The Plancherel Measure for Polygonal Graphs (*).

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Summary. – *We compute explicitely the Plancherel measure for groups acting isometrically and simply transitively on polygonal graphs.*

1. – Introduction.

A. IOZZI and M. PICARDELLO ([6], [7]) extended to the context of groups acting isometrically and simply transitively on a polygonal graph the theory of spherical functions, representations and convolution operators studied in [4], [5] in the case of free groups (and in [1] in the case of homogeneous trees). In particular, they define a commutative algebra (for the convolution product) A of radial functions and compute the Gelfand spectrum of the completion of A in the l^1 norm. If A^* is the completion in the l^2 convolutor norm, A^* is a commutative C^* algebra with unit which can be identified, via the Gelfand transform, to the space of all continuous functions on a compact subset of the real line. As in the case of free groups, this isomorphism can be extended to an isometry between l^2 radial and $L^2(S, d\mu)$, where $d\mu$ is a suitable probability measure called the Plancherel measure. The aim of this paper is to compute explicitely $d\mu$, and, as an application, to extend to polygonal graphs the Plancherel theorem proved in [4] for the free groups (see also [1]).

In the case of groups acting faithfully on homogeneous trees the explicit form of $d\mu$ was already known. The computation was carried out by CARTIER [2] and SAWYER [9] in the context of random walks on a tree, by PYTLIK [8], FIGÀ-TALAMANCA and PICARDELLO [4] and by BETORI and PAGLIACCI [1] in the context of harmonic analysis on trees. Our methods are elementary and lead to a direct computation of $d\mu$ in the general case.

2. – Notation.

A connected homogeneous graph Γ is called a polygonal graph if: *i*) there exist integers k and r (larger or equal to 2) such that every vertex v belongs to exactly r polygons, each one with k sides, contained in the graph and having no side and no

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vertex in common except v; ii) every nontrivial loop runs through all the edges of at least one polygon (see [6]). The distance between two vertex v_1 and v_2 is the block distance defined in [6]: it is the minimum number of polygons crossed by a path connecting v_1 with v_2. If $k = 2$ the polygonal graph reduces to a homogeneous tree of order r. It is not difficult to prove (e.g. using the same arguments of theorem 1.1 of [1]) that a group G_r^k acting faithfully (i.e. isometrically and simply transitively) on Γ is necessarily isomorphic to the free product of r copies of Z_k (integers modulo k) if $k > 2$, while, for $k = 2$, G_r^2 is isomorphic to the free product of t copies of Z and s copies of Z_2 $(2t + s = r)$.

Denote by $V(\Gamma)$ the set of all vertex of Γ and by E_n the set of all vertex having distance n from a vertex o fixed once for all. A function f on $V(\Gamma)$ can be identified in the obvious way with a function on G_r^k. We say that f is radial if it is constant on E_n for all $n = 0, 1, 2, \ldots$. The space of all finitely supported radial functions is denoted by A. Actually A is a commutative algebra for the convolution product. If χ_n denotes the characteristic function of E_n, we have the following identity (see [7]):

$$(1) \qquad \begin{cases} \chi_1 * \chi_n = \chi_{n+1} + (k-2)\chi_n + (r-1)(k-1)\chi_{n-1}, & \text{if } n > 1 \\ \chi_1 * \chi_1 = \chi_2 + (k-2)\chi_1 + r(k-1)\chi_0 . \end{cases}$$

Hence A is generated by χ_1 and the identity χ_0. Let A^* denote the completion of A in the $l^2(G_r^k)$ convolutor norm. Then A^* is a commutative C^* algebra whose spectrum coincides, by (1), with the spectrum of χ_1 as an element of A^*. The following inequality, which is crucial in the proof of theorem 1, was proved in [6], Theorem 1.

$$(2) \qquad \|\chi_n\|_{A^*} \leqq C_k(n+1)\|\chi_n\|_2 \quad \text{for all } n \geqq 0$$

where C_k is a constant depending only on k.

Finally, the following notation will be used throughout the paper: $\sigma = (k-2)$, $\varrho = \big((k-1)(r-1)\big)^{\frac{1}{2}}$.

3. – The spectrum of χ_1.

Let S denote the spectrum of χ_1 in the C^* algebra A^*. Then we have the following result.

THEOREM 1. – If $k \leqq r$ then $S = [\sigma - 2\varrho, \sigma + 2\varrho]$. If $k > r$ then $S = \{-r\} \cup [\sigma - 2\varrho, \sigma + 2\varrho]$.

PROOF. – For any given real γ we have to solve the equation

$$(\chi_1 - \gamma\chi_0) * \sum_{n=0}^{+\infty} a_n \chi_n = \chi_0 . \tag{3}$$

If $f = \sum a_n \chi_n$ is a solution of (3) such that

$$\sum_{n=0}^{+\infty} |a_n|(n+1)\varrho^n < \infty \tag{4}$$

then, by (2) and the fact that $\|\chi_n\|_2 = (r-1)^{-\frac{1}{2}} r^{\frac{1}{2}} \varrho^n$ if $n > 0$, $f \in A^*$ and $\gamma \notin S$. If, on the other hand, every solution f of (3) satisfies

$$\sum_{n=0}^{+\infty} |a_n|^2 \varrho^{2n} = \infty \tag{5}$$

then $\gamma \in S$. It is easily seen that, by (1), (3) is equivalent to the following finite difference equation

$$\varrho^2 a_{n+1} + (\sigma - \gamma) a_n + a_{n-1} = 0 , \qquad n \geqq 1 \tag{6}$$

with the initial condition

$$r(k-1)a_1 - \gamma a_0 = 1 . \tag{7}$$

Suppose first $|\sigma - \gamma| < 2\varrho$ and set $\cos\varphi = -(\sigma-\gamma)/2\varrho$. The general solution of (6) has the form: $a_n = \varrho^{-n}(C_1 e^{in\varphi} + C_2 e^{-in\varphi})$. Since the null solution does not satisfies (7), we have $|a_n| \geqq C\varrho^{-n}$ for some positive constant C and infinitely many n. Then (5) holds and $\gamma \in S$. Hence $[\sigma - 2\varrho, \sigma + 2p] \subseteq S$. Suppose now $|\sigma - \gamma| > 2\varrho$. The general solution of (6) has the form: $a_n = C_1\lambda_+^n + C_2\lambda_-^n$, where $\lambda_\pm = -(\sigma - \gamma)/2\varrho^2 \pm ((\sigma-\gamma)^2 - 4\varrho^2)^{\frac{1}{2}}/2\varrho^2$. Now $|\varrho\lambda_+| < 1$ if $\sigma - \gamma > 2\varrho$ and $|\varrho\lambda_+| > 1$ if $\sigma - \gamma < -2\varrho$. On the contrary, $|\varrho\lambda_-| > 1$ if $\sigma - \gamma > 2\varrho$ and $|\varrho\lambda_-| < 1$ if $\sigma - \gamma < -2\varrho$.

Therefore, if $\sigma - \gamma < -2\varrho$, (4) can be satisfied if and only if $C_1 = 0$ and $C_2(r(k-1)\lambda_- - \gamma) = 1$, i.e. if and only if $r(k-1)\lambda_- - \gamma \neq 0$, which is always true in the case considered. Hence, if $\sigma - \gamma < -2\varrho$, $\gamma \notin S$. Let us consider now the case $\sigma - \gamma > 2\varrho$. As before, we can find a solution satisfying (4) if and only if $r(k-1)\lambda_+ - \gamma \neq 0$. This time, however, the equation: $r(k-1)\lambda_+ - \gamma = 0$ has one solution, $\gamma = -r$, in the case $k > r$ (and only in this case). It follows that, if $k > r$ and $\gamma = -r$, every solution of (3) satisfies (5) and $-r \in S$. In conclusion, if $\sigma - \gamma > 2\varrho$ and $k \leqq r$ there is a solution (and only one) of (3) satisfying (4), so that $\gamma \notin S$. The same is true if $k > r$, except for the point $\gamma = -r$ which belongs to the spectrum of χ_1.

REMARK. – If $k > r$ the point $-r$ is isolated in S, while if $k = r$ it coincides with $\sigma - 2\varrho$. In the first case the above arguments show also that $-r$ belongs to the point spectrum of the convolution operator on $l^2(G_r^k)$ defined by χ_1.

The spectrum of χ_1 was computed in the case where $k = 2$ by several people (see e.g. [2], [3], [4], [8], [9]) and indipendently, by other methods, in [7] in the general case.

4. – The Plancherel measure.

For every $f \in A^*$ let $\hat{f}$ denote the Gelfand transform of f. We define a probability measure $d\mu$ on S by the rule

$$\int_S \hat{f}\, d\mu = f(o)\,. \tag{8}$$

It is immediately seen (see e.g. [8]) that the mapping $f \to \hat{f}$ extends to an isometric isomorphism between $l^2(G_r^k)$ radial and $L^2(S, d\mu)$. In this section we compute explicitely $d\mu$.

For every $n \geqq 0$ and $x \in S$ let $P_n(x) = \chi_n(x)$. Suppose $x \in [\sigma - 2\varrho, \sigma + 2\varrho]$. Then we set $x = \sigma + 2\varrho \cos\theta$ and define $Q_n(\theta) = \varrho^{-n} P_n(\sigma + 2\varrho\cos\theta)$. If $k > r$ and $x = -r$ we let $Q_n(-r) = \varrho^{-n} P_n(-r)$. This simply amount to identifying S with $[0, \pi]$ if $k \leqq r$ and S with $\{-r\} \cup [0, \pi]$ if $k > r$. Finally we introduce the functions: $X_n(\theta) = \sin\big((n+1)\theta\big)/\sin\theta$ if $n \geqq 0$ and $\theta \in [0, \pi]$, $X_n(\theta) = 0$ otherwise.

LEMMA. – For every r and k

$$Q_n(\theta) = X_n(\theta) + \varrho^{-1}\sigma X_{n-1}(\theta) - (r-1)^{-1} X_{n-2}(\theta)\,, \tag{9}$$

for all $n \geqq 0$ and $\theta \in [0, \pi]$.

Moreover, if $k > r$,

$$Q_n(-r) = -r(1-r)^{n-1}\varrho^{-n}\,. \tag{10}$$

PROOF. – Suppose first $x \in [\sigma - 2\varrho, \sigma + 2\varrho]$ and set $q_n(\theta) = \varrho^n Q_n(\theta)$. Then, by (1), q_n is the solution of the finite difference equation

$$q_{n+1} - 2\varrho\cos\theta\, q_n + \varrho^2 q_{n-1} = 0\,, \qquad n > 1 \tag{11}$$

with the initial conditions

$$\begin{cases} q_1 = \sigma + 2\varrho\cos\theta\,, \\ q_2 = (\sigma + 2\varrho\cos\theta)^2 - (\sigma + 2\varrho\cos\theta) - r(k-1)\,. \end{cases} \tag{12}$$

The general solution of (11) can be written in the form: $q_n = \varrho^n C_1(e^{in\theta} - e^{-in\theta}) + \varrho^{n-1} C_2(e^{i(n-1)\theta} - e^{-i(n-1)\theta})$. From (12) we obtain $C_1 = (\sigma + 2\varrho\cos\theta)/2i\varrho\sin\theta$ and $C_2 = -r(k-1)/2i\varrho\sin\theta$, whence (9). If $k > r$ and $x = -r$, equations (1) imply:

$$P_{n+1}(-r) + (\sigma + r)P_n(-r) + \varrho^2 P_{n-1}(-r) = 0\,, \qquad n > 1 \tag{11'}$$

with the initial conditions

$$\begin{cases} P_1(-r) = -r \\ P_2(-r) = r^2 + \sigma r - (k-1)r\,. \end{cases} \tag{12'}$$

One checks immediately that, if Q_n is given by (10), $P_n = \varrho^n Q_n$ is the solution of (11)′ satisfying the conditions (12)′.

THEOREM 2. – If $k < r$, then:

$$d\mu = \Big(\big(2k\pi(1 + \varrho^2 + \sigma - x)\big)^{-1} + (k-1)\big(2k\pi\big((k-1)^2 + \varrho^2 + (k-1)(x-\sigma)\big)^{-1}\big)\big(4\varrho^2 - (x-\sigma)^2\big)^{\frac{1}{2}}\Big)\, dx\,. \tag{13}$$

If $k = r$, then:

$$d\mu = \big(2k\pi(1 + \varrho^2 + \sigma - x)\big)^{-1}\big(4\varrho^2 - (x-\sigma)^2\big)^{\frac{1}{2}}\, dx + (k-1)\big(2k\pi\varrho(2\varrho + x - \sigma)\big)^{-\frac{1}{2}}(2\varrho + \sigma - x)^{\frac{1}{2}}\, dx\,. \tag{14}$$

If $k > r$, then:

$$d\mu = \big(2k\pi(1 + \varrho^2 + \sigma - x)\big)^{-1} + (r-1)\big(2k\pi\big((r-1)^2 + \varrho^2 + (r-1)(x-\sigma)\big)^{-1}\big)\big(4\varrho^2 - (x-\sigma)^2\big)^{\frac{1}{2}}\, dx + k^{-1}(k-r)\,\delta_{-r} \tag{15}$$

where, in the above equations, $x \in [\sigma - 2\varrho, \sigma + 2\varrho]$ and δ_{-r} is the unit mass at the point $-r$.

PROOF. – Suppose first $k \leqq r$ and denote by $d\omega$ the measure on $[0, \pi]$ corresponding to $d\mu$ under the mapping $x = \sigma + 2\varrho\cos\theta$. We set:

$$y_n = \int_0^\pi X_n(\theta)\, d\omega\,. \tag{16}$$

By (8) and (9) we obtain:

$$y_n + \varrho^{-1}\sigma y_{n-1} - (r-1)^{-1} y_{n-2} = 0\,, \qquad n \geqq 2 \tag{17}$$

and

$$\begin{cases} y_0 = 1 \\ y_1 + \varrho^{-1}\sigma y_0 = 0 \, . \end{cases} \tag{18}$$

The solution of (17) with the initial conditions (18) is given by

$$y_n = k^{-1}\varrho^{-n} + k^{-1}(k-1)(1-k)^n \varrho^{-n} \, . \tag{20}$$

Suppose first $k < r$. Then

$$\sum_{n=0}^{+\infty} y_n X_n(\theta) = k^{-1}\varrho^2(\varrho^2 + 1 - 2\varrho\cos\theta)^{-1} + k^{-1}(k-1)\varrho^2\big((k-1)^2 + \varrho^2 + 2\varrho(k-1)\cos\theta\big)^{-1} \tag{21}$$

and the convergence of the series is absolute and uniform. Denote by $g(\theta)$ the function on the right hand side of (21). Since the functions X_n form an orthonormal complete system for the measure $2\pi^{-1}\sin^2\theta \, d\theta$ on $[0, \pi]$, (16) implies

$$d\omega = 2\pi^{-1} g(\theta) \sin^2\theta \, d\theta \, .$$

Changing back the variable we obtain (13).

If $k = r$ (20) reads: $y_n = k^{-1}\varrho^{-n} + k^{-1}(k-1)(-1)^n$. This time we have:

$$k^{-1}\sum_{n=0}^{+\infty} \varrho^{-n} X_n(\theta) = k^{-1}\varrho^2(1 + \varrho^2 - 2\varrho\cos\theta)^{-1} \tag{21$'$}$$

and

$$(-1)^n = \pi^{-1}\int_0^\pi (1-\cos\theta) X_n(\theta)\, d\theta \, , \qquad n = 0, 1, 2, \ldots \, . \tag{21$''$}$$

Denoting by $h(\theta)$ the sum of the series in (21)$'$ and observing that the convergence is absolute and uniform, we have from (21)$'$ and (21)$''$: $d\omega = 2\pi^{-1}h(\theta)\sin^2\theta \, d\theta + \pi^{-1}k^{-1}(k-1)(1-\cos\theta)\, d\theta$ whence, changing back the variable, we obtain (14).

Finally let $k > r$. In this case we set $d\mu = d\mu_1 + C\delta_{-r}$, where $d\mu_1$ is concentrated on the interval $[\sigma - 2\varrho, \sigma + 2\varrho]$ and C is a constant to be determined. Let $d\omega_1$ be the measure on $[0, \pi]$ corresponding to $d\mu_1$. Set:

$$y_n^1 = \int_0^\pi X_n(\theta)\, d\omega_1 \, . \tag{22}$$

Observe now that, by (10),

$$Q_n(-r) = Y_n + \varrho^{-1}\sigma Y_{n-1} - (r-1)^{-1} Y_{n-2} \tag{23}$$

where

$$Y_n = -(k-r)^{-1}\varrho^{-n}\big((1-k)^{n+1}-(1-r)^{n+1}\big)\,, \qquad n \geqq 0 \tag{24}$$

and $Y_n = 0$ if $n < 0$. By (8), (9) and (23) we get that y_n^1 and C must satisfy the equation:

$$(y_n^1 + CY_1) + \varrho^{-1}\sigma(y_{n-1}^1 + CY_{n-1}) - (r-1)^{-1}(y_{n-2}^1 + CY_{n-2}) = 0\,, \qquad \text{for } n \geqq 2 \tag{25}$$

with the initial conditions:

$$\begin{cases} y_0^1 + CY_0 = 1 \\ (y_1^1 + CY_1) + \varrho^{-1}\sigma(y_0^1 + CY_0) = 0\,. \end{cases} \tag{26}$$

From (25) and (26) we then get: $y_n^1 + CY_n = k^{-1}\varrho^{-n} + k^{-1}(k-1)(1-k)^n\varrho^{-n}$. Since y_n^1 can not tend exponentially to infinity (by (22)), we have, remembering (24):

$$\begin{cases} y_n^1 = k^{-1}\varrho^{-n} + k^{-1}(r-1)(1-r)^n\varrho^{-n} \\ C = k^{-1}(k-r)\,. \end{cases} \tag{27}$$

Now, $\sum_{n=0}^{+\infty} y_n^1 X_n(\theta)$ converges absolutely and uniformly on $[0, \pi]$ to the function $g_1(\theta) = k^{-1}\varrho^2(1+\varrho^2 - 2\varrho\cos\theta)^{-1} + k^{-1}(r-1)\varrho^2\big((r-1)^2 + \varrho^2 + 2\varrho(r-1)\cos\theta\big)^{-1}$. Hence $d\omega_1 = 2\pi^{-1}g_1(\theta)\sin^2\theta\, d\theta$. Changing back the variable and taking the second equation of (27) into account we get (15).

5. – The Plancherel theorem.

In this section we extend to G_r^k the Plancherel theorem proved in [4] (theorem 7) for the free groups (see also the extension of [1] to homogeneous trees). We refer to [7] for all unexplained notation and definitions.

Denote by Ω the Poisson boundary of G_r^k and by $d\nu$ the quasi invariant probability measure on Ω associated to the Poisson kernel. IOZZI and PICARDELLO define, by means of the Poisson kernel, a family of representations π_z on $L^2(\Omega, d\nu)$. If $\operatorname{Re} z = \frac{1}{2}$, π_z is irreducible and unitary, while, if $\operatorname{Im} z = n\pi/\log\varrho^2$ (n an integer) and $\operatorname{Re} z \neq 0, 1$, π_z is unitarizable. The first family is called the principal series, the second one the complementary series. By means of the mapping $\gamma(z) = \varrho^{2z} + \varrho^{2(1-z)} + \sigma$ the interval $S = [0, \pi/\log\varrho^2]$ on the line $\operatorname{Re} z = \frac{1}{2}$ corresponds biunivocally to the interval $[\sigma - 2\varrho, \sigma + 2\varrho]$ and, if $k > r$, $w = (\log\varrho^2)^{-1}\log(k-1) + i\pi/\log\varrho^2$ is mapped into $-r$, so that π_w belongs to the complentary series. We denote by $(\ ,\ )_w$ an inner product on $L^2(\Omega, d\nu)$ that makes π_w unitary (e.g. like in [4], section 3, or in [7]). Define, for every finitely supported function f on G_r^k:

$$f_z = \sum_{x \in G_r^k} f(x)(\pi_z(x) u)$$

where u is the function identically 1 on Ω. Finally we identify, via the mapping γ, S with J, if $k \leqq r$, and S with $J \bigcup \{w\}$, if $k > r$. It is easy to write explicitely the expression of the Plancherel measure in the new variable. If we call dm this measure, a moment's reflection show that dm can be obtained from the measure $d\omega$ (introduced in the proof of Theorem 2 above), if $k \leqq r$, simply replacing θ by $\theta \log \varrho^2$. If $k > r$, denoting by dm_1 the restriction of dm to J, dm_1 can be obtained from $d\omega_1$ by means of the same substitution. We then have:

if $k < r$

$$\mathrm{dm} = 2\pi^{-1} k^{-1} \varrho^2 \log \varrho^2 \Big((1 + \varrho^2 - 2\varrho \cos(\theta \log \varrho^2))^{-1} + (k-1)\big((k-1)^2 + \varrho^2 + 2\varrho(k-1)\cos(\theta \log \varrho^2)\big)^{-1} \Big) \times \sin^2(\theta \log \varrho^2)\, d\theta ,$$

if $k = r$

$$dm = 2\pi^{-1} k^{-1} \varrho^2 \log \varrho^2 (1 + \varrho^2 - 2\varrho \cos(\theta \log \varrho^2))^{-1} \sin^2(\theta \log \varrho^2)\, d\theta + \pi^{-1} k^{-1}(k-1) \log \varrho^2 \big(1 - \cos(\theta \log \varrho^2)\big)\, d\theta$$

if $k > r$

$$dm = 2\pi^{-1} k^{-1} \varrho^2 \log \varrho^2 \Big((1 + \varrho^2 - 2\varrho \cos(\theta \log \varrho^2))^{-1} + (r-1)\big((r-1)^2 + \varrho^2 + 2\varrho(r-1)\cos(\theta \log \varrho^2)\big)^{-1} \Big) \times \sin^2(\theta \log \varrho^2)\, d\theta + k^{-1}(k-r)\,\delta_w .$$

THEOREM 3. – For every finitely supported function f on G_r^k we have:

i) if $k \leqq r$: $\quad \|f\|_2^2 = \int_J \|f_{\frac{1}{2}+i\theta}\|^2_{L^2(\Omega, d\nu)}\, dm ,$

ii) if $k > r$: $\quad \|f\|_2^2 = \int_J \|f_{\frac{1}{2}+i\theta}\|^2_{L^2(\Omega, d\nu)}\, dm_1 + k^{-1}(k-r)(f_w, f_w)_w .$

The proof of this theorem can be obtained exactly by the same arguments used in the proof of theorem 7 of [4].

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