Complex Interpolation and Geometry of Banach Spaces (*).

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Summary. – *The coincidence of the real and complex methods of interpolation is investigated. Positive results are established under the presence of geometrical properties which are expressed in terms of vector valued Fourier transforms. The results are applied to complex interpolation of H^p spaces and to the study of geometrical properties of Banach spaces.*

1. – Introduction.

The theory of interpolation spaces plays an important role in classical harmonic analysis. It also has important applications to PDE's, Approximation Theory, and many other areas of analysis including applications to the study of the geometry of Banach spaces.

The most important methods of interpolation that have been developed are the so called real and complex methods of interpolation. A brief synopsis is given in § 2 below.

These methods are rather different in character and produce, in general, different results. The real method allows one to derive «strong type» results from weak end point estimates, it also allows one to interpolate the very general classes of spaces (not necessarily Banach spaces) that do appear in real life. Moreover, it has the good quality of being easier to determine than its complex counterpart. The complex method is a rather powerful tool to deal with analytic families of operators (you can interpolate both the spaces and the operators!) and gives sharp norm estimates.

In this paper we are concerned with the relationship between the two methods. Following ideas of PEETRE [18], we have shown elsewhere (cf. [16], [17]) that in many instances the real method can be used successfully to determine the complex method.

The key to these results are certain geometrical properties of the spaces connecting many of the familiar scales of interpolation spaces. These properties are better expressed for interpolation purposes in terms of vector valued Fourier transforms. However, these transforms encode basic geometrical properties as we shall soon show.

(*) Entrata in Redazione il 16 maggio 1983.

In retrospect it is interesting to point out that one of the first applications of the interpolation theorem of Marcel Riesz was his unified and simplified proof of the Hausdorff-Young-Tichmarsh estimates for the Fourier transforms $f \to \hat{f}$.

Here we take vector valued versions of these classical theorems as definitions and proceed to derive a manifold of interesting applications.

The plan of the paper is as follows: in § 2 we give a brief presentation of the needed background on interpolation theory, in § 3 we introduce several notions of Fourier type and study its stability under interpolation, in § 4 we complete work by A. P. CALDERÓN, A. TORCHINSKY [5] and R. MACIAS [14] on interpolation of H^p spaces on homogeneous spaces, in § 5 we show that Fourier type spaces have norms satisfying Clarkson's inequalities, in § 6 we discuss p-Rademacher type and the Fourier type of Banach lattices.

There are no difficult proofs in the paper. Some of the results derived (as those in § 4) give solutions to open problems and it seems difficult to imagine an easier solution than the one given here in view of the technical complications that already appear in the classical $\boldsymbol{R}^n$ situation (see [10]). The applications in § 5 and § 6 provide a framework from which many classical and new results on the geometry of Banach spaces can be obtained without effort.

Acknowledgements. – I wish to thank M. CWIKEL, S. JANSON, S. KAIJSER, P. NILSSON and J. PEETRE for their valuable comments on the first version of this article.

2. – Interpolation methods.

In this section I shall review some of the basic definitions and results of the real and complex methods of interpolation. The interested reader is referred to [3] for a detailed account.

A Banach couple $\bar{A} = (A_0, A_1)$ is a couple of Banach spaces A_0, A_1, embedded in a suitable topological Hausdorff space V.

2.1. *The real method.*

Let $\bar{A}$ be a Banach couple, $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$, then let

$$W_{\theta,p}(\bar{A}) = \left\{u\colon (0, \infty) \to A_0 \cap A_1 / t^{-\theta} u(t) \in L^p\left(A_0, \frac{dt}{t}\right),\ t^{1-\theta} u(t) \in L^p\left(A_1, \frac{dt}{t}\right)\right\}$$

with,

$$\|u\|_{W_{\theta,p}} = \max\left\{\|t^{-\theta} u(t)\|_{L^p(A_0, dt/t)},\ \|t^{1-\theta} u(t)\|_{L^p(A_1, dt/t)}\right\}.$$

The interpolation spaces $\bar{A}_{\theta,p}$ are defined by

$$\bar{A}_{\theta,p} = \left\{x \in A_0 + A_1\colon \exists u \in W_{\theta,p} \text{ s.t. } x = \int_0^\infty u(t)\frac{dt}{t} \ \text{(convergence in } A_0 + A_1)\right\}$$

with

$$\|x\|_{\bar{A}_{\theta,p}} = \inf\left\{\|u\|_{W_{\theta,p}} \colon x = \int_0^\infty u(t)\,\frac{dt}{t}\right\}.$$

This is the description of the real methods of LIONS-PEETRE [13]. A similar definition can be obtained by discretization (i.e., replacing $\int$ by $\sum$ in an appropriate way).

2.2. *The complex method* (cf. CALDERÓN [4]).

Let $\bar{A}$ be a Banach couple, $0<\theta<1$, then let $\mathcal{F}_\theta(\bar{A})$ be the space of functions $f\colon 0\leqslant \operatorname{Re} z\leqslant 1 \to A_0+A_1$, such that $f(z)$ is analytic on $0<\operatorname{Re} z<1$, continuous and bounded on A_0+A_1, and $f(j+i\eta)$ is a continuous function with respect to A_j, $j=0,1$, $\eta\in\boldsymbol{R}$. Let

$$\|f\|_{\mathcal{F}_\theta(\bar{A})} = \max_{j=0,1}\{\|f(j+i\eta)\|_{L^\infty(A_j)}\} \tag{1}$$

and define $\bar{A}_\theta = \{x\in A_0+A_1\colon \exists f\in\mathcal{F}_\theta(\bar{A}) \text{ s.t. } f(\theta)=x\}$,

$$\|x\|_{\bar{A}_\theta} = \inf\{\|f\|_{\mathcal{F}_\theta(\bar{A})}\colon f(\theta)=x\}.$$

It is important to remark that in (1) we can replace the $L^\infty(A_j)$ spaces by any $L^{p_j}(A_j)$, $1\leqslant p_j\leqslant\infty$, $j=1,\infty$ and still obtain the same interpolation spaces. This was explicitly pointed out by PEETRE [18].

2.3. *Interpolation of vector valued L^p spaces.*

Let $1\leqslant\bar{p}=(p_0,p_1)\leqslant\infty$, and given a Banach couple $\bar{A}$, form the Banach couple $L^{\bar{p}}(\bar{A}) = \big(L^{p_0}(A_0), L^{p_1}(A_1)\big)$, of vector valued L^p spaces on a measure space (X,Σ,μ).

The next result summarizes the basic results on interpolation of vector valued L^p spaces.

(2.3.1) THEOREM. – Let $1\leqslant p_0\neq p_1\leqslant\infty$, $0<\theta<1$, $1\leqslant q\leqslant\infty$, $1/p_\theta=(1-\theta)/p_0+\theta/p_1$, $u=\min\{q,p_\theta\}$, $s=\max\{q,p_\theta\}$, then

(i) $L^{p_\theta}(\bar{A}_{\theta,q})\subseteq\big(L^{\bar{p}}(\bar{A})\big)_{\theta,s}$; $\big(L^{\bar{p}}(\bar{A})\big)_{\theta,u}\subseteq L^{p_\theta}(A_{\theta,q})$

(ii) $\big(L^{\bar{p}}(\bar{A})\big)_\theta = L^{p_\theta}(\bar{A}_\theta)$.

PROOF. – (i) By the «power theorem» (cf. [3], page 68) for any Banach couple $\bar{B}$ we have $(\bar{B}_{\theta,q})^{p_\theta}=(\bar{B}^{\bar{p}})_{\eta,r}$, where $r=q/p_\theta$, $\eta=\theta p_\theta/p_1$. Moreover, it is well known and easy to see that

$$K(t,f,[L^{\bar{p}}(\bar{A})]^{\bar{p}}) \sim \int_X K(t,f(x),\bar{A}^{\bar{p}})\,d\mu(x).$$

Therefore,

$$\|f\|^{p_\theta}_{(L^{\bar p}(\bar A))_{\theta,q}} \sim \left\{\int_0^\infty [t^{-\eta} K(t, f, [L^{\bar p}(\bar A)]^{\bar p})]^r \frac{dt}{t}\right\}^{1/r} \sim \left\{\int_0^\infty \left\{\int_X t^{-\eta} K(t, f(x), \bar A^{\bar p})\, d\mu(x)\right\}^r \frac{dt}{t}\right\}^{1/r}.$$

Consequently, if $q \geqslant p_\theta$, by Minkowski's inequality

$$\|f\|^{p_\theta}_{(L^{\bar p}(\bar A))_{\theta,q}} \leqslant \| \|f(x)\|_{(\bar A^{\bar p})_{\eta,r}}\|_1$$

thus by the power theorem

$$\begin{aligned}\|f\|^{p_\theta}_{(L^{\bar p}(\bar A))_{\theta,q}} &\leqslant \| \|f(x)\|^{p_\theta}_{\bar A_{q,\theta}}\|_1 = \\ &= \|f\|^{p_\theta}_{L^{p_\theta}(\bar A_{\theta,q})}.\end{aligned}$$

The norm estimates have to be reversed when $q < p_\theta$.

(ii) See [3].

(2.3.2) REMARK. – The second part of (2.3.1) is proved in CALDERÓN [4] with extra restrictions on A_0, A_1. It is also easy to give a proof of the first half using the methods of LIONS-PEETRE [13], and the definitions given in this section.

2.4. *Real vs complex.*

The real and complex methods give in general very different results. However, under the presence of additional geometrical properties of the spaces involved, they do coincide.

The relationship between the real and complex method that we shall exploit here depends on the behaviour of vector valued Fourier transforms. The original idea goes back once again to LIONS-PEETRE [13] and is developed in PEETRE [18].

(2.4.1) DEFINITION. – A Banach space X is said to be of weak $\boldsymbol{R}$-Fourier type p, $1 \leqslant p \leqslant 2$, if the vector valued Fourier transform

$$\mathcal{F}f(\xi) = \int_{\boldsymbol{R}} \exp[it\xi] f(t)\, dt$$

defines a bounded operator,

$$\mathcal{F}\colon L^p(X) \mapsto L^{p'}(X), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

PEETRE [18] derives the following

(2.4.2) THEOREM. – Let $\bar A = (A_0, A_1)$ be a Banach couple with weak $\boldsymbol{R}$-Fourier type $\bar p = (p_0, p_1)$, then

$$\bar A_{\theta,p_\theta} \subseteq \bar A_\theta, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

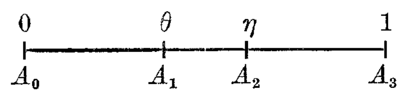
In [16], [17] we have shown that (2.4.2) can be converted in a rather powerful tool when used in conjunction with the beautiful proof of Sagher's conjecture by T. WOLFF [23]:

(2.4.3) THEOREM. – Let A_i, $i = 0, \ldots, 3$ be Banach spaces continuously embedded in a suitable large Hausdorff topological vector space. Let $0 < \theta < \eta < 1$, $\theta = \lambda\eta$, $\eta = (1-\mu)\theta + \mu$, then

(i) If $A_1 = (A_0, A_2)_{\lambda,p}$, $A_2 = (A_1, A_3)_{\mu,q}$, then $A_1 = (A_0, A_3)_{\theta,p}$, $A_2 = (A_0, A_3)_{\eta,q}$.

(ii) If $A_1 = (A_0, A_2)_\lambda$, $A_2 = (A_1, A_3)_\mu$, then $A_1 = (A_0, A_3)_\theta$, $A_2 = (A_0, A_3)_\eta$.

The picture of the situation is:



And its importance in our method is that it allows to deal with each « end point » space separately and therefore use (2.4.2) to the advantage.

A detailed study of (2.4.3) has been given in [11].

3. – Fourier type.

It will be important to extend the definition of Fourier type to a more general setting.

Let G be a locally compact abelian group with character group $\hat{G}$, provided with normalized Haar measures μ and $\hat{\mu}$ respectively. Given a Banach space X, we define the Fourier transform $\mathcal{F}$ on $L^1(G, X)$ by

$$(\mathcal{F}f)(\xi) = \int_G \xi(t) f(t)\, d\mu(t)\,, \qquad \xi \in \hat{G}\,.$$

(3.1) DEFINITION. – A Banach space X is said to be of G-Fourier, type p, $1 \leqslant p \leqslant 2$ if the operator $\mathcal{F}$ can be extended to define a bounded contraction,

$$\mathcal{F}\colon L^p(G, X) \to L^{p'}(\hat{G}, X)\,, \qquad \frac{1}{p} + \frac{1}{p'} = 1\,.$$

We shall say that X is of weak G-Fourier type p, $1 \leqslant p \leqslant 2$, if $\mathcal{F}$ can be extended to a bounded operator

$$\mathcal{F}\colon L^p(G, X) \to L^{p'}(\hat{G}, X)\,, \qquad \frac{1}{p} + \frac{1}{p'} = 1\,.$$

A Banach space X is said to be of (resp. weak) Fourier type p, $1 \leqslant p \leqslant 2$, if it is of (resp. weak) G-Fourier type p for all locally compact abelian groups.

It is clear that every Banach space X is of Fourier type 1 and every Hilbert space is of Fourier type 2. Interpolation theory provides methods to obtain Banach spaces of Fourier type p, $1 < p < 2$.

(3.2) THEOREM. – Let $\bar{A} = (A_0, A_1)$ be a Banach couple of G-Fourier type p_0 and p_1 respectively, let $0 < \theta < 1$, and $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, then

(i) $\bar{A}_{\theta,p_\theta}$ is of weak G-Fourier type p_θ.

(ii) $\bar{A}_\theta$ is of G-Fourier type p_θ.

(iii) Similar statements hold if we replace « G-Fourier type » by « (resp. weak) Fourier type » in the assumptions and the conclusions

PROOF. – By hypothesis

$$\mathcal{F}\colon \begin{cases} L^{p_0}(A_0) \to L^{p_0'}(A_0) \\ L^{p_1}(A_1) \to L^{p_1'}(A_1)\,. \end{cases}$$

Therefore, using the real and complex methods, with $\bar{p}' = (p_0', p_1')$, we get

(i) $\mathcal{F}\colon \big(L^{\bar{p}}(\bar{A})\big)_{\theta,p_\theta} \to \big(L^{\bar{p}'}(\bar{A})\big)_{\theta,p_\theta}$;

(ii) $\mathcal{F}\colon \big(L^{\bar{p}}(\bar{A})\big)_\theta \to \big(L^{\bar{p}'}(\bar{A})\big)_\theta$

using (2.3.1) we get

$$\mathcal{F}\colon \begin{cases} L^{p_\theta}(\bar{A}_{\theta,p_\theta}) \to L^{p_\theta'}(\bar{A}_{\theta,p_\theta}) \\ L^{p_\theta}(\bar{A}_\theta) \to L^{p_\theta'}(\bar{A}_\theta) \end{cases}$$

as desired.

(3.3) REMARK. – The above result is due to PEETRE [18] in the case of weak $\boldsymbol{R}$-Fourier type. Peetre's result has the additional restriction that one of the spaces (A_0, A_1) should be reflexive which is not really needed here.

The following duality theorem holds:

(3.4) THEOREM. – Let X be a Banach space with the Radon-Nikodym property (cf. [6]), then

(i) if X is of (resp. weak) G-Fourier type p, $1 < p \leqslant 2$, then X^* is of (resp. weak) G-Fourier type p;

(ii) if X is of (resp. weak) Fourier type p, $1 < p \leqslant 2$, then X^* is of (resp. weak) Fourier type p.

PROOF. – By hypothesis,

$$\mathcal{F}: L^p(G, X) \to L^{p'}(\hat{G}, X)$$

so that by duality, and using the fact that X has the Radon-Nikodym property, we get

$$\mathcal{F}: L^p(\hat{G}, X^*) \to L^{p'}(G, X^*)$$

(ii) follows by observing that $\hat{\hat{G}} = G$.

(3.5) EXAMPLES.

(3.5.1). L^p is of Fourier type $\min\{p, p'\}$ (cf. PEETRE [18]), $1 \leqslant p \leqslant \infty$.

(3.5.2). Let Ω be a minimally smooth domain of $\boldsymbol{R}^n$ (cf. [1]). Let $W_p^k(\Omega)$ denote the usual Sobolev spaces

$$W_p^k(\Omega) = \{f: D^\alpha f \in L^p(\Omega),\ |\alpha| \leqslant k\}$$

with its usual norm, then $W_p^k(\Omega)$ is of Fourier type $\min\{p, p'\}$, $1 \leqslant p \leqslant \infty$. This follows, using $W_p^k(\Omega) = [W_1^k(\Omega), W_2^k(\Omega)]_\theta$, $1 - \theta/2 = 1/p$, $1 < p \leqslant 2$ (cf. [17]) and duality to deal with the case $2 < p < \infty$.

(3.5.3). The $c_p(H)$ spaces are the spaces of compact operators on a Hilbert space H, normed with $\|A\|_p = [\mathrm{tr}\,(A^*A)^{p/2}]^{1/p}$, $1 \leqslant p < \infty$, and the usual $\|\ \|_\infty$ norm. These are non commutative versions of L^p spaces and share many of the properties of their commutative counterparts. In particular their interpolation theory and duality theory is well understood (cf. [9], [19]).

It follows that we can compute the Fourier type of $c_p(H)$ to be $\min\{p, p'\}$.

(3.5.4). The example (3.5.3) can be generalized in several directions. A theory of integration has been developed in gage spaces which allows the study of very general non commutative $\mathfrak{L}^p$ spaces including the ones described above. In particular these spaces share the same interpolation properties as the L^p spaces and we can compute their Fourier type as above (see [19] and (5.6.2) below).

4. – Applications to interpolation of H^p spaces.

In this section I shall outline the Fourier type technique as it applies to interpolation of H^p spaces.

I shall deal with H^p spaces based on Homogeneous spaces (i.e., groups equipped with an appropriate family of dilations, for example $\boldsymbol{R}^n$). The reader is referred to [8] for a full treatment. The main result obtained is an extension of results of

CALDERÓN and TORCHINSKY [5] and MACIAS [14] concerning complex interpolation of these spaces.

(4.1) THEOREM. – Let $X_0 = H^1(G)$, and let X_1 be either $L^\infty(G)$ or BMO(G), then $(X_0, X_1)_\theta = L^{p_\theta}(G)$, $1/p_\theta = 1-\theta$, where G is an homogeneous space.

PROOF. – The idea of the proof (cf. [16], [17]) is to reduce everything to the real method where the results can be easily obtained using atomic decompositions. It follows (cf. [8]) that

$$(X_0, X_1)_{\theta,q} = L(p_\theta, q)(G)\,, \qquad 1 \leqslant q \leqslant \infty\,. \tag{4.2}$$

(The FEFFERMAN-RIVIERE-SAGHER [7] arguments can be used almost verbatim here.)

Now using (4.2) and (2.4.2) we get $L^{p_\theta} = (H^1, L^2)_{\theta,p_\theta} \subseteq (H^1, L^2)_\theta$, with $1/p_\theta = 1-\theta/2$. Moreover, since $H^1 \subset L^1$, we also have $(H^1, L^2)_\theta \subseteq L^{p_\theta}$.

Finally we interpolate $(L^p, X_1)_\theta = L^{p_\theta}$, $1 < p < \infty$, using the usual arguments if $X_1 = L^\infty$ or the Fefferman-Stein proof if $X_1 =$ BMO. Summarizing we have:

$$(H^1, L^2)_\theta = L^{p_\theta}\,, \qquad \frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{2} \tag{4.3}$$

$$(L^p, X_1)_\theta = L^{p_\theta}\,, \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p}\,, \qquad 1 < p < \infty\,. \tag{4.4}$$

We can now use (2.4.3) to glue these end point results and prove $(H^1, X_1)_\theta = L^{p_\theta}$, $1/p_\cdot = 1-\theta$. Consider first the case $0 < \theta < \frac{1}{2}$. Let $1 < p < 2$ be fixed and let $\lambda = 2/p'$, $\mu = 1 - p/2$, then according to (4.3) and (4.4) we have $(H^1, L^2)_\lambda = L^p$, $(L^p, X_1)_\mu = L^2$. Therefore, by (2.4.3)

$$(H^1, X_1)_\theta = L^p\,, \qquad \theta = \frac{\lambda\mu}{(1-\lambda) + \lambda\mu} \tag{4.5}$$

and the computation gives

$$1 - \theta = \frac{1}{p}\,, \qquad 1 < p < 2\,. \tag{4.5$'$}$$

The case $\frac{1}{2} < \theta < 1$ which corresponds to $2 < p < \infty$ is obtained similarly. Using the reiteration theorem (cf. [3]) we can state

(4.6) COROLLARY. – Let X_1 be either $L^\infty(G)$ or BMO(G), then

(i) $\big(H^p(G), X_1\big)_\theta = L^{p_\theta}(G)$, $1/p_\theta = (1-\theta)/p$, $1 \leqslant p < \infty$;

(ii) $\big(H^{p_0}(G), H^{p_1}(G)\big)_\theta = L^{p_\theta}(G)$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1 \leqslant p_0 \neq p_1 < \infty$.

(4.7) REMARK. – The argument given above is rather general and applies to many concrete scales of spaces as it was shown by the author elsewhere (cf. [16], [17]).

5. – Clarkson's inequalities for Fourier type spaces.

The idea of using the Fourier transform to obtain the classical Clarkson's inequalities for L^p spaces originates in the work of WILLIAMS and WELLS [22]. Their method fits in our general framework of interpolation spaces where it produces a far reaching generalization.

In this section we study the vector valued Fourier transform with respect to the group $G_0 = \{0, 1\}$ with addition modulo 2, with its counting measure and $\hat{G}_0 = \{\varphi_0, \varphi_1\}$, where $\varphi_0 \equiv 1$, and $\varphi_1(j) = (-1)^j$, $j = 0, 1$, provided with $\frac{1}{2}$ times its counting measure.

Let X be a Banach space, and consider $L^p(G_0, X)$, $L^{p'}(\hat{G}_0, X)$, $1 \leqslant p \leqslant 2$. A simple computation shows that

(5.1) THEOREM. – X is of G_0-Fourier type p, $1 \leqslant p \leqslant 2$, if and only if $\forall x, y \in X$

$$\left(\tfrac{1}{2}\|x-y\|_X^{p'} + \tfrac{1}{2}\|x+y\|_X^{p'}\right)^{1/p'} \leqslant \left(\|x\|_X^p + \|y\|_X^p\right)^{1/p}. \tag{5.2}$$

Duality gives

(5.3) THEOREM. – If X is of G_0-Fourier type p, $1 \leqslant p \leqslant 2$, and satisfies the Radon-Nykodim property, then $\forall x, y \in X^*$

$$\left(\tfrac{1}{2}\|x-y\|_{X^*}^{p'} + \tfrac{1}{2}\|x+y\|_{X^*}^{p'}\right)^{1/p'} \leqslant \left(\|x\|_{X^*}^p + \|y\|_{X^*}^p\right)^{1/p}. \tag{5.4}$$

(5.5) REMARK. – Using the inversion formulae for the Fourier transform, we see that the inequalities (5.2) and (5.4) have to be reversed when $p \geqslant 2$.

In the particular case $X = L^p$ (5.2) gives

$$\left(\tfrac{1}{2}\|x-y\|_p^{p'} + \tfrac{1}{2}\|x+y\|_p^{p'}\right)^{1/p'} \leqslant \left(\|x\|_p^p + \|y\|_p^p\right)^{1/p}, \quad 1 < p \leqslant 2, \tag{5.2$'$}$$

and (5.4) gives

$$\left(\tfrac{1}{2}\|x-y\|_{p'}^{p'} + \tfrac{1}{2}\|x+y\|_{p'}^{p'}\right)^{1/p'} \leqslant \left(\|x\|_{p'}^p + \|y\|_{p'}^p\right)^{1/p}, \quad 1 < p \leqslant 2 \tag{5.4$'$}$$

that is the classical Clarkson's inequalities (cf. [1]).

From our point of view, however, these inequalities are consequences of the G_0-Fourier type property and can be obtained using the appropriate interpolation theorems. We present a few examples to illustrate this point.

(5.6) EXAMPLES.

(5.6.1). Consider the c_p spaces described briefly in (3.5.3). It is well known that these spaces have the expected interpolation properties (see [9], [19] for instance). Moreover, it is known (cf. [9]) that $(c_p)^* = c_{p'}$, $1/p + 1/p' = 1$. Therefore we can apply (5.1), (5.3), (5.5) to obtain the Clarkson's inequalities for these spaces.

A rather involved direct proof of these inequalities is given in [15].

(5.6.2). We can do better considering the more general non commutative $\mathfrak{L}^p$ spaces described in (3.5.4). It follows from [19] that $(\mathfrak{L}^{p_0}, \mathfrak{L}^{p_1})_{\theta,p_\theta} = \mathfrak{L}^{p_\theta}$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. The Fourier type techniques of § 4 can be used to compute the complex method. It follows that we can derive Clarkson's inequalities for these spaces.

(5.6.3.) Similar comments adply to the H^p spaces considered in § 4.

(5.6.4). The inequalities of (5.1) apply to the Sobolev spaces $W_p^k(\Omega)$, $1 \leqslant p \leqslant 2$. Note, however, that the inequalities of (5.3) do not apply in this case since the dual spaces fall out of the scale: $(W_p^k)^* = W_{p'}^{-k}$.

Similar inequalities can be obtained considering larger cyclic groups. Let $G_n = \{1, \ldots, n\}$ under addition modulo n, then a Banach space X is of G_n-Fourier type p, $1 \leqslant p \leqslant 2$ if and only if, $\forall x_1, \ldots, x_n \in X$,

$$\left(\sum_{j=1}^{n} \frac{1}{n} \left\| \sum_{k=1}^{n} \exp \frac{2\pi i j k}{n} x_k \right\|_X^{p'}\right)^{1/p'} \leqslant \left(\sum_{k=1}^{n} \|x_k\|_X^p\right)^{1/p}.$$

6. – Rademacher type and interpolation.

An interesting situation occurs when we consider the Cantor group $G_n = \bigoplus_{k=1}^{n} \{0, 1\}$. This group is generated by $e_j = (0, \ldots, 1, 0, \ldots)$, $j = 1, \ldots, n$, addition modulo 2 coordinate-wise. The character group $\hat{G}_n \equiv \left\{\gamma\colon \{1, \ldots, n\} \to \{-1, 1\}\right.$, such that for $g = \sum_{j=1}^{n} a_j e_j$, $a_j \in \{0, 1\}$, $\left.\gamma(g) = \prod_{j=1}^{n} [\gamma(j)]^{a_j}\right\}$.

Use the counting measure on G_n and $1/2^n$ times the counting measure on $\hat{G}_n$. It follows from the definitions (cf. [22]) that

(6.1) LEMMA. – A Banach space X is of G_n-Fourier type p, $1 < p \leqslant 2$, if and only if $\forall x_1, \ldots, x_n \in X$,

$$\left(\sum_{\gamma \in \hat{G}_n} \frac{1}{2^n} \left\| \sum_{j=1}^{n} \gamma(j) x_j \right\|_X^{p'}\right)^{1/p'} \leqslant \left(\sum_{j=1}^{n} \|x_j\|_X^p\right)^{1/p}. \tag{6.2}$$

In terms of the Rademacher functions $\varphi_n(t) = \operatorname{sign}(\sin 2^n \pi t)$, $t \in [0, 1]$, the inequality (6.2) takes a more familiar form. In fact for each $\gamma \in \hat{G}_n$ there exists a unique interval I_γ of length $\overline{2}^n$ such that $\varphi_j(t) = \gamma(j)$, $1 \leqslant j \leqslant n$, $t \in I_\gamma$, it follows that

$$\frac{1}{2^n} \sum_{\gamma \in G_n} \left\| \sum_{j=1}^n \gamma(j) x_j \right\|_X^{p'} = \int_0^1 \left\| \sum_{j=1}^n \varphi_j(t) x_j \right\|_X^{p'} dt$$

and consequently

(6.2) THEOREM. – If a Banach space X is of G_n Fourier type p, $1 \leqslant p \leqslant 2$, then $\forall x_1, \ldots, x_n \in X$

$$\left(\int_0^1 \left\| \sum_{j=1}^n \varphi_j(t) x_j \right\|_X^{p'} dt \right)^{1/p'} \leqslant \left(\sum_{j=1}^n \|x_j\|_X^p \right)^{1/p}.$$

In view of KAHANE's theorem (cf. [12]) (6.2) implies that if X is of G_n-Fourier type p, $1 < p \leqslant 2$, $\forall n \in \boldsymbol{N}$ then X is of Rademacher type p, $1 < p \leqslant 2$ (cf. [2]).

Applications to concrete scales of spaces provide a number of important simplifications on known results.

Consider the following examples

(6.4) EXAMPLES.

(6.4.1). The first non trivial examples are the c_p spaces. The Rademacher type of these spaces was obtained by TOMCZAK-JAEGERMANN using entirely different and involved methods in [21].

(6.4.2). H^p, W_k^p, $\mathfrak{L}^p$ spaces can be treated as examples where the theory applies.

Our final application deals with Banach lattices with specified convexity and concavity.

A deep result of PISIER [20] shows that a Banach lattice X is p convex and p' concave, $1 < p < 2$, if and only if X is of the form $X = (X_0, H)_\theta$, with X_0 a Banach space, H a Hilbert space and $1/p = 1 - \theta/2$.

It follows that a Banach lattice X that is p convex and p' concave is of Fourier type p. In particular X is of Rademacher type p according to our previous results.

(6.5) CONJECTURE. – Is the converse true? More generally we would like to characterize those Banach spaces X with Fourier type p. A natural conjecture would be: X is of Fourier type p if and only if there exist X_0 Banach space, H Hilbert space such that $X = (X_0, H)_\theta$, $1/p = 1 - \theta/2$.

REFERENCES

[1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] B. BEAUZAMY, *Espaces d'Interpolation Réels: Topologie et Géométrie*, Lecture Notes in Math., **666**, Springer, Berlin, 1978.

[3] J. BERG - J. LOFSTROM, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.
[4] A. P. CALDERÓN, *Intermediate spaces and Interpolation: the complex method*, Studia Math., **24** (1964), pp. 113-190.
[5] A. P. CALDERÓN - A. TORCHINSKY, *Parabolic maximal functions associated with a distribution II*, Advances in Math., **24** (1977), pp. 101-171.
[6] J. DIESTEL - J. UHL, *Vector measures*, Math. Surveys No. 15, Amer. Math. Soc., Providence, 1977.
[7] C. FEFFERMAN - N. M. RIVIÈRE - Y. SAGHER, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc., **191** (1974), pp. 75-81.
[8] G. B. FOLLAND - E. M. STEIN, *Hardy spaces on homogeneous groups*, Mathematical Notes No. 28, Princeton University Press, Princeton, 1982.
[9] I. C. GOHBERG - M. G. KREIN, *Introduction to the theory of linear non-self adjoint operators in Hilbert space*, Moscow, 1965.
[10] S. JANSON - P. JONES, *Interpolation between H^p spaces: The complex method*, J. Functional Anal., **48** (1982), pp. 58-80.
[11] S. JANSON - P. NILSSON - J. PEETRE, *Notes on Wolff's note on interpolation spaces*, Technical report, Lund, 1983.
[12] J. P. KAHANE, *Series de Fourier absolument convergentes*, Springer, Berlin, 1970.
[13] J. L. LIONS - J. PEETRE, *Sur une classe d'Espaces d'Interpolation*, Inst. Hautes Études Sci. Publ. Math., **19** (1964), pp. 5-68.
[14] R. MACIAS, *Interpolation theorems on generalized Hardy spaces*, Ph. D. Dissertation, Washington University, St. Louis, 1974.
[15] Ch. MCCARTHY, c_p, Israel J. Math., **5** (1967), pp. 249-271.
[16] M. MILMAN, *Interpolation of some concrete scales of spaces*, Technical report, Lund, 1982.
[17] M. MILMAN, *Fourier type and complex interpolation*, Proc. Amer. Math. Soc., **89** (1983), pp. 246-248.
[18] J. PEETRE, *Sur la transformation de Fourier des functions à valeurs vectorielles*, Rend. Sem. Mat. Univ. Padova, **42** (1969), pp. 15-26.
[19] J. PEETRE - G. SPARR, *Interpolation and non-commutative integration*, Ann. Mat. Pura Appl., **104** (1975), pp. 187-207.
[20] G. PISIER, *Some applications of the complex interpolation method to Banach lattices*, Journal D'Analyse Math., **35** (1979), pp. 264-280.
[21] N. TOMCZAK-JAEGERMANN, *The moduli of smoothness and convexity and the Rademacher averages of trace classes* $S_p (1 \leqslant p < \infty)$, Studia Math., **50** (1974), pp. 163-182.
[22] L. R. WILLIAMS - J. H. WELLS, *L^p inequalities*, J. Math. Anal. Appl., **64** (1978), pp. 518-529.
[23] T. WOLFF, *A note on interpolation spaces*, Lecture Notes in Math., **908**, Springer, Berlin, 1982, pp. 199-204.