On a Wave Theory for the Operator $\varepsilon\partial_t(\partial_t^2 - c_1^2\Delta_n) + \partial_t^2 - c_0^2\Delta_n$ (*).

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Summary. – *The paper deals with a rigorous analysis of the* « wave hierarchie » *related to the operator* $\mathbf{L}_n$ *quoted in the title. Whatever the number* $n = 1, 2, 3$ *of space dimensions may be, the fundamental solutions* E_n *are constructed. These distributions are tempered positive Radon measures associated with positive value functions which have numerous basic properties. So the Cauchy problem* $\mathscr{P}_n$ $(n = 1, 2, 3)$ *with quite arbitrary data is explicitly solved. As another example, also the solution of the signaling problem* $\mathscr{H}$ *is established. Then, various basic aspects of the wave behavior such as diffusion, asymptotic properties, maximum principles and the generalized Huyghens principle are evaluated. Moreover, singular perturbation problems as* $\varepsilon \to 0$, *with estimates of the remainder terms uniformly valid for all* $t \geq 0$, *are discussed too.*

0. – Introduction.

Let $\mathbf{L}_n$ be the strictly-hyperbolic operator

$$\mathbf{L}_n = \varepsilon\partial_t(\partial_t^2 - c_1^2\Delta_n) + \partial_t^2 - c_0^2\Delta_n \tag{0.1}$$

where $n = 1, 2, 3$ is the number of space dimensions, $x \in R^n$, t is the time and Δ_n is the Laplace operator in R^n. Further ε, c_0^2, c_1^2, are three positive constants, with $c_0^2 < c_1^2$ as it results in many usual physical systems.

We propose a rigorous analysis of various basic aspects of wave propagation in dissipative media characterized by $\mathbf{L}_n$. For this we will discuss the general behavior of the solutions of some boundary-value problems related to (0.1), such as the initial-value problem $\mathscr{P}_n$ in all the three cases $n = 1, 2, 3$ and the one-dimensional half-space problem $\mathscr{H}$.

Typical examples of physical phenomena related to $\mathbf{L}_n$ can be found in dynamic of relaxing gases [1 ... 4], in magnetohydrodynamics [1, 10], in hereditary electromagnetism [11, 12] and in isotropic viscoelasticity, where $\mathbf{L}_n$ describes the motions of the standard linear solid [5 ... 9]. In all these models ε is a « *small parameter* »

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indicative of the dissipative cause (e.g. a relaxation time or an absorption coefficient), while c_0 and c_1 are characteristic speeds depending on the material properties of the medium.

There is an extensive literature on the operator $\mathbf{L}_n$ (see the references in [1 ... 9] and [13 ... 18]). The main concern has been the study of one-dimensional problems with very particular boundary data. But, even in these cases, some of the results—at different levels of mathematical rigor—are often incomplete in two respects. The solutions computed by means of series expansions or integral representations lead to very untractable expressions. Many *formal* approximations are not rigorous, as estimates of the remainder terms are missing.

In [16] we succeeded in constructing the fundamental solution E_3 of the operator $\mathbf{L}_3$ in terms of a C^∞ rapidly decreasing and positive-value function $F(r, t)$ $(r = |x|)$ which has other basic properties.

In this paper, extending such analysis, at first we prove that also the fundamental solutions E_n of $\mathbf{L}_n$ for $n = 1, 2$ are expressible by means of the only F. Consequently, for any n the distributions E_n are tempered positive Radon measures associated with positive-value functions (Sect. 3). Then, on the basis of these explicit formulae for E_n, the distribution and the classic solutions of problems $\mathscr{P}_n$ $(n = 1, 2, 3)$ are established for quite arbitrary data (Sect. 4). As another example also the problem $\mathscr{H}$ is explicitly solved (Sect. 5).

Successively, to obtain a rigorous and sufficiently exhaustive evaluation of the various wave phenomena connected to $\mathbf{L}_n$ (see Sect. 1), a qualitative analysis of the solutions of $\mathscr{P}_n$ and $\mathscr{H}$ is given. This survey concerns with various basic aspects of wave behaviour such as maximum principles (Sect. 7), diffusion of waves, asymptotic properties as t or $|x| \to \infty$ (Sect. 8) and generalized Huyghens principle (Sect. 9). Moreover, singular perturbation problems as $\varepsilon \to 0$, with pointwise estimates of the remainder uniformly valid for all $t \geqslant 0$, are also examined (Sect. 10).

On this subject it is worth noting that when $0 \leqslant t < T$ $(T < \infty)$ and the initial data are elements of Sobolev spaces, well-known theorems of modern methods in partial differential equations [23] imply a priori integral estimates. The transition to pointwise estimates can be obtained by means of the Sobolev inequalities [24]. But unfortunately these results hold for hyperbolic operators only in the slab $0 \leqslant t < T$, while we are interested also in the case $T = \infty$. In order to achieve uniform estimates for all $t \geqslant 0$, we shall prove that the solutions of $\mathscr{P}_n$ and $\mathscr{H}$ have the same asymptotic properties of the solution of the Cauchy problem (with suitable datum) for the heat equation (see Sect. 1).

Lastly we observe that the explicit construction of E_n permits us to solve also the *general* Cauchy problem with data prefixed on a hypersurface in xt-space. This explicit solution, together with maximum principles, is of primary importance to analyse also *unilateral problems* for the equation we deal with; such application, for $n = 1$, has been discussed in [29].

In the next section we state the problem and the main results.

1. – Wave hierarchies. Statement of the main results.

The operator $\mathbf{L}_n$ is a typical noticeable example of the situation when waves of different orders and with different speeds c_0, c_1, appear in the same problem (*Wave hierarchies*). According to the celebrated qualitative analysis of Whitham on linear and non-linear waves [1, 2], to obtain a rigorous and exhaustive picture of the wave motion, the following questions must be examined.

a) To analyse the roles of the « highest order waves » (with speed c_1) and the « lowest order waves » (with speed c_0), so to establish which set of waves is the most important and will be really observed.

b) Generally, the speed at which the main signal travels differs from the speed c_1 of the wave-front. Indeed, the dissipation often determines a diffusion of waves which is connected with the characteristic speed c_0 of the lowest order operator and represents, at large t, the main part of the disturbance.

c) When singular perturbation problems related to $\mathbf{L}_n$ for $\varepsilon \to 0$ are discussed, attention must be paid to the *boundary or interior layers* which can appear in dependence of the various boundary-value problems which one deals with. Further, when $\varepsilon \to 0$, a singularity at $t = \infty$ might be too. In fact the lowest order operator $\partial_t^2 - c_0^2 \Delta_n$, which one deduces from $\mathbf{L}_n$ putting formally $\varepsilon = 0$, is typical of undamped motions, while $\mathbf{L}_n$ generally represents damped wave motions.

These various wave phenomena can be rigorously evaluated only if one establishes also the time-intervals when each of them prevails. Moreover the remainder terms of the approximations proposed must be estimated.

Some of the main results of our analysis can be outlined briefly as follows.

1.1. *Classic solution of $\mathscr{P}_n$ and maximum principles.*

Referring to the half-space

$$Y_+^{n+1} = \{(x, t) : x \in R^n, \, t > 0\}, \tag{1.1}$$

by means of an appropriate linear substitution on x, the classic forward Cauchy problem $\mathscr{P}_n$ can be given the form

$$\mathbf{L}_n u = \varepsilon \partial_t (\partial_t^2 - \Delta_n) u + (\partial_t - c^2 \Delta_n) u = f, \qquad (x, t) \in Y_+^{n+1} \tag{1.2}$$

$$\partial_t^i u(x, 0) = f_i(x), \qquad (i = 0, 1, 2), \qquad x \in R^n \tag{1.3}$$

with $c^2 = c_0^2 / c_1^2 < 1$. Therefore in what follows we will refer to $\mathscr{P}_n$ where $c_1 = 1$ and $c_0 = c < 1$.

If the source term $f(x, t)$ and the initial data $f_i(x)$ are sufficiently smooth, $\mathscr{P}_n$ has a unique regular solution (Sect. 4) which—*in all the three cases* $n = 1, 2, 3$—is given by

$$(1.4)\qquad u = \varepsilon\int_0^t rF(r, t)\bar{f}_2(x, r)\, dr + (1 + \varepsilon\partial_t)\int_0^t rF(r, t)\bar{f}_1(x, r)\, dr + \\ + (\varepsilon\partial_t^2 - \varepsilon\Delta_n + \partial_t)\int_0^t rF(r, t)\bar{f}_0(x, r)\, dr + u_f .$$

In (1.4) $\bar{f}_i(x, r)$ are the classic *mean values* of the data defined by (2.4a, b, c) according to the number n of space dimensions (Sect. 2) and u_f is the Duhamel's integral related to the inhomogeneous case (see (4.6)). Further, the kernel F is a positive value C^∞-function which is given by (3.2) and has numerous basic properties (see Theorems 3.2 and 6.1 and 6.3).

On the basis of these properties, at first one deduces some maximum principles (Theorems 7.1 and 7.2) which permit us to estimate u in terms of the data. Some consequences of these theorems are (Sect. 7):

A) If $f\geqslant 0$, $f_i\geqslant 0$ $(i = 0, 1, 2)$ (and $\Delta_n f_0\geqslant 0$ only when $n = 2, 3$), everywhere in Y_+^{n+1} one has

$$(1.5)\qquad u(x, t)\geqslant 0 , \qquad (x, t) \in Y_+^{n+1} .$$

B) In the homogeneous case ($f = 0$), referring to the mean values $\bar{f}_i(x, r)$ with $r \in [0, t]$, everywhere in Y_+^{n+1} $(n = 1, 2, 3)$ it results

$$(1.6)\qquad |u|\leqslant \sup_r |\partial_r(r\bar{f}_0)| + c^{-1}\sum_{i=1}^{2} \varepsilon^{i-1} \sup_r |r\bar{f}_i| , \qquad (0\leqslant r\leqslant t) .$$

C) In the inhomogeneous case, Theorem 7.2 provides explicit bounds for the estimate of remainder terms in iterative approximation methods.

These estimates generally imply rigorous evaluations of u for all t between 0 and a prefixed $T < \infty$. Thus, for example, one can prove that the first signal—when $0\leqslant t\leqslant \varepsilon$—appears as a «small» *precursor wave* propagating obviously with the faster speed of wave-front (see Sect. 8). Moreover, in this time-interval $[0, \varepsilon]$ one can estimate u with a degree of accuracy however prefixed (Sect. 8).

1.2. *Diffusion of waves and asymptotic behavior.*

When t is large compared with ε, i.e. $t \in]\varepsilon, \infty[$, the analysis of the wave behavior is obviously a more difficult question. However, very useful theorems for this analysis can be inferred from the basic Theorem 6.3 on the asymptotic properties of the kernel F. This theorem proves that *when* $t/\varepsilon > 1$ *the main part* *of F coincides with the fundamental solution of the classic heat equation*

$$u_t = \frac{1-c^2}{2}\,\varepsilon u_{yy}\,, \tag{1.7}$$

where the space variable $|y| = |r - ct|$ is connected with the boundary of the forward characteristic cone

$$\Lambda_c \equiv \{(t, x): t > 0,\ |x| \leqslant ct\}\,, \qquad (x \in R^n) \tag{1.8}$$

related to the lowest order operator $\partial_t^2 - c^2\Delta_n$. Consequently, if one puts $[2(1-c^2)]^{-1} = b^2$ and

$$\Psi(x, r) = \partial_r(r\bar{f}_0) + c^{-1}r(\bar{f}_1 + \varepsilon\bar{f}_2)\,, \tag{1.9a}$$

$$\|\Psi\| = \sup_{x,r} |\Psi(x, r)|\,, \qquad (x \in R^n,\ 0 \leqslant r \leqslant t) \tag{1.9b}$$

considering the function u_h given by

$$u_h(x, t) = \frac{b}{\sqrt{\pi\varepsilon t}}\int_{-\infty}^{+\infty} \exp\left[-\frac{b^2}{\varepsilon t}(r - ct)^2\right]\Psi(x, r)\,dr\,, \tag{1.10}$$

the following theorem can be stated (Sect. 8).

Theorem 1.1. – *If the data $f_i(x)$ are such that the function $\Psi(x, r)$ is bounded for $x \in R^n$ and $r \in [0, t]$ then, as $t > \varepsilon$, the solution u of $\mathscr{P}_n$ (with $f = 0$), in all the three cases $n = 1, 2, 3$, is approximated by the solution u_h of the heat equation* (1.7) *defined by* (1.10). *This approximation is uniformly valid in $x \in R^n$ and is given by*

$$|u - u_h| \leqslant \text{const}\ \|\Psi\| \sqrt{\frac{\varepsilon}{t}}\,, \qquad x \in R^n,\ t > \varepsilon\,, \tag{1.11}$$

where the constant depends only on c.

Therefore, according to what foreseen by Whitham [1, 2], these results imply rigorously the following basic conclusions.

For small values of t (i.e. $0 \leqslant t \leqslant \varepsilon$) a small precursor wave which propagates with the faster speed of wave-front appears. But, when t is large compared with ε $[0 < \varepsilon < t)$, the main signal is related to the speed c of the lowest order waves and propagates into the medium as a diffusion process.

Theorem 1.1, besides proving and evaluating diffusion at large t, achieves obviously the asymptotic analysis of u by means of well-known properties of the comparison function u_h given by (1.10). Clearly this analysis can be improved further on when the data are specified. Some typical cases are discussed in Section 8.2. So e.g. the solution of $\mathscr{P}_1$ with periodic initial data and $f = 0$, unlike the classic case, vanishes as $t \to \infty$ uniformly in x. Referring to singular perturbation problems, this proves the singularity of the perturbation as $t \to \infty$.

1.3. *Generalized Huyghens principle.*

As $\boldsymbol{L}_n$ characterizes dissipative media, when $n = 3$ the « Huyghens principle in the strong form » does not hold, owing to the diffusion of waves above stated. However we will show that, also for the dissipative media we deal with, a substantial difference exists between the case $n = 3$ and the other cases $n = 1$, $n = 2$. In fact, when $n = 3$, there is a time-instant t_1 after which the effect of initial localized disturbances—though not vanishing—*decays exponentially in time.* This result does not subsist when $n = 2$ or $n = 1$ (see Sect. 9).

Suppose that initially the disturbance is localized in a ball $B(0, \varrho)$ of radius ϱ and center 0. Then, if one puts $t_1 = c^{-1}(|x| + \varrho)$, one has

THEOREM 1.2. – *When $x \in R^3$ and the initial data are of compact support, for all $t > t_1$ and uniformly in $x \notin B(0, \varrho)$ the solution u of $\mathscr{P}_3$ (with $f = 0$) satisfies the following estimate*

$$|u(x, t)| < \text{const.} \exp[-\alpha^2(1 - t_1/t)^2 t], \qquad x \in R^3, \ t > t_1. \tag{1.12}$$

1.4. *Singular perturbation problems.*

The results above stated imply also a correct analysis of singular perturbations which appear as $\varepsilon \to 0$. To show the possibility of various singularities which the perturbation as $\varepsilon \to 0$ can exhibit, two one-dimensional problems are discussed (Sect. 10). The former concerns with the signaling problem where an *interior layer* appears near to the subcharacteristic $|x| = ct$ related to the speed $c < 1$. A theorem like Theorem 1.1 provides an approximation uniformly valid for all $t \geq 0$ and $x > \varepsilon$ (see Theorem 10.1). The latter deals with the problem $\mathscr{P}_1$ defined by periodic initial disturbances (Sect. 10.2). In this case, as $\varepsilon \to 0$, the perturbation shows two singularities: one at $t = 0$, where $\partial_t^2 u_0(x, 0) \neq f_2(x)$, and another at $t = \infty$ where $\lim\limits_{t\to\infty} u = 0$ (see Example 8.1). Theorem 10.2 provides an approximation of u uniformly valid for all $t \geq 0$ and $x \in R$.

Referring, as an example, to the viscoelastic case (Sect. 10.3), Theorem 10.2 implies for $\varepsilon \to 0$ an uniform representation such as

$$u = \exp[-b_0 \varepsilon t] u_0(x, ct) + \exp[-t/\varepsilon] z(x, t) + \varepsilon \varrho \tag{1.13}$$

where u_0 is the *pure elastic wave* which one has when $\varepsilon = 0$ and is associated with the lowest speed c. Further, $z(x, t)$ is another pure wave related to the speed of the wave-front and *the remainder* ϱ *is uniformly bounded for all* $t \geqslant 0$ *and* $x \in R$.

Consequently the evolution of viscoelastic waves induced by periodic initial disturbance is characterized, when $\varepsilon \to 0$, by the rapidly damped signal $\exp[-t/\varepsilon]z$ related to the *fast-time* t/ε and by the main signal represented by slowly damped oscillating wave $\exp[-b_0 \varepsilon t]u_0$ associated with the *slow-time* εt. Further the formula (1.13) enables us to evaluate also the times of validity of elastic or viscous behavior. In fact

1) When $t \in [0, \varepsilon]$, there is a *quasi-elastic behavior* described by the pure wave u_0 with speed c, in front of which there is a damped precursor propagating at the faster speed of the wave-front.

2) When $t \in [\varepsilon, \varepsilon^{-1}]$, the main signal related to the elastic wave u_0 with speed c prevails.

3) When $t > \varepsilon^{-1}$, there is a *viscous behavior* with rapidly damped signal.

REMARK 1.1. – Clearly, the rigorous approximations proposed can be iterated as one wants. Then the remainder term is the solution of the inhomogeneous case; so Theorem 7.2 gives explicit bounds for this solution (see e.g. the proof of Theorem 10.2).

2. – Notations.

Let $\mathcal{A} \subset R^n$ be an open set; $\mathscr{D}'(\mathcal{A})$ is the set of all distributions on $\mathcal{A}$, while $\mathscr{E}'$ is the subspace of $\mathscr{D}'$ consisting of all distributions with bounded supports. The value of a functional $E \in \mathscr{D}'$ at a function ψ of the space $C_0^\infty(\mathcal{A})$ will be denoted by $E[\psi]$ or by $\langle E, \psi \rangle$. The symbol $\mathscr{S}(R^n)$ will denote the class of rapidly decreasing functions and $\mathscr{S}'(R^n)$ the totality of tempered distributions on R^n. In connection with $\mathscr{S}$, denote by $\mathscr{S}_k^m(R^n)$ the space of functions $\psi \in C^m(R^n)$ such that for any $\eta > 0$ there exists a $B_0 > 0$ fulfilling the inequality

$$(1 + |x|^2)^k |D^\alpha \psi(x)| < \eta \qquad \text{for } |x| > B_0, \ |\alpha| \leqslant m, \tag{2.1}$$

with m, k positive integers. The norm $\| \ \|$ in $\mathscr{S}_k^m$ is

$$\|\psi\|_{m,k} = \sum_{|\alpha| \leqslant m} \sup_x (1 + |x|^2)^k |D^\alpha \psi(x)|, \qquad \psi \in \mathscr{S}_k^m. \tag{2.2}$$

Referring to the D'Alembertian $\partial_t^2 - \Delta_n$ with the fastest speed, we will denote by Λ_n the forward characteristic cone

$$\Lambda_n \equiv \{(x \in R^n, t) \colon t \geqslant 0, \ |x| \leqslant t\}, \qquad (n = 1, 2, 3) \tag{2.3}$$

and by $\partial \Lambda_n$ the boundary of Λ_n.

Lastly, the classical *mean values* $\bar{f}_i$, $\bar{f}$ of the data $f_i(x)$, $f(x, t)$ of the problem $\mathscr{P}_n$ are defined as follows, according to the number n of space dimensions.

a) Case $n = 3$:

$$\bar{f}_i(x, r) = \frac{1}{4\pi r^2}\iint\limits_{|x-y|=r} f_i(y)\, dS_y; \qquad \bar{f}(x, r; t) = \frac{1}{4\pi r^2}\iint\limits_{|y-x|=r} f(y, t)\, dS_y\,. \tag{2.4a}$$

In this case the domain of dependence of $\bar{f}_i(x, r)$ is the sphere of radius r about the point (x_1, x_2, x_3).

b) Case $n = 2$:

$$\left\{\begin{aligned} \bar{f}_i(x, r) &= \frac{1}{2\pi r}\iint\limits_{|y-x|\leqslant r} \frac{f_i(y)}{\sqrt{r^2 - |x-y|^2}}\, dy\,; \\ \bar{f}(x, r; t) &= \frac{1}{2\pi r}\iint\limits_{|y-x|\leqslant r} \frac{f(y\,, t)}{\sqrt{r^2 - |x-y|^2}}\, dy\,. \end{aligned}\right. \tag{2.4b}$$

Here the domain of dependence of $\bar{f}_i(x, r)$ consists of the whole circular disc of radius r about the point (x_1, x_2).

c) Case $n = 1$:

$$\bar{f}_i(x, r) = \frac{1}{2r}\int\limits_{x-r}^{x+r} f(y)\, dy\,; \qquad \bar{f}(x, r; t) = \frac{1}{2r}\int\limits_{x-r}^{x+r} f(y, t)\, dy\,. \tag{2.4c}$$

3. – Fundamental solutions of the operators L_n. Distribution solutions.

The operator $\mathbf{L}_n$ is strictly hyperbolic as it verifies the Garding's condition; then there exists one and only one fundamental solution with support contained in the half-space $\bar{Y}_+^{n+1}$ $(n = 1, 2, 3)$. The importance of explicit fundamental solutions of given operators is well known. In [16] we have constructed the fundamental solution of the operator $\mathbf{L}_3$; to achieve explicit formulae also when $n = 1$ and $n = 2$, at first we briefly refer to some results stated in [16]. Let $|x| = r$ be and

$$\left\{\begin{aligned} &\lambda^2 = \frac{(1-c^2)}{2}; \qquad \frac{c^2}{\varepsilon} = k^2; \qquad \gamma^2 = \frac{1+c^2}{2\varepsilon}; \qquad \eta = \frac{\lambda^2(t-r)}{2\varepsilon} \\ &\xi = 2\lambda c\sqrt{r(t-r)}\,\varepsilon^{-1}\,, \qquad \omega = \lambda^2\sqrt{t^2 - r^2}\,\varepsilon^{-1}\,. \end{aligned}\right. \tag{3.1}$$

If I_k is the modified Bessel function of first kind, let

$$\begin{aligned} F(r, t) = \varepsilon^{-1}\exp\left[-\gamma^2 t + k^2 r\right][I_0(\omega) + \\ + \int\limits_0^1 \exp\left[\eta v^2\right][4\eta v I_0(\xi v) + \xi I_1(\xi v)] I_0(\omega\sqrt{1-v^2})\, dv]\,. \end{aligned} \tag{3.2}$$

Then, if we refer e.g. to the space $\mathscr{S}(R^n)$ of rapidly decreasing function χ, the fundamental solution $\langle E_3, \chi\rangle$ of the operator $\mathbf{L}_3$, is a distribution associated with the function

$$E_3(x,t) = \frac{1}{4\pi|x|} H(t-|x|)\,F(|x|,t)\,, \tag{3.3}$$

where $H(t)$ is the Heaveside function. In fact, if we consider the following distribution-valued function

$$[0,\infty) \ni t \to E_3^{(t)}[\chi] = \int_0^t F(r,t)\, r\bar{\chi}(r,t)\, dr \in \mathscr{S}'(R^4) \tag{3.4}$$

where $\bar{\chi}(r,t)$ is the spherical mean of χ over the sphere with center 0 and radius r in the hyperplane $t = \text{const}$, it can be shown [16]:

THEOREM 3.1. – *For any real $t > 0$, let $E_3^{(t)}$ be the distribution defined by* (3.4). *Then, the functional*

$$\langle E_3, \chi\rangle = \int_0^\infty E_3^{(t)}[\chi(t,\cdot)]\, dt\,, \qquad \chi \in \mathscr{S}(R^4) \tag{3.5}$$

is a tempered positive Radon measure which represents the only fundamental solution of the operator $\mathbf{L}_3$ with support contained in $\overline{Y}_4^+$.

These results are consequences of various properties of the function F given by (3.2) and defined in

$$\Omega \equiv \{(t,r) \in R^2 \colon t > 0,\ 0 < r < t\}\,. \tag{3.6}$$

According to [16] one has

THEOREM 3.2. – *The kernel $F(r,t)$ has the following basic properties:*

$$F(r,t) \in C^\infty(\Omega) \quad \text{and} \quad \mathbf{L}_1 F = 0 \quad \text{in } \Omega \tag{3.7a}$$

$$F(r,t) > 0\,, \qquad (r,t) \in \overline{\Omega} \tag{3.7b}$$

$$\int_0^t F(r,t)\, dr \leqslant c^{-1}\,, \qquad t \geqslant 0 \tag{3.7c}$$

$$F(t,t) = \varepsilon^{-1} \exp[-\lambda^2 t/\varepsilon]\,, \qquad F(0,t) = \varepsilon^{-1}\exp[-k^2 t]\,. \tag{3.7d}$$

Furthermore, if $\mathscr{L}_t$ denotes the Laplace transform and s is the parameter of the transformation, it must be also remarked that

THEOREM 3.3. – *In the half-plane* $\mathrm{Re}(s) > -c^2/\varepsilon$ *the Laplace integral* $\mathscr{L}_t H(t - r) F(r, t)$ *converges absolutely and one has:*

$$\mathscr{L}_t H(t-r)F(r,t) = \frac{\exp[-rs\sqrt{(\varepsilon s+1)/(\varepsilon s+c^2)}]}{\varepsilon s+c^2} = \hat{F}(r,s)\,. \tag{3.8}$$

Now we are going to prove that this analysis can be applied to obtain also E_1 and E_2. For this, let

$$F_1(r,t) = \int_r^t F(z,t)\,dz\,, \qquad F_2(r,t) = \int_r^t \frac{F(z,t)}{\sqrt{z^2-r^2}}\,dz \tag{3.9}$$

$$E_1 = \frac{1}{2} H(t-|x|)F_1(|x|,t)\,, \qquad E_2 = \frac{1}{2\pi} H(t-|x|)F_2(|x|,t)\,. \tag{3.10}$$

If we consider the following functionals associated with E_k $(k = 1, 2)$

$$\langle E_k, \chi\rangle = \int_0^\infty dt \int_{R^k} \chi(t,y) E_k(y,t)\,dy\,, \qquad \chi \in \mathscr{S}(R^{k+1}) \tag{3.11}$$

one has

THEOREM 3.4. – *In the cases* $n = 1$ *and* $n = 2$, *like when* $n = 3$, *the functionals* $\langle E_n, \chi\rangle$ *defined by* (3.11) *are tempered positive distributions of order zero. Furthermore* $\langle E_n, \chi\rangle$ *is the only fundamental solution of the operator* $\mathbf{L}_n$ *with support contained in* $\overline{Y}_+^{n+1}$.

PROOF. – By means of Fourier and Laplace's operators, it is possible to verify that E_1 and E_2 are *formally* defined by the symbolic relations

$$\mathscr{L}_t E_1 = \hat{E}_1(x,s) = (2\sigma)^{-1}\hat{F}(|x|,s)\,, \qquad x \in R \tag{3.12a}$$

$$\mathscr{L}_t E_2 = \hat{E}_2(x,s) = [2\pi(\varepsilon s + c^2)]^{-1} K_0(|x|\sigma)\,, \qquad x \in R^2 \tag{3.12b}$$

where $\hat{F}(r, s)$ is the Laplace transform defined in (3.8), K_0 is the modified Bessel function of second kind and $\sigma = [s^2(\varepsilon s + 1)/(\varepsilon s + c^2)]^{\frac{1}{2}}$.

Now, being the inverse transform of $\hat{F}$ constructed (see (3.8)), E_1 and E_2 can be computed too. In fact, on the basis of Theorem 3.2 and applying the Fubini-Tonelli theorem, in the half-plane $\mathrm{Re}(s) > 0$ one has

$$\mathscr{L}_t E_1(r,t) = \tfrac{1}{2}\int_r^\infty \exp[-st]F_1(r,t)\,dt = \tfrac{1}{2}\int_r^\infty \hat{F}(z,s)\,dz = (2\sigma)^{-1}\hat{F}(r,s)\,.$$

As for E_2, we observe that

$$\mathscr{L}_t E_2(r,t) = (2\pi)^{-1}\int_r^\infty \exp[-st]\,dt\int_r^t \frac{F(z,t)}{\sqrt{z^2-r^2}}\,dz = \frac{1}{2\pi}\int_r^\infty \frac{\hat{F}(z,s)}{\sqrt{z^2-r^2}}\,dz = \frac{K_0(r\sigma)}{2\pi(\varepsilon s + c^2)}\,.$$

The functions $F_k(r,t)$, achieved by means of this heuristic formal analysis, are $C^\infty(\Omega)$ positive value functions (see (3.9)-(3.7*b*)) both expressible in terms of the kernel F which satisfies (3.7*c*). Therefore the distribution valued functions

$$[0,\infty)\in t \to E_k^{(t)}[\chi] = \int_{R^k} \chi(y,t)\,E_k(y,t)\,dy\,, \qquad \chi\in\mathscr{S}(R^{k+1})$$

are of class C^∞ and bounded for all $t\geqslant 0$. Consequently the distributions induced (3.11) are positive tempered Radon measures.

Furthermore, according to the results of [16] related to the case $n=3$, it is easy to prove that

$$\mathbf{L}_k[E_k] = \delta\,, \qquad \delta\in\mathscr{D}'(R^{k+1})\,, \tag{3.13}$$

where δ is the Dirac measure in R^{k+1}. At last, as $\operatorname{supp} E_k = \Lambda_k$ $(k=1,2)$, the uniqueness of the fundamental solution in the class of distributions with support in $\overline{Y}_+^{k+1}$ is a consequence of well known theorems of the theory of distributions.

REMARK 3.1. – Obviously, the construction of E_1 and E_2 can be achieved also by means of the classical Hadamard's method of descent.

Finally, if the symbol $\circledast$ denotes the convolution in R^{n+1} of two distributions $\in\mathscr{D}'(R^{n+1})$, by Theorems 3.1-3.4 it follows:

THEOREM 3.5. – *Let $f\in\mathscr{D}'(R^{n+1})$ be a distribution with support in $\overline{Y}_+^{n+1}$. Then the functional $u_f = E_n \circledast f$ $(n=1,2,3)$ is the unique distribution solution of the equation $\mathbf{L}_n u = f$ with support in $\overline{Y}_+^{n+1}$. Moreover, if $f\in\mathscr{E}'(R^{n+1})$, then $u_f\in\mathscr{S}'(R^{n+1})$ $(n=1,2,3)$.*

PROOF. – As

$$\operatorname{supp} f\subset \overline{Y}_+^{n+1}\,, \qquad \operatorname{supp} E_3 = \partial\Lambda_3\,, \qquad \operatorname{supp} E_k = \Lambda_k\,, \qquad (k=1,2) \tag{3.14}$$

the convolution $E_n\circledast f$ exists and is a distribution solution of (1.2). Moreover, as consequence of (3.14), $\operatorname{supp} u_f\subset \overline{Y}_+^{n+1}$ and this condition characterizes the uniqueness. At last, when $f\in\mathscr{E}'(R^{n+1})$, u_f is the convolution of two tempered distributions, one of which has bounded support; consequently also u_f is a tempered distribution.

4. – The forward Cauchy problem. Classic solutions.

For functions $u(x, t)$, with $x \in R^n$ $(n = 1, 2, 3)$, we use the differentiation symbols

$$D_0 = \partial_t\,, \quad D = (D_1, \ldots, D_n) = (\partial x_1, \ldots, \partial x_n)\,, \quad (n = 1, 2, 3)\,,$$

where D is the gradient vector with respect to the space variables. By means of the Schwartz notations we have

$$\varepsilon^{-1}\mathbf{L}_n = P_n(D_0, D) = D_0^3 + \varepsilon^{-1}D_0^2 - D^2 D_0 - c^2\varepsilon^{-1}D^2 \tag{4.1}$$

and we can write the problem (1.2)-(1.3) in the form

$$\varepsilon P_n(D_0, D)u = f\,, \qquad t > 0,\ x \in R^n\ (n = 1, 2, 3) \tag{4.2}$$

$$D_0^i u(x, 0) = f_i(x)\,, \qquad (i = 0, 1, 2)\,. \tag{4.3}$$

As P_n is an hyperbolic operator with constant coefficients, the classic initial value problem $\mathscr{P}_n$, as it is well known, is a typical well-posed problem. Obviously, by a solution of $\mathscr{P}_n$ we shall mean a function which in Y_+^{n+1} is of class C^3 and satisfies (4.2), while in $\overline{Y}_+^{n+1}$ is of class C^2 and verifies (4.3) (as $t \to 0^+$).

Of course, the foregoing construction of the E_n's enables us to determine the solution of $\mathscr{P}_n$ for general data f and f_i and in all the three cases $n = 1, 2, 3$ of the space dimensions. In fact, when the data are sufficiently regular functions, on the basis of the properties of E_n (Sect. 3), the convolutions of the E_n's with the data exist and are smooth functions. Consequently, referring to the polynomial $P_n(D_0, D)$ arranged in (4.1) according to powers of D_0, by means of well known techniques (see e.g. [24]), one easily verifies that the solution u of problem $\mathscr{P}_n$ is given by the formula $(x \in R^n)$:

$$u = \varepsilon E_n * f_2 + (\varepsilon D_0 + 1)E_n * f_1 + (\varepsilon D_0^2 - \varepsilon D^2 + D_0)E_n * f_0 + E_n \circledast f\,, \tag{4.4}$$

where the symbol $*$ denotes the convolution with respect to the space variables x for fixed t.

However, we now want to prove that, in all the three cases for n, these convolutions are expressible in a more concrete form in terms of the only kernel $F(r, t)$ given by (3.2) and of the classical *mean values* $\bar{f}_i, \bar{f}$ of the data f_i, f defined by (2.4a, b, c), according to the number n.

In fact we will prove that when the data are smooth functions the convolutions

$E_n * f_i$ and $E_n \circledast f$ are given by the integrals

$$E_n * f_i = u_{f_i} = \int_0^t rF(r, t)\bar{f}_i(x, r)\, dr \tag{4.5}$$

$$E_n \circledast f = u_f = \int_0^t d\tau \int_0^\tau rF(r, t)\bar{f}(x, r; t-\tau)\, dr \tag{4.6}$$

and consequently we will obtain:

THEOREM 4.1. – *Let $x \in R^n$ and $t \geqslant 0$. If the source term $f(x, t) \in C^3(Y_+^{n+1})$ and the initial data $f_i(x) \in C^{4-i}(R^n)$ $(i = 0, 1, 2)$, then the initial-value problem $\mathscr{P}_n$ has a unique solution $u \in C^3(Y_+^{n+1})$ which—whatever the number $n = 1, 2, 3$ may be—admits the same following representation*

$$u = u_f + \varepsilon u_{f_2} + (1 + \varepsilon \partial_t) u_{f_1} + (\varepsilon \partial_t^2 - \varepsilon \Delta_n + \partial_t) u_{f_0}\,, \tag{4.7}$$

with u_{f_i} and u_f given by (4.5)-(4.6).

PROOF. – At first, let us consider the case when the initial data f_i are vanishing. If we put $f(x, t) = 0$ in

$$Y_-^{n+1} = \{(t, x) : t < 0,\ x \in R^n\}\,, \tag{4.8}$$

by Theorem 3.5 the convolution $E_n \circledast f$ is the only distribution solution of (4.2) vanishing in Y_-^{n+1} and in addition is a C^3-function, as f is of class $C^3(Y_+^{n+1})$. Consequently $E_n \circledast f$ is also a classic solution of (4.2) and is given by

$$E_n \circledast f = \int_0^t E_n(x, \tau) * f(x, t-\tau)\, d\tau\,, \qquad (t \geqslant 0)\,. \tag{4.9}$$

Now, when $n = 3$, by (3.3) easily one has

$$E_3(x, \tau) * f(x, \cdot) = \int_0^\tau rF(r, \tau)\bar{f}(x, r; \cdot)\, dr\,, \tag{4.10}$$

where $\bar{f}(x, r; \cdot)$ is the spherical mean of $f(x, \cdot)$ defined in (2.4*a*). Thus, the formulae (4.9)-(4.10) imply (4.6) when $n = 3$.

As for $n = 2$, by $(3.10)_2$ and $(3.9)_2$ with standard computations it draws

$$E_2 * f = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^\tau f(x_1 - \varrho\cos\theta, x_2 - \varrho \operatorname{sen}\theta; \cdot)\, \varrho\, d\varrho \int_\varrho^\tau \frac{F(r, \tau)\, dr}{\sqrt{r^2 - \varrho^2}} = \int_0^\tau rF(r, \tau)\bar{f}(x, r; \cdot)\, dr\,, \tag{4.11}$$

with $\bar{f}(x, r; \cdot)$ defined in (2.4*b*). By (4.9)-(4.11) also for $n = 2$ one deduces (4.6). At last, when $n = 1$, by $(3.10)_1$-$(3.9)_1$ one obtains (4.6) again, with $\bar{f}$ defined according to (2.4*c*).

By (4.6), in any case, it is easy to verify that

$$D_0^i u_f(x, 0) = 0 , \quad (i = 0, 1, 2), \ x \in R^n .$$

Analogously, when $f = 0$ and $f_i \neq 0$, as the initial data $f_i(x) \in C^{4-i}(R^n)$ then the convolutions $E_n * f_i$ are the smooth functions given by (4.5) (see (4.10)-(4.11) with f_i in the place of f). Thus, the foregoing statements prove that the formula (4.7) defines a classic solution of the problem $\mathscr{P}_n$ for $n = 1, 2, 3$. The uniqueness in the class of sufficently regular solutions is a consequence of well-known theorems on hyperbolic equations with constant coefficients. Thus the proof is complete.

REMARK 4.1. – It is interesting to remark that in (4.7) the unique element which differentiates the three cases for n is the function $w(x, r) = r\bar{f}_i(x, r)$ which represents the solution of the standard problem associated with the n-dimensional D'Alembertian

$$\partial_r^2 w - \Delta_n w = 0 \tag{4.12}$$

$$w(x, 0) = 0 , \quad \partial_r w(x, 0) = f_i(x) . \tag{4.13}$$

Analogous remark holds for the convolution $E_n \circledast f$ related to the inhomogeneous case. Consequently the basic properties of F, together with the well-known behaviour of $w = r\bar{f}_i$, will enable us to make in the next Section a wide analysis of the solution of $\mathscr{P}_n$ for any n.

5. – The one-dimensional half-space problem.

The kernels F and F_1 which define the fundamental solutions analysed previously allow us to obtain also explicit solution of other boundary-value problems related to $\mathbf{L}_n$. As an example, we will consider the onedimensional half-space or signaling problem $\mathscr{H}$ defined by

$$\mathbf{L}_1 v = 0 , \qquad x \in R^+, \ t > 0 \tag{5.1}$$

$$\partial_t^i v(x, 0) = 0 , \quad x \in R^+, \ (i = 0, 1, 2) \tag{5.2}$$

$$v(0, t) = \Phi(t) , \quad v(x, t) \to 0 \quad \text{as } x \to \infty \ (t > 0) . \tag{5.3}$$

By means of an inversion formula of Laplace transforms such as (3.8), we can see that the kernel E_0 which characterizes the explicit solution of problema $\mathscr{H}$ is (see [27])

$$E_0 = \exp[-\lambda^2 x/\varepsilon]\delta(t-x) + H(t-x)F_0(x,t)\,, \qquad x \in R^+ \tag{5.4}$$

where δ is the Dirac measure in R and F_0 is the $C^\infty(\Omega)$ function given by

$$F_0(x,t) = (\varepsilon\partial_t + c^2)F(x,t)\,, \qquad (0 \leqslant x \leqslant t)\,. \tag{5.5}$$

Consequently one can prove that

THEOREM 5.1. – *When the datum $\Phi(t) \in C^3(R^+)$ and is such that $\Phi^{(i)}(0) = 0$ for $i = 0, 1, 2$, then the problem $\mathscr{H}$ has a unique regular solution given by*

$$v(x,t) = H(t-x)[\exp[-\lambda^2 x/\varepsilon]\Phi(t-x) + \int_x^t F_0(x,\tau)\Phi(t-\tau)\,d\tau]\,. \tag{5.6}$$

A qualitative analysis of this solution will be dealt with successively (Sections 7 and 10.1), after establishing other properties of the kernels F, F_1, and F_0 (Sect. 6). Now we observe only that applying the Laplace transform to (5.4)-(5.5), by means of Theorem 3.3, one deduces the symbolic relation (see (3.8))

$$\mathscr{L}_t H(t-x)F_0(x,t) = \hat{F}_0(x,s) = \exp[-x\sigma] - \exp[-xs - \lambda^2 x/\varepsilon]\,, \tag{5.7}$$

where the Laplace integral F_0 converges absolutely also when $\operatorname{Re}(s) = 0$. Consequently, one can state rigorously that

$$\int_x^\infty F_0(x,t)\,dt = \hat{F}_0(x,0) = 1 - \exp[-\lambda^2 x/\varepsilon]\,. \tag{5.8}$$

6. – Basic properties of the fundamental solutions.

Now we deal with some basic properties of the kernels F, F_1, F_0 defined above. We begin by proving that

THEOREM 6.1. – *The $C^\infty(\Omega)$ positive value functions F and F_1 are such that everywhere in $\bar{\Omega}$ one has:*

$$0 < \partial_t F_1 \leqslant F\,, \qquad \partial_t(\varepsilon\partial_t + 1)F_1 > 0 \tag{6.1a}$$

$$F_0 = (\varepsilon\partial_t + c^2)F \geqslant 0\,, \qquad (\varepsilon\partial_t + 1)F \geqslant 0 \tag{6.1b}$$

$$[\varepsilon\partial_t^2 - \varepsilon\partial_r^2 + (1-c^2)\partial_t]F_1 \geqslant 0\,. \tag{6.1c}$$

PROOF. - These inequalities are substantially consequences of the properties and recurrence formulae of the modified Bessel functions which appear in the definitions (3.2)-(3.9) of F and F_1. So, letting

$$G = \varepsilon^{-1} \exp[-\gamma^2 t + k^2 r]\Big[I_0(\omega) + \xi\int_0^1 \exp[\eta v^2] I_1(\xi v) I_0(\omega\sqrt{1-v^2})\, dv\Big], \tag{6.2}$$

one has

$$\partial_t F = -\partial_r G, \quad G(t, t) = F(t, t) \tag{6.3}$$

hence

$$\partial_t F_1 = \int_r^t \partial_t F(z, t)\, dz + F(t, t) = G > 0, \quad (r, t) \in \bar{\Omega}. \tag{6.4}$$

Further, by (6.2) easily it draws

$$(\varepsilon\partial_t + 1)\partial_t F_1 = \varepsilon G_t + G > 0 \tag{6.5}$$

and, being $F \geqslant G$ (see (3.2)-(6.2)), by (6.4)-(6.5) we obtain (6.1a). Now, referring to the kernel F_0 which solves the problem $\mathscr{H}$ and is defined by (5.5), we observe that by means of integrations by parts one has

$$\begin{aligned} F_0 = \lambda^2 x(\varepsilon^2\xi\omega)^{-1} \exp[-\gamma^2 t + k^2 r]\Big[\lambda^2\xi I_1(\omega) + 2c^2\omega \exp[\eta] I_1(\xi) + \\ + \lambda^2\xi^2\int_0^1 \exp[\eta v^2] \frac{I_1(\omega\sqrt{1-v^2})}{2x\sqrt{1-v^2}}[2x + (t-x)v^2] I_1(\xi v)\, dv\Big], \end{aligned} \tag{6.6}$$

which demonstrates $F_0 \geqslant 0$ everywhere in $\bar{\Omega}$. Then, as

$$(\varepsilon\partial_t + 1)F = (1 - c^2)F + F_0 > 0,$$

also (6.16) is proved. As for (6.1c), the foregoing relations and standard computations enable us to obtain finally

$$\begin{aligned} [\varepsilon\partial_t^2 - \varepsilon\partial_r^2 + (1 - c^2)\partial_t]F_1 = c^2 F - c^2\partial_t F_1 + \\ + \exp[-\gamma^2 t + k^2 r]\int_0^1 v \exp[\eta v^2] I_0(\omega\sqrt{1-v^2})[4\eta v I_1(\xi v) + \xi I_0(\xi v)]\, dv, \end{aligned} \tag{6.7}$$

which implies (6.1c) being $F \geqslant \partial_t F_1$ (see (6.1a)).

REMARK 6.1. – As it will be shown later (Sect. 7), being $F = -\partial_r F_1$, the operator which in (4.7) is applied to u_{f_0} is expressible in terms of

$$R = (\varepsilon\partial_t^2 - \varepsilon\partial_r^2 + \partial_t)F_1 \geqslant 0\,, \qquad (r, t) \in \bar{\Omega}\,, \tag{6.8}$$

which is a positive value function everywhere in $\bar{\Omega}$, as it is obvious by (6.7). Further, by the definition (6.8) of R and by (6.7) one has

$$R(t, t) = c^2 F(t, t) \tag{6.9a}$$

$$R(0, t) = k^2 \exp[-\gamma^2 t][(1 + 2\lambda^4 t\varepsilon^{-2})I_0(\lambda^2 t/\varepsilon) + 2\lambda^4 t\varepsilon^{-2} I_1(\lambda^2 t/\varepsilon)] \leqslant \leqslant \varepsilon^{-1}(1 + t/\varepsilon)\exp[-k^2 t]\,. \tag{6.9b}$$

To complemente the Theorem 6.1, setting

$$h_2(t) = \int_0^t \exp[-\gamma^2\tau] I_0(\lambda^2\tau/\varepsilon)\, d\tau \tag{6.10a}$$

$$h_1(t) = \varepsilon^{-1} h_2(t) + h_2'(t) \tag{6.10b}$$

we prove now the following theorem

THEOREM 6.2. – *The $C^\infty(\Omega)$ positive value functions F and R defined by* (3.2) *and* (6.8) *are such that*

$$\int_0^t F(r, t)\, dr = \varepsilon^{-1} h_2(t)\,, \qquad \int_0^t R(r, t)\, dr = 1 - \exp[-k^2 t] \tag{6.11}$$

$$(1 + \varepsilon\partial_t)\int_0^t F(r, t)\, dr = h_1(t) \tag{6.12}$$

where for h_1 and h_2 the following estimates hold

$$1 \leqslant h_1(t) \leqslant c^{-1}\,, \qquad 0 \leqslant h_2(t) \leqslant \varepsilon c^{-1}\,, \qquad (t \geqslant 0)\,. \tag{6.13}$$

PROOF. – The formulae (6.11) can be verified by means of Theorem 3.3. In fact

$$\mathscr{L}_t \int_0^t F(r, t)\, dr = \int_0^\infty \hat{F}(r, s)\, dr = s^{-1}[(\varepsilon s + 1)(\varepsilon s + c^2)]^{-\frac{1}{2}}\,,$$

hence, as it follows by known Laplace integrals (see [19], pag. 239), one has $(6.11)_1$ and as consequence (6.12). Likewise one obtains $(6.11)_2$. Further one can see easily that h_1 and h_2 are increasing functions such that $h_1(0) = 1$, $h_1(\infty) = c^{-1}$ and $h_2(0) = 0$, $h_2(\infty) = \varepsilon c^{-1}$; consequently also (6.13) are proved.

In order to evaluate the integrals of convolution u_{f_t} such as (4.5), consider now the asymptotic behaviour of the kernels F, F_0, R as $t \to \infty$. For this, let $d = d_0(\varepsilon/t)^{\frac{1}{3}}$ be, where d_0 is an arbitrary real positive constant such that $d_0 < \min(c, 1-c)$. Then, if $\varepsilon < t$ and observing that in (4.5) it is $0 \leqslant r \leqslant t$, one has

$$Q_t \equiv \{r \in [(c-d)t, (c+d)t]\} \subset [0, t] .$$

Further, let E_h be the fundamental solution of the heat equation (1.7) where $|y| = |r - ct|$; i.e.

$$E_h(r, t) = \frac{b \exp[-b^2(r-ct)^2/\varepsilon t]}{\sqrt{\pi \varepsilon t}}, \qquad (b^2 = (4\lambda^2)^{-1}) . \tag{6.14}$$

Then, the following basic theorem can be proved.

THEOREM 6.3. – *Let $(r, t) \in \Omega$ and $0 < \varepsilon < t$. Then, for any $r \in Q_t$, it results*

$$F(r, t) = c^{-1}(1 + \varrho_0')E_h + \varrho_0'' \tag{6.15a}$$

where the remainder terms ϱ_0', ϱ_0'' satisfy the estimates

$$|\varrho_0'| < k_0\left(1 + \frac{|r-ct|}{\sqrt{\varepsilon t}}\right)^3 \sqrt{\frac{\varepsilon}{t}} \tag{6.15b}$$

$$|\varrho_0''| < k_0 \exp[-\beta_1^2(t/\varepsilon)^{\frac{1}{3}}] \tag{6.15c}$$

with k_0, β_1^2 constants depending only on c.

PROOF. – See Appendix.

REMARK 6.2. – The estimates of Theorem 6.3 hold also for the kernel F_0, R, $\varepsilon F_t + F$. In fact, by means of the method of proof of Theorem 6.3, one can verifies that for $r \in Q_t$ and $0 < \varepsilon < t$ it is

$$F_0 = c(1 + \varrho_1')E_h + \varrho_1'', \qquad R = (1 + \varrho_2')E_h + \varrho_2'' \tag{6.16a}$$

$$(\varepsilon\partial_t + 1)F = c^{-1}(1 + \varrho_3')E_h + \varrho_3'' \tag{6.16b}$$

where the remainder terms satisfy estimates such as (6.15b, c).

Let now

$$\alpha_0^2 = c^2 + \lambda^2 - \sqrt{c^4 + \lambda^4}, \qquad \alpha_1^2 = c^2/4 . \tag{6.17}$$

As another consequence of the method of proof of Theorem 6.3, one can deduces also an estimate uniformly valid for any $t \geqslant 0$ and $r \in [0, t]$. In fact

THEOREM 6.4. – *For all $t\geqslant 0$ and $r\in[0,t]$, F verifies the following inequality*

$$\varepsilon F(r,t)<k_1\exp\left[-\alpha_0^2 t/\varepsilon\right]+k_1(t/\varepsilon)\exp\left[-\alpha_1^2(r-ct)^2/\varepsilon t\right], \tag{6.18}$$

where α_0 and α_1 are the constants defined in (6.17).

PROOF. – See Appendix.

REMARK 6.3. – When $t>\varepsilon$ and in a neighbourhood of the boundary of the forward cone Λ_c, the kernels F, F_0 and R are therefore approximated by means of the fundamental solution E_h of the heat equation (1.7).

7. – Maximum principles.

At first we consider the Cauchy problem $\mathscr{P}_n$ $(n=1,2,3)$. Let h_1 and h_2 be the functions defined in (6.10a, b) and let $\bar f_i(x,r)$ the mean values of the data $f_i(x)$ given by (2.4a, b, c) according to the number n. We put

$$u^-(x,t)=\inf_r\,\partial_r(r\bar f_0)+\sum_{i=1}^{2}h_i\inf_r\,(r\bar f_i),\qquad 0\leqslant r\leqslant t \tag{7.1}$$

$$u^+(x,t)=\sup_r\,\partial_r(r\bar f_0)+\sum_{i=1}^{2}h_i\sup_r\,(r\bar f_i),\qquad 0\leqslant r\leqslant t \tag{7.2}$$

where $h_i(t)$ are positive value and bounded functions (see (6.13)).

THEOREM 7.1. – *Whatever the number $n=1,2,3$ of space dimensions may be, the solution of the problem $\mathscr{P}_n$ (with $f=0$) satisfies the following maximum principle*

$$u^-\leqslant u(x,t)\leqslant u^+,\qquad x\in R^n,\ t\geqslant 0. \tag{7.3}$$

PROOF. – The proof is a consequence of Theorems 6.1 and 6.2. Consider the representation (4.7) of u and observe that, being the source term f vanishing, one has

$$u=\varepsilon u_{f_2}+(1+\varepsilon\partial_t)u_{f_1}+u_* \tag{7.4}$$

where u_{f_i} are defined by (4.5) and

$$u_*=(\varepsilon\partial_t^2-\varepsilon\Delta_n+\partial_t)u_{f_0}.$$

As $F>0$ everywhere in $\bar\Omega$, by $(6.11)_1$ it is

$$h_2\inf_r\,(r\bar f_2)\leqslant\varepsilon u_{f_2}\leqslant h_2\sup_r\,(r\bar f_2),\qquad (0\leqslant r\leqslant t).$$

Further

$$(1+\varepsilon\partial_t)u_f = \varepsilon F(t,t)t\bar{f}_1(x,t) + \int_0^t r\bar{f}_1(x,r)(1+\varepsilon\partial_t)F(r,t)\,dr \tag{7.5}$$

hence, applying (6.1b) and (6.12), it draws

$$h_1 \inf_r (r\bar{f}_1) \leqslant (1+\varepsilon\partial_t)u_{f_1} \leqslant h_1 \sup_r (r\bar{f}_1)\,, \qquad (0\leqslant r\leqslant t)\,.$$

As for u_*, being $F = -\partial_r F_1$, by means of standard computations and integrating by parts (see (3.7d)), it results

$$u_* = \exp[-k^2 t] f_0(x) + \int_0^t R(r,t)\,\partial_r[r\bar{f}_0(x,r)]\,dr\,, \tag{7.6}$$

where R is the positive value function given by (6.8). Observe now that

$$f_0(x) = [\partial_r(r\bar{f}_0(x,r)]_{r=0}$$

and that $(6.11)_2$ holds. Consequently also for u_* one has

$$\inf_r \partial_r(r\bar{f}_0) \leqslant u_* \leqslant \sup_r \partial_r(r\bar{f}_0)\,, \qquad (0\leqslant r\leqslant t)$$

and the proof is complete.

In view of other applications, we observe explicitly that from (7.4)-(7.5)-(7.6) it is evident that when $f=0$ u can be expressed (see (3.7d)) also as follows ($x\in R^n$, $t\geqslant 0$)

$$\begin{aligned} u = \varepsilon u_{f_2} + \exp[-\lambda^2 t/\varepsilon]t\bar{f}_1(x,t) + \int_0^t r\bar{f}_1(x,r)(1+\varepsilon\partial_t)F(r,t)\,dr\, + \\ + \exp[-k^2 t]f_0(x) + \int_0^t R(r,t)\,\partial_r[r\bar{f}_0(x,r)]\,dr\,, \end{aligned} \tag{7.7}$$

with R given by (6.8).

Obviously various consequences can be obtained from Theorem 7.1, as show the following examples.

EXAMPLE 7.1. - By (7.2)-(6.13), for $x\in R^n$ and $t\geqslant 0$ immediately it follows

$$|u| \leqslant \sup_r |\partial_r(r\bar{f}_0)| + c^{-1}\sum_{i=1}^{2}\varepsilon^{i-1}\sup_r |r\bar{f}_i|\,, \qquad (0\leqslant r\leqslant t)\,. \tag{7.8}$$

EXAMPLE 7.2. - When all the data have a same constant sign, also the solution u has constant sign. More precisely, if

a) $f_i(x)\geqslant 0$ $(i=0,1,2)$ when $n=1$

b) $f_i(x)\geqslant 0$ $(i=0,1,2)$ and $\Delta_n f_0 \geqslant 0$ when $n=2,3$

$x\in R^n$

then

$$u(x,t)\geqslant 0\,,\qquad x\in R^n,\ t\geqslant 0\,, \tag{7.9}$$

which gives also a very simple minimum principle when $f_0(x)=0$.

In the inhomogeneous case the formula (7.9) holds too when, in addition, $f(x,t)\geqslant 0$ in Y_+^{n+1}.

EXAMPLE 7.3. – For $n=1$ and $f_i(x)\geqslant 0$ $(i=0,1,2)$ easily one has

$$\inf_x f_0(x)\leqslant u\leqslant \sup_x f_0(x)+(2c)^{-1}\int_{x-t}^{x+t}[f_1(y)+\varepsilon f_2(y)]\,dy\,.$$

Furthermore, these results can be used also to state comparison theorems and to obtain appropriate approximations of u with the determination of an explicit bound for the error. It suffices to apply well known methods of the theory on maximum principles (see e.g. [30]).

Consider now the case with $f\neq 0$ and $f_i=0$ $(i=0,1,2)$. Then the solution (4.7) of $\mathscr{P}_n$ reduces it self to the term u_f defined by (4.6). So, if we put

$$f_*(x,t,\tau)=\sup_{0\leqslant r\leqslant\tau} r\bar{f}(x,r;t-\tau)\,,\qquad x\in R^n,\ 0\leqslant\tau\leqslant t \tag{7.10}$$

from the properties of F it draws:

THEOREM 7.2. – *For $x\in R^n$ $(n=1,2,3)$ and $t>0$, the solution u_f of the problem $\mathscr{P}_n$ with initial data vanishing satisfies the following estimate*

$$|u_f|\leqslant c^{-1}\int_0^t|f^*(x,t,\tau)|\,d\tau\,,\qquad (x,t)\in Y_+^{n+1} \tag{7.11}$$

where f_* is given by (7.10).

PROOF. – Referring to the definition (4.6) of u_f, it suffices to observe that $F>0$ and the inequality (3.7c) holds; then

$$\int_0^\tau F(r,\tau)\,r\bar{f}(x,r;t-\tau)\,dr\leqslant c^{-1}f_*(x,t,\tau)\,,$$

hence obviously one deduces (7.11).

Finally consider the solution v of the signaling problem $\mathscr{H}$ (Sect. 5). The properties (6.1b) and (5.8) of the kernel F_0 enable us to prove the following maximum principle for v.

THEOREM 7.3. – *If for all $t > 0$ the datum $\Phi(t)$ has a constant sign, also v has the constant sign of Φ. Furthermore, in any case, it results*

$$|v(x, t)| \leqslant \sup_{0 \leqslant \tau \leqslant t-x} |\Phi(\tau)| , \qquad t \geqslant x \geqslant 0 . \tag{7.12}$$

PROOF. – The first part of Theorem is evident on the basis of (5.6)-(6.1*b*). As for (7.12) we observe that, being $F_0 \geqslant 0$, by (5.8) one has

$$|v| \leqslant \sup_{\tau} |\Phi| \left[\exp[-\lambda^2 x/\varepsilon] + \int_x^\infty F_0(x, \tau)\, d\tau \right] = \sup_{\tau} |\Phi| .$$

8. – General behaviour of the solutions of $\mathscr{P}_n$.

Now we deal with a qualitative analysis of the solution u of $\mathscr{P}_n$ $(n = 1, 2, 3)$ for all $t \geqslant 0$.

At first we observe that the kernel $F(r, t)$ depends on $r = |x|$, t, ε by means of the ratios r/ε and t/ε (with $r \leqslant t$) and that the modified Bessel functions which appear in the definition (3.2) of F are series of power of $(r/\varepsilon)(t - r)/\varepsilon$ and $(t^2 - r^2)/\varepsilon^2$. Consequently, when $t/\varepsilon < 1$ (hence also $r/\varepsilon < 1$), there is not problem; well known properties of the Bessel functions enable us to estimate F with a degree of accuracy however prefixed, particularly near to the wave-front $r = t$ and near $r = 0$.

As an example, let us consider the case $f_1 = f_2 = 0$, $f = 0$ and let (see (6.9*b*))

$$h_0(t) = R(0, t) \leqslant \varepsilon^{-1}(1 + t/\varepsilon) \exp[-k^2 t] . \tag{8.1}$$

As $|R(r, t) - R(0, t)| < \text{const}\,(t/\varepsilon)$, when f_0 is a smooth function and $t/\varepsilon < 1$, by (7.7) one deduces

$$u = u_1 + O(t^2/\varepsilon^2) , \qquad (t/\varepsilon < 1) ,$$

with

$$u_1 = \exp[-k^2 t] f_0(x) + h_0(t)\, t \bar{f}_0(x, t) .$$

Therefore, being h_0 of the order of $\exp[-t/\varepsilon]$, the first signal appears as exponentially damped small precursor wave propagating with the speed of the wave-front.

8.1. *Diffusion of waves. Proof of Theorem* 1.1.

On the contrary, when $t/\varepsilon > 1$, the analysis is more difficult. Obviously the asymptotic properties of u depend on the behavior of the data $f_i(x)$. For example, constant data can imply that $u \to \infty$ as $t \to \infty$; in fact the problem $\mathscr{P}_1$, with

$f_0 = f_1 = f = 0$ and $f_2 = A_2$ (const) admits the unbounded solution

$$u = A_2 \varepsilon t - A_2 \varepsilon^2 (1 - \exp[-t/\varepsilon]) .$$

Theorem 6.3 on the asymptotic behavior of the kernel F is basic for this analysis. By means of this theorem one deduces that u and the solution u_h of the Cauchy problem related to heat equation (1.7) with datum Ψ (see (1.9a)) have the same asymptotic properties. Therefore, referring to the Section 1.2, we prove now Theorem 1.1.

PROOF OF THEOREM 1.1. - By (7.7) and (6.16a, b) one deduces

$$u = \int_0^t E_h(r, t)\, \Psi(x, r)\, dr + \varrho_h$$

where Ψ is defined by (1.9a) and E_h is the fundamental solution of (1.7) given by (6.14). Further, let us consider the estimates (6.15b, c) and Remark 6.2; being Ψ a bounded function by hypothesis, one has easily

$$|\varrho_h| \leqslant \text{const } \|\Psi\| (\varepsilon/t)^{\frac{1}{2}} .$$

Lastly, by definitions (1.10) and (6.14) of u_h and E_h it is obvious that

$$\left|u_h - \int_0^t E_h(r, t)\, \Psi(x, r)\, dr\right| < \text{const} \exp[-\lambda_0^2 t/\varepsilon]$$

hence (1.11) follows.

REMARK 8.1. - Obviously the hypothesis of Theorem 1.1 that Ψ is a bounded function can be attenuated. In fact in any case, when $t/\varepsilon > 1$, u_h represents the main part of u.

REMARK 8.2. - Let us observe that, by means of well-known properties of the comparison function u_h and with appropriate hypotheses on Ψ, *for all* $t > 0$ by (1.11) one has

$$\lim_{\varepsilon \to 0} u(x, t) = \lim_{\varepsilon \to 0} \Psi(x, ct) = \partial_r(r\bar{f}_0) + c^{-1} r \bar{f}_1$$

which is the classic solution of the problem deduced from $\mathscr{P}_n$ setting formally $\varepsilon = 0$.

Therefore, to draw a conclusion, one can assemble all the times in two intervals such as $[0, \varepsilon]$ and $]\varepsilon, \infty[$, so the results foreseen by Whitham and advanced in Sect. 1.2 are rigorously proved and evaluated.

8.2. *Asymptotic properties.*

Theorem 7.1, besides proving and evaluating diffusion at large t/ε, can be applied also to obtain the asymptotic behavior of u. Clearly, this analysis can be improved when the data are specified. For this we deal with some examples.

EXAMPLE 8.1. – *Problem $\mathscr{P}_1$ with periodic initial data.*

If $f_i(x) = A_i \operatorname{sen} \omega_i x$ one has

$$r\bar{f}_i(x, r) = (A_i/\omega_i) \operatorname{sen}(\omega_i x) \operatorname{sen}(\omega_i r), \qquad (i = 0, 1, 2) \tag{8.2}$$

and these functions verify the hypothesis of Theorem 1.1. Consequently, being (see [19], p. 158)

$$\frac{b}{\sqrt{\pi\varepsilon t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{b^2 v^2}{\varepsilon t}\right] \cos(\omega_i v)\, dv = \exp[-\lambda^2 \omega_i^2 \varepsilon t], \qquad (\lambda^2 = (4b^2)^{-1})$$

by (1.10)-(1.9a) one obtains

$$u_h = [\partial_r(r\bar{f}_0)]_{r=ct} \exp[-\lambda^2 \omega_0 \varepsilon t] + \sum_{i=1}^{2} \varepsilon^{i-1} t\bar{f}_i(x, ct) \exp[-\lambda^2 \omega_i^2 \varepsilon t]. \tag{8.3}$$

Therefore, unlike the classic case, when the initial disturbances are periodic and $f = 0$, the solution of $\mathscr{P}_1$ vanishes as $t \to \infty$ uniformly in x.

REMARK 8.3. – For periodic initial data, the asymptotic formula (8.3) exhibits the *slow-time* εt which accounts the singularity of the perturbation as $\varepsilon \to 0$ and $t \to \infty$ (see Sect. 10.2).

EXAMPLE 8.2. – *Problem $\mathscr{P}_3$ with initial data $f_i(x) \in \mathscr{S}_2^m(R^3)$.*

When the initial data are suitable vanishing as $|x| \to \infty$, it must needs apply directly Theorem 6.4 for asymptotic estimates. So, in the case we deal with, when $f = 0$ one obtains

$$f_i(x) \in \mathscr{S}_2^{u}(R^3) \Rightarrow u \in \mathscr{S}_1^m(\Upsilon_+^4).$$

In fact, we will prove that

THEOREM 8.1. – *Let $f_0 \in \mathscr{S}_2^1(R^3)$ be and $f_1, f_2 \in \mathscr{S}_2^0(R^3)$. Then, for large t and uniformly in $x \in R^3$, the solution of $\mathscr{P}_3$ (with $f = 0$) satisfies the following asymptotic estimate*

$$|u| \leq \frac{\text{const.}\, t}{1 + |x|^2 + t^2} (\|f_0\|_{12} + \|f_1\|_{02} + \|f_2\|_{02}) \tag{8.4}$$

with the constant depending only on c.

PROOF. - Being $f_i \in \mathscr{S}_2^m(R^3)$ $(m = 0, 1)$, one has

$$|r\bar{f}_i(x, r)| \leqslant \frac{\|f_i\|_{m2}}{4\pi r} \iint\limits_{|y-x|=r} \frac{dS_y}{(1+y^2)^2}, \qquad x \in R^3 .$$

Further

$$\iint\limits_{|y-x|=r} \frac{dS_y}{(1+y^2)^2} = 4\pi r^2[1 + 2|x|^2 + 2r^2 + (|x|^2 - r^2)^2]^{-1} \leqslant 4\pi r^2(1 + |x|^2 + r^2)^{-1}$$

hence

$$|u_{f_i}| \leqslant \|f_i\|_{m2} \int_0^t \frac{rF(r, t)}{1 + |x|^2 + r^2}\, dr , \qquad x \in R^3 .$$

So, applying Theorem 6.4, easily it draws

$$|u_{f_i}| \leqslant \text{const}\, \|f_i\|_{m2} t(1 + |x|^2 + t^2)^{-1}$$

and, estimating in the same way all the therms of u given by (7.7), the proof is complete.

9. - Generalized Huyghens principle. Proof of theorem 1.2.

Consider now the case that the initial disturbances are localized in a ball $B(0, \varrho)$ of radius ϱ and center 0.

Then, according to what has been advanced in Section 1.3, we will show that when $n = 3$ there is a time instant $t_1 = c^{-1}(|x| + \varrho)$ after which u—though not vanishing—*decays exponentially in time*. Further we will prove that *this results does not subsist when $n = 2$ or $n = 1$*.

A) *Case* $n = 3$. Proof of Theorem 1.2.

As it is well known, when the initial disturbance is localized in the ball $B(0, \varrho)$ for any $x \in R^3$ such that $x \notin B(0, \varrho)$ it results:

$$r\bar{f}_1(x, r) = 0 \quad \text{for any } r \notin [|x| - \varrho, |x| + \varrho] . \tag{9.1}$$

Further for large r the function $r\bar{f}_i(x, r)$, while its support expands, decays being at most of the order of $1/r$ (see e.g. [24], p. 109). Consequently, for $t \geqslant |x| + \varrho$ one has

$$u_{f_i}(x, t) = \int_{|x|-\varrho}^{|x|+\varrho} F(r, t)\, r\bar{f}_i(x, r)\, dr , \qquad x \in R^3 \tag{9.2}$$

and by Theorem 1.1 easily one can see that u_{f_i} is always such that

$$|u_{f_i}| \leqslant \text{const.}\, t^{-1}\,, \qquad t > |x| + \varrho\,. \tag{9.3}$$

But at the later instants such that

$$t \geqslant t_1 = (|x| + \varrho)/c > |x| + \varrho$$

one has $r \leqslant |x| + \varrho < ct$ and therefore, if we put $\alpha^2 = b^2c^2/\varepsilon$, by Theorem 6.3 easily it draws

$$|u_{f_i}| \leqslant \frac{\text{const}}{\sqrt{\varepsilon t}} \int\limits_{|x|-\varrho}^{|x|+\varrho} \exp\left[-\frac{b^2}{\varepsilon t}(ct - r)^2\right] dr \leqslant \text{const} \exp\left[-\alpha^2(1 - t_1/t)^2 t\right]. \tag{9.4}$$

Applying this formula to all the terms u_{f_i} which appear in the representation (7.7) of u, easily (1.12) follows.

REMARK 9.1. – As the proof of Theorem 1.2 shows, the main part of the signal appears as a diffusione process in the time-interval $[(|x| - \varrho)/c, (|x| + \varrho)/c]$ related to the speed c of the lowest order waves.

B) Case $n = 2$ *or* $n = 1$.

Consider now the case $n = 2$ where, it is well known, the effect of the initial disturbances appear in $x \notin B(0, \varrho)$ at the instant $t_0 = |x| - \varrho$ and is observed there at any later instant. So one has

$$u_{f_i} = \int\limits_{|x|-\varrho}^{t} F(r, t)\, r \bar{f}_i(x, r)\, dr\,, \qquad x \in R^2 \tag{9.5}$$

and $[|x| - \varrho, t]$ is—for any t—a neighbourhood of the point of maximum $r = ct$ of the function E_h approximating $F(r, t)$ when t is large. Consequently in this case *there is not time-instant* t_1 *such that the estimate* (1.12) *holds*. Analogous result holds when $n = 1$.

In two dimensions it is possible to prove, by means of the various estimates above stated, that the rate of decay with time t is of order $1/t$, like the classic case.

10. – Analysis of singular perturbations.

In this Section we will show the application of theorems above stated to the analysis of singular perturbations for the one-dimensional Cauchy problem $\mathscr{P}_1$ and the signaling problem $\mathscr{H}$. The generalization to the three-dimensional and bi-dimensional cases is straight-forward. In what follows, u_ε and v_ε denote the solutions of $\mathscr{P}_1$ and $\mathscr{H}$ related to $\boldsymbol{L}_1$, while u_0 and v_0 are the solutions of the limit problems which one deduces from $\mathscr{P}_1$ and $\mathscr{H}$ putting formally $\varepsilon = 0$ in $\boldsymbol{L}_1$.

On this subject we recall that the present boundary layer problems fall within a class of abstractly defined problems for which J. L. LIONS demonstrated the convergence of u_ε to u_0 in the sense of a suitable weak topology for a finite interval of time [21].

At first we observe that the subcharacteristics of $\boldsymbol{L}_1$ are $x \pm ct = \text{const}$; so, when $c^2 < 1$ it is reasonable to conceive of u_0 as a limit of the exact solution as $\varepsilon \to 0$. On the contrary, when $c^2 > 1$ the subcharacteristics through the point (x, t) lie outside the domain of dependence of u_ε. The speed of the disturbances associated with the subcharacteristics is greater than that of the characteristics of $\boldsymbol{L}_1$. In this case one cannot expect u_0 as the limit of the exact solution u_ε [26]. This strange behavior is connected with the fact that the condition $c^2 > 1$ is not reasonable from the physical point of view.

Thus, for example, in viscoelasticity one has $c^2 = c_0^2/c_1^2 = g(\infty)/g(0)$, where $g(0)$ and $g(\infty)$ are, respectively, the determinations for $t = 0$ and $t = \infty$ of the decreasing relaxation function $g(t)$. In thermochemistry c_1 is the *frozen sound speed* and c_0 is the *equilibrium sound speed*; well-known results of thermodynamic stability show that $c_0 < c_1$ [4].

10.1. *Signaling problem. Interior layer.*

The signaling problem $\mathscr{H}$—defined by (4.1)-(5.2)-(5.3)—has the solution v_ε given by (5.6). Now, when $\Phi^{(i)}(0) \neq 0$, there is a real discontinuity in the function v_ε and in its derivatives along the characteristic curve $x = t$; in fact v_ε is identically zero for $x > t$. On the contrary

$$v_0(x, t) = H(t - x/c)\,\Phi(t - x/c)$$

has a discontinuity on the subcharacteristic $x = ct$; such a discontinuity is not permitted to v_ε (when $\varepsilon > 0$). Consequently, the limit solution u_0 cannot be an uniformly valid approximation and an *interior layer* on the particular subcharacteristic $\zeta = x - ct = 0$ appears.

The *formal* computation of asymptotic series can be done by means of the matching asymptotic expansions (see [25], [26]). But a rigorous analysis and an exact estimate of this singular behavior can be only given by a careful examination of the solution (5.6). The question is solved by means a theorem such as Theorem 1.1. If

$$\beta^2 = c^3 b^2\,, \qquad \|\Phi\| = \sup_\tau |\Phi(\tau)|\,, \qquad 0 < \tau < t - x\,,$$

on the basis of what was illustrated at Section 8, it draws the following uniformly valid approximation to v_ε.

THEOREM 10.1. – *Let $\Phi(\tau)$ be bounded in $[0, \infty[$ and let $0 < \varepsilon < x$. Then, for any $x \in R^+$ and $t \geqslant x$, one has*

$$v_\varepsilon(x, t) = \frac{\beta}{\sqrt{\pi\varepsilon x}} \int_{-\infty}^{t-x/c} \exp\left[-\frac{\beta^2 u^2}{\varepsilon x}\right] \Phi\left(t - \frac{x}{c} - u\right) du + v_1 \tag{10.1}$$

where the remainder term v_1 satisfies the estimate uniformly in $t \geqslant x$

$$|v_1| < \text{const}\, \|\Phi\| (\varepsilon/x)^{\frac{1}{2}} \tag{10.2}$$

with the constant depending only on c.

REMARK 10.1. – This theorem shows rigorously that the *interior layer equation*, according to what foreseen by WHITHAM [1-2], is the heat equation

$$u_T = \beta_1 u_{YY} \quad \text{with } T = x,\ Y = t - x/c$$

where the diffusion coefficient β_1 is given by $\varepsilon/4\beta^2$. The essentially diffusive character of the phenomenon, when $x > \varepsilon > 0$, is revealed. The maximum disturbance, for every positive $\varepsilon/x < 1$, travels with the speed c of the lowest order waves and *diffuses* with a characteristic diffusion width defined by $x - ct = (B/a)\sqrt{\varepsilon x}$, where B is a prefixed constant. The interior layer tickness here is an order of magnitude larger than in the initial boundary layer.

10.2. *Periodic initial data. Singularity at $t = \infty$.*

Consider now the problem $\mathscr{P}_1$ with $f = 0$ and

$$f_i(x) = A_i \cos \omega_i x\,, \qquad (i = 0, 1, 2)\ (A_i \text{ const})\,. \tag{10.3}$$

By Example 8.1 one has uniformly in x

$$\lim_{t\to\infty} u_\varepsilon(x, t) = 0\,, \qquad x \in R\,,$$

while $u_0(x, t)$ does not admit limit as $t \to \infty$. Further $\partial_t^2 u_0(x, 0) \neq f_2(x)$, so in this case the perturbation for $\varepsilon \to 0$ is singular at $t = 0$ and $t = \infty$.

The explicit formula (8.3) for the comparison function u_h related to this case, together with the estimate (1.11), give us a rigorous asymptotic representation as $(\varepsilon/t) \to 0$. But this approximation does not account the singularity at $t = 0$. This question can be solved easily by means of theorems previously stated. For this let

$$b_k = \frac{1 - c^2}{2}\omega_k \quad (k = 0, 1)\,, \qquad 2\gamma_0 = c^2\omega_0^2 - \varepsilon^2 b_0^2$$

$$g_0 = \tfrac{1}{2} f_2 - \varepsilon f_1 + (b_0 + \gamma_0) f_0$$

and let $\exp[-t/\varepsilon]z(x,t)$ a *boundary-layer function* with z given by

$$z(x,t)=\tfrac{1}{2}\int_{x-t}^{x+t}[\varepsilon b_0 f_0(y)+tg_0(y)]\,dy\,. \tag{10.4}$$

Further, let

$$u_0(x,ct)=ct\bar{f}_1(x,ct)+\partial_t[ct\bar{f}_0(x,ct)]\,. \tag{10.5}$$

Referring to these notations we will prove that

THEOREM 10.2. – *Let u_ε be the solution of the problem $\mathscr{P}_1$ defined by $f=0$ and by the initial conditions* (10.3). *As $\varepsilon\to 0$ u_ε, for all $t\geqslant 0$ and $x\in R$, is approximated by*

$$u_\varepsilon=\exp[-b_0\varepsilon t]\,\partial_t[ct\bar{f}_0(x,ct)]+\exp[-b_1\varepsilon t][ct\bar{f}_1(x,ct)]+ \\ +\exp[-t/\varepsilon]z(x,t)+\varepsilon r_1\,, \tag{10.6}$$

with the remainder term r_1 uniformly bounded by a constant depending only on c.

PROOF. – The proof is a consequence of Theorem 7.2. In fact, the remainder term r_1 is the solution of a problem $\mathscr{P}_1$ with initial data vanishing and a source term $f=f'+f''$ with

$$f'=-\varepsilon^{-1}\mathbf{L}_1(\exp[-t/\varepsilon]z)$$

$$f''=-\varepsilon^{-1}\mathbf{L}_1[\exp[-b_0\varepsilon t]\,\partial_t(ct\bar{f}_0)+\exp[-b_1\varepsilon t](ct\bar{f}_1)]\,.$$

Now, referring to the function f_* defined by (7.10) and observing that $b_i=\omega_i^2(1-$ $-c^2)/2$, it is easy to verify that

$$|f_*(x,\tau,t)|\leqslant\text{const}\,\{\varepsilon\exp[-\varepsilon\bar{b}(t-\tau)]+\varepsilon^{-1}\exp[-(t-\tau)/\varepsilon][1+(t-\tau)+(t-\tau)^2]\}\,,$$

where $\bar{b}=\min(b_0,b_1)$. So, by Theorem 7.2 (see (7.11)) one has

$$|r_1|=|u_f|\leqslant\text{const}\,,\qquad (x,t)\in Y_+^2$$

with the constant depending only on c.

REMARK 10.2. – Unlike the method expounded in [14], this analysis can be done in the general case that $f_i(x)$ are given by their Fourier series.

10.3. *An example. The viscoelastic case.*

As an example, consider the application of Theorem 10.2 to the viscoelastic case where the equation (1.2) characterizes the motions of the *standard linear solid* whose memory function $g(t)$ is

$$g(t) = g(\infty) + [g(0) - g(\infty)] \exp[-t/\varepsilon],$$

with ε a relaxation time and $g(\infty) < g(0)$. In this case the two characteristic speeds c_0 and c_1 are

$$c_1^2 = \varrho^{-1} g(0), \qquad c_0^2 = \varrho^{-1} g(\infty), \qquad (c_0^2 < c_1^2)$$

and ϱ is the mass density. Obviously the case $\varepsilon = 0$ corresponds to the elastic waves propagating with the speed c_0.

To deal with a simple case, we will refer to problem $\mathscr{P}_1$ with periodic initial data, assuming that $\omega_2 = \omega_1 = \omega_0$. Then by (10.6) one has

$$u_\varepsilon = \exp[-b_0 \varepsilon t] u_0(x, c_0 t) + \exp[-t/\varepsilon] z(x, c_1 t) + \boldsymbol{\varepsilon} r_1 \tag{10.7}$$

where $u_0(x, c_0 t)$ is the pure elastic wave defined by (10.5) and associated with the speed c_0, while z is the wave represented by (10.4) and related to the faster speed c_1.

By (10.7) the conclusions advanced in Section 1.4 obviously follow.

11. – Appendix: proof of theorem 6.3.

Consider the function $F(r, t)$ which is defined in Ω and put $t/\varepsilon = \tau$, $r = tz$ ($z \in [0, 1]$). According to (3.1), if

$$w = \lambda^2 \sqrt{1 - z^2}, \qquad p = 2c\lambda \sqrt{z(1-z)}, \qquad q = \lambda^2(1 - z)/2,$$

one has $\omega = \tau w$, $\xi = \tau p$, $\eta = \tau q$. Setting

$$g(z, \tau, v) = \tau \exp[-[\lambda^2 + c^2(1 - z) - qv^2]\tau] I_0(\tau w \sqrt{1 - v^2})[4qvI_0(\tau pv) + \\ + pI_1(\tau pv)] \tag{11.1}$$

the function εF, which depends only on z, τ, is

$$\varepsilon F(z, \tau) = \exp[-\lambda^2 \tau - c^2(1 - z)\tau] I_0(\tau w) + \int_0^1 g(z, \tau, v)\, dv. \tag{11.2}$$

Further let $Q \equiv \{(z, v) \in [0, 1]^2\}$ be.

Preliminarly we observe that by the well-known relation (y real positive)

$$I_n(y) \leqslant \exp(y)\,, \qquad (y \geqslant 0) \tag{11.3}$$

one has, for all $\tau \geqslant 0$ and $(z, v) \in Q$

$$g(z, \tau, v) \leqslant \exp[-\tau h(z, v)]\,, \tag{11.4}$$

with

$$h(z, v) = \left(c\sqrt{1-z} - \lambda v\sqrt{z}\right)^2 + (\lambda^2/2)\left[\sqrt{1-z} - \sqrt{(1+z)(1-v^2)}\right]^2. \tag{11.5}$$

This function h plays a basic role in our analysis. First of all we observe that

REMARK 11.1. – The function h, given by (11.5) and defined in Q, has an absolute minimum at $z_0 = c$, $v_0 = [2c/(1+c)]^{\frac{1}{2}}$, where vanishes.

Moreover it can be shown that

LEMMA 11.1. – *For all $(z, v) \in Q$ it results*

$$h \geqslant (c^2/4)(z - z_0)^2; \qquad h \geqslant (c\lambda^2/4)(v - v_0)^2. \tag{11.6}$$

PROOF. – In [16] we have already proved $(11.6)_1$. As for $(11.6)_2$ we observe that if

$$H_1 = \left(c\sqrt{1-z} - \lambda v\sqrt{z}\right)^2, \qquad H_2 = (\lambda^2/2)\left[\sqrt{1-z} - \sqrt{(1+z)(1-v^2)}\right]^2$$

we have

a) for $(z, v) \in [0, c] \times [0, v_0]$ *or* $(z, v) \in [c, 1] \times [v_0, 1]$

$$H_1(z, v) \geqslant H_1(z_0, v) = c\lambda^2(v - v_0)^2;$$

b) for $(z, v) \in [0, c] \times [v_0, 1]$ *or* $(z, v) \in [c, 1] \times [0, v_0]$

$$H_2(z, v) \geqslant H_2(z_0, v) = (\lambda^2/2)(1 + c)\left(\sqrt{1-v^2} - \sqrt{1-v_0^2}\right)^2.$$

Now, being $\left|\sqrt{1-v^2} - \sqrt{1-v_0^2}\right| \geqslant (v_0/2)|v - v_0|$ and $(1+c)v_0^2 = 2c$, it follows also

$$H_2(z, v) \geqslant (c\lambda^2/4)(v - v_0)^2.$$

Therefore for all $(z, v) \in Q$ the inequality $(11.6)_2$ holds.

By (11.4) and Lemma 11.1 one deduces:

LEMMA 11.2. – *For any real positive constant a_0 such that $[v_0 - a_0, v_0 + a_0]$ is enclosed in $[0, 1]$, for all $\tau \geqslant 0$ and $z \in [0, 1]$ the following estimate holds*

$$0 < \varepsilon F(z, \tau) - \int_{v_0 - a_0}^{v_0 + a_0} g(z, \tau, v)\, dv \leqslant k_2(1 + \tau) \exp[- c\lambda^2 a_0^2 \tau/4] \tag{11.7}$$

where the constant k_2 depends only on c.

PROOF. – Applying (11.3) to the finite term of (11.2), one has

$$\exp[(c^2 z - \gamma^2)\tau] I_0(\tau w) \leqslant \exp[- \tau\alpha(z)] \leqslant \exp[- \tau\alpha_0^2] \tag{11.8}$$

where

$$\alpha_0^2 = c^2 + \lambda^2 - \sqrt{c^4 + \lambda^4} > (c/4)\lambda^2 a_0^2\,, \tag{11.9}$$

is the absolute minimum of the function $\alpha(z) = c^2(1 - z) + \lambda^2 - \lambda^2\sqrt{1 - z^2}$. Besides, also the terms

$$\left(\int_0^{v_0 - a_0} + \int_{v_0 + a_0}^{1}\right) g(z, \tau, v)\, dv$$

satisfies (11.7) according to (11.4) and $(11.6)_2$.

These results suggest, on the basis of the classical Laplace's method, that the asymptotic behaviour of F, when $\tau > 1$, depends on a suitable neighbourhood of (z_0, v_0). To determine a rigorous estimate of the remainder term we must consider

$$\theta(\tau) = \theta_0 \tau^{\frac{1}{3}}\,, \qquad \chi(\tau) = \chi_0 \tau^{\frac{1}{3}} \tag{11.10}$$

where θ_0 and χ_0 are two arbitrary real positive constant such that $\theta_0 < \min(z_0, 1 - z_0)$ and $\chi_0 < \min(v_0, 1 - v_0)$. So,

$$Q_0 \equiv \{(z, v) \in [z_0 - \theta, z_0 + \theta] \times [v_0 - \chi, v_0 + \chi]\} \subset Q$$

and consequently the asymptotic expansion of Bessel functions can be applied for all $\tau > 1$ and $(z, v) \in Q_0$. It is well known that for y real, large and positive [20] one has

$$I_n(y) = \frac{e^y}{\sqrt{2\pi y}}(1 + r_0)\,, \qquad (y > 1) \tag{11.11}$$

with $|r_0(y)| < \mathrm{const}\, y^{-1}$. Therefore applying this formula to the function g when $\tau > 1$ and $(z, v) \in Q_0$, one obtains

$$g(z, v, \tau) = \mu(z, \tau) \exp[-\tau h(z, v)](1 + r_1)\,, \qquad \big(\tau > 1,\ (z, v) \in Q_0\big) \tag{11.12}$$

with

$$\mu(z, v) = \frac{4qv + p}{2\pi\sqrt{pvw}\,(1 - v^2)^{\frac{1}{2}}} \tag{11.13}$$

and $|r_1| < \mathrm{const}\, \tau^{-1}$, the constant depending only on c, θ_0, χ_0.

LEMMA 11.3. – *For all $\tau > 1$ and $(z, v) \in Q_0$, it results*

$$\exp[-\tau h(z, v)] = (1 + r_2) \exp[-a^2(v - v_0)^2\tau - b^2(z - c)^2\tau]\,, \tag{11.14}$$

with $a^2 = c(1 + c)$, $b^2 = (4\lambda^2)^{-1}$ and the remainder term r_2 such that

$$|r_2| < k_3\big(|v - v_0| + |z - z_0|\big)^3 \tag{11.15}$$

where k_3 is a constant depending only on c.

PROOF. – Evaluate h by Taylor's formula observing that $h_{zv}(z_0, v_0) = 0$. This gives

$$h = a^2(v - v_0)^2 + b^2(z - z_0)^2 + h_* \tag{11.16}$$

with

$$h_* = \frac{1}{3!}[(z - z_0)\,\partial_z + (v - v_0)\,\partial_v]^{(3)} h(z_*, v_*)$$

where $(z_*, v_*) \in Q_0$; hence easily one deduces $|h_*| \leqslant \mathrm{const}\,\big(|z - z_0| + |v - v_0|\big)^3$ with the constant depending only on c, θ_0 and χ_0: Moreover, choosing suitably θ_0 and χ_0 according (11.10) one has

$$\tau|h_*| \leqslant \mathrm{const}\,(\theta_0 + \chi_0)^3 < 1\,. \tag{11.17}$$

Consequently

$$|\exp[-\tau h_*] - 1| \leqslant \tau|h_*| \exp[\tau|h_*|] \leqslant e\tau|h_*| \leqslant \mathrm{const}\,\tau\big(|z - z_0| + |v - v_0|\big)^3\,. \tag{11.18}$$

Thus (11.14)-(11.15) are consequences of (11.16)-(11.17) and the proof is complete.

PROOF OF THEOREM 6.3. - Referring to (11.13), by means of Taylor's formula one has

$$|\mu(z, v) - \mu(z_0, v_0)| < \text{const}\left(|v - v_0| + |z - z_0|\right), \qquad (z, v) \in Q_0$$

with $\mu(z_0, v_0) = \mu_0 = ab/(\pi c)$. Therefore Lemmas 11.2 and 11.3 with formulae (11.12)-(11.14) and (11.15) give

$$\varepsilon F = \frac{ab}{\pi c} \exp[-b^2\tau(z-c)^2] \int\limits_{v_0-\chi}^{v_0+\chi} \exp[-a^2\tau(v-v_0)^2](1+r_3) + r_4\,,$$

with the remainder terms r_3, r_4, such that

$$|r_3| < \text{const}\left[\tau^{-1} + \tau\left(|v - v_0| + |z - z_0|\right)^3\right], \qquad |r_4| < \text{const}\exp[-\alpha_0^2\chi^2\tau]\,.$$

At this point, by means of standard computations, the proof is complete.

REMARK 11.2. - The proof of Theorem 6.4 is an obvious consequence of formulae (11.8) and $(11.6)_1$.

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