Hölder Continuity of the Gradient of Solutions of Uniformly Parabolic Equations with Conormal Boundary Conditions (*) (**).

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Summary. – *We study solutions of the conormal derivative problem for uniformly parabolic equations in divergence form. Under weak regularity hypotheses on the operator, the global Hölder continuity of the gradient of a weak solution is established. The method of proof is based on* [5] *and the results extend those in* [7, *Section* V.7].

1. – Introduction.

For D a smooth bounded domain in $\boldsymbol{R}^n$ $(n \geqslant 1)$ and $T > 0$, set

$$\Omega = D \times (0, T), \qquad B\Omega = D \times \{0\}, \qquad S\Omega = \partial D \times (0, T),$$

let γ denote the inner normal to ∂D and write $X = (x, t)$ for a generic point in $\boldsymbol{R}^n \times [0, T]$. We consider the problem

$$\begin{aligned} &- u_t + \operatorname{div} A(X, u, Du) + B(X, u, Du) = 0 \quad \text{in } \Omega, \\ &A(X, u, Du)\cdot\gamma + \psi(X, u) = 0 \quad \text{on } S\Omega, \qquad u = \varphi \quad \text{on } B\Omega \end{aligned} \tag{1.1}$$

for a vector function A and scalar functions B, ψ, and φ. To state our results, we use the function spaces

$$V_2(\Omega) = \{u \in L^2(\Omega) \colon \operatorname*{ess\,sup}_{0<t<T} \|u(\cdot, t)\|_D + \|Du\|_\Omega < \infty\},$$

$$V_2^*(\Omega) = \{u \in V_2(\Omega) \colon \|u(\cdot, t + h) - u(\cdot, t)\|_D \to 0 \text{ as } h \to 0 \text{ for all } t \in [0, T]\}$$

(where $\| \ \|$ denotes the L^2 norm), and $H_{1+\beta}$ the space of functions with finite norm

$$|u|_{1+\beta} = |u|_0 + |Du|_0 + [Du]_\beta + \langle u \rangle_{1+\beta}$$

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where

$$|u|_0 = \sup_{\Omega} |u| ,$$

$$[u]_\beta = \sup_{x \neq y \in \Omega} |u(x) - u(Y)|/|X - Y|^\beta ,$$

$$\langle u \rangle_{1-\beta} = \sup_{\substack{x \in \Omega \\ 0<t<s<T}} |u(x, t) - u(x, s)|/|t - s|^{(1+\beta)/2} ,$$

$$|(x, t) - (y, s)| = \Big(\sum_{i=1}^{n} (x^i - y^i)\Big)^2 + |t - s|^{\frac{1}{2}} .$$

In terms of these spaces, we have the following result.

THEOREM 1.1. – Let $u \in V_2^*$ be a bounded weak solution of (1.1) and suppose there are positive constants $\beta < 1, \lambda, \Lambda, \mu_1, \mu_2, \mu_3$ such that

$$|A(X, z, 0)| \leqslant \mu_1 , \tag{1.2a}$$

$$|A(x, t, z, p) - A(y, t, w, p)| \leqslant \mu_2(1 + |p|)[|x - y|^\beta + |z - w|^\beta] \tag{1.2b}$$

$$|B(X, z, p)| \leqslant \mu_3(1 + |p|^2) \tag{1.2c}$$

for all $(X, z, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ with $|z| \leqslant M = \sup |u|$ and all $(y, w) \in D \times [-M, M]$. Suppose also that

$$|A(x, t, z, p) - A(x, s, z, p)| \leqslant \mu_2 |t - s|^{\beta/2}(1 + |p|) , \tag{1.3}$$

$$|\psi(x, t, z) - \psi(y, s, w)| \leqslant \mu_2(|x - y|^2 + |t - s| + |z - w|^2)^{\beta/2} \tag{1.4}$$

for all $(X, z, p) \in S\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ and all $(y, s, w) \in S\Omega \times \boldsymbol{R}$ with $|z|, |w| \leqslant M$. Suppose further that the matrix $(a^{ij}) = (\partial A^i/\partial p_j)$ satisfies

$$a^{ij}(X, z, p)\xi_i\xi_j \geqslant \lambda|\xi|^2 , \qquad |a^{ij}(X, z, p)| \leqslant \Lambda \tag{1.5}$$

for all $(X, z, p) \in \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n$ with $|z| \leqslant M$ and all $\xi \in \boldsymbol{R}^n$. Suppose finally that $\partial D \in H_{1+\beta}$, that $\varphi \in H_{1+\beta}(D)$, and that

$$A(x, 0, \varphi(x), D\varphi(x)) \cdot \gamma + \psi(x, 0, \varphi(x)) = 0 \quad \text{on } \partial D . \tag{1.6}$$

Then there is a constant $\delta = \delta(n, \lambda, \Lambda, \beta) > 0$ such that $u \in H_{1+\delta}$ with

$$|u|_{1+\delta} \leqslant C(\lambda, \lambda, \Lambda, M, \mu_1, \mu_2, \mu_3, |\varphi|_{1+\beta}, D, T) . \tag{1.7}$$

THEOREM 1.2. – In addition to the hypotheses of Theorem 1.1, suppose there is a function $\omega(\tau, N)$, increasing in both variables with $\lim_{\tau \to 0+} \omega(\tau, N) = 0$ for each N, such that

$$|a^{ij}(X, z, p) - a^{ij}(X, z, q)| \leqslant \omega(|p - q|, N) \tag{1.8}$$

for all $(X, z, p) \in \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n$ and $q \in \boldsymbol{R}^n$ with $|z| \leqslant M$, $|p|$, $|q| \leqslant N$. If also

$$(1.9) \qquad |a^{ij}(x, t, z, p) - a^{ij}(x, s, z, p)| \leqslant \big(|t-s|^{\frac{1}{2}}, N\big)$$

or

$$(1.9)' \qquad |A(x, t, z, p) - A(x, s, z, p)| \leqslant \omega(1, N)|t-s|^{\beta/2}$$

for all $(X, z, p) \in \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n$ and $s \in [0, T]$ with $|z| \leqslant M$, $|p| \leqslant N$, then (1.6) holds with $\delta = \beta$ and C depending also on ω.

Note that (1.8) quantifies the assumption that a^{ij} is uniformly continuous on any bounded subset of $\Omega \times [-M, M] \times \boldsymbol{R}^n$, and similarly for (1.9) and (1.9)'.

We begin in Section 2 with estimates on $|u|_0$ and on various Hölder norms of u. Of particular interest is Lemma 2.4, which estimates $\langle u \rangle_{1+\beta}$ in terms of the other terms in $|u|_{1+\beta}$; this lemma is based on a similar result for linear parabolic equations [6]. Next we study mean oscillations of solutions of linear parabolic conormal problems in Section 3. This section follows the corresponding results for elliptic equations [1] and [4, Chapter III], but the parabolic results presented here seem to be new. Theorem 1.1 and 1.2 are proved in Section 4 using a modified version of the arguments in [5] for the elliptic Dirichlet problem; some of the ideas from Section 2 reappear in a more general form. We close with some remarks on the elliptic conormal problem.

2. – Pointwise estimates on the solution.

We suppose initially that $\partial D \in C^1$; later on this assumption will be strengthened. An important consequence of this assumption is the following Sobolev inequality.

LEMMA 2.1. – There is a constant $K = K(D)$ such that

$$(2.1) \qquad \left(\iint_{\Omega} |h|^{2(n+2)/n}\, dx\, dt\right)^{n/(n+2)} \leqslant K\left[\iint_{\Omega} (|Dh|^2 + h^2)\, dx\, dt + \operatorname*{ess\,sup}_{0<t<T} \int_D h(x, t)^2\, dx\right].$$

for all $h \in V_2^*$.

PROOF. – As in [10, Lemma 1.3] (with u there identically one), we need only check that

$$\int_{\partial D} g\, ds \leqslant K_0(D) \int_D (|Dg| + g)\, dx$$

for all nonnegative Lipschitz functions g. But the assumed smoothness of ∂D and a simple partition of unity argument yield the desired inequality. ∎

Note that part of the conclusion of Lemma 2.1 is that V_2^* is embedded in $L^{2(n+2)/n}(\Omega)$.
In addition to the function spaces V_2 and V_2^*, we introduce

$$W_2^{1,1}(\Omega) = \{u \in L^2 \colon u_t \in L^2,\ Du \in L^2\}\ ,$$

and $\overset{0}{W}{}_2^{1,1}$, the closure in $W_2^{1,1}$ of the set of C^∞ functions which vanish at $t = 0$. A weak solution of (1.1) is defined to be any function $u \in V_2^*$ which satisfies the integral identity

$$(2.2a)\qquad \int_D u(x, T_0)\eta(x, T_0)\,dx + \int_0^{T_0}\!\!\int_D -u\eta_t + D\eta\cdot A(X, u, Du) - B(Xu, Du)\eta\,dx\,dt = \int_0^{T_0}\!\!\int_{\partial D}\psi(X, u)\eta(X)\,ds\,dt$$

for all bounded $\eta \in \overset{0}{W}{}_2^{1,1}(\Omega)$ and all $T_0 \in (0, T)$ with

$$(2.2b)\qquad \lim_{t\to 0}\|u(\cdot, t) - \varphi\|_D = 0\ .$$

In fact we shall often need to consider weak solutions of a slightly more general identity, namely

$$(2.2a)'\qquad \int_D u(x, T_0)\eta(x, T_0)\alpha(x)\,dx + \int_0^{T_0}\!\!\int_D -u\eta_t\alpha(x) + D\eta\cdot A - B\eta\,dx\,dt = \int_0^{T_0}\!\!\int_{\partial D}\psi(X, u)\eta(X)\,ds\,dt$$

with α a nonnegative function.

We now estimate the maximum of u, following [10, Lemmata 3.1 and 3.2] and [11, Theorem 4.1].

LEMMA 2.2. – Let $a_0, a_1, b_0, b_1, c_0, M$ be nonnegative constants with $a_0 > 0$ and $\operatorname{ess\,sup}_D |\varphi| \leqslant M$, and suppose that the conditions

$$(2.3)\qquad p\cdot A(X, z, p) \geqslant |p|^2 - a_1|z|^2\ ,$$

$$(2.4)\qquad zB(X, z, p) \leqslant b_0|p|^2 + b_1|z|^2\ ,$$

$$(2.5)\qquad z\psi(X, z) \leqslant c_0|z|^2$$

are satisfied wherever $|z| \geqslant M$ (and for all X, p for which the functions are defined). Suppose also that

$$(2.6)\qquad \alpha \geqslant a_0 \quad \text{in } D$$

and set $a = a_1 + b_1 + c_0^2 + c_0$. Then any bounded weak solution u of $(2.2a)'$, $(2.2b)$ satisfies

$$(2.7)\qquad \sup_\Omega |u| \leqslant C\big(a_0, b_0, D, a(T + |D|^{2(/n+2)})\big)M\ .$$

PROOF. – Let $q\geqslant 2+2b_0$, set

$$\left(1-\frac{M}{|u|}\right)_+=\max\left\{\left(1-\frac{M}{|u|}\right),0\right\},$$

suppose initially that $u\in W_2^{1,1}$ and choose

$$\eta=|u|^{q-2}\left[\left(1-\frac{M}{|u|}\right)_+\right]^{(n+2)q-n-1}u$$

in $(2.2a)'$. Writing

$$U=\int\limits_M^{|u|}\xi^{q-1}\left(1-\frac{M}{\xi}\right)_+^{(n+2)q-n-1}d\xi\,,$$

$$q_1=[(n+2)q-n-1]\frac{M}{|u|}+(q-1)\left(1-\frac{M}{|u|}\right)$$

(q_1 was given incorrectly in [10, p. 225]), we arrive at the identity

$$\int\limits_D\alpha(x)\,U(x,T_0)\,dx+\int\limits_0^{T_0}\int\limits_D Du\cdot Aq_1\left(1-\frac{M}{|u|}\right)_+^{(n+2)(q-1)}|u|^{q-2}\,dx\,dt-$$
$$-\int\limits_0^{T_0}\int\limits_D uB\left(1-\frac{M}{|u|}\right)_+^{(n+2)q-n-1}|u|^{q-1}\,dx\,dt=\int\limits_0^{T_0}\int\limits_{\partial D}u\psi\left(1-\frac{M}{|u|}\right)_+^{(n+2)q-n-1}|u|^{q-2}\,ds\,dt\,,$$

and a standard Steklov averaging argument, as in [7, (2.16) of Chapter III], shows that this identity is also valid for u merely in V_2^*. An easy calculation gives a lower bound for U from which it follows that

$$\sup_{0<t<T}\int\limits_{D\times\{t\}}\left(1-\frac{M}{|u|}\right)_+^{(n+1)q-n}|u|^q\,dx+\iint\limits_\Omega|Du|^2\left(1-\frac{M}{|u|}\right)_+^{(n+2)q-n-2}|u|^{q-2}\,dx\,dt\leqslant$$
$$\leqslant c_1(a_0,D)\,aq^2\iint\limits_\Omega\left(1-\frac{M}{|u|}\right)_+^{(n+2(q-n-2}|u|^q\,dx\,dt\,.$$

Then a standard iteration scheme, which involves Lemma 2.1, gives

$$\sup_Q|u|\leqslant 2M+c_2(a_0,b_0,D)\left(a^{(n+2)/2}\iint\limits_\Omega|u|^q\,dx\,dt\right)^{1/q}\tag{2.8}$$

for all $q\geqslant 1$.

To bound the integral of $|u|^q$, we assume first that $u\in W_2^{1,1}$ and set

$$\eta=(|u|^{q-1}-M^{q-1})_+u/|u|$$

for $q>1$, $q\geqslant 1+b_0$. Now we obtain

$$\int_{D(T_0)}|u|^q\,dx\leqslant c_3(D,q)\,a\int_0^{T_0}\int_{D(t)}|u|^q\,dx\,dt+c_4(q)|D|M^q$$

where $D(t)=\{x\in D\colon |u(x,t)|>M\}$ and therefore

$$\iint_{\Omega_M}|u|^q\,dx\,dt\leqslant c_4\exp(c_3aT)|D|M^q\,. \qquad \blacksquare \tag{2.9}$$

Note that C in (2.7) depends on D only through the constants K and K_0 from Lemma 2.1 and the explicit display of its measure.

A careful examination of the proof of Lemma 2.2 shows that the boundedness of u can be relaxed to $u(\cdot,t)\in L^r(D)$ for some $r>1+b_0$ and all $t\in(0,T)$. Specifically, in deriving (2.9), we replace $|u|^{q-1}$ by $\max\{|u|^{q-r},N^{q-r}\}|u|^{r-1}$ for large N, assuming without loss of generality that $r\geqslant 2$; as in [15, p. 846], we send $N\to\infty$ to infer that $\iint|u|^q\,dx\,dt$ and $\iint|Du|^2|u|^{q-1}\,dx\,dt$ are finite for all q. An alternative proof, using a different iteration scheme, can be achieved with the test functions from [7, Section V.3]. As in that section, conditions (2.3)-(2.5) can be weakened but we shall not be concerned with that here.

For our next estimates, we introduce for $X_0=(x_0,t_0)\in\mathbf{R}^{n+1}$ and $R>0$ the sets

$$\begin{aligned}
&Q_R=Q_R(X_0)=\{X\in\mathbf{R}^{n+1}\colon |x-x_0|<R,\ t_0-R^2<t<t_0\}\,,\\
&Q_R^+=\{X\in Q_R\colon x^n<x_0^n\}\,,\qquad Q_R^0=\{X\in Q_R\colon x^n=x_0^n\}\,,\\
&Q_R^*=\{X\in\partial Q_R\colon x^n\geqslant x_0^n,\ t<t_0\}\,.
\end{aligned}$$

Note that $Q_R^*\cup Q_R^+=PQ_R^+$, the parabolic boundary of Q_R^+. We also introduce the sets $\tilde Q_R,\tilde Q_R^+,\tilde Q_R^0$, and $\tilde Q_R^*$ by replacing the inequalities $t_0-R^2<t<t_0$ by $t_0<t<+t_0+R^2$ in the corresponding definitions. We then have the following Hölder estimate.

LEMMA 2.3. – Let M, a_0-a_4, b_0, b_1, c_0 be nonnegative constants with $a_4\geqslant a_0>0$. Suppose that

$$p\cdot A(X,z,p)\geqslant|p|^2-a_1\,,\qquad |A(X,z,p)|\leqslant a_2|p|+a_3 \tag{2.10}$$

$$|B(X,z,p)|\leqslant b_0|p|^2+b_1\,,\qquad a_0\leqslant\alpha(x)\leqslant a_4\,, \tag{2.11}$$

$$|\psi(X,z)|\leqslant c_0 \tag{2.12}$$

for $|z|\leqslant M$. Then there are constants C and σ depending only on a_0, a_4, b_0M such that any bounded weak solution of

$$-\alpha(x)u_t+\operatorname{div}A(X,u,Du)+B(X,u,Du)=0\quad\text{in } Q_R^+ \tag{2.13}$$

with $|u|\leqslant M$ obeys the estimate

$$\operatorname*{osc}_{Q_r^+} u \leqslant C[\operatorname*{osc}_{Q_R^+} u + (a_1 + a_3)R + b_1R^2](r/R)^\sigma + C \operatorname*{osc}_{Q_R^0} u \quad \text{for } 0 < r < R . \tag{2.14}$$

If also

$$A^n(X, u, Du) + \psi(X, u) = 0 \quad \text{on } Q_R^0 , \tag{2.15}$$

then

$$\operatorname*{osc}_{Q_r^+} u \leqslant C[\operatorname*{osc}_{Q_R^+} u + (a_1 + a_3 + c_0)R + b_1R^2](r/R)^\sigma \quad \text{for } 0 < r < R . \tag{2.16}$$

These results are also valid with Q_R^+ replaced by $\tilde{Q}_R^+$, etc., if $u = 0$ for $t = t_0$.

PROOF. – (2.14) follows from [18, Theorem 4.2] (see also [7, Theorem V.1.1]). To prove (2.16), we reduce to the case $\psi \equiv 0$ by replacing A^n with

$$A^n(X, z, p) + \psi(x', 0, t, u(x', 0, t)) .$$

Then the test function arguments of [7, Theorem V.1.1.] and [18, Theorem 2.2] can be applied. ■

Note that Lemma 2.3 and appropriate change of variables allow us to obtain a global Hölder modulus of continuity for any bounded weak solution of (1.1). Specifically under the change of variables Ψ: $(x, t) \to (y, t)$ (note that y may also be a function of t), (1.1) goes over to

$$-\alpha v_t + \operatorname{div} \bar{A}(Y, v, Dv) + \bar{B}(Y, v, Dv) = 0 \quad \text{in } \Psi(\Omega) ,$$
$$\bar{A}[Y, v, Dv)\cdot\bar{\gamma} + \bar{\psi}(Y, v) = 0 \quad \text{on } S\Psi(\Omega) , \qquad v = \bar{\varphi} \quad \text{on } B\Psi(\Omega)$$

where

$$\alpha = \det(\partial x/\partial y) , \quad \bar{A}^j(Y, w, q) = \alpha A^i(X, w, p)\, \partial y^j/\partial x^i , \quad \bar{B}(Y, w, q) = \alpha B(X, w, p) ,$$
$$\bar{\psi}(X, z) = \alpha\psi(X, z) , \quad \bar{\varphi}(y) = \varphi(x) , \quad v(y) = u(x) , \quad \text{and } p_i = q_j\, \partial y^j/\partial x^i .$$

Our final lemma connects the temporal and spatial regularity of u.

LEMMA 2.4. – Let $M, \mu_0 - \mu_5, \lambda, \Lambda, \sigma, R$ be positive constants with $\sigma \leqslant 1$. Suppose A is weakly differentiable with respect to p and suppose that the conditions

$$a^{ij}(X, z, p)\xi_i\xi_j \geqslant \lambda|\xi|^2 \quad \text{for all } \xi \in \boldsymbol{R}^n , \tag{2.17}$$

$$\sum_{i,j} |a^{ij}(X, z, p)| \leqslant \Lambda, \tag{2.18}$$

$$|A(X, z, 0) - A(X_0, w, 0)| \leqslant \mu_1(|X - X_0|^2 + |z - w|^2)^{\sigma/2} \tag{2.19}$$

$$\alpha(x) \geqslant \mu_0 \tag{2.20}$$

hold for all $(X, z, p) \in \tilde{Q}_R^0 \cup \tilde{Q}_R^+ \times \mathbf{R} \times \mathbf{R}^n$ with $|z| \leqslant M$ and $|w| \leqslant M$. Suppose also that

$$|\psi(X, z) - \psi(X_0, w)| \leqslant \mu_2(|X - X_0|^2 + |z - w|^2)^{\sigma/2}. \tag{2.21}$$

Then any solution u of (2.13), (2.15) with $|u| \leqslant M$ and

$$|Du(X)| \leqslant \mu_3, \quad |Du(X) - Du(X_0)| \leqslant \mu_4 |X - X_0|^\sigma, \quad |B(X, u, Du)| \leqslant \mu_5 \tag{2.22}$$

for all $X \in \tilde{Q}_R^+$ obeys the estimate

$$|u(x_0, t) - u(x_0, t_0)| \leqslant C(\lambda, \Lambda, \mu_0 - \mu_4, \mu_5 R^{1-\sigma}, n)|t - t_0|^{(1+\sigma)/2} \tag{2.23}$$

for $t_0 \leqslant t \leqslant t_0 + R^2$.

PROOF. – We imitate the argument of [6]. Fix $t_1 \in (t_0, t_0 + R^2)$, set

$$\varrho = [4\Lambda(t_1 - t_0)/\mu_0]^{\frac{1}{2}}, \qquad s = \sup_{t_0 < t < t_1} [u(x_0, t) - u(x_0, t_0)],$$

and assume without loss of generality that $s > 0$ and $\varrho \leqslant \lambda R/8\Lambda$. Now define

$$\|x - x_0\| = (\lambda^2 |x' - x_0'|^2 / 64\Lambda^2 + |x^n - x_0^n|^2)^{\frac{1}{2}}$$

and introduce the sets N^+, N^0, N^* by replacing $|x - x_0|$ with $\|x - x_0\|$ in the definitions of $\tilde{Q}_\varrho^+$, $\tilde{Q}_\varrho^0$, Q_ϱ^*, respectively.

We now consider the function

$$v(x, t) = \left(\mu_5 + \frac{2\Lambda s}{\varrho^2}\right)\frac{(t - t_0)}{\mu_0} + \frac{s\|x - x_0\|^2}{\varrho^2} + \mu_4 \varrho^{1+\sigma} + \\ + \left[\frac{\mu_2}{\lambda}\left\{(1 + \mu_3)\frac{8\Lambda}{\lambda}\varrho\right\}^\sigma + \frac{s}{4\varrho}\right](\varrho - x^n + x_0^n) + Du(X_0)\cdot(x - x_0).$$

Because v is smooth we have

$$-\alpha v_t + \operatorname{div} A(X_0, u(X_0), Dv) = -\alpha v_t + a^{ij}(X_0, u(X_0), Dv) D_{ij} v \leqslant -\mu_5 \quad \text{in } N^+,$$

$$A^n(X_0, u(x_0), Dv) = -\psi(X_0, u(X_0)) + \\ + \int_0^1 a^{nj}(X_0, u(X_0)), \sigma Dv + (1 - \sigma) Du(X_0))\, d\sigma (D_j v - D_j u(X_0)) \leqslant \\ \leqslant -\psi(X_0, u(X_0)) - \mu_2\{(1 + \mu_3)(1 + 16\Lambda/\lambda)\varrho\}^\sigma \quad \text{on } N^0.$$

Hence if we set

$$\bar{a}^{ij} = \int_0^1 a^{ij}(X_0, u(X_0), \sigma Du(X_0) + (1-\sigma) Dv)\, d\sigma ,$$

$$f^i = A^i(X, u, Du) - A^i(X_0, u(X_0), Du(X_0)) ,$$

we see that $w = u - u(x_0, t_0) - v$ is a weak solution of

$$-\alpha w_t + D_i(\bar{a}^{ij} D_j w + f^i) \geq 0 \quad \text{in } N^+ ,$$

$$\bar{a}^{nj} D_j w + f^n \geq 0 \quad \text{on } N^0 , \qquad w \leq 0 \quad \text{on } N^* .$$

A simple variant of Lemma 2.2 with $M = \varrho \sup |f|$, $a_1 = \varrho^{-2}$, $b_0 = b_1 = c_0 = 0$ and K and K_0 independent of ϱ shows that

$$\begin{aligned} w &\leq C_1(\lambda, \mu_0, n, \delta) \varrho \sup |f| \\ &\leq C_2(\lambda, \Lambda, \delta, \delta, \sigma, \mu_0 - \mu_4) \varrho^{1+\sigma} \end{aligned}$$

Evaluating this inequality at $x = x_0$ yields

$$u(x_0, t) \leq v(x_0, t) + C_2 \varrho^{1+\sigma}$$

and taking the supremum over $t \in (t_0, t_1)$ yields

$$s \leq C\varrho^{1+\sigma} + \mu_5(t_1 - t_0) + s\left[\frac{2\Lambda}{\mu_0 \varrho^2}(t_1 - t_0) + \frac{1}{4}\right] \leq C\varrho^{1+\sigma} + \frac{1}{2}s$$

from which an upper bound for $u(x_0, t) - u(x_0, t_0)$ follows easily. A similar argument gives a lower bound. ∎

The limiting case $\sigma = 0$ in Lemma 2.4 will also be important in what follows. In this case, we obtain

$$|u(x_0, t) - u(x_0, t_0)| \leq C(\lambda, \Lambda, \mu_0, \mu_1, \mu_2, \mu_5 R, n)(1 + \mu_3)|t - t_0|^{\frac{1}{2}} \tag{2.23}$$

by removing the term $Du(X_0)\cdot(x - x_0)$ from v. More generally, if

$$|B(X, z, p)| \leq \mu_5 + \lambda \mu_6 |p|^2 , \qquad \mu_6 > 0 ,$$

we can reduce to the case $\mu_6 = 0$ by considering

$$u^{\pm} = \pm [\exp(\pm 2\mu_6 u)]/2\mu_6$$

in place of u to infer $(2.23)'$ with C now depending also on $\mu_6(M + \mu_1^2 R)$.

Clearly analogous results hold for solutions of (2.13) in $\tilde{Q}_R$, and in $\tilde{Q}_R \cap \Omega$ if the restriction of u to $S\Omega \cap Q_R$ satisfies an appropriate Hölder condition in time. Hence we need only prove the Hölder continuity of Du to infer that $u \in H_{1+\sigma}$ for suitable $\sigma > 0$.

3. – Estimates for linear problems.

In this section we prove some estimates for solutions of some simple linear parabolic conormal problems which extend the corresponding results of CAMPANATO [1] for the elliptic Dirichlet problem. Our exposition follows (although not closely) [4, Chapter III] which in turn is based on [1]. The reader is referred also to [2] for some related parabolic estimates.

Our first estimate is for constant coefficient, homogeneous problems.

LEMMA 3.1. – Let (A^{ij}) be a constant matrix satisfying

$$A^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbf{R}^n, \qquad \max_{i,j} |A^{ij}| \leqslant \Lambda \tag{3.1}$$

for positive constants λ and Λ. Then any weak solution v of

$$v_t = D_i(A^{ij}D_jv) \quad \text{in } Q_R^+, \qquad A^{nj}D_jv = 0 \quad \text{on } Q_R^0 \tag{3.2}$$

is in $C^\infty(Q_R^+ \cup Q_R^0)$ and there is a constant $C = C(\lambda, \Lambda, n)$ such that

$$\max_{t_0-(R/2)^2<t<t_1} \int\limits_{B_{R/2}^+\times\{t\}} v^2\,dx + \int\limits_{Q_{R/2}^+} |Dv|^2\,dx\,dt \leqslant CR^{-2}\iint\limits_{Q_R^+} v^2\,dx\,dt, \tag{3.3}$$

$$\iint\limits_{Q_\varrho^+} v^2\,dx\,dt \leqslant C(\varrho/R)^{n+2}\iint\limits_{Q_R^+} v^2\,dx\,dt, \tag{3.4}$$

$$\iint\limits_{Q_\varrho^+} |v - \{v\}_\varrho|^2 \leqslant C(\varrho/R)^{n+4}\iint\limits_{Q_R^+} |v - \{v\}_R|^2\,dx\,dt \tag{3.5}$$

for all $\varrho < R$ where

$$\{v\}_\varrho = \iint\limits_{Q_\varrho^+} v\,dx\,dt \Big/ \iint\limits_{Q_\varrho^+} dx\,dt, \qquad B_R^+ = \{x \in \mathbf{R}^n\colon |x| < R,\ x^n > 0\}. \tag{3.6}$$

PROOF. – The smoothness of v is well-known.

To prove (3.3), we let ζ be a $C^1(\overline{Q_R^+})$ function with

$$\zeta \equiv 1 \quad \text{in } Q_{R/2}^+ , \qquad \zeta \equiv 0 \quad \text{on } Q_R^* , \qquad 0 \leqslant \zeta \leqslant 1 \quad \text{on } Q_R^+ ,$$
$$|D\zeta|^2, |\zeta_t| \leqslant 16R^{-2} \qquad \text{in } Q_R^+ .$$

Using $\eta = \zeta^2 v$ as a test function leads readily to (3.3).

Clearly

$$\iint\limits_{Q_\varrho^+} v^2\, dx\, dt \leqslant c(n)\varrho^{n+2}(\sup_{Q_{R/2}^+} v)^2$$

for $\varrho \leqslant R/2$ and the proof of (2.9) (with $M = 0$ and η multiplied by $\zeta^{(n+2)q-n-1}$; cf. [10, Lemma 4.2]) implies that

$$\sup_{Q_{R/2}^+} v \leqslant C(n, \lambda, \Lambda)\left(R^{-n-2}\iint\limits_{Q_R^+} v^2\, dx\, dt\right)^{\frac{1}{2}} .$$

The combination of these estimates gives (3.4) for $\varrho \leqslant R/2$, while (3.4) is obvious for $\varrho > R/2$.

Next we use the Poincaré inequality to infer that

$$\iint\limits_{Q_\varrho^+} |v - \{v\}_\varrho|^2\, dx\, dt \leqslant C(n)\varrho^2\Big[\varrho^2\iint\limits_{Q_\varrho^+} (v_t)^2\, dx\, dt + \iint\limits_{Q_\varrho^+} |Dv|^2\, dx\, dt\Big] .$$

Using the differential equation for v and (3.1), we obtain

$$\iint\limits_{Q_\varrho^+} (v_t)^2\, dx\, dt \leqslant C(\lambda, \Lambda, n)\iint\limits_{Q_\varrho^+} |Dv|^2\, dx\, dt \leqslant C(\lambda, \Lambda, n)\Big[\iint\limits_{Q_{\varrho}^+} |D(D'v)|^2 + |D(A^{nj}D_j v)|^2\, dx\, dt\Big] .$$

We now assume that $\varrho \leqslant R/8$, since otherwise (3.5) is clear. Since (3.3) and (3.4) are also valid for each component of $D'v$ and for $A^{nj}D_j v$ (because the latter function vanishes on Q_R^0), we obtain

$$\iint\limits_{Q_\varrho^+} |v - \{v\}_\varrho|^2\, dx\, dt \leqslant C\varrho^2\iint\limits_{Q_{2\varrho}^+} |Dv|^2\, dx\, dt \leqslant C\varrho^2(\varrho/R)^{n+2}\iint\limits_{Q_{R/2}^+} |Dv|^2\, dx\, dt .$$

The proof is completed by noting that (3.3) is valid also for $v - \{v\}_R$. ∎

We remark that (3.3) and (3.4) remain valid if the matrix (A^{ij}) is merely bounded and measurable.

For constant coefficient, inhomogeneous problems, the following result is valid.

LEMMA 3.2. – With A^{ij}, Λ, λ as in Lemma 3.1, suppose u is a continuous weak solution of

$$u_t = D_i(A^{ij}D_j u + f^i) \quad \text{in } Q_R^+, \qquad A^{nj}D_j u + f^n = 0 \quad \text{on } Q_R^+ \tag{3.7}$$

for some $f \in L^2(Q_R^+)$, and set

$$I(u;\varrho) = \iint\limits_{Q_\varrho^+} |u - \{u\}_\varrho|^2\,dx\,dt + \varrho^2 \iint\limits_{Q_\varrho^+} |Du|^2\,dx\,dt\,. \tag{3.8}$$

If $\varrho < R$, then

$$I(u;\varrho) \leqslant C(n,\lambda,\Lambda)\Big[(\varrho/R)^{n+4} I(u;R) + R^2 \iint\limits_{Q_R^+} |f|^2\,dx\,dt\Big]\,. \tag{3.9}$$

PROOF. – Let v be the classical solution of (3.2) satisfying

$$u = v \quad \text{on } Q_R^*,$$

given by [13, Theorem 3] and note that $v \in V_2^*(Q_r^+)$ for all $r < R$. Set $w = u - v$ and observe that

$$\int\limits_{B_R^+ \times \{t_0\}} w\eta\,dx = \iint\limits_{Q_R^+} (A^{ij}D_j w + f^i) D_i \eta\,dx\,dt + \iint\limits_{Q_R^+} w\eta_t\,dx\,dt$$

for all $\eta \in W_2^{1,1}$ which vanish in a neighborhood of Q_R^*. In particular we can take $\eta = \eta_\varepsilon = (u - v - \varepsilon)_+ - (v - u - \varepsilon)_+$ (and use a Steklov averaging argument) to infer that

$$\iint\limits_{\{\eta_\varepsilon \neq 0\}} |Dw|^2\,dx\,dt \leqslant CR^{-1} \iint\limits_{Q_R^+} |f|^2\,dx\,dt$$

because $D\eta = Dw$ where $\eta \neq 0$. By sending ε to zero, we conclude that $w \in V_2^*$ so $v \in V_2^*$. Hence we can apply (3.3) to w, (3.4) to Dv, and (3.5) to v. Adding the resulting inequalities yields (3.9). ∎

Finally we consider a variable coefficient homogeneous problem.

COROLLARY. – Let a^{ij} be defined in Q_R^+ with

$$\sum_{i,j} |a^{ij}(X) - a^{ij}(X_0)| \leqslant \omega(|X - X_0|) \tag{3.10}$$

for some nonnegative increasing function ω and let $A^{ij} = a^{ij}(X_0)$ be as in Lemma 3.1. Then any weak solution u of

$$u_t = D_i(a^{ij}D_j u) \quad \text{in } Q_R^+, \qquad a^{nj}D_j u = 0 \quad \text{on } Q_R^0 \tag{3.11}$$

satisfies

$$\iint_{Q_\varrho^+} |u - \{u\}_\varrho|^2\,dx\,dt \leqslant C(n, \lambda, \Lambda)[(\varrho/R)^{n+4} + \omega(R)] \iint_{Q_R^+} |u - \{u\}_R|^2\,dx\,dt\,.$$

PROOF. – Assume without loss of generality that $\varrho < R/2$, and apply Lemma 3.2 with $f^i = [a^{ij}(X) - a^{ij}(X_0)]D_j u$ in $Q_{R/2}^+$ to infer that

$$I(u;\,\varrho) \leqslant C(n, \lambda, \Lambda)\Big[(\varrho_4 R)^{n+4} I(u;\,R_4 2) + \omega(R)R^2 \iint_{Q_{R/2}^+} |Du|^2\,dx\,dt\Big]\,.$$

Now we throw away the gradient term on the left hand side of this inequality and estimate the gradient terms on the right via (3.3), taking advantage of the fact that

$$\iint_{Q_\varrho^+} |u - \{u\}_\varrho|^2\,dx\,dt = \min_{L\in \mathbf{R}} \iint_{Q_\varrho^+} |u - L|^2\,dx\,dt\,. \tag{3.12}$$ ■

Notice that we can replace Q_R^+ by $\tilde{Q}_R^+$, etc., if v, u and f^i are zero for $t = t_0$.

4. – Proof of Theorems 1.1 and 1.2.

We now prove the gradient estimates of Theorems 1.1 and 1.2. The key to these results is an existence and regularity result under simpler hypotheses.

LEMMA 4.1. – Let λ, Λ, t_0, R be positive constants, let $\varphi \in C(\overline{Q_R^+})$, and suppose A^i only depends on t and p. If

$$a^{ij}(t, p)\xi_i\xi_j \geqslant \lambda|\xi|^2\,, \qquad |a^{ij}(t, p)| + |A^i(t, 0)| \leqslant \Lambda \tag{4.1}$$

for all $(t, p) \in (t_0 - R^2, t_0) \times \mathbf{R}^n$, then there is a positive constant $\sigma = \sigma(n, \lambda, \Lambda)$ and a (unique) function $v \in C(\overline{Q_R^+}) \cap \bigcap_{r<R} H_{1+\sigma}(Q_r^+)$ which is a weak solution of

$$v_t = \operatorname{div} A(t, Dv) \text{ in } Q_r^+\,, \quad A^n(t, Dv) = 0 \text{ on } Q_r^+\,, \quad v = \varphi \text{ on } Q_R^* \tag{4.2}$$

for all $r < R$. Moreover

$$\sup_{Q_r^+} |D_k v| \leqslant C(n, \lambda, \Lambda)\Big(r^{-n-2} \iint_{Q_{2r}^+} |D_k v|^2\,dx\,dt\Big)^{\frac12} \tag{4.3}$$

$$\operatorname*{osc}_{Q_r^+} D_k v \leqslant C(n, \lambda, \Lambda)(r/R)^\sigma \operatorname*{osc}_{Q_{R/2}^+} D_k v \tag{4.4}$$

for $k = 1, \ldots, n = 1$ and $r < R/2$, and

$$\sup_{Q_r^+} |D_n v| \leqslant C(n, \lambda, \Lambda)\left[\left(r^{-n-2}\iint_{Q_2^+} |D_n v|^2\,dx\,dt\right)^{\frac{1}{2}} + \sup_{Q_{2r}^0} |D_n v|\right] \tag{4.5}$$

$$\operatorname*{osc}_{Q_r^+} D_n v \leqslant C(n, \lambda, \Lambda)\left((r/R)^\sigma \operatorname*{osc}_{Q_{R/2}^+} D_n v + \operatorname*{osc}_{Q_{R/2}^0} D_n v\right). \tag{4.6}$$

PROOF OF LEMMA. – We solve (4.2) by an approximation argument. Since the approximating solutions satisfy (4.3)-(4.6) uniformly, these inequalities are valid also for the limit function.

Via a suitable mollification we may assume that $A \in C^\infty(Q_R^+)$ and that $\varphi \in H_\theta(Q_R^+)$ for some $\theta \in (0, 1)$. Also we replace A by $\eta A + (1-\eta)p$ where $\eta \in C^\infty(Q_R)$ with $0 \leqslant \eta \leqslant 1$, $\eta = 0$ on PQ_R, $\eta = 1$ in $Q_{(1-\varepsilon)R}$ for $\varepsilon \in (0, 1)$. After so doing we have an approximating problem

$$v_t = \operatorname{div} A(x, t, Dv) \text{ in } Q_R^+, \qquad A^n(x, t, Dv) = 0 \text{ on } Q_R^0, \qquad v = \varphi \text{ on } Q_R^*. \tag{4.7}$$

We shall show that (4.7) has a classical solution in the space $H_{2+\theta}^{(-\theta)}$ defined in [13]. We define a map $P\colon H_{2+\theta}^{(-\theta)} \to H_\theta^{(2-\theta)} \times H_{1+\theta}^{(1-\theta)}(Q_R^+) \times H_\theta(Q_R^*)$ by

$$Pv = \left(v_t - \operatorname{div} A(x, t, Dv), A^n(x, t, Dv), v\right).$$

It is easy to see that the Fréchet derivative of P is always invertible and that P has closed range by virtue of [13, Theorem 4(*b*)] and the uniform boundedness of v, Dv, and $[Dv]_\sigma$ on Q_r^+ for any $r < R$. Standard nonlinear functional analysis (e.g. [9, Lemma 4]) then implies that P is surjective. ∎

Note that this lemma remains valid with Q replaced by $\tilde{Q}$ if $\varphi = 0$ for $t = 0$ and $A^n(0, 0) = 0$.

PROOF OF THEOREM 1.1. – Our goal is to show that

$$\iint_{Q_R \cap \Omega} |Du - \{Du\}_R|^2\,dx\,dt \leqslant CR^{n+2+2\delta} \tag{4.8}$$

for all cylinders Q_R. Then [3, Theorem 3.1 (*b*)] implies the desired estimate. When Q_R meets $B\Omega$ or Q_R does not meet $S\Omega$, (4.8) is proved by a straightforward modification of the proof when Q_R meets $S\Omega$ but not $B\Omega$. Thus we consider only the latter case.

We fix a point $(x_0, t_0) \in S\Omega$ with $t_0 > 1$ and assume without loss of generality that $\psi(x_0, t, u(x_0, t)) = 0$ for $t_0 - 1 < t < t_0$ and that $Q_1 \cap \Omega = Q_1^+$ with $Q_1 \cap \partial\Omega = Q_1^0$. Suppose also that $|Du|$ is bounded and set $N = |Du|_0$. (The finiteness of this quantity will be proved later.) By using an $H_2^{(-1-\beta)}$ change of variables in our flattening, we may assume that there is a function $\alpha \in H_\beta(B_1^+)$ with

$$|D\alpha| \leqslant C(x^n)^{\beta-1}, \quad \alpha \geqslant 1/C > 0, \quad |\alpha|_\beta \leqslant C \text{ in } B_1^+$$

such that u is a weak solution of

$$\alpha u_t = \operatorname{div} A(X, u, Du) + B(X, u, Du) \quad \text{in } Q_R^+,$$
$$A^n(X, u, Du) + \psi(X, u) = 0 \quad \text{on } Q_R^0$$

for all $R < 1$, with A, B, ψ satisfying the hypotheses of Theorem 1.1. Setting

$$\tilde{A} = A/\alpha, \quad \tilde{B} = B - A\cdot D\alpha/\alpha^2, \quad \tilde{\psi} = \psi/\alpha,$$

we see that u is also a weak solution of

$$u_t = \operatorname{div}\tilde{A} + \tilde{B} \quad \text{in } Q_R^+, \quad \tilde{A}^n + \tilde{\psi} = 0 \quad \text{on } Q_R^0$$

with $\tilde{A}$ and $\tilde{\psi}$ satisfying (1.2a, b), (1.3), (1.4), (1.5) and $\tilde{B}$ satisfying

$$|\tilde{B}(X, z, p)| \leqslant \mu_3\big(1 + |p|^2 + (1 + |p|)(x^n)^{\beta-1}\big).$$

(Although we shall not dwell on the matter, it is worth noting that the fact that $Du \in L^\infty$ is needed for some of our test function arguments to work because otherwise $|Du|(x^n)^{\beta-1}$ might not be integrable.)

After performing all these simplifications and suppressing the tildes, we fix $R < 1$ and let v be the solution of

$$v_n = \operatorname{div} A(x_0, t, u(x_0, t), Dv) \quad \text{in } Q_R^+,$$
$$A^n\big(x_0, t, u(x_0, t), Dv\big) = 0 \quad \text{on } Q_R^0, \quad v = u \text{ on } Q_R^+$$

given by Lemma 4.1. Then (3.12), (4.3), and (4.4) imply that

$$(4.9) \qquad \iint\limits_{Q_{\tau R}^+} |D'v - \{D'v\}_{\tau R}|^2\, dx\, dt \leqslant C(\tau R)^{n+2}(\operatorname*{osc}_{Q_{\tau R}^+} D'v)^2 \leqslant$$
$$\leqslant C\tau^{2\sigma}(\tau R)^{n+2}(\operatorname*{osc}_{Q_{R/4}^+} D'v)^2 \leqslant C\tau^{2\sigma}(\tau R)^{n+2} \sup_{Q_{R/4}^+} |D'v - \{D'v\}_{R/2}|^2 \leqslant$$
$$\leqslant C\tau^{n+2+2\sigma}\iint\limits_{Q_{R/2}^+} |D'v - \{D'v\}_{R/2}|^2\, dx\, dt$$

for $0 < \tau < 1/4$. Next, using $u - v$ as test function in the integral identities for u and v (cf. Lemma 3.2), we infer that $v \in V_2^*(Q_R^+)$ and that, for B_R^+ as defined in (3.6),

$$\int\limits_{B_R^+\times\{t_0\}} (u - v)^2\, dx + \iint\limits_{Q_R^+} D(u - v)\cdot\big\{A\big(x_0, t, u(x_0, t), Du\big) - A\big(x_0, t, u(x_0, t), Dv\big)\big\}\, dx\, dt =$$
$$= \iint\limits_{Q_R^+} D(u - v)\cdot\big\{A\big(x_0, t, u(x_0, t), Du\big) - A(X, u, Du)\big\}\, dx\, dt +$$
$$+ \iint\limits_{Q_R^+} B(X, u, Du)(u - v)\, dx\, dt + \iint\limits_{Q_R^0} \psi(X', u)(u - v)\, dx'\, dt.$$

The left hand side of this equation is no smaller than

$$\lambda \iint_{Q_R^+} |D(u-v)|^2\, dx\, dt\,.$$

To estimate the right hand side of this equation, first we use a variant of Lemma 2.2 to infer that

$$\sup_{Q_R^+} |v-\{u\}_R| \leqslant C(R + \sup_{Q_R^+} |u-\{u\}_R|)\,.$$

From the limiting case of $\sigma = 0$ in Lemma 2.4, we also infer that $|u-v| \leqslant C(1+N)R$ in Q_R^+. We also need a sharper estimate on $u-v$. Namely, let $\delta, \varepsilon > 0$ with $\delta < 1/2$ and choose $R_0 \leqslant \varepsilon^{1/(1-2\delta)}$ so that $\operatorname{osc}_{Q_R^+} u \leqslant \varepsilon^{1/(1-2\delta)}$ for $R < R_0$. (That we can choose such an R_0 follows from the (Hölder) continuity of u, obtained by referring to the remarks after Lemma 2.3.) If $R \leqslant R_0$, we have

$$|u-v| < C[\varepsilon^{1/(1-2\delta)}]^{1-2\delta}[(1+N)R]^{2\delta} \quad \text{in } Q_R^+\,,$$

while

$$|u-v| \leqslant C(1+|u|_0) R_0^{-2\delta} R^{2\delta} \quad \text{in } Q_R^+$$

for $R > R_0$. Hence for any $\delta, \varepsilon > 0$ with $\delta < 1/2$, we have

$$\operatorname{osc}_{Q_R^+} u,\ \sup_{Q_R^+} |u-v| \leqslant C_\varepsilon R^{2\delta} + \varepsilon(NR)^{2\delta}\,.$$

Therefore the first integral on the right hand side of the equation is no larger than

$$(C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\delta} + \frac{\lambda}{4} \iint_{Q_R^+} |D(u-v)|^2\, dx\, dt\,.$$

Similarly

$$\iint_{Q_R^+} [1+|Du|^2]|u-v|\, dx\, dt \leqslant (C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\delta}\,.$$

Moreover

$$\iint_{Q_R^+} (1+|Du|)(x^n)^{\beta-1}|u-v|\, dx\, dt \leqslant C(1+N^2)R \iint_{Q_R^+} (x^n)^{\beta-1}\, dx\, dt = C(1+N^2)R^{n+2+\beta}\,.$$

Finally, by applying the divergence theorem to $|u-v|$, we obtain

$$\iint_{Q_R^0} \psi(u-v)\, dx'\, dt \leqslant CR^\beta(1+N) \iint_{Q_R^0} |u-v|\, dx'\, dt \leqslant C(1+N) R^\beta \iint_{Q_R^+} |D(u-v)|\, dx\, dt \leqslant$$

$$\leqslant C(1+N^2) R^{n+2+2\beta} + \frac{\lambda}{4} \iint_{Q_R^+} |D(u-v)^2|\, dx\, dt\,.$$

Combining all these estimates yields

$$\iint_{Q_R^+} |D(u-v)|^2\,dx\,dt \leqslant (C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\delta} \tag{4.10}$$

for $\delta \leqslant \beta/2$. We now fix $\delta \leqslant \beta/2$ so that $\delta < \sigma$, and we set

$$\Phi_1(R) = \iint_{Q_R^+} |D'u - \{D'u\}_R|^2\,dx\,dt\,.$$

Then (4.9) and (4.10) imply that

$$\Phi_1(\tau R) \leqslant C_\varepsilon[\tau^{n+2+2\sigma}\Phi_1(R) + R^{n+2+2\delta}] + \varepsilon N^{2+2\delta} R^{n+2+2\delta}$$

for all R and τ in $(0, 1)$. Because $\Phi_1(1) \leqslant c(n) N^2$, a standard algebraic argument implies that

$$\Phi_1(R) \leqslant C_\varepsilon R^{n+2+2\delta} + \varepsilon N^{2+2\delta} R^{n+2+2\delta}\,. \tag{4.11}$$

In particular $D'u(X_0) = \lim_{R\to 0} \{D'u\}_R$ exists, and also

$$\iint_{Q_R^+} |D'v|^2\,dx\,dt \leqslant C(1+N^2) R^{n+2}\,,$$

$$\iint_{Q_R^+} |D'v - \{D'v\}_R|^2\,dx\,dt \leqslant (C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\sigma}\,,$$

$$|D'u(X_0) - D'v(X_0)| \leqslant (C_\varepsilon + \varepsilon N^{1+\delta}) R^\delta\,.$$

Now we observe that

$$\bar{a}^{nj} D_j v(x,t) = -A^n(x_0, t, u(x_0, t), 0)\,,$$

$$\bar{a}^{nj}[D_j v(y,s) - D_j v(x,t)] = A^n(x_0, t, u(x_0, t), Dv(y,s)) - A^n(x_0, s, u(x_0, s), Dv(y,s))$$

for appropriate integral averages $\bar{a}^{nj}$ of a^{nj} and hence

$$\sup_{Q^0_{R/2}} |D_n v| \leqslant C(1 + \sup_{Q^0_{R/2}} |D'v|) \leqslant C(1+N)\,,$$

$$\operatorname*{osc}_{Q^0_{R/2}} D_n v \leqslant (C_\varepsilon + \varepsilon N^{1+\delta}) R^\delta\,.$$

(The derivation of this last inequality is the only place where the Hölder continuity with respect to time of A and ψ is used.) Recalling (4.5) and (4.6) we infer

$$\iint_{Q_{\tau R}^+} |D_n v - D_n v(0)|^2\,dx\,dt \leqslant$$

$$\leqslant C_\varepsilon\Big[\tau^{n+2+2\sigma}\iint_{Q_{R/2}^+} |D_n v - D_n v(0)|^2\,dx\,dt + R^{n+2+2\delta}\Big] + \varepsilon N^{2+2\delta} R^{n+2+2\delta}$$

Since there is a unique number U_0 such that $A^n(X_0, u(X_0), D'u(X_0), U_0) = 0$, it follows that

$$\iint_{Q_R^+} |D_n u - U_0|^2 \, dx \, dt \leqslant (C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\delta} . \tag{4.12}$$

Recalling (3.12), we infer from (4.11) and (4.12) that

$$\Phi(R) \leqslant (C_\varepsilon + \varepsilon N^{2+2\delta}) R^{n+2+2\delta} ,$$

so [3, Theorem 3.1(*b*)] gives

$$[Du]_{\delta;\Omega} \leqslant C_\varepsilon + \varepsilon(|Du|_0)^{1+\delta} .$$

Now we choose $X_1 = (x_1, t_1) \in \Omega$ so that $|Du(X_1)| \geqslant \frac{1}{2}|Du|_0$. Thus $w = u(\cdot, t_1)$ satisfies

$$[Dw]_{\delta;D} \leqslant C_\varepsilon + \varepsilon(|Du|_0)^{1+\sigma} \leqslant C_\varepsilon + \varepsilon C(D)|w|_0^\delta(|w|_\delta + [Dw]_\delta) \tag{4.13}$$

by a standard interpolation inequality. By choosing ε sufficiently small (depending only on $|u|_0$ and D), we obtain the desired bound on $|Du|_0$, $[Du]_\delta$ and $\langle u\rangle_{1+\delta}$ from Lemma 2.4. The proof of Theorem 1.1 is now complete for the case when $|Du|$ is bounded.

To prove Theorem 1.1 in its full generality, we note that

$$B(X, u, Du) = f(X)(1 + |Du|^2)$$

for some bounded measurable f with $|f| \leqslant \mu_3$. A simple variant of Lemma 4.1 and the estimates just derived show that the problem

$$w_t = \operatorname{div} A(X, u, Dw) + f(X)(1 + |Dw|^2) \quad \text{in } \Omega ,$$
$$A(X, u, Dw)\cdot\gamma + \psi(X, u) = 0 \quad \text{on } S\Omega , \qquad w = \varphi \quad \text{on } B\Omega$$

has a weak solution $w \in H_{1+\delta}$. Then, because

$$\big||Du|^2 - |Dw|^2\big| \leqslant |D(u-w)|^2 + 2|Dw||D(u-w)| ,$$

we see that $u - w$ is the solution of a boundary value problem of the type we have been considering and the conditions (2.3), (2.4), (2.5), (2.6) are satisfied with

$$a_1 = 0 , \quad b_0 = \mu_3(\sup|u-w| + 4\sup|Dw|^2) , \quad b_1 = 1 , \quad c_0 = 0 , \quad a_0 = 1/\lambda ,$$

and $M = 0$. Thus Lemma 2.2 implies that $u \equiv w$ and hence $u \in H_{1+\delta}$. ∎

Note that Theorem 1.1 is really a local result because the proof of (4.13) also establishes a corresponding inequality between certain weighted norms; see, e.g., [19, p. 761]. Moreover we can relax (1.2c) to

$$|B(X, z, p)| \leqslant b(X) + \mu_3|p|^2$$

if

$$\iint\limits_{Q_R \cap \Omega} b^2\, dx\, dt \leqslant \mu_3 R^{n+2\beta} \qquad \text{for all cylinders } Q_R;$$

in particular if $b \in L^q$, $q > n + 2$.

Note that condition (1.2c) can be generalized to

$$(1.2c)' \qquad |B(X, z, \varrho)| \leqslant \mu_3\big(1 + |p|^2 + (1 + |p|^\beta)d^{\beta-1}\big)$$

if a modulus of continuity is known for u and Du is assumed bounded. The modulus of continuity is not needed provided we replace $|p|$ by $o(|p|)$ in (1.2b) and (1.2c)$'$. With this change, Theorem 1.1 is also valid in noncylindrical domains with weak solutions being suitably redefined.

We point out that our proof follows [5] only in broadest outline. There the estimate corresponding to our (4.9) is proved differently and a different version of our (4.10) is used which avoids the interpolation argument. Obviously no manipulations peculiar to the time dependence appear in [5], but our rewriting of the differential equation as $u_t = \operatorname{div} A + B$ in Q_R^+ seems to be new in the parabolic literature.

PROOF OF THEOREM 1.2. – Assume first that (1.9) holds. Then there is a positive increasing function ω_1 (determined by ω and the quantities on the right hand side of (1.6)) with $\lim\limits_{R\to 0} \omega_1(R) = 0$ such that

$$|a^{ij}(x_0, t_0, u(x_0, t_0), Dv(x_0, t_0)) - a^{ij}(x_0, t, u(x_0, t), Dv(x, t))| \leqslant \omega_1(R)$$

for all $(x, t) \in Q_R^+$. Using the corollary to Lemma 3.2, we infer

$$(4.14) \qquad \iint\limits_{Q_{\tau R}^+} |D'v - \{D'v\}_{\tau R}|^2\, dx\, dt \leqslant C\big(\tau^{n+4} + \omega_1(R)^2\big) \iint\limits_{Q_{R/2}^+} |D'v - \{D'v\}_{R/2}|^2\, dx\, dt$$

in place of (4.9).

Now we note that $|Du| \leqslant C$, so

$$\iint\limits_{Q_R^+} (1 + |Du|^2)|u - v|\, dx\, dt \leqslant C \iint\limits_{Q_R^+} |u - v|\, dx\, dt \leqslant CR^{n+4} + \frac{\lambda}{4} \iint\limits_{Q_R^+} |D(u - v)|^2\, dx\, dt$$

by using Cauchy's inequality and then Poincaré's inequality on B_R^+. Hence in place of (4.10) we obtain

$$\iint\limits_{Q_R^+} |D(u-v)|^2\,dx\,dt \leqslant C[R^{n+2+2\beta} + R^{n+2}(R^{\beta-1}\sup|u-v|)] \leqslant \leqslant C[R^{n+2+2\beta} + R^{n+2}(R^{\beta}\sup|D(u-v)|)]\,.$$

But

$$|Du(X_0) - Dv(X_0)|^2 \leqslant C[R^{2\beta} + R^{\beta}\sup|D(u-v)|] \leqslant \leqslant C[R^{2\beta} + R^{\beta}(|Du(X_0) - Dv(X_0)| + [D(u-v)]_\delta R^\delta)]$$

for any $\delta \in (0,1)$. Assuming that $[D(u-v)]_\delta$ is finite, we obtain

$$|Du(X_0) - Dv(X_0)|^2 \leqslant C(R^{2\beta} + [D(u-v)]_\delta R^{\delta+\beta})$$

and hence

$$\iint\limits_{Q_R^+} |D(u-v)|^2\,dx\,dt \leqslant C(T^{n+2+2\beta} + R^{n+2+\beta+\delta}[D(u-v)]_\delta)\,. \tag{4.15}$$

Combining (4.14) and (4.15) (for sufficiently small R) via the arguments used in proving Theorem 1.1, we conclude that $Du \in H_{(\beta+\delta)/2}$ with

$$[Du]_{(\beta+\delta)/2} \leqslant C(1 + [Du]_\delta^{\frac{1}{2}})\,.$$

(and C independent of δ). A easy iteration starting with, say, $\delta = 0$, yields

$$[Du]_\varepsilon \leqslant C^2$$

for all $\varepsilon < \beta$ and hence $[Du]_\beta \leqslant C$.

In case (1.9)′ holds, we proceed as before except that (after obtaining a bound for Du) we consider v a solution of

$$v_t = \operatorname{div} A(X_0, u(X_0), Dv) \quad \text{in } Q_R^+\,, \qquad A^n(X_0, u(X_0), Dv) = 0 \quad \text{on } Q_R^0\,,$$
$$u = v \quad \text{on } Q_R^*\,.$$

Lemma 2.4 provides the appropriate time behavior for u.

5. – Elliptic problems.

The uniformly elliptic version of problem (1.1) is simpler than the parabolic form. In particular, the Hölder assumptions on A and ψ can be relaxed to Dini assumptions.

For this result, we recall that a continuous, increasing function ζ is a Dini function if $\zeta(0) = 0$ and

$$I(\zeta)(t) = \int_0^t \zeta(s)\frac{ds}{s} < \infty \quad \text{for all } t \in (0, \infty)\,.$$

If also

$$\zeta(s)/s^\alpha \leqslant \zeta(t)/t^\alpha \quad \text{for all } 0 < t \leqslant s$$

and some $\alpha \in (0, 1]$, we say that ζ is α-increasing.

THEOREM 5.1. – Let $u \in W^{1,2}$ be a bounded weak solution of

$$\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0 \quad \text{in } \Omega\,, \qquad A(x, u, Du)\cdot\gamma + \psi(x, u) = 0 \quad \text{on } \partial\Omega \tag{5.1}$$

with $\partial\Omega \in C^1$. Suppose that there are positive constants $\alpha \leqslant 1, \lambda, \Lambda, \mu_1, \mu_2, \mu_3$ and an α-increasing Dini function ζ such that

$$|A(x, z, 0)| \leqslant \mu_1\,, \qquad |B[x, z, p)| \leqslant \mu_2(1 + |p|^2)\,, \tag{5.2}$$

$$|A(x, z, p) - A(y, w, p)| \leqslant \zeta(|x - y|)(1 + |p|) + \zeta(|z - w|)(1 + |p|) \tag{5.3}$$

for all $(x, z, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ and $(y, w) \in \Omega \times \boldsymbol{R}$ with $|z|$, $|w| \leqslant M = |u|_0$. Suppose also that

$$|\psi(x, z) - \psi(y, w)| \leqslant \zeta(|x - y| + |z - w|)\,, \qquad |\psi(x, z)| \leqslant \mu_3 \tag{5.4}$$

for all (x, z) and (y, w) in $\partial\Omega \times \boldsymbol{R}$ with $|z|$, $|w| \leqslant M$ and that

$$a^{ij}(x, z, p)\xi_i\xi_j \geqslant \lambda|\xi|^2\,, \qquad |a^{ij}(x, z, p)| \leqslant \Lambda \tag{5.5}$$

for all $\xi \in \boldsymbol{R}^n$ and (x, z, p) in $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ with $|z| \leqslant M$. Suppose finally that $\partial\Omega \in H_{1+Z}$, that is, there is a function $f \in C^1(\boldsymbol{R}^n)$ with $|Df| \neq 0$ on $\partial\Omega$, $\Omega = \{x \in \boldsymbol{R}^n\colon f(x) > 0\}$, and

$$|Df(x) - Df(y)| \leqslant \zeta(|x - y|) \quad \text{for all } x, y \text{ in } \boldsymbol{R}^n\,. \tag{5.6}$$

Then $u \in C^1(\bar{\Omega})$ and there is a constant $\beta = \beta(n, \Lambda/\lambda, \alpha) \in (0, 1]$ such that

$$|Du(x) - Du(y)| \leqslant C(\alpha, n, \Lambda/\lambda, \zeta, \mu_1, \mu_2, \mu_3, M, \Omega) I(\zeta)(|x - y|^\beta) \tag{5.7}$$

for all x, y in Ω.

PROOF. – Now we can reduce to the case with Ω and $\partial\Omega$ replaced by B_1^+ and B_1^0, respectively, with A, B, ψ satisfying (5.2), (5.3), (5.4), (5.5).

Using the obvious definition for v, we obtain

$$\int_{B_{\tau R}^+} |D'v - \{D'v\}_{\tau R}|^2\, dx \leqslant C\tau^{n+2\sigma} \int_{B_{R/2}^+} |D'v - \{D'v\}_R|^2\, dx$$

in place of (4.9) and, from the Hölder estimate with exponent θ for u,

$$\int_{B_R^+} |D(u - v)|^2\, dx \leqslant CR^{n+\theta}(1 + N^2) + CR^n \zeta(R)^2$$

in place of (4.10). If β is chosen so that $\zeta(t^\beta)$ is ε-increasing for some $\varepsilon < \min\{\sigma, \theta/2\}$, the proof of Theorem 1.1 leads to

$$\Phi(R) \leqslant C(1 + N^2) R^n \zeta(R^\beta)$$

and then [16, (6.3)] yields

$$|Du(x) - Du(y)| \leqslant C(1 + N) I(\zeta)(|x - y|^\beta) \qquad \text{for all } x, y \text{ in } \Omega\,.$$

the proof is completed by using the interpolation inequality [16, (10.1)].

Note that when ζ is a power function (and hence everything is Hölder continuous), we can avoid the interpolation inequality by proceeding as in [5]. In the case $\zeta(t) \equiv t$, Theorem 5.1 was essentially proved by LADYZHENSKAYA and URAL'TSEVA [8, Chapter X]. Also Theorem 5.1 is true locally and the structure condition on B can be relaxed to

$$|B(x, z, p)| \leqslant b(X) + \mu_2 |p|^2$$

with

$$\int_{B_R \cap \Omega} b^2\, dx \leqslant \mu_2 R^{n-2} \zeta(R^2) \qquad \text{for all balls } B_R\,,$$

in particular for $b \in L^q$, $q > n$.

Finally if a^{ij} is continuous with respect to p on bounded subsets of $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ and if $\alpha < 1$, then (5.7) is valid with $\beta = 1$. The proof is simpler than in the parabolic case because now we have

$$\int_{B_R^+} |D(u - v)|^2\, dx \leqslant CR^n \zeta(R)^2\,.$$

We close by mentioning that related results (with $C^{1,\alpha}$ boundary) for two-dimensional problems appear in [14] and [17].

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