array} \begin{pmatrix} 1 & r-1 \\ * & * \\ 0 & * \end{pmatrix} \right\} \qquad A_2 = \left\{ \begin{array}{c} \\ r-1 \\ 1 \end{array} \begin{pmatrix} r-1 & 1 \\ * & * \\ 0 & * \end{pmatrix} \right\}$$

we have that A_1 and A_2 are not conjugate in M.

But they cannot be conjugate also in G, since if φ is a field automorphism $A_i^\varphi = A_i$, for $i = 1, 2$.

So this case does not appear, since G must belong to D_π.

In the case II), the Hall π-subgroups of M are either the normalizers of some Sylow t-subgroup (where t is an odd prime, $t \in \pi(q+1)$) or the Sylow 2-subgroups (if q is Mersenne); so, with the same argument as in [4], we can prove that $M \in D_\pi$, as we needed to show. The Lemma is proved.

REMARK. – We actually characterize all simple groups in $D_{\pi,\pi'}$ in the Section 3.

We can now prove:

THEOREM 1.4. – *Let G be a group such that $G \in D_{\pi,\pi'}$, and let M be a normal subgroup of G, then $M \in D_{\pi,\pi'}$.*

PROOF. – We proceed by induction on $|G| + |M|$.

Let N be a minimal normal subgroup of G contained in M.

If $N < M$, then, by induction, $N \in D_{\pi,\pi'}$. Further $G/N \in D_{\pi,\pi'}$, and so, since $|G/N| < |G|$ and $M/N \trianglelefteq G/N$, we have, by induction, $M/N \in D_{\pi,\pi'}$. But then, from $N \in D_{\pi,\pi'}$, $M/N \in D_{\pi,\pi'}$, and $G \in D_{\pi,\pi'}$, by [2 Lemma 4.2], we get $M \in D_{\pi,\pi'}$, as we claimed.

Thus we may assume $N = M$, so that M is a minimal normal subgroup of G. In particular, we may assume that M is the direct product of non abelian simple isomorphic groups

$$M = S_1 \times S_2 \times \ldots \times S_n$$

where $S_j \cong S$, where S is a group in the list of [3 Theorem 1.1], and $j = 1, \ldots, n$.

Assuming that $2 \in \pi$, π' consists only of odd primes, so, by [3], $M \in D_{\pi'}$. It is enough to prove then that $M \in D_\pi$.

By Lemma 1.2, we may assume G/M is a π'-group.

Now let A_1 and A_1^* be Hall π-subgroups of S_1.

For $1 \leqslant i \leqslant n$, there exists $g_i \in G$ such that $S_i = S_1^{g_i}$, and choose $g_1 = 1$.

Let $A_i = A_1^{g_i}$ and $A_i^* = A_1^{*g_i}$. Let $H = \langle A_i : 1 \leqslant i \leqslant n \rangle = A_1 \times A_2 \times \ldots \times A_n$,

$$H^* = \langle A_i^*, 1 \leqslant i \leqslant n \rangle = A_1^* \times A_2^* \times \ldots \times A_n^*.$$

Then H and H^* are Hall π-subgroups of G and so, since $G \in D_\pi$, there exists $g \in G$ such that $H^* = H^g$.

Now, g must permute $S_1, \ldots, S_n$ and so $S_i^g = S_1$, for some i.

For this i, $g_i g \in N_G(S_1)$ and $A_1^* = H^* \cap S_1 = H^g \cap S_i^g = (H \cap S_i)^g = A_i^g = A_1^{g_i g}$.

Hence A_1^* and A_1 are conjugate in $N_G(S_1)$. It now follows that

$$\big(N_G(S_1)/(S_2 \times \ldots \times S_n)\big)/\big((S_1 \times \ldots \times S_n)/(S_2 \times \ldots \times S_n)\big)$$

is a π'-group.

Since every π-subgroup of S_1 is contained in some Hall π-subgroup of S_1, it follows that $N_G(S_1)/(S_2 \times \ldots \times S_n) \in D_\pi$.

Since $C_G(S_1) \geqslant S_2 \times \ldots \times S_n$, we have that $N_G(S_1)/C_G(S_1) \in D_\pi$.

But $S_1 \leqslant N_G(S_1)/C_G(S_1) \leqslant \text{Aut}\, S_1$.

Since M satisfies $D_{\pi'}$, S_1 satisfies $D_{\pi'}$ (see [2]), further $(N_G(S_1)/C_G(S_1))/S_1$ is a π'-group.

So, by Lemma 1.1, $N_G(S_1)/C_G(S_1) \in D_{\pi'}$.

It now follows, by Lemma 1.3, that $S_1 \in D_{\pi,\pi'}$. But then $M \in D_{\pi,\pi'}$, and Theorem 1.4 is proved.

2. – The D_π-property and the simple groups $PSL(2, q)$.

In this Section, we expose some results on the property D_π in the groups $PSL(2, q)$; some of which will be applied in the third Section.

All trough this Section, no requirements on the $D_{\pi'}$-property have been done.

LEMMA 2.1. – *Let $G = PSL(2, q)$, where $q = p^n$. Let us assume that $G \in D_\pi$ and $2, 3 \in \pi$. Then $\pi(G) \subseteq \pi$.*

PROOF. – We can obviously assume $q > 3$. Let H be a Hall π-subgroup of G and let $2, 3 \in \pi$. Let us examine the possibilities for H, looking at the Dickson's list (see [5]).

1st STEP. – *H cannot be cyclic.*

In fact if H were cyclic of order t, t should divide $(q \pm 1)/\varepsilon$ where $\varepsilon = (2, q-1)$. But there is an involution normalizing H so that H cannot be Hall.

2nd STEP. – *H cannot be dihedral of order $2t$ where t divides $(q \pm 1)/\varepsilon$, $\varepsilon = (2, q-1)$.*

Since $q > 3$, A_4 is a subgroup of G and so A_4 should be contained in a dihedral group and this is not possible.

3rd STEP. – *H cannot be isomorphic to A_4.*

If $H \cong A_4$, then for $p = 2$ we get $G \cong PSL(2, 4) \cong A_5$, which does not satisfy $D_{\{2,3\}}$. For $p > 3$, then there exists in G a dihedral group of order 6 which cannot be contained in any copy of A_4.

4th STEP. – *H cannot be isomorphic to S_4.*

If S_4 were a Hall π-subgroup of G, then $G \notin C_\pi$, since there would be two conjugacy classes of subgroups isomorphic to S_4 in G (see [5 pg. 202]).

5th STEP. – *H cannot be isomorphic to A_5.*

Let $H \cong A_5$, then $\{2, 3, 5\} \subseteq \pi$.

If $q = 5, 4$ then $G \cong A_5$ and in this case the theorem holds.

If $q > 5$, let $5/(q-1)$. If $2, 3$ do not divide $(q-1)/2$ then $2, 3$ divide $q+1$ and so there exists in G a dihedral subgroup of order 12, that cannot be contained in any copy of A_5. So either 2 or 3 divide $(q-1)/2$. Then there exists a cylic group of order either 10 or 15, that cannot be contained in any copy of A_5.

The same if $5/(q+1)$.

6th STEP. – *H cannot be a Frobenius group of order $q(q-1)/\varepsilon$ (or in general of order qt, where $t/((q-1)/\varepsilon)$, $\varepsilon = (2, q-1)$).*

Since in G there exists a dihedral group of order $2(q-1)$ (resp. $2t$), it should be $q = 2^k$. But the dihedral group of order $2(2^k - 1)$ is maximal in G and so it cannot be contained in any copy of H.

So if $2, 3 \in \pi$, we get $H = G$.

REMARK 2.2. – Suppose $G = PSL(2, 2^k)$ and $G \in D_\pi$ for a set of primes π such that $\pi(G) \not\subseteq \pi$. If $2 \in \pi$, then $\pi = \{2\}$.

THEOREM 2.3. – *Let G be a group such that $M \leqslant G \leqslant \mathrm{Aut}(M)$, where $M = PSL(2, q)$, $q = p^m$ (p a prime). If $M \in D_\pi$, for some set π, then $G \in D_\pi$.*

REMARK. – The following proof is based on the proof of Theorem 2.2 [4], for that reason we omit those steps that can be found in [4].

PROOF. – The proof is, by induction on $|G:M| + |M|$.

Let G be a minimal counterexample to the Theorem.

As in [4], $G \in C_\pi$. Let H be a Hall π-subgroup of G. Since $G \notin D_\pi$, there exists a π-subgroup K, such that $K \not\leqslant H^x$, for every $x \in G$. We can assume K π-maximal. As in [4] we can assume $G = HM$ and so G/M π-group. Further we observe that if T is a solvable subgroup of M, since $N_G(T)/N_M(T)$ is solvable and $N_M(T)$ is solvable, $N_G(T)$ is solvable too.

So as in [4] we get $G = KM$ and $K \cap M \leqslant H \cap M$.

Suppose first $K \cap M = 1$. Then if $(|K|, |M|) = 1$ we can proceed as in [4], once we observe that H is solvable.

So we can assume r is a prime dividing $|M|$ and $|K|$. Let y be an element of order r in K and let R be a Sylow r-subgroup of G containing $\langle y \rangle$. $C_G(y)$ has as a subgroup an elementary abelian subgroup of order r^2. Further $K \leqslant C_G(y)$, since K is abelian. Now $C_G(y) < G$, so either $C_G(y)$ is solvable or

$$C_G(y) \cap M \cong \begin{cases} PSL(2, p^r) \\ PGL(2, p^s) \end{cases} \quad \text{where } p^r/p^m = q \text{ or } p^{2s}/p^m = q\,.$$

If $C_G(y)$ is solvable then we get the contradiction as in [4].

So suppose the other case holds. Let $C = C_G(y)$.

Then $C_C(C \cap M)$ is solvable and $C/C_C(C \cap M) \lesssim \operatorname{Aut}\begin{pmatrix} PSL\,(2, p^r) \\ PGL\,(2, p^s) \end{pmatrix}$ and $C/C_C(C \cap M) \in D_\pi$. It follows $C \in D_\pi$. As before we get a contradiction.

So we may assume that $K \cap M \neq 1$ and $K \cap M < H \cap M$.

Since H and K are both solvable and K is a Hall π-subgroup of every solvable subgroup of G in which it is contained, it follows that K is a Hall π-subgroup of $N_G(K \cap M)$.

It follows then $N_G(K \cap M) \cap (H \cap M) = N_{H \cap M}(K \cap M) = K \cap M$.

So, since $K \cap M$ is properly contained in $H \cap M$, we can exclude the possibilities $H \cap M$ cyclic and $H \cap M$ a Sylow r-subgroup of M (in general $H \cap M$ nilpotent).

If $H \cap M$ is dihedral, then we can proceed as in [4], with just the remark that the normalizer in G and so the centralizer of a solvable subgroup of M is solvable and so it satisfies D_π, and, further, that 3 cannot be in π in this case, by Lemma 2.1.

So $H \cap M$ must be a Frobenius group of order a divisor of $q(q-1)$.

Since $K \cap M$ is selfnormalizing in $H \cap M$, $K \cap M$ can be neither a p-subgroup nor a subgroup of order a proper divisor of $\big((q-1)/(2, q-1), |H \cap M|\big)$. Further $K \cap M$ cannot have as order $|H \cap M|/q$.

In fact in this case, $K \cap M$ would be a Hall π_0-subgroup of $H \cap M$ with $\pi_0 = \pi - \{p\}$. Since H is solvable, by Frattini's argument we would have

$$H = (H \cap M)\, N_H(K \cap M)\,.$$

But $N_H(K \cap M) \cap H \cap M = K\, M$, so it would follow $|N_H(K \cap M)| = |K|$.

Further $N_G(K \cap M) \geqslant K, N_H(K \cap M)$.

Since $N_G(K \cap M)$ is solvable and K is a Hall π-subgroup of it, K would be conjugate to $N_H(K \cap M)$ and so we would have a contradiction. So $K \cap M$ must have a composite order $p^r t$ where t is a divisor of $p^r - 1$, $r \leqslant n$.

Let H_0 be the Frobenius kernel of $H \cap M$ and let K_0 that of $K \cap M$.

The Sylow p-subgroups of $PSL(2, q)$ are TI-sets. Hence $K \cap M$ must normalize the entire Sylow p-subgroup of $PSL(2, q)$. So $H_0 = K_0$.

But then $H, K \leqslant N_G(K_0)$ and with the same argument we get the final contradiction.

3. – We now apply the results of Theorem 1.4 and Theorem 2.3 to obtain a characterization of finite groups in the class $D_{\pi,\pi'}$.

The following Theorem 3.2 is a generalization of Theorem 2.2 [4].

First we need to characterize simple groups in $D_{\pi,\pi'}$ in the following:

LEMMA 3.1. – *If G is a simple group in $D_{\pi,\pi'}$, whose order is divisible by primes in π and in π', then $G = PSL(2, q)$, where $q > 3$, $q(q-1) \equiv 0(3)$, $q \equiv -1(4)$ and $\pi(q+1) \subseteq \pi$, $\pi\big(q(q-1)/2\big) \subseteq \pi'$.*

PROOF. – Since the following proof is the revisioned version of the proof of Theorem 1.1 [4], we omit the parts of the proof that can be found in [4]. Further we adopt the same notation.

As in [4] G is one of the groups in the Arad-Fisman list [3], $G = AB$, where A is a Hall π-subgroup and B is a Hall π'-subgroup.

If $G \cong A_r$ or $G \cong PSL(2, q)$ $q \in \{11, 29, 59\}$ or $G \cong PSL(r, q)$ r odd with $(r, q-1) = 1$, or $G \cong PSL(5, 2)$, then $G \notin D_{\pi,\pi'}$ and the proof is in [4].

If $G \cong M_{11}$ then we have two possible cases *a*) $A \cong M_{10}$; *b*) A solvable.

In case *a*) $\pi = \pi(A) = \{2, 3, 5\}$; $\pi' = \{11\}$. S_5 is a subgroup of G and it is maximal. So $G \notin D_\pi$ in this case.

In case *b*) $\pi = \pi(A) = \{2, 3\}$; $\pi' = \{5, 11\}$. $GL(2, 3)$ is a subgroup of G and it is not contained in any Hall π-subgroup of G (which is the normalizer of a Sylow 3-subgroup of G), since $GL(2, 3)$ is maximal (see [6]).

So $G \notin D_\pi$ also in this case.

If $G = M_{23}$, then we have to consider the following two cases: *a*) $A = M_{22}$ and B of order 23; *b*) B Frobenius group of order 11.23.

In case *a*) the proof is the same as in [4] and we get $G \notin D_{\pi,\pi'}$:

In case *b*) A is a split extension of an elementary abelian group of order 2^4 by A_7. Since A_8 is contained in M_{23} and it is maximal we get $G \in D_\pi$ (see [6]).

It only remains the case $G = PSL(2, q)$ where $3 < q \not\equiv 1(4)$ and A solvable. As in [4] the unique possible factorization for G is with A dihedral of order $q+1$ and B Frobenius group of order $q(q-1)/2$.

As in [4] we have 3 divides $q(q-1)$ and in such hypothesis, we can prove $G \in D_{\pi,\pi'}$. (By Remark 2.2 we can exclude q a power of 2).

Now we can prove:

THEOREM 3.2. – *Let G be a group. Then $G \in D_{\pi,\pi'}$ if and only if the composition factors of G are of the following types*: 1) *π-groups*; 2) *π'-groups*; 3) *simple groups $PSL(2, q)$, where $q > 3$, $q(q-1) \equiv 0(3)$, $q \equiv -1(4)$, $\pi(q+1) \subseteq \pi$ and $\pi(q(q-1)/2) \subseteq \pi'$.*

PROOF. – If $G \in D_{\pi,\pi'}$, then, by Theorem 1.4 every subnormal subgroup of G belong to $D_{\pi,\pi'}$. It follows that, if M/N is a composition factor of G, M/N is a π-group or a π'-group or M/N is isomorphic to a simple group $PSL(2, q)$ with the required properties by Lemma 3.2.

Viceversa follows using induction, Theorem 2.3 and [2, Th. 4.6].

BIBLIOGRAPHY

[1] P. HALL, *Theorems like Sylow's*, Proc. London Math. Soc., (3) **6** (1956), pp. 286-304.
[2] F. GROSS, *On the existence of Hall subgroups*, J. of Algebra, **98** (1986), pp. 1-13.
[3] Z. ARAD - E. FISMAN, *On finite factorizable groups*, J. of Algebra, **86** (1984), pp. 522-548.

[4] A. L. GILOTTI - L. SERENA, *A generalization of π-separability in finite groups*, Arch. Math., **97** (1986), pp. 301-308.
[5] B. HUPPERT, *Endliche Gruppen, I*, Springer-Verlag, Berlin-Heidelberg, 1967.
[6] S. A. SISKIN, *Abstract properties of the simple sporadic groups*, Russian Math. Surveys, **35** (1980) II, pp. 209-246.