Duality in Manifolds (*).

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Summary. – *For the purpose of presenting general Alexander-duality-theorems* (§ 7), *strong-shape-homology* $\bar{\boldsymbol{E}}_*$ *and cohomology* $\bar{\boldsymbol{E}}^*$ *with coefficients in an arbitrary spectrum* $\boldsymbol{E}$ *turns out to be the appropriate implement. Therefore the main properties of* $\bar{\boldsymbol{E}}_*$ *and* $\bar{\boldsymbol{E}}^*$ *are (analogously to those of ordinary (co-) homology with coefficients in* $\boldsymbol{E}$) *developed* (§§ 3-5). *In order to be able to perform the necessary constructions, strong-shape-theory and in particular two different kinds of smash-products in this shape-category are treated* (§§ 1,2, *appendix*). *All previously known Alexander-duality theorems appear as special cases of the main theorems of this paper* (§ 8).

0. – Introduction.

The objective of this paper is to establish a proof of the following Alexander-Pontrjagin duality theorem (theorem 7.4.).

Let M^n be a compact manifold which is $\mathcal{F}$-orientable for a CW-ring-spectrum $\mathcal{F}$. Then for any $\mathcal{F}$-module spectrum $\mathcal{E}$ and any pair (X, A), $A \subset X \subset M^n$, we have an isomorphism which is natural with respect to inclusions

$$\bar{\gamma}_u\colon \bar{\mathcal{E}}_p(X, A) \approx \mathcal{E}^{n-p}(M^n - A, M^n - X)\,, \quad p \in \mathbb{Z}\,. \tag{1}$$

Here $\bar{\mathcal{E}}^p$ denotes shape homology with compact support (§ 3) while $\mathcal{E}^*$ is Čech cohomology (§ 5) with coefficients in a CW-spectrum $\mathcal{E} = \{E_k\}$. If we restrict ourselves to compact pairs (X, A), then we have an isomorphism (theorem 7.3.)

$$\gamma_u\colon \bar{\mathcal{E}}_p(X, A) \approx \bar{\mathcal{E}}^{n-p}(M^n - A, M^n - X)\,, \quad p \in \mathbb{Z} \tag{2}$$

for any spectrum $\mathcal{E} = \{E_k\}$ (being still an $\mathcal{F}$-module spectrum) and Čech-shape-cohomology (§ 5).

These theorems are proved in § 7 by arguments which are based on a construction laid down in § 9. It turns out that all difficulties arise already in the case $M^n = S^n$ (theorem 7.1.). The step from $M^n = S^n$ to an arbitrary manifold is treated in complete analogy to the classical case (cf. [1] or [14]). What one needs are Mayer-Vietoris sequences for (co)homology. These questions are settled in § 3 and § 5.

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In order to extend (1) from compact to arbitrary spaces we need lemma 5.2. which goes back to K. SITNIKOV (who discovered the tautness of ordinary Čech cohomology with coefficients in an abelian group [2], [15]).

His proof does not extend to our case. Our proof is due to J. DUGUNDJI and depends on propositions A1, A2 in the appendix. Here, the coefficient spectrum $\mathcal{E}$ being a CW-spectrum turns out to be crucial for the validity of Sitnikov's lemma.

All existing duality theorems of the Alexander-Pontrjagin type are special cases of (1) or (2). This is explained in § 8 where we also indicate that, at least for compact spaces, (2) can be considered as a first step towards general S-duality (theorem 8.2.).

Strong shape homology $\overline{\mathcal{E}}_p(X)$ displays some interesting properties which are explained in § 4: For a special kind of coefficient spectra $\mathcal{E}$ one has $\overline{\mathcal{E}}_p(X) \approx \mathcal{E}_p(|\overline{S}(X)|)$, where $|\overline{S}(\ )|$ denotes the shape singular complex (cf. [3]). This applies in particular to Eilenberg-MacLane spectra $\boldsymbol{K}(G)$ for finitely generated groups. As a consequence we have a universal coefficient theorem for these homology groups.

The basic ideas of strong shape theory are recollected and considerably extended in § 1 where, in particular, we need a second kind of smash product $X\overline{\wedge}Y$ for topological spaces. In proposition 1.4. the question under what conditions $X\overline{\wedge}Y$ and $X\wedge Y$ are equivalent in the shape category $\overline{\mathcal{K}}$ is settled.

The second section is devoted to the problem of finding smash products for mappings in the shape category.

The investigations of § 1 are extended in the appendix by presenting the explicit construction of a shape mapping (proposition A3) together with all the necessary machinery. Moreover we repeat the proof, that every inclusion $i: A \subset B$ (A compact) of metric spaces becomes a cofibration in $\overline{\mathcal{K}}$ (proposition A9, A10). All this together makes the present paper independent of earlier papers on strong shape theory ([3], [5]).

Finally § 6 is devoted to slant products, preparing the statements and the proof of the main theorems in § 7.

The reader is assumed to be familiar with classical (co-)homology theory with coefficients in a spectrum $\mathcal{E}$. This material can be found for example in [1].

1. – Shape constructions.

In order to keep this paper independent of [5] and because we want to present some additional material, it seems to be necessary to include a section on shape theory. The kind of shape theory we are using is called *strong shape theory*. Our policy will be to give access to a definition of the strong shape category $\overline{\mathcal{K}}$ in this section, referring to a more detailed discussion of some concepts used for that purpose (like 2-homotopies, 2-categories and 2-functors) in the appendix (10.2).

Before we are able to give the definition of $\overline{\mathcal{K}}$ let us make some preliminary remarks:

1) Let $\mathcal{K}$ be any category of based spaces and continuous, base-point preserving maps. We have the concept of a homotopy between two mappings $H: f_0 \simeq f_1$:

$X \to Y$. This is a mapping $H: X \times I_n \to Y$, where $I_n = [0, n]$, $n = 0, 1, 2, \ldots$ such that $Hi_0 = f_0$, $Hi_n = f_1$, $i_t(x) = (x, t)$.

Composition of homotopies is obvious: Let $H': X \times I_m \to Y$ be a mapping such that $Hi_n = H'i_0$, then we have a $G = H' \circ H: X \times I_{n+m} \to Y$ defined by $G|I_n = H$, $G|[n, n+m] = H'$. We abbreviate I_1 by I and introduce the following two relations:

11) Let $F: X \times I_n \to Y$ be any and $G: X \times I \to Y$ a stationary homotopy (i.e. one which is independent of t) such that $G \circ F$ (resp. $F \circ G$) is defined, then we set $G \circ F = F$ (resp. $F \circ G = F$). We will denote G by 1_f, $f(x) = G(x, t)$.

12) Let $F: X \times I \to Y$ be any homotopy and let $G = F^{-1}: X \times I \to Y$ be defined by $G(x, t) = F(x, 1-t)$, then we require that $F^{-1} \circ F = 1_f$ (with $f = Fi_0$).

A homotopy of the form $F: X \times I \to Y$ will be called *elementary*. So we can easily deduce from our definition:

1.1. LEMMA. – Every homotopy $\omega \in \mathcal{K}(f_0, f_1)$, $f_i \in \mathcal{K}(X, Y)$ allows a unique reduced decomposition

$$\omega = \varepsilon_1 \ldots \varepsilon_k$$

where ε_i is an elementary homotopy and (analogously as in group theory) $\varepsilon_i \neq 1$ and $\varepsilon_i \neq \varepsilon_{i-1}^{-1}$.

REMARK. – 1) We have to define homotopies this way because the ordinary concept of a homotopy does not turn $\mathcal{K}(X, Y)$ into a category: Composition is neither associative nor does there exist an identity.

2) In addition we need the concept of a *homotopy between homotopies* (a 2-homotopy): (cf. § 10.2)).

Let $\omega_0, \omega_1: X \times I_n \to Y$ be two homotopies between the same maps $f_0, f_n: X \to Y$. Then we consider classes of mappings $\xi = [A]$

$$A: X \times I_n \times I_m \to Y$$

such that

$$A(x, t, i) = \omega_i(x, t), \quad i = 0, m$$
$$A(x, i, s) = f_i(x), \quad i = 0, n$$

with an equivalence relation, which is thoroughly discussed and defined in the appendix. We call $\xi = [A]$ *elementary* whenever we have $m = 1$.

As already mentioned, the kind of shape theory which we are now going to define is *strong shape theory*. This has to be distinguished from *weak shape theory* which is readily treated in [7]. Although it can be proved that at least for compact metric spaces, two spaces X and Y are of the same (weak) shape if and only if they are homotopy equivalent in the strong shape category, strong shape theory provides a much richer structure. All questions of algebraic topology require the concept of a strong rather than that of a weak shape unless one is willingly to work with pro-objects (like pro-groups instead of groups). The difference between these constructions can from the structural point of view be compared with the difference between Čech homology and Steenrod-Sitnikov homology (the first is generally not exact, while the second one is always exact). Alternatively one may compare the Spanier-Whitehead category with Boardman's category: The first being obtained by performing a stabilization process at a homotopy category, while Boardman's category is the stabilized version of a topological category with continuous mappings (rather than their homotopy classes) as morphisms. The price which one has to pay for the advantages of strong shape theory is, like in the other examples, a higher degree of complexity.

Meanwhile there appeared various other approaches to the *homotopy category* $(\overline{\mathcal{K}})_h$ which appearantly lead to the same result for compact metric spaces (cf. e.g. J. DYDAK, J. SEGAL, F. W. CATHEY). Because we use individual mappings, 1-homotopies and then 3-homotopy classes of 2-homotopies, this is a «*3-stage-strong shape category*». One could extend this to 4-, 5- and finally some kind of ∞-stage shape construction. However for compact metric spaces, the 3-stage approach turns out to be sufficient.

We are now ready to define our shape category $\overline{\mathcal{K}}$ depending on a full subcategory $\mathcal{T}$ of **Top** (resp. $\mathbf{Top}_0$) of good spaces (which in our example will be the category of ANEs).

Let $Y \in \mathcal{K}$ be any object, then we define $\mathcal{T}_Y$ to be the following 2-category:

1) The objects are mappings $g \in \mathbf{Top}_0(Y, P)$, $P \in \mathcal{T}$.

2) The 1-morphisms $(r, \omega): g_1 \to g_2$ are pairs where $g_i: Y \to P_i$ is an object of $\mathcal{T}_Y$ and $r: P_1 \to P_2$ a continuous mapping, while $\omega: rg_1 \simeq g_2$ is a fixed homotopy.

3) A 2-morphism $(\nu, \xi): (r_1, \omega_1) \simeq (r_2, \omega_2): g_1 \to g_2$ is a pair, where

$$\nu: r_1 \simeq r_2$$

is a homotopy and

$$\xi: \omega_2 \circ \nu g_1 \simeq \omega_1$$

a homotopy between homotopies. More precisely we have to work with homotopy classes $[\xi]$ of 2-homotopies (all between the *same* 1-homotopies) instead of individual 1-homotopies. By an abuse of notation however we will continue to write ξ instead of $[\xi]$ (cf. § 10.2.2) for more details).

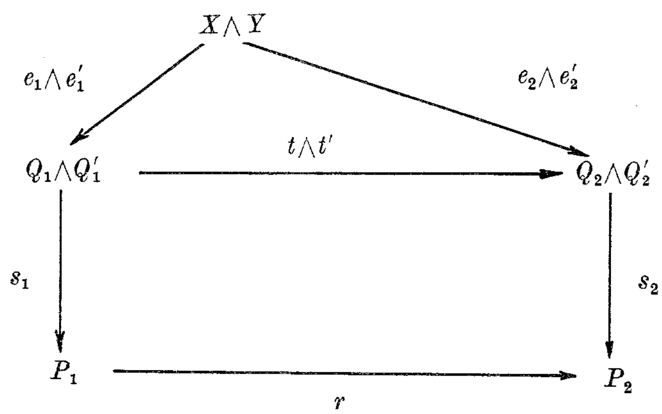
We are not going to present all different kinds of compositions in $\mathcal{T}_Y$ rendering it into a 2-category. Moreover one could define more involved (n-)categories $\mathcal{T}_Y$

by taking into account higher (n-)homotopies (rather than only 1- and 2-homotopies). However we are not going to pursue this aspect further.

From now on we require that all spaces are k-spaces and that consequently all other operations (which possibly do not automatically produce k-spaces, like the $\wedge$-product) are, if necessary followed by the functor k, which turns every space into a k-space.

Let $X, Y \in \mathfrak{K}$ be fixed spaces, then we define a category $\mathfrak{T}_X \overline{\wedge} \mathfrak{T}_Y$. Let to this end $g \in \mathfrak{T}_{X\wedge Y}$ be such that it allows a decomposition $g = s(e\wedge e')$, $e: X \to Q \in \mathfrak{T}$, $e': Y \to Q' \in \mathfrak{T}$, $s: Q\wedge Q' \to P \in \mathfrak{T}$, then an object of $\mathfrak{T}_X \overline{\wedge} \mathfrak{T}_Y$ is defined to be a fixed decomposition $g = s(e\wedge e')$. A 1-morphism between two such objects consists of a diagram

(1)



together with given homotopies $\omega: te_1 \simeq e_2$, $\omega': t'e_1' \simeq e_2'$, $\delta: rs_1 \simeq s_2\,(t\wedge t')$. In a next step we complete the definition of 1-morphisms by declaring all morphisms of the form (1) with $r = $ identity and $\delta = 1$ as invertible (i.e. we form the corresponding quotient category, cf. [6]). The 2-morphisms are defined analogously.

We have a natural 2-functor $\alpha: \mathfrak{T}_X \overline{\wedge} \mathfrak{T}_Y \to \mathfrak{T}_{X\wedge Y}$ which forgets the given decompositions of certain $g: X\wedge Y \to P$, resp. for the 1- and 2-morphisms. The definition of a 2-functor is recorded in the appendix.

All this can be easily extended to a finite number of factors, obtaining categories

$$\Pi = \mathfrak{T}_{X_1} \overline{\wedge} \ldots \overline{\wedge} \mathfrak{T}_{X_n}$$

and 2-functors $\alpha: \Pi \to \mathfrak{T}_{X_1\wedge\ldots\wedge X_n}$. For $n = 1$, we define α to be the identity $1: X \to X$.

We denote Π by $X_1 \overline{\wedge} \ldots \overline{\wedge} X_n$ and consequently $\mathfrak{T}_X$ by X. The category $\overline{\mathfrak{K}}$ has these $\overline{\wedge}$-products of spaces as objects. The morphism $[\bar{f}] \in \overline{\mathfrak{K}}(X_1 \overline{\wedge} \ldots \overline{\wedge} X_n, Y_1 \overline{\wedge} \ldots \overline{\wedge} Y_m)$ are classes of 2-functors

$$\bar{f}: \mathfrak{T}_{Y_1} \overline{\wedge} \ldots \overline{\wedge} P_{Y_m} \to \mathfrak{T}_{X_1} \overline{\wedge} \ldots \overline{\wedge} \mathfrak{T}_{X_n}$$

satisfying the following conditions:

a) $\left(s(\wedge e^{(i)})\colon \wedge Y_i \to P \in \mathfrak{T}\right) \Rightarrow \bar{f}(s(\wedge e^{(i)})\colon \wedge X_i \to P.$

b) Let μ be a 1-morphism of the form (1), then $\bar{f}(\mu)$ has the same $r\colon P_1 \to P_2$.

c) Let μ' be a 2-morphism with $\nu\colon r_1 \simeq r_2$, then $\bar{f}(\mu')$ has the same $\nu\colon r_1 \simeq r_2$ as first component.

For the formulation of the next condition for $\bar{f}$, recall that according to the definition of a general 2-functor (cf. 10.2.3), definition of a 2-functor 3 *a*) we have for any two 1-morphisms, which can be composed (i.e. for which $\mu_2 \circ \mu_1$ exists) a connecting 2-morphism

$$\varkappa = (\pi, \beta)\colon \bar{f}(\mu_2\mu_1) \to \bar{f}(\mu_2)\bar{f}(\mu_1)\,.$$

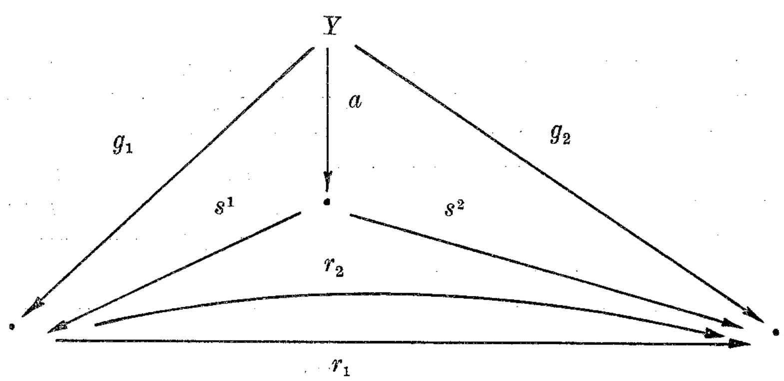
We require:

d) The first component $\pi\colon r_2 r_1 \simeq r_2 r_1$ in $\varkappa$ is the identity.

Here r_i is of course the mapping r in (1) now for μ_i.

The last condition turns out to be very convenient although it is not explicitely used in this paper. For the sake of brevity we write it down only for the case $m = n = 1$. The general formulation can be easily provided.

Let



be a 2-morphism $(\nu, \xi)\colon (r_1, \omega_1) \to (r_2, \omega_2)$ with $\xi = \bar{\zeta}$ a, then we have

e) $\bar{f}(\nu, \xi) = \left(\nu, \delta_2(\bar{\zeta}\bar{f}(a)(r_1\delta_1^{-1}))\right)$ where δ_i stems from $\bar{f}(s^i, 1) = (s^i, \delta_i)$.

Two such functors $\bar{f}, \bar{f}'$ are equivalent whenever one has $\alpha\bar{f} = \alpha\bar{f}'$, $\alpha\colon \mathfrak{T}_{X_1} \overline{\wedge} \dots \overline{\wedge}\, \mathfrak{T}_{X_m} \to \mathfrak{T}_{X_1 \wedge \dots \wedge X_m}$ being the previously defined natural transformation. By an abuse of notation we will always write $\bar{f}$ instead of $[\bar{f}]$.

In particular for $m = n = 1$ (i.e. in case of topological spaces) conditions *a*) - *d*) reduce to:

a) $g\colon Y \to P \in \mathcal{T} \Rightarrow \bar{f}(g)\colon X \to P$;

b) $(r, \omega)\colon g_1 \to g_2 \Rightarrow \bar{f}(r, \omega) = (r, \omega_1)$ for suitable ω_1;

c) $(\nu, \xi)\colon (r_1, \omega_1) \to (r_2, \omega_2) \Rightarrow \bar{f}(\nu, \xi) = (\nu, \xi_1)$ for a suitable 2-homotopy ξ_1;

d) in the connecting morphism

$$\varkappa = (\pi, \beta)\colon \bar{f}\big((r_2, \omega_2)(r_1, \omega_1)\big) \to \bar{f}(r_2, \omega_2)\bar{f}(r_1, \omega_1)$$

one has $\pi = 1$.

Moreover, since α becomes the identity, two 2-functors are equivalent whenever they are equal.

It should be kept in mind that the products $X_1 \overline{\wedge} \ldots \overline{\wedge} X_n$ are not related to spaces. So the objects of $\overline{\mathcal{K}}$ are particular 2-categories, some of them can be represented by topological spaces (namely those of the form $\mathcal{T}_X$).

An example of a shape morphism of a space into a $\overline{\wedge}$-product is furnished by the previously described

$$\alpha \in \overline{\mathcal{K}}(X_1 \wedge \ldots \wedge X_n,\ X_1 \overline{\wedge} \ldots \overline{\wedge} X_n)\,,$$

which we denote by the same letter α. However we have to observe that α *as a shape morphism* points into the reverse direction, compared to α *as a 2-functor*.

We will very soon (see proposition 1.4.) study under what additional assumptions α becomes an equivalence in $\overline{\mathcal{K}}$.

The extension of the category $\overline{\mathcal{K}}$ by introducing these new objects is the price one has to pay for the existence of arbitrary smash products $\bar{f} \overline{\wedge} 1_Z\colon X \overline{\wedge} Z \to Y \overline{\wedge} Z$ where $\bar{f}\colon X \to Y$ is a given shape morphism. This will be treated in the next section.

Let $f \in \mathcal{K}(X, Y)$ be a continuous mapping, then we have a shape morphism $h(f) \in \overline{\boldsymbol{K}}(X, Y)$ which is defined by

$$h(f)(g) = gf$$

and similar for the 1- and 2-morphisms.

This furnishes a functor $h\colon \mathcal{K} \to \overline{\mathcal{K}}$ (see [5]).

As in [3], [5] we have for each $\bar{f} \in \overline{\mathcal{K}}(X, Y)$, $Y \in \mathcal{T}$ the assignment $h'(\bar{f}) \in \mathcal{K}(X, Y)$ defined by

$$h'(\bar{f}) = \bar{f}(1) \qquad (1\colon Y \to Y)\,.$$

One has ([5] 2.3.*a*), 2.4.):

1.2. PROPOSITION. – *a*) For each $f \in \mathcal{K}(X, Y)$, $Y \in \mathcal{T}$ we have

$$h'h(f) = f\,.$$

b) For each $\bar{f} \in \overline{\mathcal{K}}(X, Y)$, $Y \in \mathcal{T}$ we have a natural homotopy in $\overline{\mathcal{K}}$

$$hh'(\bar{f}) \simeq \bar{f}\,.$$

Here homotopies in $\overline{\mathcal{K}}$ are defined in complete analogy to the concept of a homotopy in $\mathcal{K}$ by using shape morphisms $\bar{H}\colon X \times I \to Y$ in $\overline{\mathcal{K}}$.

In what follows, we will in most cases surpress the functor h in our notation, writing $f \in \mathcal{K}(X, Y)$ for a continuous f (instead of $h(f) \in \mathcal{K}(X, Y)$).

As a consequence of 1.2. we have the following important:

1.3. COROLLARY. – To any $\bar{f} \in \overline{\mathcal{K}}(X, Y)$, $Y \in \mathcal{T}$ there exists a continuous $f \in \in \mathcal{K}(X, Y)$ such that $h(f) \simeq \bar{f}$ (cf. [5], corollary 2.5.).

At this point we have to specify our category $\mathcal{K}$. Here we are confronted with the following difficulty: For general spaces we would like to confine ourselves to metrizable (shortly: metric) spaces, while the good spaces are supposed to be ANE spaces. Recall that an ANE space P is defined by the property that a continuous mapping $f\colon A \to P$ of a closed subspace A of a metric space admits an extension $F\colon U \to P$ for a suitable open neighbourhood U of A. Any CW-space is an ANE [10], but not metric. Since the spaces E_k appearing in our spectra $\mathcal{E} = \{E_k\}$ are sometimes CW-spaces (and therefore «good»), we are obliged to find a category which contains metric and ANE spaces as well. Hence we take for $\mathcal{K}$ the full subcategory of $\mathbf{Top}_0$ consisting of *a*) all metric spaces, *b*) all ANEs and *c*) all finite $\wedge$-products of spaces in *a*) or in *b*). The category $\mathcal{T} \subset \mathcal{K}$ turns out to be a full subcategory of $\mathcal{K}$ (containing all ANR spaces as well as all CW-spaces).

We have

1.4. PROPOSITION. – The transformation $\alpha\colon X \wedge Y \to X \overline{\wedge} Y$ for spaces X, Y becomes an equivalence in $\overline{\mathcal{K}}$, whenever one of the following conditions is fulfilled:

a) X and Y are compact (from now on, we will by an abuse of notation simply write «compact» whenever we mean «compact *and metrizable*»);

b) X (or Y) is a compact ANE, the other arbitrary in $\mathcal{K}$;

c) X and Y are ANE spaces (e.g. CW-spaces).

The proof is prepared by the following

1.5. LEMMA. – Let $g\colon X \wedge Y \to P \in \mathcal{T}$ be any continuous map, X compact and suppose Y be embedded in a metric space M as a closed subset. Then there exists an extensions $g'\colon X \wedge U \to P$ of g over $X \wedge U \supset X \wedge Y$ where U is open in M. In other words we have $g'i = g$ with $i\colon X \wedge Y \subset X \wedge U$, the inclusion.

PROOF. – We give a proof for the $\times$-product and the unbased case. The assertion for based X, Y and the $\wedge$-product then follows immediately. Since $Y \subset M$ is metric and P an ANE, we find an extension $f\colon V \to P$ of g over an open V, $X \times Y \subset \subset V \subset X \times M$. However to each $(x, y) \in X \times Y$ there are open sets $U_{x,y}(x) \ni x$, $W_{x,y}(y) \ni y$ with $U_{x,y} \times W_{x,y} \subset V$. Because X is supposed to be compact, finitely many $U_i = U_{x_i,y}$ cover X and we finally come down with a $W_y = \cap W_{x_i,y}$ such that $X \times W_y \subset V$. The desired neigbourhood U is

$$U = \bigcup_{y \in Y} W_y \,.$$

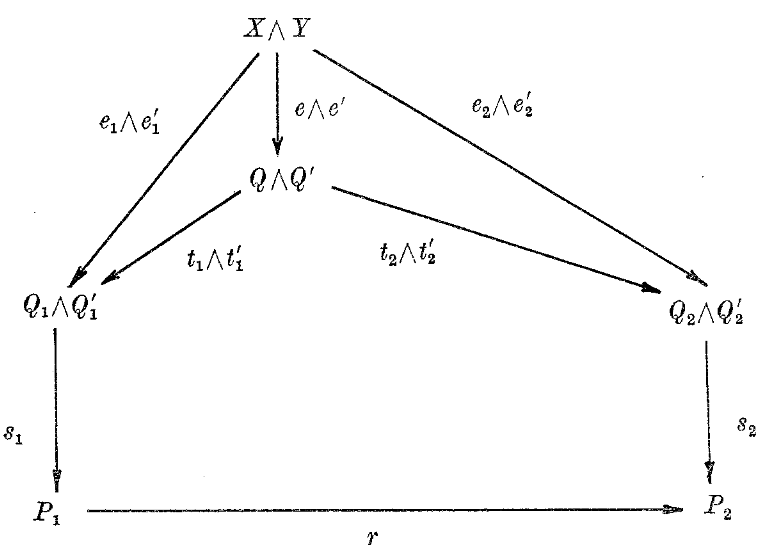
As a corollary we can deduce the well-known fact (cf. [7] for ANR spaces):

1.6. COROLLARY. – Let $P \in \mathfrak{T}$ be fixed and $X \in \mathfrak{K}$ compact. Then the function space P^X is contained in $\mathfrak{T}$.

PROOF. – We must prove that given a metric space M and a closed subspace Y, we can extend any continuous $f\colon Y \to P^X$ over an open neighbourhood U of Y in M. This can be deduced from 1.5. by going over to $g = f'\colon X \times Y \to P$, the adjoint of f.

PROOF OF 1.4. – *Ad a)* We embed X, Y in two different copies of a Hilbert cube Q, hence $X \wedge Y \subset Q \wedge Q \approx Q$. By applying 1.5. twice, we obtain for any continuous $g\colon X \wedge Y \to P \in \mathfrak{T}$ open neighbourhoods (rel. Q) $U = U(X)$, $V = V(Y)$ as well as an extension $g'\colon U \wedge V \to P$ of g. Since U, V are ANRs (cf. [6], p. 96) we have furnished a factorization $g = g'(a_1 \wedge a_2)$, $a_1\colon X \subset U$, $a_2\colon Y \subset V$ as required. So a functor $\beta\colon \mathfrak{T}_{X \wedge Y} \to \mathfrak{T}_X \overline{\wedge} \mathfrak{T}_Y$ can be defined on the objects by choosing a fixed decomposition $\beta(g) = s(e \wedge e')$ (for example by setting $s = g'$, $e = a_1$, $e' = a_2$). Now let $(r, \omega)\colon g_1 \to g_2$ be given, then lemma A6 ensures the existence of a diagram

(2)



with commutative upper triangles and a homotopy $\gamma\colon s_1(t_1\wedge t_1') \simeq s_2(t_2\wedge t_2')$, such that $\gamma(e\wedge e') = \omega$. Observe that for $i = 1, 2$, $a\colon s_i(e_i\wedge e_i') \approx s_i(t_i\wedge t_i')(e\wedge e')$ is an isomorphism by definition. This enables us to define $\beta(r, \omega)$ on the 1-morphisms by (2). On the 2-morphisms we proceed similarly.

This completes the construction of a functor β having the property that $\alpha\beta =$ = identity. On the other hand this implies that $\beta\alpha$ is equivalent to the identity. This completes the proof of *a*).

Ad b) Let $g\colon X\wedge Y \to P \in \mathcal{T}$ be continuous, then we have the adjoint $g' = r\colon Y \to$ $\to P^X \in \mathcal{T}$ (because of 1.6.). Thus we have established a factorization $g = e(1\wedge r)$, where $e\colon X \to P^X \in \mathcal{T}$ denotes the evaluation map. The construction of a functor β is in this case immediate.

Ad c) This is trivial because any $g\colon X\wedge Y \to P \in \mathcal{T}$ factors over the identity $1\wedge 1\colon X\wedge Y \to X\wedge Y$.

REMARK. – Proposition 1.4. *b*) makes sure that we can talk about suspensions $S^1\wedge X \approx S^1\overline{\wedge}X$ as well as about cones $CX = (I, \{1\})\wedge X \approx (I, \{1\})\overline{\wedge}X$ for any $X \in \mathcal{K}$ without being obliged to distinguish between these two kinds of products.

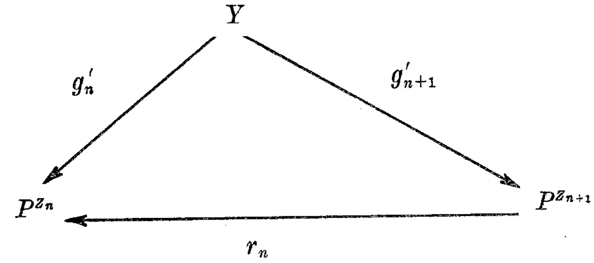
2. – Smash products of morphisms.

We start with the definition of $\bar f\wedge 1_Z \in \overline{\mathcal{K}}(X\wedge Z, Y\wedge Z)$ for given $\bar f \in \overline{\mathcal{K}}(X, Y)$, X, Y metric and Z compact metric: Let $g \in \mathcal{T}_{Y\wedge Z}$ be fixed, $g\colon Y\wedge Z \to \boldsymbol{P} \in \mathcal{T}$, then we have the adjoint $g'\colon Y \to P^Z$. Recall that according to corollary 1.6. P^Z is an object of $\mathcal{T}$ and so we define ([3], § 4):

$$(\bar f\wedge 1_Z)(g) = \bar f(g')'\colon X\wedge Z \to P\,. \tag{1}$$

We are now establishing $\bar f\wedge 1_Z$ for F_σ-spaces Z. By this we mean spaces $Z = \cup Z_n$, Z_n a compact CW-complex, such that every compact $K \subset Z$ is contained in some Z_n. Without loss of generality we assume that $Z_n \subset Z_{n+1}$ for any n. Let $g\colon Y\wedge Z \to$ $\to P \in \mathcal{T}$ be continuous, $i_n\colon Z_n \subset Z_{n+1}$ the inclusion and $g_n = g|Y\wedge Z_n$, then we have the commutative diagram:

(2)



$r_n = P^{i_n}$

which under $\bar{f}$ is transformed into a homotopy commutative triangle

$$\bar{f}(g_n') \simeq r_n \bar{f}(g_{n+1}') . \tag{3}$$

This yields a homotopy commutative diagram

$$\begin{array}{ccc} X \wedge Z_n & \xrightarrow{(\bar{f}\wedge Z_n)(g_n)} & \\ \downarrow{\scriptstyle 1\wedge i_n} & & P \\ X \wedge Z_{n+1} & \xrightarrow{(\bar{f}\wedge Z_{n+1})(g_{n+1})} & \end{array} \tag{4}$$

Now we form the telescope space $\tilde{Z} = \sum_{n=1}^{\infty} Z_n \times I/\sim$, where $Z_n \times \{1\} \subset Z_n \times I$ is identified with $Z_n \times \{0\} \subset Z_{n+1} \times I$. Because of our assumption we have continuous mappings $p: \tilde{Z} \to Z$ (the projection) and $s: Z \to \tilde{Z}$ (which can be easily constructed inductively by a cofiber argument). These two mappings are homotopy inverses to each other.

Diagram (4) provides us with a mapping $e: X \wedge \tilde{Z} \to P$ so that we can define

$$(\bar{f}\wedge 1_Z)(g) = e(1_X \wedge s) .$$

This establishes a $(\bar{f}\wedge 1_Z) \in \overline{\mathcal{K}}(X\wedge Z, Y\wedge Z)$ as can be easily checked. Moreover we have for two shape morphisms

$$\bar{f}_1 \in \overline{\mathcal{K}}(X, Y) , \quad \bar{f}_2 \in \overline{\mathcal{K}}(Y, Y') \quad (\bar{f}_2\bar{f}_1)\wedge 1_Z = (\bar{f}_2\wedge 1_Z)(\bar{f}_1\wedge 1_Z) .$$

REMARK. – The assumption that all Z_n are CW-spaces is introduced for convenience and can be weakened. We give the following examples of such F_σ-sets:

1) Every open subset of some n-sphere S^n.
2) Every CW-space Z, all whose n-skeletons for all n are compact.
3) In particular all Eilenberg-MacLane spaces $K(G, n)$ for finitely generated abelian group G.

In addition to smash products of this kind we must deal with smash products of shape morphisms $\bar{f} \in \overline{\mathcal{K}}(A, B\overline{\wedge} C)$ with a F_σ-set Z, resulting in a $\bar{f}\wedge 1_Z \in \overline{\mathcal{K}}(A\wedge Z, B\overline{\wedge}(C\wedge Z))$.

Observe that $(B\overline{\wedge} C)\wedge Z$ is not defined!. The construction of $\bar{f}\wedge 1_Z$ is accomplished in complete analogy to the previous case: One defines $\bar{f}\wedge 1_Z$ for compact Z (in this case we do not need any assumption on a CW structure) then one goes over to $\bar{f}\wedge 1_{\tilde{Z}}$ for a F_σ-set Z and proceed as in the former case.

As an application we will encounter mappings $\bar{f}: S^{p+1} \to E_l \bar{\wedge} X$ for compact $X \subset S^n$ and $Z = Y = S^n - X$. Letting $u: \Sigma^k X \wedge Y \to S^{n+k-1}$ be the « duality map » which stems from $u(x, y) = (x - y)/|x - y|$, we obtain a shape map

$$\begin{array}{ccc} \gamma_u(\bar{f}): S^{p+l} \wedge \Sigma^k Y & \longrightarrow & E_l \wedge S^{n+k-1} \simeq \Sigma^{n+k-1} E_l \\ \| & & \downarrow \\ \Sigma^{p+l+k} Y & & E_{n+k+l-1} \end{array}$$

whose existence will be crucial for the duality theorems.

We can construct $\bar{\wedge}$-products $1 \bar{\wedge} \bar{f}: E \bar{\wedge} X \to E \bar{\wedge} Y$ for fixed $\bar{f} \in \bar{\mathcal{K}}(X, Y)$ and any $E \in \mathcal{K}$ in the following way:

Let $s(e \wedge e')$ be a given decomposition of a $g: E \wedge Y \to P \in \mathcal{K}$ hence an object of $\mathcal{T}_E \bar{\wedge} \mathcal{T}_Y$, then we set

$$(1 \wedge \bar{f})\big(s(e \wedge e')\big) = s\big(e \wedge \bar{f}(e')\big) .$$

On the 1- and 2-morphisms we proceed similarly. Observe that a morphism of the form § 1(1) with $r =$ identity, $\delta = 1$ is tranformed into a morphism of the same kind, so that $1 \wedge \bar{f}$ is well-defined on the quotient categories.

We summarize:

2.1. PROPOSITION. – There exists a smash-product $1 \bar{\wedge} \bar{f}: E \bar{\wedge} X \to E \bar{\wedge} Y$ for any spaces $E, X, Y \in \mathcal{K}$ and for any $\bar{f} \in \bar{\mathcal{K}}(X, Y)$. This smash-product is functorial in X, Y:

1) $X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} Z \Rightarrow 1 \bar{\wedge} (\bar{g}\bar{f}) = (1 \bar{\wedge} \bar{g})(1 \bar{\wedge} \bar{f})$.

2) $1_E \bar{\wedge} 1_X = 1_{E \bar{\wedge} X}$.

Moreover for compact metric E we have a homotopy commutative diagram:

$$\begin{array}{ccc} E \bar{\wedge} X & \xrightarrow{1 \bar{\wedge} \bar{f}} & E \bar{\wedge} Y \\ \alpha \uparrow & & \uparrow \alpha \\ E \wedge X & \xrightarrow[\bar{f} \wedge 1]{} & E \wedge Y \end{array} \tag{5}$$

where α stems from 1.4.

PROOF. – Only (5) needs proof: Let $a_1 \wedge a_2 = a: E \wedge Y \to Q_1 \wedge Q_2$, $Q_i \in \mathcal{T}$ be given, then we have a map

$$\eta: Q_2 \to (Q_1 \wedge Q_2)^E \in \mathcal{T}$$

such that

$$\eta a_2 = a' ,$$

which is defined by $\eta(q_2) = \big(e \to (a_1(e), q_2)\big)$. Let $\varepsilon\colon E \wedge Q^E \to Q$ be the evaluation map, then we conclude:

$$\bar{f}(a')' = \varepsilon\big(1 \wedge \bar{f}(a')\big) = \varepsilon\big(1 \wedge \bar{f}(\eta a_2)\big) \simeq \big(1 \wedge \eta \bar{f}(a_2)\big) = a_1 \wedge \bar{f}(a_2) .$$

This provides us easily with the required homotopy.

REMARK. – 1) All the results concerning $\wedge$- and $\overline{\wedge}$-products could also be established for the $\times$-products instead of the $\wedge$-product. We could in particular define an object $X \overline{\times} Y$ and the related maps. This case of free spaces and maps will be used in § 4.

2) Let Z be a retract of a F_σ-set Z', then one is still able to define $\bar{f} \wedge 1_Z$ by

$$\bar{f} \wedge 1_Z = (1_Y \wedge r)(\bar{f} \wedge 1_{Z'})(1_X \wedge i)$$

where $i\colon Z \subset Z'$, $r\colon Z' \to Z$ denote the inclusion resp. the retraction.

3. – The homology functor.

A spectrum $\mathcal{E} = \{E_n,\ n \in \mathbb{Z},\ \sigma_n\colon \Sigma E_n \to E_{n+1}\}$ is a collection of based spaces (which are assumed to be either metric of lying in $\mathcal{T}$) together with continuous mappings $\sigma_n\colon \Sigma E_n \to E_{n+1}$. Observe that according to proposition 1.4. *b*) we do not have to distinguish between $S^1 \wedge E_n$ and $S^1 \overline{\wedge} E_n$. By an abuse of notation we will denote the iteration of σ_n:

$$\Sigma^k E_n \xrightarrow{\Sigma^{k-1}\sigma_n} \Sigma^{k-1} E_{n+1} \to \dots \to \Sigma E_{n+k-1} \to E_{n+k}$$

also by σ.

Moreover by an abuse of notation we call a spectrum $\mathcal{E} = \{E_n\}$ such that all $E_n \in \mathcal{T}$, a *CW spectrum*. In particular, every spectrum with the property that all E_n are CW complexes is a CW spectrum.

We prefer this notation in order to be in accordance with the general usage.

Let $A, B \in \mathcal{K}$ be two based spaces, then we write $[A, B]$ instead of $\overline{\mathcal{K}}_h(A, B)$. This extends to more general objects like $A_1 \overline{\wedge} A_2$ resp. $B_1 \overline{\wedge} B_2$.

We are defining (reduced) *homology groups* for based compact metric spaces $X = (X, x_0)$

$$\bar{\mathcal{E}}_n(X) = \varinjlim_k\, [S^{n+k}, E_k \overline{\wedge} X] \tag{1}$$

with coefficients in a spectrum $\mathcal{E}$.

The bonding maps in the direct system are defined as usual:

$$[S^{n+k}, E_k\overline{\wedge}X] \to [S^{n+k+1}, \Sigma E_k\overline{\wedge}X] \to [S^{n+k+1}, E_{k+1}\overline{\wedge}X]\,.$$

Our results in § 1, § 2 guarantee that everything (like suspensions for spaces and mappings) is well-defined. The group strusture in $[S^{n+k}, E_k\overline{\wedge}X]$ is induced by the comultiplication

$$\varkappa\colon S^{n+k} \to S^{n+k}\vee S^{n+k}$$

in the following way (similar to that in ordinary topology): Let $\bar{f}_1, \bar{f}_2 \in [S^{n+k}, E_k\overline{\wedge}X]$ be given morphisms and $s(e\wedge e') \in \mathcal{S}_{E_k}\overline{\wedge}\mathcal{S}_X$ any object, then we define

$$(\bar{f}_1+\bar{f}_2)(s(e\wedge e')) = \varphi_P\big(\bar{f}_1(s(e\wedge e'))\vee\bar{f}_2(s(e\wedge e'))\big)\varkappa\,,$$

with $\varphi_P\colon P\vee P \to P$ being the ordinary folding map for the space in $\mathcal{S}$ into which $s(e\wedge e')$ is mapping. For the 1- and 2-morphisms, one proceeds analogously.

More generally we have for any object $X_1\overline{\wedge}\ldots\overline{\wedge}X_k \in \overline{\mathcal{K}}$ the concept of *homotopy groups*

$$\bar{\pi}_n(X_1\overline{\wedge}\ldots\overline{\wedge}X_n) = [S^n, X_1\overline{\wedge}\ldots\overline{\wedge}X_k]$$

with natural group structure defined in the same way as in the previous case.

Let $\boldsymbol{Com}_0 \subset \mathcal{K}$ be the full subcategory of $\mathcal{K}$ consisting of based compact metric spaces and $h(\boldsymbol{Com}_0) = \overline{\boldsymbol{Com}}_0$ the full subcategory of $\overline{\mathcal{K}}$ determined by compact metric spaces. We explained in § 2 (proposition 2.1.) how to define for a $\bar{f} \in \overline{\boldsymbol{Com}}_0(X, Y)$ the smash product $1\overline{\wedge}\bar{f}\colon E_k\overline{\wedge}X \to E_k\overline{\wedge}Y$. As a result we obtain induced maps $\bar{\mathcal{E}}_*(\bar{f}) = \bar{f}^*\colon \bar{\mathcal{E}}_*(X) \to \bar{\mathcal{E}}_*(Y)$, turning $\bar{\mathcal{E}}_*\colon \overline{\boldsymbol{Com}}_0 \to \mathcal{AB}^{\mathbf{Z}}$ into a functor.

For arbitrary spaces $X = (X, x_0) \in \mathcal{K}$ we define $\bar{\mathcal{E}}_*$ with compact supports:

$$\bar{\mathcal{E}}_n(X) = \varinjlim_{K\subset X} \bar{\mathcal{E}}_n(K)\,, \quad K \subset X \text{ compact, } n \in \mathbb{Z} \tag{2}$$

resp. for morphisms, induced by continuous $f \in \mathcal{K}(X, Y)$.

REMARKS. – 1) Actually we have defined a functor $\bar{\mathcal{E}}_*\colon \mathcal{K} \to Ab^{\mathbf{Z}}$ which coincides on $\mathbf{Com}_0$ with $\bar{\mathcal{E}}_*h$. We do *not* define $\bar{\mathcal{E}}_*(\bar{f})$ for an arbitrary $\bar{f} \in \overline{\mathcal{K}}(X, Y)$, because for compact $K \subset X$, in general $\bar{f}|K$ does not factorize over a compact $K' \subset Y$. According to our convention not to write down the functor h explicitely, we do not distinguish between $\bar{\mathcal{E}}_*h$ and $\bar{\mathcal{E}}_*$ in our notation. The fact that $\bar{\mathcal{E}}_*$ on $\boldsymbol{Com}_0$ allows an extension over $\overline{\boldsymbol{Com}}_0$ is needed in order to ensure that one has $\bar{\mathcal{E}}_*(X) \approx \bar{\mathcal{E}}_*(Y)$ for any based compacta (X, x_0), (Y, y_0) being equivalent in $\overline{\boldsymbol{Com}}_{0h}$ (cf. 3.2.) (resp. for the unreduced case).

2) One could try to define (1) by using the $\wedge$-product instead of the $\overline{\wedge}$-product. However this does not necessarily lead to a functor because we do not know how to define induced maps unless we agree to impose restrictive conditions upon the spectrum $\mathcal{E}$.

– We are now going to verify the Eilenberg-Steenrod axioms for the reduced theory $\mathcal{E}_n$ (with the exception of the dimension axiom).

Let $\mathcal{E}$ be any spectrum, then we have:

3.1. THEOREM. – On the category $\boldsymbol{Com}_0$ the homology $\overline{\mathcal{E}}_*$ fulfills the following axioms:

1) For $f_0, f_1 \in \boldsymbol{Com}_0(X, Y)$ and $f_0 \simeq f_1$ we have

$$\overline{\mathcal{E}}_*(f_0) = \overline{\mathcal{E}}_*(f_1) .$$

Moreover we have for $\bar{f}_0, \bar{f}_1 \in \overline{\boldsymbol{Com}}_0(X, Y)$, $\bar{f}_0 \simeq \bar{f}_1$ in $\overline{\boldsymbol{Com}}_0$

$$\overline{\mathcal{E}}_*(\bar{f}_0) = \overline{\mathcal{E}}_*(\bar{f}_1) .$$

2) Let $(A, a_0) \subset (X, x_0)$ be any inclusion, $(X \cup CA, *)$ the pointed space, where $C\ldots$ denotes the unreduced cone with top vertex $*$, then the following sequence is exact:

$$\overline{\mathcal{E}}_*(A) \xrightarrow{i_*} \overline{\mathcal{E}}_*(X) \xrightarrow{j_*} \overline{\mathcal{E}}_*(X \cup CA) , \tag{3}$$

where $i: A \subset X$, $j: X \subset X \cup CA$ are inclusions.

3) There exists a natural isomorphism:

$$\sigma_n: \overline{\mathcal{E}}_n(X) \approx \overline{\mathcal{E}}_{n+1}(\Sigma X) , \tag{4}$$

where Σ denotes the reduced suspension.

REMARK. – 1) For the sake of completeness we will present an independent proof of theorem 3.1. although the proof of the corresponding result in [1] could be immediately translated to our case.

2) In $\overline{\boldsymbol{Com}}_0 = h(\boldsymbol{Com}_0) \subset \mathcal{K}$ every inclusion is a cofibration (see proposition A9 in the appendix). Consequently we have in $\overline{\boldsymbol{Com}}{}_0^2$ a homotopy equivalence between the pair (X, A) and $(X/A, *)$ whenever A is contractible in $\overline{\boldsymbol{Com}}_0$.

This has some important implications:

3.2. PROPOSITION. – *a*) For any $(X, x_0) = X \in \boldsymbol{Com}_0$, we have a homotopy equivalence in $\overline{\boldsymbol{Com}}_0$ between 1) the pairs

$$(\tilde{\Sigma} X, x_0 \times I) \simeq (\Sigma X, *) \tag{5}$$

where $\tilde{\Sigma}$ denotes unreduced suspension and 2) the pairs

$$(X/A, *) \simeq (X \cup CA, CA) \tag{6}$$

for any inclusions $(A, a_0) \subset (X, x_0)$ in $\boldsymbol{Com}_0$ where it finally does not matter whether we use the reduced or the unreduced cone.

3.3. COROLLARY. – Assertion 2) in theorem 2.1. is equivalent to the contention:

2′) the sequence

$$\bar{\mathcal{E}}_*(A) \xrightarrow{i_*} \bar{\mathcal{E}}_*(X) \xrightarrow{p_*} \bar{\mathcal{E}}_*(X/A)$$

is exact, where $p: (X, x_0) \to (X/A, *)$ denotes the projection.
The proof is well-known.

PROOF OF THEOREM 3.1. – *Ad* 1) For $\bar{f}_0 \simeq \bar{f}_1$ in $\overline{\boldsymbol{Com}}_0$ we have due to the shape constructions in §§ 1, 2

$$\bar{f}_{0*} = \bar{f}_{1*}\colon [S^{n+k}, E_k \overline{\wedge} X] \to [S^{n+k}, E_k \overline{\wedge} Y]$$

this proves the second contention. The first follows as a special case ($\bar{f}_i = h(f_i)$ for $f_i \in \boldsymbol{Com}_0(X, Y)$).

Ad 2) We have $ji \simeq 0$ in $\boldsymbol{Com}_0$ and therefore $ji \simeq 0$ in $\overline{\boldsymbol{Com}}_0$ (omitting the functor h from our notation). This implies $j_* i_* = 0$.

Let on the other hand $\xi \in \bar{\mathcal{E}}_n(X)$ be such that $i_* \xi = 0$. Without loss of generality we may assume that there is given a $\bar{f} \in \bar{\mathcal{K}}_h(S^{n+k}, E_k \overline{\wedge} X)$, $\bar{f} \in \xi$ having the property

$$(1 \wedge j)\bar{f} \simeq 0 .$$

Thus we find a shape mapping

$$\bar{f}'\colon (D^{n+k+1}, S^{n+k}) \to (E_k \overline{\wedge} (X \cup CA), E_k \overline{\wedge} X) .$$

On the other hand we have an extension of $\bar{f}$ over $CS^{n+k} = D^{n+k+1}$

$$\bar{f}''\colon (D^{n+k+1}, *) \to (E_k \overline{\wedge} CX, *)$$

which agrees with $\bar{f}'$ on S^{n+k}. Setting

$$S^{n+k+1} = D_+^{n+k+1} \cup D_-^{n+k+1} ,$$

$$D_+^{n+k+1} \cap D_-^{n+k+1} = S^{n+k}$$

we see that we can glue both shape mappings together, obtaining in this way a

$$\bar{F}\colon (S^{n+k+1}, *) \to (E_k \overline{\wedge} (CX \cup CA), *) .$$

However we have in $\overline{\boldsymbol{Com}}_0$ an equivalence

$$(CA \cup CX, *) \simeq (\Sigma A, *)$$

and consequently a morphism in $\overline{\mathcal{K}}$:

$$\bar{g}\colon (S^{n+k+1}, *) \to (E_k \overline{\wedge} \Sigma A, *) .$$

We denote the stable class of $\bar{g}$ by η and claim that $i_* \eta = \xi$. More precisely:

$$(7) \qquad (1 \overline{\wedge} i)\bar{g} \simeq \Sigma \bar{f} .$$

PROOF. – Let $\Sigma X = C^+X \cup C^-X$, $X = C^+X \cap C^-X$, where $C^{\pm}X$ are two copies of the reduced cone over X and consider the inclusion $q\colon CA \cup CX \subset C^+X \cup C^-X$, where $CX = C^-X$, $CA \subset C^+X$. We conclude on one hand that by construction

$$(1 \overline{\wedge} q)\bar{F} \simeq (1 \overline{\wedge} \Sigma i)\bar{g}$$

while on the other hand $(1 \overline{\wedge} q)\bar{F}$ extends $\bar{f}\colon (S^{n+k}, *) \to (E_k \overline{\wedge} X, *)$ over ΣS^{n+k} into $E_k \overline{\wedge} (C^+X \cup C^-X) = E_k \overline{\wedge} X$.

This readily proves (7) and completes the verification of the exactness of (3).

Ad 3) The proof runs entirely as in the classical case. This completes the proof of theorem 3.1.

Up to this moment we were concerned with reduced homology. However it is well-known how to go over to unreduced homology which will be denoted by the same letter. As a consequence we can deduce from theorem 3.1. and proposition A9 the following:

3.4. THEOREM. – On the category $\boldsymbol{Com}^2$ of compact metric pairs there exists a homology theory $\bar{\mathcal{E}}_*$ which fulfills the Eilenberg-Steenrod axioms (with the exception of the dimension axiom) with a strong excision:

$$\bar{\mathcal{E}}_n(X, A) \approx \bar{\mathcal{E}}_n(X/A, *) .$$

For a polyhedral pair $(X, A) \in \boldsymbol{Com}^2$ one has

$$\bar{\mathcal{E}}_n(X, A) \approx \mathcal{E}_n(X, A) .$$

Because all inclusions in $\boldsymbol{Com}$ are cofibration in $\overline{\boldsymbol{Com}}_0$, we can derive from 3.2. a general *Mayer-Vietoris sequence* (without any restrictions concerning « excisive couples »):

3.5. THEOREM. – Let $(X, A), (X, B) \in \boldsymbol{Com}^2$ be any two pairs which are contained in some $(Z, C) \in \boldsymbol{Com}^2$, then the following sequence is exact:

$$\begin{aligned} &\ldots \to \bar{\mathcal{E}}_n(X \cap Y, A \cap B) \xrightarrow{\alpha} \bar{\mathcal{E}}_n(X, A) \oplus \bar{\mathcal{E}}_n(Y, B) \xrightarrow{\beta} \\ &\ldots \to \bar{\mathcal{E}}_n(X \cup Y, A \cup B) \xrightarrow{\Delta} \bar{\mathcal{E}}_{n-1}(X \cap Y, A \cap B) \to \ldots \end{aligned} \tag{8}$$

The homomorphisms α, β, Δ are defined as usual (cf. [16]).

4. – $\bar{\mathcal{E}}_*$ as simplicial homology.

In order to investigate $\mathcal{E}_*$ more closely, we recall the construction of the ***shape singular complex*** $\bar{S}(X)$ of a topological space X, which is formed in complete analogy to the ordinary singular complex $S(X)$ by taking all singular simplexes $\bar{\sigma}^n: \Delta^n \to X$ in $\bar{\mathcal{K}}$. Analogously we define $\bar{S}(X \bar{\wedge} Y)$.

In [3] (theorem 4.7.) we dealt with the problem of finding a homotopy equivalence between $|\bar{S}(X \wedge Y)|$ and $|\bar{S}(X)| \wedge |\bar{S}(Y)|$. The proof recorded there, contained a gap in so far, as we worked with the $\times$-product instead of the $\wedge$-product. Here we are going to prove:

4.1. THEOREM. – Let X, Y be spaces, X a compactum, $\bar{\pi}_0(X) = 0$, and Y shape simply connected (i.e. $\bar{\pi}_i(Y) = 0$, $i = 0, 1$). Moreover we assume that either: *a*) Y is compact metric or *b*) Y is a CW-complex such that all n-skeletons are compact.

Then we have a homotopy equivalence

$$|\bar{S}(X)| \wedge |\bar{S}(Y)| \simeq |\bar{S}(X \bar{\wedge} Y)| . \tag{1}$$

PROOF. – We start with case *a*) and as in [3] with the $\times$-product: There exists a mapping

$$\varphi^\times: |\bar{S}(X)| \times |\bar{S}(Y)| \to |\bar{S}(X \bar{\times} Y)| \tag{2}$$

in $\mathcal{S}_E$, the category of Kan complexes, which is defined by

$$\varphi^\times(\bar{\sigma}^n, \bar{\tau}^n) = (\bar{\sigma}^n \times \tau^n) d$$

($d: \Delta^n \to \Delta^n \times \Delta^n$, the diagonal map). There is a corresponding map

$$\varphi^\wedge: \bar{S}(X) \wedge \bar{S}(Y) \to \bar{S}(X \wedge Y) . \tag{3}$$

The $\times$- resp. the $\wedge$-products $\bar{\sigma}\times\bar{\tau}(\bar{\sigma}\wedge\bar{\tau})$ are readily defined in view of the results in § 2. The homotopy inverse of φ^X is as in [3] furnished by the map

$$\psi(\bar{\eta}^n) = (p_X\bar{\eta}^n, p_Y\bar{\eta}^n)\,, \qquad \bar{\eta}^n \in \bar{S}(X\wedge Y)$$

where p_X, p_Y are the projections. The fact that for the geometric realization one has

$$|\varphi^X||\psi| \simeq 1\,, \qquad |\psi||\varphi^X| \simeq 1$$

is elementary and can be proved as in [3] (proof of theorem 4.7.).

We form $\varphi = |\varphi^\wedge|: |\bar{S}(X)|\wedge|\bar{S}(Y)| \to |\bar{S}(X\wedge Y)|$; this map is well defined and we are going to prove that φ is a weak homotopy equivalence. Because Y is supposed to be shape simply connected, $|\bar{S}(Y)|$, $|\bar{S}(X)|\wedge|\bar{S}(Y)|$ are simply connected. Moreover, we can prove:

1) If X, Y are compact metric based spaces, Y shape simply connected, then $X\wedge Y$ is shape simply connected.

PROOF. – Let $g_1\wedge g_2: X\wedge Y \to P\wedge Q$ be an object in $\mathfrak{T}_{X\wedge Y}$ (which category is isomorphic to $\mathfrak{T}_X\overline{\wedge}\mathfrak{T}_Y$ because of proposition 1.4.).

We can assume without loss of generality that Q is a compact polyhedron and that g_2 is an inclusion. Therefore Q can be assumed to be connected (because Y cannot have more than one component).

This allows us to go over to a CW space with trivial 0-skeleton (consisting solely of the basepoint). Now let $\bar{\sigma}^1: (S^1, *) \to (X\wedge Y, *)$ be a shape loop.

We claim that $\bar{\sigma}^1$ is homotopic to zero (in $\bar{\mathfrak{K}}$). To this end we evaluate $\bar{\sigma}^1(g_1\wedge g_2)$ and observe that this loop is (naturally) homotopic to a simplicial loop (p, γ^1), $\gamma^1: (S^1, *) \to (Q, q_0)$ for fixed $p\in P$. Because $Q^0 = q_0$ the 1-skeleton of P does not contribute to the 1-skeleton of $P\wedge Q$.

We are going to construct a $\bar{\gamma}^1: (S^1, *) \to (Y, y_0)$ such that $\bar{\sigma}^1 \simeq \bar{\zeta}^0\wedge\bar{\gamma}^1$, using the results in the appendix (proposition A3): Let $i_1\wedge i_2: P\wedge Q \to P'\wedge Q'$ be an inclusion, and $\gamma'^1: (S^1, *) \to (Q', q_0)$ be the corresponding loop, i.e. $\bar{\sigma}^1(g_1\wedge g_2) \simeq (p', \gamma'^1)$.

We have a homotopy between $(i_1(p), \gamma^1)$ and (p', γ'^1), which is determined by a singular 2-cell $\eta^2: I\times I = D^2 \to P'\wedge Q'$. This η^2 is homotopic (rel. boundary) to either a 2-cell of the form (p_1, τ^2), $\tau^2: I\times I \to Q'$, $p_1\in P'$ or to a 2-cell (α^1, β^1) with $\alpha^1: I \to P'$, $\beta^1: I \to Q'$. The latter case means that $i_2\gamma^1 = \gamma'^1$, while in the first case we obtain a homotopy between γ^1 and γ'^1 in Q'. Applying this to $\mathfrak{T}'_Y \subset \mathfrak{T}_Y$ (cf. proposition A5) yields a loop $\bar{\gamma}^1$ such that $\bar{\sigma}^1 \simeq \bar{\zeta}^0\wedge\bar{\gamma}^1$, $\bar{\zeta}^0 \in \bar{\mathfrak{K}}((S^0, *), (X, x_0))$. Because $\bar{\pi}_1(Y) = 0$ we have a $\bar{\delta}^2: (D^2, *) \to (Y, y_0)$ with $\bar{\delta}^2|bd\, D^2 = \bar{\gamma}^1$ $(D^2 = I\times I)$. This in turn implies $\bar{\sigma}^1 \simeq 0$ because we have $(\bar{\zeta}^0\wedge\bar{\delta}^2)|bd D^2 = \bar{\sigma}^1$. This completes the proof of 1).

As a corollary we have

2) $|\bar{S}(X\wedge Y)|$ is simply connected.

PROOF. – Since $|\bar{S}(X\wedge Y)|$ is a CW-space, we have

$$\pi_i(|\bar{S}(X\wedge Y)| \approx \bar{\pi}_i(X\wedge Y) \quad \text{for all } i$$

which implies the assertion.

3) Let $\bar{f}\in\bar{\mathcal{K}}(A, B)$ be any shape map for shape simply connected A, B such that $\bar{H}_n(\bar{f})$ becomes an isomorphism for all n, then $\bar{\pi}_n(\bar{f})$ is an isomorphism for all n. Here $\bar{H}_*(X)$ is defined as $H_*(|\bar{S}(X)|)$.

This Whitehead theorem in shape theory is proved in the same way as in ordinary topology (using now the functor $\bar{S}$ instead of S). Compare [4] theorem 1.3.

The homology $\bar{H}_*$ which was introduced in [3], turns out to be exact. Moreover we have for each compact pair (A, B) (with $\bar{\pi}_0(A)=\bar{\pi}_0(B)=0$, cf. [3], lemma 7.3., where in fact this assumption is needed) an isomorphism

$$\bar{H}_*(A, B)\approx\bar{H}_*(A/B, *)\,.$$

Now we can complete the proof that φ is a weak homotopy equivalence by showing that $\bar{H}_*(\varphi)$ is an isomorphism. Observe that

$$\bar{H}_*(\varphi)=H_*(\varphi)$$

because φ maps a CW-space into a CW-space.

Thus this isomorphism will be established by an easy exactness argument using the following facts:

4) $\bar{H}_*(|\bar{S}(X\wedge Y)|)\approx\bar{H}_*(X\wedge Y)\approx\bar{H}_*(X\times Y, X\wedge Y)$;

5) $\bar{H}_*(|\bar{S}(X)|\wedge|\bar{S}(Y))\approx H_*(|\bar{S}(X)|\times|\bar{S}(Y)|, |\bar{S}(X)|\wedge|\bar{S}(Y)|)$;

6) $\bar{H}_*(\varphi^X)$ is an isomorphism;

7) $\bar{H}_*(|\bar{S}(X)|\vee|\bar{S}(Y)|)\approx\bar{H}_*(X)\oplus\bar{H}_*(Y)\approx\bar{H}^*(X\vee Y)$.

The last isomorphism in 7) is a consequence of the axioms. Now we can establish the isomorphism

$$\bar{H}_*(|\bar{S}(X)|\times|\bar{S}(Y)|, |\bar{S}(X)|\vee|\bar{S}(Y)|)\to\bar{H}_*(X\times Y, X\vee Y)$$

induced by φ^X, which follows from 4) - 7).

This assures us that φ is a weak and therefore also a strong homotopy equivalence establishing the assertion *a*) in the theorem.

In order to prove *b*) we claim that every singular simplex $\bar{\sigma}^n\in\bar{S}(\overline{X\wedge Y})$ for a CW-space Y with compact m-skeletons factors over $X\wedge Y_n$ up to a natural homotopy. This follows immediately by applying the simplicial approximation process (which can be performed in a canonical way). So the map

$$\varphi\colon |\bar{S}(X)|\wedge|\bar{S}(Y)| \to |\bar{S}(X\overline{\wedge}Y)|$$

has a homotopy inverse because every $X\wedge Y_n$ is compact and *a*) applies.

This completes the proof of the theorem.

As an application we can formulate the following assertion:

4.2. THEOREM. – Let $\mathcal{E} = \{E_n\}$ be a CW-spectrum, such that every m-skeleton of every E_n is compact and E_n is simply connected for sufficiently large n.

Then one has for any compact metric X with $\bar{\pi}_0(X) = 0$

$$\bar{\mathcal{E}}_*(X) \approx \mathcal{E}_*(|\bar{S}(X)|)\,.$$

PROOF. – We have

$$\begin{aligned}\bar{\mathcal{E}}_n(X) &= \varinjlim_k\,[S^{n+k}, E_k\overline{\wedge}X] \approx\\ &\approx \varinjlim_k\,[S^{n+k}, |\bar{S}(E_k\overline{\wedge}S)|] \approx\\ &\approx \varinjlim_k\,[S^{n+k}, E_k\wedge|\bar{S}(X)|] \approx\\ &\approx \mathcal{E}_n(|\bar{S}(X)|)\,.\end{aligned}$$

4.3. COROLLARY. – Let G be a finitely generated abelian group, then one has an isomorphism

$$\overline{\boldsymbol{K}(\boldsymbol{G})}_*(X) \approx H_*(|\bar{S}(X)|; G)\,.$$

In particular there exists a natural universal coefficient sequence

$$0 \to \overline{\boldsymbol{K}(\boldsymbol{G})}_n(X)\otimes G \to \overline{\boldsymbol{K}(\boldsymbol{G})}_n(X) \to \overline{\boldsymbol{K}(\boldsymbol{Z})}_{n-1} * G \to 0\,. \tag{4}$$

PROOF. – The Eilenberg-MacLane spaces $K(G, n)$ fulfill the requirements of theorem 4.2. The existence of (4) follows from homological algebra.

5. – Cohomology.

It is well known how to define a *cohomology theory* based on a coefficient spectrum $\mathcal{E}$:

Let $(X, x_0) \in \mathcal{K}$ be any space, then we set

$$\bar{\mathcal{E}}^n(X) = \varinjlim_k\,[\Sigma^k X, E_{n+k}]\,, \tag{1}$$

where the brackets are to be understood in $\overline{\mathcal{K}}$. The induced maps are also well-defined. It follows immediately from [5] theorem 2.4. or from corollary 1.3. that $\mathcal{E}^n(X)$ is isomorphic to $\overline{\mathcal{E}}^n(X)$ (where all maps and homotopies are now continuous) whenever $\mathcal{E}$ is a CW-spectrum.

The proof of the following theorem is routine and therefore omitted (cf. [1] for the classical model):

5.1. THEOREM. – The cohomology functor $\overline{\mathcal{E}}^*\colon \overline{\mathcal{K}} \to Ab^Z$ fulfills the following conditions:

1) $\bar{f}_0 \simeq \bar{f}_1 \Rightarrow \overline{\mathcal{E}}^*(\bar{f}_0) = \overline{\mathcal{E}}^*(\bar{f}_1)$.

2) Let $i\colon A \subset X$ be any inclusion, then the sequence

$$\overline{\mathcal{E}}^n(X \cup CA) \xrightarrow{j^*} \overline{\mathcal{E}}^n(X) \xrightarrow{i^*} \overline{\mathcal{E}}(A)$$

is exact.

3) There exists a natural equivalence

$$\sigma\colon \overline{\mathcal{E}}^n(X) \approx \overline{\mathcal{E}}^{n+1}(\Sigma X) .$$

REMARK. – The first remark in § 3 following (2) carries over to the case of $\overline{\mathcal{E}}^*$.

In case $\mathcal{E}$ is a CW-spectrum we have the following assertion which is originally due to K. SITNIKOV [14]:

5.2. LEMMA. – Let (X, A) be any pair of spaces such that $A \subset X \subset M$, M being a compact manifold, then we have for any CW-spectrum $\mathcal{E}$ an isomorphism:

$$\mathcal{E}^n(X, A) \approx \varinjlim_{(U,V)} \mathcal{E}^n(U, V)$$

where $(U, V) \supset (X, A)$ is a (rel. to M) open pair.

PROOF. – We prove the lemma for the absolute case and deduce the relative case from the exact cohomology sequence by naturality. Here, we have again, as in the case of homology turned over from reduced to unreduced cohomology. This is a classical process, which is well-known and does not deserve further mentioning.

Let $\{\zeta_U\}$ be contained in $\varinjlim \mathcal{E}^n(U)$ for $X \subset U \subset M$ an open set, then we set $i_{U*}\zeta_U = \zeta \in \mathcal{E}^n(X)$ ($i_U\colon X \subset U$ being the inclusion). Observe that ζ is independent of the choice of U. This furnishes a homomorphism

$$\psi\colon \varinjlim \mathcal{E}^n(U) \to \mathcal{E}^n(X) .$$

Let on the other hand $f\colon \Sigma^k X \to E_{n+k}$ be continuous, then proposition A1 assures that $\{f\} = \psi\{\zeta_v\}$ for suitable $\{\zeta_v\} \in \varinjlim \mathcal{E}^n(U)$. Therefore ψ is an epimorphism. On the other hand proposition A2 confirms that the kernel of ψ is trivial. Thus ψ is an isomorphism.

5.3. PROPOSITION. – Let $\mathcal{E}$ be an arbitrary spectrum and let (X, A), (Y, B) be pairs of open subsets of a compact manifold M^k. Then there exists an exact Mayer-Vietoris sequence

$$(2) \qquad \ldots \leftarrow \bar{\mathcal{E}}^n(X, A) \oplus \bar{\mathcal{E}}^n(Y, B) \overset{\alpha^*}{\leftarrow} \bar{\mathcal{E}}^n(X \cup Y, A \cup B) \overset{\Delta^*}{\leftarrow} \bar{\mathcal{E}}^{n-1}(X \cap Y, A \cap B) \overset{\beta^*}{\leftarrow} \ldots$$

with the usual homomorphism $\alpha^*, \beta^*, \Delta^*$ (see [1], [16]).

PROOF. – We claim that any triple $(X; A, B)$, $X \subset M = M^k$ with open (rel. to X) A, B is excisive (rel. to $\bar{\mathcal{E}}^*$). This means that the inclusion $i\colon (B, A \cap B) \subset (X, A)$ induces an isomorphism

$$\bar{\mathcal{E}}^*(X, A) \approx \bar{\mathcal{E}}^*(B, A \cap B)\,.$$

We assume without loss of generality that $A, B \neq \emptyset$ and $X \neq A, B$, because in the remaining cases the assertion turns out to be trivial.

To this end let $\bar{f} \in \bar{\mathcal{K}}(\Sigma^r(B \cup C(A \cap B), E_{n+r})$ be a shape morphism, $g\colon E_{n+r} \to E'_{n+r} \in \mathcal{T}$ be continuous and $j\colon \Sigma^r(B \cup C(B \cap A)) \subset \Sigma^r(X \cup CA)$ the inclusion. M is compact, hence $X \subset M$ a normal space. This implies the existence of an Urysohn function $\varphi\colon X \to I$ such that $\varphi|X - A = 0$, $\varphi|X - B = 1$. This provides us with an extension f' of $\bar{f}(g)\colon \Sigma^r(B \cup C(A \cap B)) \to E'_{n+r}$ up to homotopy over $\Sigma^r(X \cup CA)$ in the following way: We are obliged to define f' only on the points of the cone CA, so let 1) $y \in \Sigma^r C(X - B) \subset \Sigma^r CA$ be mapped into the basepoint $* \in E'_{n+r}$ and 2) a point

$$y = (y', t_0, t_1, \ldots, t_r) \in \Sigma^r(B \cup C(A \cap B))\,, \qquad y' \in B$$

be mapped into

$$f'(y) = \bar{f}(g)\big(y', \max(\varphi(y'), t_0), t_1, \ldots, t_r\big) \in E'_{n+r}\,.$$

For y' close to $X - B$ and $t_0 < 1$, we have $\max(\varphi(y'), t_0) = \varphi(y')$ and for $y' \in X - A$ we have $\max(\varphi(y'), t_0) = t_0$: The function $\max(s\varphi(y'), t_0)$, $0 \leqslant s \leqslant 1$ provides us with a homotopy $f' \circ j \simeq \bar{f}(g)$, which is of course basepoint preserving (the base point being as usual the vertex of the cone).

The Urysohn function φ and consequently the extension process is independent of g, hence we can apply the results of the appendix to the effect, that there exists an extension $\bar{f}'$ of $\bar{f}$, up to homotopy (rather than an extension of the individual $\bar{f}(g)$). This assures us that the inclusion i induces an epimorphism

$$i^*\colon \bar{\mathcal{E}}^*(X, A) \to \bar{\mathcal{E}}^*(B, A \cap B)\,.$$

The fact that i^* is also a monomorphism is established similarly by applying the same procedure to a homotopy $\bar{E}\colon \bar{f}j \simeq 0$, where now $\bar{f} \in \overline{\mathcal{K}}(\Sigma^r(X \cup CA), E_{n+r})$ is a shape mapping.

After having established this, proposition 5.3. follows by standard arguments (see [1]).

REMARKS. – 1) The proof of lemma 5.2. is substantially based on the assumption that $\mathcal{E}$ is supposed to be a CW-spectrum (we use the fact that all spaces E_k in $\mathcal{E}$ are ANEs).

2) The groups $\bar{\mathcal{E}}^*(X)$ which are defined in (1) are (even for CW-spectra $\mathcal{E}$) not isomorphic to the spectrum-cohomology introduced in [1]. Only for CW-spectra we have an isomorphism between these two concepts of a cohomology if one of the two conditions hold: 1) $\mathcal{E}$ is an Ω-spectrum or alternatively 2) X is a compact CW-space.

This is due to the fact that, unlike F. ADAMS in [1], we do not work in the Boardman-category $\mathcal{B}$. However every CW-spectrum $\mathcal{E}$ turns out to be in $\mathcal{B}$ equivalent to a suitable Ω-spectrum $\mathcal{E}'$.

For this reason we must, in comparing our results with another duality theorem in § 8 3), require that $\mathcal{E}$ is an Ω-CW–spectrum. In § 8 3) we are dealing with a duality theorem, where this difference at the end, surpisingly drops out.

6. – Products.

In order to establish the definition of a slant product ./. we use the definition of a ringspectrum $\mathcal{F}$ and of a $\mathcal{F}$-module spectrum $\mathcal{E}$: To this end we refer to [1] where these concepts are readily defined. In practice we need simply mappings

$$\mu = \mu_{rl}\colon E_l \overline{\wedge} F_r \to E_{r+l} \tag{1}$$

fulfilling the ordinary compatibility conditions (involving the mapping $\sigma\colon E_n \to E_{n+1}$ resp. for $\mathcal{F}$). Details can be found in [1].

In order to avoid difficulties with the smash product we will always assume that $\mathcal{F}$ is a CW-spectrum and therefore according to 1.4. $F_l \wedge F_r = F_l \overline{\wedge} F_r$.

Let X be a F_σ-space in $\mathfrak{T}$ (cf. § 2) and Y compact metric, then we define a slant product

$$/\colon \mathcal{F}^n(Y \wedge X) \otimes \bar{\mathcal{E}}_p(Y) \to \bar{\mathcal{E}}^{n-p}(X)$$

by formally repeating the construction of a slant product in classical topology: Take

$$\zeta = \{[f]\} \in \varinjlim\, [Y \wedge \Sigma^k X, F_{n+k}] = \mathcal{F}^n(X \wedge Y)$$

and

$$\eta = \{[\bar{g}]\} \in \varinjlim \overline{\mathcal{K}}_h(S^{p+l}, E_l \overline{\wedge} Y) ,$$

then we can assume that f is in fact a continuous mapping because $\mathcal{F}$ is supposed to be a CW-spectrum.

We define

$$\zeta/\eta = \{\bar{e}\} \in \varinjlim \overline{\mathcal{K}}_h(\Sigma^r X, E_{n-p+r}) \tag{2}$$

with $r = k + p + l$ as the composition

$$\begin{array}{l} \Sigma^{k+p+l} X = \Sigma^r X \\ \quad \int\int \\ S^{p+l} \wedge \Sigma^k X \xrightarrow{\bar{g}\wedge 1} E_l \overline{\wedge} Y \wedge \Sigma^r X \xrightarrow{1 \overline{\wedge} f} \bar{E}_l \overline{\wedge} F_{n+k} \to E_{n+k+l} = E_{n-p+r} . \end{array}$$

Observe that $\bar{e}$ is well defined because $\bar{g} \wedge 1$ and $1 \overline{\wedge} f$ exist according to our results in § 2, in particular proposition 2.1.

Whenever $\mathcal{E}$ is a CW-spectrum, we can find a map $[e] \in [\Sigma^r X, E_{n-p+r}]$ such that $h(e) \simeq \bar{v}$.

This definition is completely analogous to the ordinary definition of the slant product. Therefore this product is homomorphic and natural in the usual sense, as long as it is defined. We will use this slant product for X being an open subset of a compact manifold M^n. Since M^n allows an embedding into a euclidean space of sufficiently high dimension, X is a retract of a F_σ-set. (cf. § 2, final remark 2)).

We need a definition of $\mathcal{F}$-orientability of a manifold M^n, which is identical with the concept of $\mathcal{F}$-orientability given in any modern textbook on topology (see e.g. [1]):

An element $u \in \mathcal{F}^n(M^n \times M^n, M^n \times M^n - \Delta)$ (Δ = diagonal in $M^n \times M^n$) is an $\mathcal{F}$-orientation whenever for any point $x \in M^n$ the element

$$\begin{aligned} i_x^*(u) \in\ & \mathcal{F}^n\big(x \times M^n, x \times M^n - \{x \times x\}\big) \approx \\ & \approx \mathcal{F}^n\big(M^n, M^n - \{x\}\big) \approx \\ & \approx \mathcal{F}^n\big(\mathbb{R}^n, \mathbb{R}^n - \{0\}\big) \approx \\ & \approx \mathcal{F}^n(S^n, *) \end{aligned}$$

is a generator of $\mathcal{F}^*(S^n, *)$ as a $\pi_*(\mathcal{F})$-module. Here

$$i_x \colon \big(x \times M^n, x \times M^n - \{x \times x\}\big) \subset (M^n \times M^n, M^n \times M^n - \Delta)$$

is of course the inclusion. Since this concept is sufficiently treated in the literature, we do not have to go into the details.

The duality theorem deals primarily with an n-dimensional compact manifold M^n and a pair (Y, B) such that Y, B are compact, $Y \subset M^n$. We have an inclusion:

$$i\colon \big((M-B)\times Y,\ (M-Y)\times Y \cup (M-B\times B\big) = \\ = (M-B,\ M-Y)\times(Y, B) \subset (M\times M,\ M\times M - \Delta)$$

where we wrote M for M^n.

Assume that M^n is $\mathcal{F}$-orientable with orientation class u, then we have for each $\mathcal{F}$-module spectrum $\mathcal{E}$ a mapping

$$\gamma_u(\cdot) = i_* u/.\colon \bar{\mathcal{E}}_p(Y, B) \to \bar{\mathcal{E}}^{-n-p}(M-B,\ M-Y)\,, \qquad \cdot \in \bar{\mathcal{E}}_p(Y, B) \quad \text{and } p \in \mathbb{Z}\,.$$

Because of the related properties of the slant product (cf. [1], [9], [16]) this mapping is a homomorphism and natural for inclusions $(Y', B') \subset (Y, B)$.

By an abuse of notation we will during the proof of theorem 7.1. also write $\gamma_u(\bar{g}) = \bar{f}$ for the individual maps (rather than for the stable classes $\{\bar{g}\}$, $\{\bar{f}\}$).

More precisely: Set

$$L_k(X) = \Sigma^k\big(M^n \cup C(M^n - X)\big)$$

then for given $\bar{g}\colon S^{p+l} \to E_l \overline{\wedge} X^+$ and suitable representative of $i^* u$ denoted by an abuse of notation by

$$u^k\colon X^+ \wedge \Sigma^k\big(M^n \cup C(M^n - X)\big) \to F_{n+k}$$

we obtain a

$$\gamma_u(\bar{g}) = \bar{f}\colon L_r(X) \to E_{n-p+r}$$

where we deal with an (unbased) space $X = (X, \emptyset)$ and set $r = p + k + l$. The map $\bar{f}$ has the form

$$\bar{f} = \mu(E_l \overline{\wedge} u^k)(\bar{g} \wedge L_k(X))$$

with the map

$$\mu\colon E_l \overline{\wedge} E_{n+k} \to E_{n+k+l} = E_{n-p+r}\,.$$

As usual, the basepoint of $L_0(X)$ is the vertex of the cone $C(\ldots)$.

REMARK. – In theorem 7.1. and § 8 4) we will have to deal with $M^n = S^n$. Every spectrum $\mathcal{E} = \{E_k\}$ is a $\mathcal{S}$-module spectrum, $\mathcal{S} = \{S^p\}$, because we have the obvious mappings

$$E_k \wedge S^q \to E_{k+q} .$$

Furthermore S^n is obviously $\mathcal{S}$-orientable. This enables us to work in this case with any spectrum $\mathcal{E}$.

For a CW-spectrum $\mathcal{E}$ we can do a little more: According to lemma 5.2. we have for any $X \subset S^n$ (not necessarily compact)

$$\mathcal{E}^{n-p}(S^n, S^n - X) \approx \mathcal{E}^{n-p}\big((S^n, S^n - X) \approx \\ \approx \varinjlim_{K \subset X} \mathcal{E}^{n-p}(S^n, S^n - K)\big)$$

and by definition

$$\overline{\mathcal{E}}_p(X) = \lim_{K \subset X} \overline{\mathcal{E}}_p(K) , \qquad K \text{ compact} .$$

Therefore we have in this case a homomorphism

$$\bar{\gamma}_u \colon \overline{\mathcal{E}}_p(X) \to \mathcal{E}^{n-p}(S^n, S^n - X) \tag{4}$$

for any $X \subset S^n$. More generally let (X, Y) be any pair $A \subset X \subset M^n$, M^n a compact $\mathcal{F}$-orientable manifold with orientation class u, then we argue similarly, obtaining a

$$\bar{\gamma}_u \colon \overline{\mathcal{E}}_p(X, A) \to \mathcal{E}^{n-p}(M^n - A, M^n - X) . \tag{5}$$

On the mapping level we can of course also define $\gamma_u(\bar{g})$ for any $\bar{g} \colon D^{p+l+1} \to \to E_l \overline{\wedge} X^+$, $\gamma_u(\bar{g}) = \bar{f} \colon CL_r(X) \to E_{n-p+r}$. We will use this in § 9 but only for finite polyhedra X, $M^n = S^n$ and a CW-spectrum $\mathcal{E}$. The definition is analogous to the previous one:

$$\gamma_u(\bar{g}) = \mu(E_l \overline{\wedge} u^k)(\bar{g} \wedge 1_{L_r(X)}) .$$

We make the simple observation that γ_u (on the mapping level as well as on the level of stable homotopy classes) is natural with respect to mappings of spectra $\varphi \colon \mathcal{E} \to \mathcal{E}'$, where of course $\varphi = \{\varphi_n \colon E_n \to E_n'\}$ is a family of continuous mappings rendering the squares

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma \varphi_n} & \Sigma E_n' \\ \sigma \downarrow & & \downarrow \sigma' \\ E_{n+1} & \xrightarrow{\varphi_{n+1}} & E_{n+1}' \end{array}$$

commutative.

Finally we must settle the following problem:

Let $M^n = S^n$, $X \subset S^n$ be a compact space, $S^n \supset K_1 \supset K_2 \supset \ldots$ a decreasing sequence of compact polyhedra with

$$\cap K_m = X .$$

In the applications we will assume that K_m is the union of finitely many n-dimensional convex disks in S^n (a so-called disk space, cf. § 9).

We denote the inclusions by

$$i^m_{m-1}\colon K_m \subset K_{m-1}$$

resp.

$$i_m\colon X \subset K_m .$$

They induce mappings like

$$i^{m*}_{m-1}\colon L_r(K_{m-1}) \to L_r(K_m) .$$

Now suppose that $\bar{g} \in \bar{\mathcal{K}}(S^{p+l}, E_l \overline{\wedge} X^+)$ ($\mathcal{E}$ a CW-spectrum) and a $f \in \mathcal{K}(L_r(X), E_{n-p+r})$ are fixed maps such that for each m one has a natural homotopy

$$\omega_m\colon \gamma_u\big((1 \overline{\wedge} i^+_m)\bar{g}\big) = f_m \simeq f i^*_m$$

(i.e. we require $\omega_m | L_r(K_{m-1}) = \omega_{m-1}$).

Then we can prove:

6.1. LEMMA. – We have

$$\gamma_u(\{\bar{g}\}) = \{f\} .$$

PROOF. – We can enlarge each K_m to a $\tilde{K}_m \supset K_m$ such that 1) the inclusion is a homotopy equivalence, 2) $\tilde{K}_m$ is open in S^n such that $L_r(\tilde{K}_m)$ is compact polyhedron and 3) $\cap \tilde{K}_m = X$. All this implies that

$$L_r(X) = \cup L_r(\tilde{K}_m)$$
$$L_r(\tilde{K}_1) \subset L_r(\tilde{K}_2) \subset \ldots$$

where all $L_r(\tilde{K}_i)$ are compact. Due to the definition of

$$S^{p+l} \wedge L_r(X) \xrightarrow{\bar{g} \wedge 1} E_l \overline{\wedge} (X^+ \wedge L_r(X))$$

in § 2 we form $\bar{g}\wedge 1$ at first for compact polyhedral spaces $L_i \subset L_r(X)$,

$$L_1 \subset L_2 \subset \dots, \qquad \cup L_i = L_r(X)$$

and apply a cofiber argument, which provides us with a homotopy equivalence between $L_r(X)$ and the telescope $\widetilde{L_r(X)} = \sum_{i=1}^{\infty} L_i \times I/\sim$ (cf. § 2). In our present case we put $L_i = L_r(\tilde{K}_i)$.

There are homotopies

$$\nu_m \colon \gamma_u\big((1\wedge \tilde{i}_m^+)\bar{g}\big) \simeq \gamma_u(\bar{g})|L_r(\tilde{K}_i)$$

where $\tilde{i}_m^+ \colon X^+ \subset \tilde{K}_m^+$ denotes the inclusion, such that $\nu_{m+1}|L_m = \nu_m$.

These homotopies together with the given homotopies ω_m yield the required homotopy

$$\omega \colon \gamma_u(\bar{g}) \simeq f\,.$$

7. – Duality theorems.

In this section we prove the main *duality theorems* (7.1. - 7.4.). This will be accomplished by an excessive use of Mayer-Vietoris arguments and by using the results of § 9.

We begin with duality theorems for spheres S^n resp. for euclidean spaces $\mathbb{R}^n$:

7.1. THEOREM. – Let $\mathcal{E} = \{E_k\}$ be any spectrum (cf. § 3) then we have for any compact based $X = (X, x_0)$, $X \subset S^n$ a, with respect to inclusions natural isomorphism

$$\gamma_u \colon \bar{\mathcal{E}}_p(X, x_0) \approx \bar{\mathcal{E}}^q\big(S^n - \{x_0\}, S^n - X\big) \tag{1}$$

$p + q = n$, $p, q \in \mathbb{Z}$.

PROOF. – We know that $\bar{\mathcal{E}}^q(A, B)$ is isomorphic to $\bar{\mathcal{E}}^q(A \cup CB, *)$ where as usual the upper vertex of the cone serves as the basepoint. Therefore (1) reduces to the isomorphism

$$\gamma_p \colon \bar{\mathcal{E}}_p(X, x_0) \approx \bar{\mathcal{E}}^q\big(\mathbb{R}^n \cup C(\mathbb{R}^n - \{x_0\})\big)\,, \tag{2}$$

where γ_u has already been defined in § 6.

We consider the unbased case and deal consequently with unreduced homology, proving for a $X \subset R^n$ the isomorphism

$$\gamma_u \colon \bar{\mathcal{E}}_p(X) \approx \bar{\mathcal{E}}^q\big(\mathbb{R}^n \cup C(\mathbb{R}^n - X)\big)\,. \tag{3}$$

The isomorphism (2) follows from (3) by well-known classical arguments. Thus we are only obliged to verify (3). This is accomplished by using the results of § 9.

Let $X = \cap U_i$, $U_1 \supset U_2 \supset \dots$ be the intersection of a descending sequence of disk spaces (cf. § 9) and

$$\bar{f}\colon \Sigma^r(\mathbb{R}^n \cup C(\mathbb{R}^n - X)) \to E_{n-p+r}$$

a shape mapping. We set $T(X) = \Sigma^r(\mathbb{R}^n \cup C(\mathbb{R}^n - X))$.

Let $\alpha_i\colon X \subset U_i$ be the inclusion and $j\colon E_{n+p+r} \to E'_{n+p+r} \in \mathfrak{T}$ be any continuous map, then we have the induced map

$$\alpha_i^* \bar{f}\colon T(U_i) \to E_{n-p+r}$$

and

$$f_i = (\alpha_i^* \bar{f})(j)\colon T(U_i) \to E'_{n+p+r}\,.$$

The results of § 9 provide us with a $g_i = \psi(f_i)\colon S^{p+1} \to E_l \wedge U_i$, $p + l = n + r$ and we set

$$\bar{g}(j \wedge \alpha_i) = g_i\,. \tag{4}$$

In view of A7, A8 in the appendix, propositions 9.2., 9.5. assure us that (4) establishes in fact a $\bar{g} \in \overline{\mathcal{K}}(S^{p+l}, E_l \overline{\wedge} X^+)$. Moreover 9.2.-9.5. guarantee that:

1) One has $\gamma_u\{\bar{g}\} = \{\bar{f}\}$, thus γ_u is surjective.

2) γ_u is injective (which is an immediate consequence of lemma 9.4.).

This completes the proof of Theorem 7.1.

In case $\mathcal{E}$ is a CW-spectrum, we can rephrase theorem 7.1. for arbitrary (not necessarily compact) X:

7.2. THEOREM. – Assume $\mathcal{E} = \{E_k\}$ being a CW-spectrum, then we have for any based $X = (X, x_0)$, $X \subset S^n$ a, with respect to inclusions, natural isomorphism

$$\bar{\gamma}_u\colon \bar{\mathcal{E}}_p(X, x_0) \approx \mathcal{E}^q(S^n - \{x_0\}, S^n - X)$$

$p + q = n$, $p, q \in \mathbb{Z}$.

PROOF. – This follows immediately with the definition of $\bar{\gamma}_u$ ((4), § 6), from lemma 5.2. and theorem 7.1.

Let $M = M^n$ be a compact n-dimensional manifold, which is $\mathcal{F}$-orientable with orientation class u and let $\mathcal{E}$ be any $\mathcal{F}$-module spectrum (cf. § 6). Recall that $\mathcal{F}$ is always supposed to be CW-spectrum.

We have again a pair of theorems:

7.3. THEOREM. – For any compact pair (X, A), $A \subset X \subset M^n$, we have a, with respect to inclusions of pairs, natural isomorphism

$$\gamma_u\colon \bar{\mathcal{E}}_p(X, A) \approx \bar{\mathcal{E}}^q(M^n - A,\, M^n - X)\,, \quad p + q = n,\ p, q \in \mathbb{Z}\,.$$

PROOF. – We verify 7.3. in the following three steps:

1) $M^n = \mathbb{R}^n$ (although $\mathbb{R}^n$ is not compact), $A = \emptyset$.

This has already been accomplished in the proof of 7.1.

2) M^n arbitrary, $A = \emptyset$.

3) M^n arbitrary, A not necessarily empty.

Case k) follows from case $k-1$) by classical Mayer-Vietoris arguments (cf. [1], [16]).

7.4. THEOREM. – Let $\mathcal{E}$ be a CW-spectrum, then we have for any pair (X, A), $A \subset X \subset M^n$, a with respect to inclusions of pairs, natural isomorphism

$$\bar{\gamma}_u\colon \bar{\mathcal{E}}_p(X, A) \approx \mathcal{E}^q(M^n - A,\, M^n - X)\,, \quad p + q = n,\ p, q \in Z\,.$$

PROOF. – This assertion follows from 7.2. in the same way as 7.3. was deduced from 7.1.

REMARK. – The crucial point in the proofs of all these theorems is embodied in proposition 9.2. where we construct an inverse ψ to γ_u in the case $X =$ disk space (i.e. for a finite polyhedron).

On the level of stable homotopy classes, this could be accomplished by a simple Mayer-Vietoris argument (inductively, with respect to the number of cells in X). We need however an assignment ψ *on the mapping level* rather than for the whole classes. Here Mayer-Vietoris arguments do not suffice because it may very well happen, that the suspension level (l depending on the given r) increases indefinitely with increasing number of cells.

It is worth mentioning that our (co-) homologies depend functorially on the coefficient spectrum in the usual, expected way. Consequently the isomorphisms γ_u, $\bar{\gamma}_u$ are natural with respect to coefficient spectra. Details are immediate and left to the reader.

8. – Applications.

Theorems 7.1. - 7.4. imply as special cases all different kinds of duality theorems for compact manifolds which are hitherto known.

1) *The Steenrod-Sitnikov duality theorems.*

Let $X \subset S^n$ be any (not necessarily compact) space, then there exists a (with respect to inclusions) natural isomorphism

$$H^s_p(X) \approx \check{H}^q(S^n - X)\,, \quad p + q = n - 1,\ p, q \in \mathbb{N}\,. \tag{1}$$

Here H^s_* stands for reduced *Steenrod-Sitnikov homology* (cf. [2], [3], [14]) while $\check{H}^*$ denotes reduced *Čech cohomology*. Theorem 7.7. [3] establishes for a shape connected compactum X an isomorphism between $H^s_p(X)$ and $H_p(|\bar{S}(X)|)$. Therefore 4.3. implies that H^s_p and $\overline{\boldsymbol{K}(\mathbb{Z})}_p(X)$ are isomorphic. On the other hand one has in this case an isomorphism between $\boldsymbol{K}(\mathbb{Z})^*(Y)$ and $\check{H}^*(Y)$ (Y paracompact cf. [13]). Since H^s_* and $\overline{\boldsymbol{K}(\mathbb{Z})}_*$ are both defined with compact support for non-compact spaces, we deduce (1) as a special case of 7.2. by a standard exactness argument (setting $\mathcal{E} = \boldsymbol{K}(\mathbb{Z})$). However K. SITNIKOV has a duality theorem of this kind for *any* abelian coefficient group G:

$$H^s_p(X; G) \approx \check{H}^q(S^n - X; G)\,, \tag{2}$$

p, q as above. Since we have again ([13]):

$$\boldsymbol{K}(G)^*(\ ) \approx \check{H}^*(\ ; G)\,,$$

we come to the conclusion that

$$\overline{\boldsymbol{K}(G)}_p(\ ) \approx H^s_p(\ ; G)$$

for any abelian group G and all subspaces of a S^n. This and corollary 4.3. imply in particular that H^s_p admits a universal coefficient theorem for a finitely generated abelian group G.

Since (1) or (2) imply all previous versions of duality theorems for *ordinary* (co-)homology (for example for compact $S^n - X$), all these theorems are recognized as implications of theorem 7.2.

The discovery of (2) in the early fifties represented a remarkable progress in this field.

We are not going to discuss the historical details (like Steenrod's duality theorem as an immediate predecessor of (2)) as well as the numerous geometrical applications of this theorem and refer to [2].

2) *The duality theorem of D. S. Kahn, J. Kaminker and C. Schochet.*

In [12] these authors establish a duality theorem for an arbitrary spectrum $\mathcal{E}$ and compact subsets $X \subset S^n$. According to our remark at the end of § 5 we will assume that $\mathcal{E}$ is an Ω-spectrum.

They discover an isomorphism

$$(3) \qquad {}^s\mathcal{E}_p(X) \approx {}^s\mathcal{E}^q(S^n - X)\,, \quad p + q = n - 1\,,$$

where ${}^s\mathcal{E}_*$ is a reduced homology functor, which they call « Steenrod homology with coefficient spectrum $\mathcal{E}$ ». Concerning the details, we refer to [12]. For $\mathcal{E}$ being an Ω-CW-spectrum their ${}^s\mathcal{E}^*$ is our $\mathcal{E}^*$.

Although we are treating the relations between ${}^s\mathcal{E}_*$ and $\overline{\mathcal{E}}_*$ (resp. ${}^s\mathcal{E}_*$ and $\mathcal{E}^*$) in a subsequent paper in more detail (cf. remark at the end of § 8 2)), we include here:

8.1. PROPOSITION. – For Ω-CW-spectra $\mathcal{E}$ and compact $X \subset S^n$, we have an isomorphism

$${}^s\mathcal{E}_*(X) \approx \overline{\mathcal{E}}_*(X)\,.$$

PROOF. – We deduce from theorem 7.1. and (3) the isomorphisms:

$$\begin{aligned}\overline{\mathcal{E}}_p(X) &\approx \overline{\mathcal{E}}^{n-p}(S^n - \{x_0\}, S^n - X) \approx \\ &\approx \mathcal{E}^{n-p}(S^n - \{x_0\}, S^n - X) \approx \\ &\approx \mathcal{E}^{n-p-1}(S^n - X) \approx {}^s\mathcal{E}_p(X)\,.\end{aligned}$$

The functor $h\colon \mathcal{K} \to \overline{\mathcal{K}}$ in § 1 induces a natural transformation $\varphi\colon {}^s\mathcal{E}^* \to \overline{\mathcal{E}}^*$ (for arbitrary Ω-spectra $\mathcal{E}$). This allows us to recognize ${}^s\mathcal{E}_*(X)$ in the following way: Take all shape mappings

$$\bar{g}\colon S^{p+l} \to E_l \overline{\wedge} X^+$$

which are transformed under γ_u into a $\bar{f}$ which lies in the image of the functor h. Proceed analogously with the homotopies. Then the group ${}^s\mathcal{E}_*(X)$ is generated by all stable homotopy classes of these particular mappings.

As for the class of spectra which are admitted in this paper (see § 3) the duality theorem (3) appears therefore as a special case of theorem 7.1.

REMARK. – The relations between ${}^s\mathcal{E}_p(X)$ and $\overline{\mathcal{E}}_p(X)$ for arbitrary spectra $\mathcal{E}$, in particular the existence of a natural isomorphism between both kinds of homology for Ω-CW-spectra has meanwhile been thoroughly treated in the article « *Under what conditions are shape homology* $\overline{\mathcal{E}}_*$ *and Steenrod homology* ${}^s\mathcal{E}_*$ *isomorphic?* » (Shape theory and geometric topology, Proceedings, Dubrovnik 1981, Lecture Notes in Mathematics Vol. 870, pp. 186-214).

3) *G. W. Whitehead's duality theorem with coefficients in a spectrum.*

This theorem is formulated in [1], p. 259 (theorem 10.6.). Here a manifold M^n is allowed which is not necessarily compact and which may have a boundary bd $M^n \neq \emptyset$. Let $\mathcal{E}$ be any CW-$\mathcal{F}$-module spectrum and suppose M^n being $\mathcal{F}$-orientable, then one has the isomorphism:

$$D\colon {}^s\mathcal{E}_p(M^n - L,\ M^n - K) \approx \mathcal{E}^{n-p}(K, L) \tag{4}$$

where (K, L) is a compact pair, $K \subset M^n$ such that $K \cap \text{bd}\, M^n \subset L$.

The homology ${}^s\mathcal{E}_p$ in [1] (not identical with ${}^s\mathcal{E}_*$ in the preceding case 2)) is defined by:

$${}^s\mathcal{E}_p(X, Y) = \mathcal{E}_p(X', Y') \tag{5}$$

for any CW-pair (X', Y'), weakly homotopy equivalent to (X, Y). The existence of such a pair is well-known, moreover (5) does not depend on the particular choice of such a pair.

We come to the definition of Čech cohomology (in the sense of F. ADAMS): Let (K, L) be any compact pair, $K \subset M^n$, then we define

$$\check{\mathcal{E}}^p(K, L) \approx \varinjlim {}^s\mathcal{E}^p(U, V) \tag{6}$$

where $(U, V) \supset (K, L)$ is an open pair, $U \subset M^n$ and ${}^s\mathcal{E}^p$ denotes singular cohomology, defined in the same way as singular homology in (5). It turns out that this kind of cohomology does not depend on the particular manifold M^n.

Let M^n be a compact manifold without boundary. We claim that for a compact pair (K, L) one has

$$\bar{\mathcal{E}}_p(M^n - L,\ M^n - K) \approx {}^s\mathcal{E}_p(M^n - L,\ M^n - K)\,.$$

This follows from the following observations:

1) $M^n - L$ and $M^n - K$ are as open subspaces of the ANR M^n (cf. [7]) also ANRs. Hence we have

$$\bar{\mathcal{E}}_*(M^n - L) \approx \mathcal{E}_*(M^n - L) \approx {}^s\mathcal{E}_*(M^n - L)$$

resp. for K.

2) $\bar{\mathcal{E}}_*$ and ${}^s\mathcal{E}_*$ are exact, hence one can deduce:

$$\bar{\mathcal{E}}_*(M^n - L,\ M^n - K) \approx {}^s\mathcal{E}_*(M^n - L,\ M^n - K)$$

by using the exact homology sequences.

We claim for the same compact pair (K, L) the existence of an isomorphism:

$$\mathcal{E}^p(K, L) \approx \check{\mathcal{E}}^p(K, L) . \tag{8}$$

To this end we embed M^n in a euclidean space $\mathbb{R}^N$ and find open pairs $(U, V) \supset \supset (K, L)$. The same argument which worked in the case of homology ensures that for pairs (U, V) having the homotopy type of a finite polyhedral pair we have (cf. remark at the end of § 5):

$$^{s}\mathcal{E}^p(U, V) \approx \mathcal{E}^p(U, V) .$$

So we have (because for compact (K, L) these pairs are cofinal in the family of all open pairs $(U, V) \supset (K, L)$):

$$\mathcal{E}^p(K, L) \approx \varinjlim \mathcal{E}^p(U, V) \approx \varinjlim {}^{s}\mathcal{E}^p(U, V) \approx$$
$$\approx \check{\mathcal{E}}^p(K, L)$$

by applying lemma 5.2.

Theorem 7.4. ensures that

$$\bar{\mathcal{E}}_p(M^n - L, M^n - K) \approx \mathcal{E}^{n-p}(K, L) .$$

Therefore (7), (8) implies that (4) is a corollary of theorem 7.4.

REMARK. - 1) Unlike the situation in 2) we do not have to assume that $\mathcal{E}$ is a Ω-spectrum, because in formulating (4) the original difference between the cohomology concepts (in [1], resp. in § 5) finally disappears.

4) *S-Duality.*

We are unable to develop a general S-duality for arbitrary subspaces of an n-sphere S^n using our kind of shape theory. However theorem 7.1. can be reformulated in such a way that it becomes a kind of first step towards S-duality.

Let P be the category whose objects are pairs $(X_1 \bar{\wedge} \ldots \bar{\wedge} X_k, m)$, where X_i is a based subspace of some S^{n_i} and $m \in \mathbb{Z}$. The morphisms are defined by

$$\varinjlim_{q} \bar{\mathcal{K}}_h(\Sigma^{m+q}(X_1 \bar{\wedge} \ldots \bar{\wedge} X_k), \Sigma^{n+q}(Y_1 \bar{\wedge} \ldots \bar{\wedge} Y_1)) .$$

Here $\Sigma(X_1 \bar{\wedge} X_2)$ is of course defined by $(S^1 \wedge X_1) \bar{\wedge} X_2$ and it is immediate that this is naturally isomorphic to $X_1 \bar{\wedge} (S^1 \wedge X_2)$.

An object (X, m) where X is a space is called *regular*. We write simply X instead of $(X, 0)$, abbreviate (X, m) by $\mathfrak{X}$ and $\mathsf{P}(\mathfrak{X}, \mathfrak{Y})$ by $\{\mathfrak{X}, \mathfrak{Y}\}$. An object $\mathfrak{X} = (X, m)$ is called compact whenever X is compact.

Hence P is the shape analogue of the ordinary S-category (cf. [6], [9]).

As usual we adopt the equivalence $(X, n) = (\Sigma X, n-1)$. There exist a smash product

$$(A, m)\overline{\wedge}(B, n) = (A\overline{\wedge}B, m+n)$$

where A and B are not necessarily regular (observe that the smash-product is associative). In particular we have a suspension

$$\Sigma(X, n) = (X\wedge S^1, n) = (X, n)\overline{\wedge}(S^1, 0)\,.$$

Every object (X, m) admits a desuspension

$$\Sigma^{-1}(X, m) = (X, m-1)\,,$$

hence P is a *stable category*.

To any $X = (X, x_0)$, $X\subset S^n$ we set

$$D_n X = \big(S^n - \{x_0\}\big)\cup C(S^n - X)\,,$$

with the top vertex of $C\ldots$ as basepoint.

Theorem 7.1. allows the following reformulation:

8.2. THEOREM. – Let $X\subset S^n$ be compact, with basepoint, $p+q=n$, $p, q\in\mathbb{Z}$ and $Z=(Z,0)\in$ P regular. Then there exists an isomorphism

$$\{S^p, X\overline{\wedge}Z\}\approx\{\Sigma^{-q}D_n X, Z\}\,,$$

which is natural in Z (with respect to all morphisms in P) and in X with respect to inclusions.

PROOF. – We must translate the assertion into the language of (co-)homology with coefficients in a suspension-spectrum $\mathcal{E} = \{\Sigma^i Z\}$:

$$\bar{\mathcal{E}}_p(X) = \{S^p, X\overline{\wedge}Z\}$$
$$\bar{\mathcal{E}}^q(Y) = \{\Sigma^{-q}Y, Z\}\,.$$

Now 8.2. follows from 7.1.

9. – The construction of ψ.

This section has a totally auxiliary character: it is devoted to the construction of some kind of inverse to γ_u. Before we give details let us introduce the following conventions:

Let $X \subset \mathbb{R}^n$, $r \in N$, $p \in \mathbb{Z}$, $q = n - p$ be given, then we set

$$L_r(X) = \Sigma^r(\mathbb{R}^n \cup C(\mathbb{R}^n - X)) .$$

The basepoint is the top vertex of the unreduced cone. Take two mappings $f_r: L_r(X) \to E_{q+r}$, $f_s: L_s(X) \to E_{q+s}$ and suppose $r \leqslant s$, then we obtain a unique representative $f'_r: L_s(X) = \Sigma^{s-r} L_r(X) \to E_{r+s}$, $f'_r = \Sigma^{s-r}\sigma^{s-r} f_r$, in the stable class $\{f_r\}$. By an abuse of notation we will write $\Sigma^{s-r} f_r$ for f'_r.

We agree to introduce a more general concept of a homotopy by defining

$$f_r \simeq f_s$$

whenever $f'_r \simeq f_s$. The same convention will be adopted for mappings

$$g_l: S^{p+l} \to E_l \wedge X^+ , \qquad g_m: S^{p+m} \to E_m \wedge X^+ .$$

Observe that $f_r \simeq f_s$ implies $\{f_r\} = \{f_s\}$ but not conversely. For $r = s$ we have $f_r \simeq f_s$ in the new sense if and only if $f_r \simeq f_s$ in the ordinary sense. Hence there is no danger of confusion. However this general homotopy is not necessarily transitive.

We have to deal with the function spaces:

$$F(L_r(X), E_{n-p+r}) = F_1 = F_1(X)$$
$$F(S^{p+s}, E_s \wedge X^+) \ = F_2 = F_2(X) ,$$

both equipped with the constant maps as basepoints.

All kinds of function spaces $F(A, B) = B^A$ are equipped with the k-topology. This can be achieved by eventually applying the functor k: *Top* $\to$ *k-spaces* (cf. [6], Ch. 2).

In this section we are working with so-called *disk-spaces* $X = \bigcup_{i=1}^{N} B_i$, where $B_i \subset \mathbb{R}^n$ is a convex, closed n-dimensional disk.

We need these disk spaces for the approximation of arbitrary compact spaces $A \subset R^n$. This enables us to impose some restrictions on the special character of these disks B_i: Take $1/k$ nets T_k, $T_1 \subset T_2 \subset \ldots$ $k \in \mathbb{N}$ in $\mathbb{R}^n$ and consider all closed $2/k$ balls around these $x_k \in T_k$. The set of all these balls is denoted by D_k. A disk space

$$X = \bigcup_{i=1}^{N} B_i$$

is now supposed to be the union of disks B_i which are finite intersections of disks in a fixed D_k.

The main goal of this section is the construction of a continuous based mapping

$$\psi = \psi_X: F_1(X) \to F_2(X) \tag{1}$$

which exhibits certain naturality properties. The crucial point is the construction of a universal functions $s = \alpha(r, p)$ (for fixed p also denoted by $\alpha(r)$ which depends only on r, p and not upon the given disk space X or the spectrum $\mathcal{E}$). Our $\alpha(r, p)$ has the form

$$s = \alpha(r, p) = n - p + r . \tag{2}$$

The required naturality properties of ψ are:

$\psi 1)$ Let $i: X \subset Y$ be an inclusion of disk spaces then there exists for each $f \in F_1(X)$ a homotopy

$$\omega_i = \omega: i_{\#}\psi_X(f) \simeq \psi_Y i^{\#}(f) \tag{3}$$

which continuously (with respect to the topology in $F_1(X)'$) depends upon f, where $i^{\#} = F_1(i^*)$, $i^*: L_r(Y) \subset L_r(X)$, $i_{\#} = F_2(i)$ are the induced mappings.

Let furthermore $X \xrightarrow{i} Y \xrightarrow{j} Z$ be two inclusions of disk spaces, then there exists a homotopy between homotopies (cf. § 1), continuously depending upon f

$$\xi: \omega_{ji} \simeq \omega_j \circ j_{\#}\omega_i . \tag{4}$$

$\psi 2)$ For any $f \in F_1(X)$ there exists a homotopy

$$\nu_f = \nu: \gamma_u \psi(f) \simeq f \tag{5}$$

which depends continuously (with respect to the topology in $F_1(X)'$) upon f. This homotopy is natural with respect to the inclusions of disk spaces in the sense that for the homotopies involved one has

$$(\nu_{i^{\#}f})^{-1}(i^{\#}\nu_f) = \gamma_u(\omega_i) . \tag{6}$$

Notice that $i^{\#}\gamma_u = \gamma_u i_{\#}$.

Now we can prove:

9.1. LEMMA. – There exists for $s = l = n - p + r$, any CW-spectrum $\mathcal{E}$ and for all disk spaces X consisting of one single disk $X = B$ a continuous

$$\psi: F_1(X) \to F_2(X)$$

which fulfils $\psi 1)$ (for inclusions $i: X \subset Y$, Y also a n-disk) and $\psi 2)$.

PROOF. – We construct homotopy equivalences

$$h_{1X} = h_1: S^{n+r} \to L_r(X)$$

$$h_{2X} = h_2: E_l \to E_l \wedge X^+$$

in the following way: Since we can assume all disk spaces lying in a large ball $K \subset \mathbb{R}^n$, we have a deformation retraction of $L_r(X)$ onto $S^{n+r} = \operatorname{bd} K \subset L_r(X)$. We denote by h_1 the inclusion $S^{n+r} \subset L_r(X)$.

Let $b \in B = X$ be a fixed point, then we have a well defined homotopy equivalence $E_l \approx E_l \wedge S^0 \subset E_l \wedge B^+$ which stems from the inclusion $S^0 = \{*, b\} \subset B^+$. This gives the required h_2.

For any inclusion $i: X \subset Y$ ($Y = B'$ again a disk) we have

$$i^* h_{1Y} = h_{1X} \tag{7}$$

but (because the two points $b \in B$ and $b' \in B' = Y$ might be different) a homotopy

$$(1 \wedge i^+) h_{2X} \simeq h_{2Y} \tag{8}$$

which is well-defined by the straight line, connecting b and b'.

For any $f \in F_1(X)$ we set

$$\psi(f) = h_2 f h_1 .$$

This assignment is obviously continuous and respects basepoints:

$$\psi(0_1) = 0_2 , \quad 0_i \in F_1(X) ,$$

the constant maps. – We deduce from (7) and (8) for an inclusion $i: X \subset Y$

$$i_\# \psi_X(f) = (1 \wedge i^+) h_{2X} f h_{1X} \simeq h_{2Y} f h_{1X}$$
$$\psi_Y i^\#(f) = h_{2Y} f i^* h_{1Y} = h_{2Y} f h_{1X} .$$

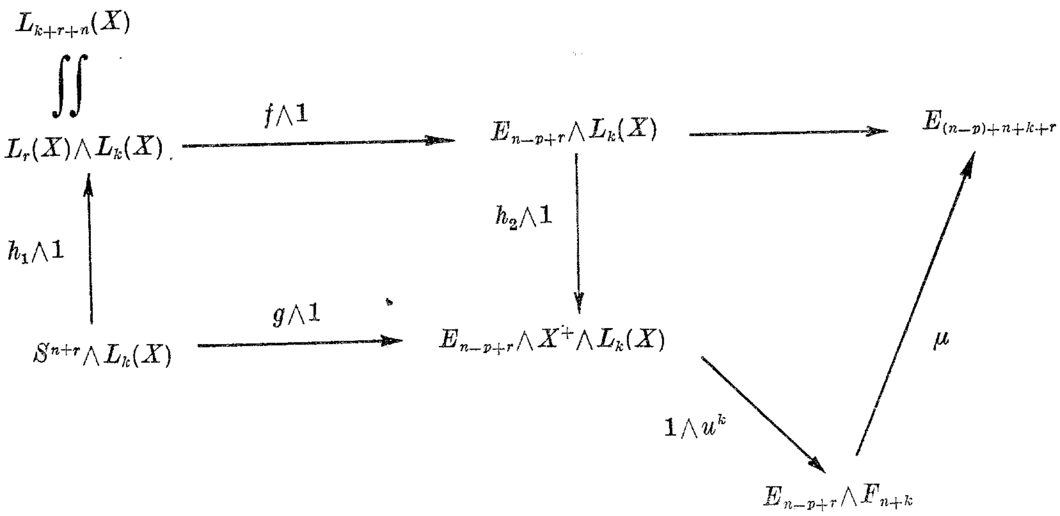
This provides us with a well-defined homotopy ω_i, which depends solely on the connecting line between b and b' but not on the individual f. Therefore ω_i depends continuously on f.

Let $X \xrightarrow{i} Y \xrightarrow{j} Z$ be two inclusions of convex n-disks, $Z = B'' \ni b''$, then the required homotopy between homotopies is induced by the triangle $\Delta(b, b', b'') \subset B''$ is an elementary way. The properties of ξ in $\psi 1)$ follow immediately.

In order to verify $\psi 2)$ we recall the definition of γ_u for compact polyhedra X:

$$\gamma_u(g) = \mu(1 \wedge u^k)(g \wedge L_k(X)) ,$$
$$g: S^{p+l} \to E_l \wedge X^+, \; u^k: X^+ \wedge L_k(X) \to F_{n+k} ,$$
$$\mu: E_{n-p+r} \wedge F_{n+k} \to E_{(n-p)+n+k+r} .$$

Since u^k stems from an $\mathcal{F}$-orientation ($\mathcal{E}$ is supposed to be an $\mathcal{F}$-module spectrum; later on, we will have to assume that $n + k \geqslant 2|p|$, cf. proof of proposition 9.3), we have homotopy commutativity in the following diagram (with $g = \psi(f)$):



This homotopy depends upon h_2.

So the proof of lemma 9.1. is complete.

Lemma 9.1. serves as a basis for an inductive process. Let to this end $X = A \cup B$, $A = \bigcup_{i=1}^{N-1} B_i$, $B = B_N$ be given and note that $D = A \cap B$ consists of less than N disks.

We will construct

$$\psi = \psi_X \colon F(L_r(X), E_{n-p+r}) \to F(S^{p+s}, E_s \wedge X^+)$$

such that $\psi 1)$, $\psi 2)$ hold, by assuming that we have already

$$\psi_Y \colon F(L_r(Y), E_{n-p+r}) \to F(S^{p+s}, E_s \wedge Y^+)$$

for $Y = A$, B and D and all $r \in \mathbb{N}$, $p \in \mathbb{Z}$ with $s = n - p + r$.

Incidentally we call a decomposition $X = A' \cup B'$ of disk spaces (in D_m) *admissible* whenever A' and B' are built up of disks lying in the same D_m.

Let $L_r(A)$, $L_r(B) \subset L_r(A \cap B)$ be the obvious inclusions, then we have the following identities resp. homotopy equivalences:

$$L_r(X) = L_r(A) \cap L_r(B)$$

$$L_{r-1}(D) = L_{r-1}(A) \cup L_{r-1}(B)$$

$$L_r(X) \overset{\simeq}{\Rightarrow} CL_{r-1}(A) \cup CL_{r-1}(B)\,.$$

While the first two relations are immediate, the third one follows by a simple cofiber argument. As a consequence we have mappings

$$d\colon L_{r-1}(A\cap B)\to L_r(X)$$

$$i_A\colon L_{r-1}(A)\to L_r(X)\,.$$

Because $i_A\simeq 0$, we have for any $f\in F_1(X)$ a homotopy $H_A\colon fi_A\simeq 0$ and because $\psi(fi_A)\simeq 0$ a mapping

$$G_A\colon D^{p+s}\to E_s\wedge A^+$$

with $s=\alpha(r-1,p-1)=n-p+r$ such that

$$G_A|S^{p+s-1}=G_A|\mathrm{bd}\,D^{p+s}=\psi(fi_A)\,.$$

In the same way we obtain a G_B. Let $\varrho_A\colon D\subset A$, $\varrho_B\colon D\subset B$ be the inclusions, then we have

$$d\varrho_A^*=i_A\,,\qquad d\varrho_B^*=i_B$$

and consequently, due to our inductive hypothesis, homotopies

$$(9)\qquad\begin{cases}\psi(fi_A)=\psi\big(\varrho_A^\#(fd)\big)\simeq(1\wedge\varrho_A^+)\,\psi(fd)\,.\\ \psi(fi_B)=\psi\big(\varrho_B^\#(fd)\big)\simeq(1\wedge\varrho_B^+)\,\psi(fd)\,.\end{cases}$$

Hence G_A, G_B and the connecting homotopies in (9) can be pasted together yielding a map

$$\psi(f)\colon S^{p+s}\to E_s\wedge(A\cup B)^+=E_s\wedge X^+\,.$$

Observe that this construction works for any spectrum $\mathcal{E}$.

We are now going to verify the different properties of ψ:

1) The assignment ψ is continuous and basepoint preserving.

PROOF. – The mappings G_A (resp. G_B) originate from the homotopies $i_A\simeq 0$ (resp. $i_B\simeq 0$) which are independent of f. Hence G_A, G_B depend continuously upon f. The connecting homotopies (9) stem from $\psi 1)$ by induction. They are therefore continuously depending on fi_A and fi_B, hence on f itself. – Let $f=0_1\in F_1(X)$, then G_A, G_B and the connecting homotopies are trivial. Therefore $\psi(f)$ is the constant map.

2) Let $i\colon X\subset Y$ be an inclusion of disk spaces, then $\psi 1)$ holds.

PROOF. – We are proving this in two steps

2.1.) Let $X = A \cup B = A' \cup B'$ be two admissible decompositions of X into diskspaces (consisting of less than N disks) then the two resulting construction $\psi(f)$, $\psi'(f)$ differ by a homotopy which continuously depends upon f.

PROOF. – We start with the case $A = A'$, $B' \supset B$. Let $j_B \colon B \subset B'$ be the inclusion, then $G_B(1 \wedge j_B^+)$ and $G_{B'}$ are according to the inductive hypothesis (applied to each « point » of the paths $G_B(1 \wedge j_B^+)$ and $G_{B'}$, separately) homotopic. Let $p \colon A \cap B \subset A \cap B' = D'$ be the inclusion, then by the same assumption, we have various homotopies

$$\eta_A \colon \psi(fi_A) \simeq (1 \wedge \varrho_A^+)\psi(fd)$$
$$\eta'_A \colon \psi(fi_A) \simeq (1 \wedge \varrho_A'^+)\psi(fd')$$
$$\zeta_A \colon (1 \wedge \varrho_A^+)\psi(fd) = (1 \wedge \varrho_A'^+)(1 \wedge p^+)\psi(fd) \simeq (1 \wedge \varrho_A'^+)\psi(fd')$$
$$\varrho'_A \colon D' \subset A, \ d' \colon L_{r-1}(D') \to L_r(X)$$

and a homotopy between homotopies

$$\xi_A \colon \zeta_A \eta_A \simeq \eta'_A ,$$

resp. for B:

$$\zeta_B \colon (1 \wedge \varrho_B^+)\psi(fd) \simeq (1 \wedge \varrho_{B'}^+)\psi(fd')$$
$$\eta_B \colon \psi(fi_B) \simeq (1 \wedge \varrho_B^+)\psi(fd)$$
$$\eta_{B'} \colon \psi(fi_{B'}) \simeq (1 \wedge \varrho_B'^+)\psi(fd')$$
$$\xi_B \colon \zeta_B \eta_B \simeq \eta_{B'} ,$$
$$\varrho_{B'} \colon D' \subset B' .$$

As a result we can combine all these homotopies obtaining a homotopy between $\psi(f)$ (which stems from G_A, η_A, η_B and G_B) and $\psi'(f)$ (which stems from G_A, η'_A, $\eta_{B'}$ and $G_{B'}$).

The general assertion 2.1) follows easily by considering successively series of decompositions of the kind

$$X = \bigcup_{i=1}^{N-1} B_i \cup B_N = \bigcup_{i=1}^{N-1} B_i \cup (B_{N-1} \cup B_N) = \ldots = \bigcup_{i \neq j} B_i \cup B_j$$

which gives access to any admissible decomposition $X = A' \cup B'$.

2.2) Let $i \colon X \subset Y$ be any inclusion (X and Y disk spaces of the form described at the beginning of this section) then there exist admissible decompositions $X =$

$= A \cup B$, $Y = A' \cup B'$ such that $A \subset A'$, $B \subset B'$. This is clear due to the general restrictions which we imposed on the nature of disk spaces.

Now we are ready to establish a proof of 2) in its general form: In the same way as in 21) we find «pointwise» homotopies between paths G_A, $G_{A'}$ resp. G_B and $G_{B'}$.

Since $D' = A' \cap B' \supset A \cap B = D$, we can proceed as before.

Let $X \xrightarrow{i} Y \xrightarrow{j} Z$, $X = A \cup B$, $Y = A' \cup B'$ and $Z = A'' \cup B''$, $A \subset A' \subset A''$, $B \subset B' \subset B''$ be given inclusions and admissible decompositions, then the existence of a homotopy between homotopies as required in $\psi 1$) follows because between the constituencies $G_A, G_{A'}, G_{A''}$, (resp. for B) one has by induction these homotopies between homotopies which can be composed to the required ξ.

The continuity with respect to f of the homotopy ω_i as well as the homotopy between homotopies follows as in 1) because all intermediate steps in the construction are continuous by induction.

3) There exists a homotopy between homotopies

$$\nu \colon \gamma_u \psi(f) \simeq f \tag{10}$$

which depends continuously on f and which behaves naturally in the sense of $\psi 2$).

PROOF. – Observe that the assignment $\gamma_u \colon F_2(X) \to F_1(X)$, now for $r = k + p + 1$ and suitably chosen k is continuous and basepoint preserving. We have the homotopy $H_A \colon f i_A \simeq 0$ (whose ψ-image is G_A) and by induction a homotopy $\gamma_u \psi(f i_A) \simeq f i_A$ which continuously depends on f. For the corresponding paths in the function spaces we have therefore

$$H_A \simeq \psi_u \psi H_A = \gamma_u G_A .$$

We obtain f by glueing together H_A and H_B, while G_A, G_B and the connecting homotopies (9) give us $\psi(f)$. This establishes (10). The continuity of ν follows again from the continuous dependence of all steps involved. Observe that a change of the decomposition results in a change of ν by the same homotopy which we discovered in 2.1).

Let $i \colon X \subset Y$ be an admissible inclusion, then we are considering the sequence of homotopies:

$$\begin{array}{ccc}
i^{\#} \gamma_u \psi f & \xrightarrow[\simeq]{i^{\#} \nu_f} & i^{\#} f \\
\Vert & & \uparrow \wr\, \nu_{i^{\#} f} \\
\gamma_u i_{\#} \psi(f) & \xrightarrow[\gamma_u \omega_i]{\simeq} & \gamma_u \psi i^{\#} f
\end{array}$$

which is commutative because we have this by induction for each «point» in H_A, H_B resp. $H_{A'}$, $H_{B'}$ and because we can assume that for ν as well as for ω_i we can take the same decompositions $A \cup B = X$, $Y = A' \cup B'$, $A \subset A'$, $B \subset B'$.

We summarize:

9.2. PROPOSITION. – Let $s = \alpha(r, p)$ be the function (2), then there exists a continuous function $\psi: F_1(X) \to F_2(X)$ satisfying $\psi 1)$, $\psi 2)$.

We need to know something more about the assignment ψ:

9.3. PROPOSITION. – For any $g \in F_2(X)$ there exists a homotopy

$$\varepsilon_g: \Sigma^{|p|}\psi\gamma_u g \simeq g \tag{11}$$

which continuously depends upon g.

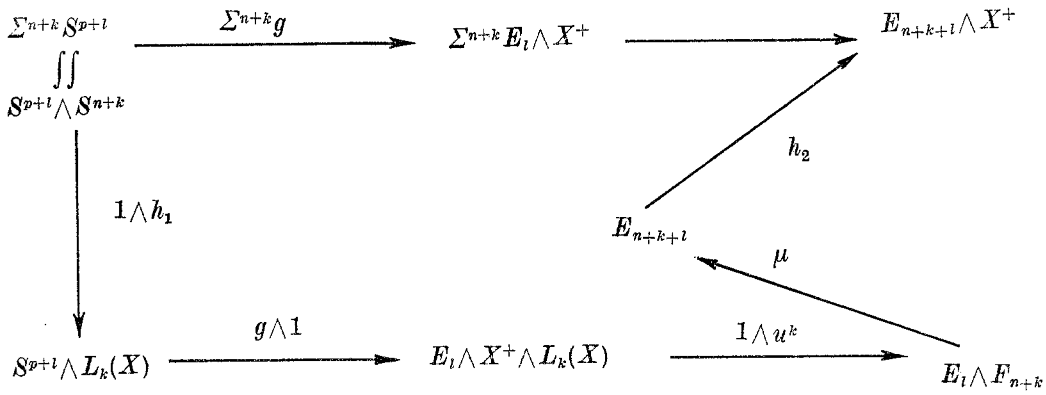
PROOF. – We will verify a slightly different assertion which obviously implies (11): There exists a homotopy

$$\varepsilon'_g: \Sigma^{p}\psi\gamma_u g \simeq g \tag{11'}$$

which continuously depends upon g. – Here Σ^p for $p < 0$ is interpreted as desuspension: $\Sigma^p f$ means the existence of a mapping f' together with a well-defined homotopy

$$\Sigma^{-p} f' \simeq f\,.$$

In order to prove (11′), we again proceed inductively: Let $X = B$ be a disk space, constisting of a single disk, then we have a homotopy commutative diagram



where h_1, h_2 stem from the proof of lemma 9.1. Without loss of generality we can assume from the beginning that

$$n + k \geqslant 2|p|\,.$$

The assertion

$$\psi\gamma_u g \simeq g$$

(which is evidently stronger than (11′) for $p \geqslant 0$) follows in the same way as 2) in lemma 9.1. Moreover we can obviously desuspend $\psi\gamma_u g$ at least $|p|$ times by assumption.

Now we assume that (11′) has already been proved for spaces X consisting of less than N disks. We proceed as in the proof of 9.2.: Let $g\colon S^{p+l} \to E_l \wedge X^+$ be continuous, then there are mappings

$$C_+ S^{p-1+l+1} = C_+ S^{p+l} \xrightarrow{G_A} E_l \wedge (C_+ X \cup C_- A) \simeq \Sigma E_1 \wedge A^+ \to E_{l+1} \wedge A^+$$
$$C_- S^{p-1+l+1} = C_- S^{p+l} \xrightarrow{G_B} E_l \wedge (C_- X \cup C_+ B) \simeq \Sigma E_l \wedge B^+ \to E_{l+1} \wedge B^+$$

where the $\pm$-notation is used to distinguish between two different copies of the associated cones. As a result we obtain Σg by glueing together G_A and G_B. By the inductive hypothesis we have (for each point in the paths G_A, G_B and therefore by continuity)

$$\Sigma^{p-1} \psi\gamma_u G_A \simeq G_A\,, \qquad \Sigma^{p-1} \psi\gamma_u G_B \simeq G_B$$

and finally

$$\Sigma^{p-1} \psi\gamma_u g \simeq \Sigma g\,.$$

But since we have $\psi\gamma_u \Sigma = \psi\Sigma\gamma_u = \Sigma\psi\gamma_u$ (the first identity follows immediately from the definition of γ_u, the second from the continuity of ψ) we conclude

$$(11'') \qquad \Sigma^p \psi\gamma_u g \simeq \Sigma g\,.$$

For $p \geqslant 0$, the suspension level of $\Sigma^p \psi\gamma_u g$ is evidently higher than that of Σg, so that (11″) follows. However for $p < 0$ we also conclude with

$$\psi\gamma_u g\colon S^{p+s} \to E_s \wedge X^+$$

that

$$s = n - p + r$$

and

$$r = k + p + l$$

hence

$$s = n + k + l\,.$$

Since $l \geqslant 0$ and $n + k \geqslant 2|p|$ we have again that the suspension level of $\Sigma^p \psi\gamma_u g$ is still higher than that of Σg, so that (11″) implies (11′). This completes the proof of (11′) and thereby of (11).

Now we are ready to settle the main problem of this section, namely the detection of an inverse to

$$\gamma_u\colon \bar{\mathcal{E}}_p(X) \to \mathcal{E}^q(L_0(X)), \quad p + q = n, \tag{12}$$

$X \subset S^n$ compact, $\mathcal{E}$ a CW-spectrum.

Let $X = \cap K_m$, $K_1 \supset K_2 \supset \dots$ be a decreasing of disk spaces with inclusions

$$i_m^{m+1}\colon K_{m+1} \subset K_m \qquad i_m\colon X \subset K_m .$$

We have $i_m^*\colon L_r(K_m) \subset L_r(X)$, $i_m^\#\colon F_1(X) \to F_1(K_m)$, $i_{m\#}\colon F_2(X) \to F_2(K_m)$.

To any continuous $f \in \mathcal{K}(L_r(X), E_{n-p+r})$ we set

$$f_m = i_m^* f \in \mathcal{K}(L_r(K_m), E_{n-p+r}) \quad \text{and} \quad g_m = \psi(f_m) .$$

According to $\psi 1)$ we have homotopies

$$\zeta_m\colon i_{m\#}^{m+1} g_{m+1} \simeq g_m . \tag{13}$$

Propositions A3, A7 in the appendix guarantee that we have obtained a $\bar{g} = \psi(f) \in \overline{\mathcal{K}}(S^{p+\alpha(r)}, E_{\alpha(r)} \overline{\wedge} X^+)$ (with $\alpha(r) = \alpha(r,p)$ for fixed p) by defining

$$\bar{g}(i_m) = g_m .$$

Furthermore $\psi 1)$, $\psi 2)$ ensures that lemma 6.1. can be applied to the effect that

$$\gamma_u\{\bar{g}\} = \gamma_u \psi\{f\} = \{f\} .$$

This settles the surjectivity of γ_u.

We come to the injectivity: Let to this end $\bar{g} \in \overline{\mathcal{K}}(S^{p+t}, E_t \overline{\wedge} X^+)$ be a fixed morphism, then we deduce from 9.3. the existence of a homotopy

$$\varepsilon_m\colon \Sigma^{|p|} \psi \gamma_u \bar{g}(i_m) \simeq \bar{g}(i_m) .$$

In other words we have a map

$$H_m\colon S^{p+\alpha(r)+t} \times I \to E_{\alpha(r)+t} \wedge K_m^+$$

for sufficiently large t such that $H_m | S^{\cdot\cdot} \times \{0\}$ $(H_m | S^{\cdot\cdot} \times \{1\})$ are suspensions of $\psi\gamma_u \bar{g}(i_m)$ (resp. of $\bar{g}(i_m)$); the corresponding suspension levels (which can be easily written down explicitely) are independent of m.

By applying $\psi\gamma_u$ to (13) and because of $\psi 1)$ we have homotopies

$$\zeta_m'\colon i_m^{m+1\#} \Sigma^{|p|} \psi\gamma_u \bar{g}(i_{m+1}) \simeq \Sigma^{|p|} \psi\gamma_u \bar{g}(i_m) .$$

On the other hand the homotopy (11) is continuous, thus we have for each $t \in I$ a homotopy between $\zeta'_m(t)$ and a sufficiently high suspension of $\zeta_m(t)$. Reinterpreting this situation furnishes a homotopy between $i_m^{m+1} H_{m+1}$ and H_m, hence altogether a homotopy $\bar{H} \in \overline{\mathcal{K}}(S^{p+\alpha(r)+t} \times I, E_{\alpha(r)+t} \overline{\wedge} X^+)$ such that $\bar{H}|S^{\cdot\cdot} \times \{0\}$ $(\bar{H}|S^{\cdot\cdot} \times \times \{1\})$ are suspensions of $\psi\gamma_u \bar{g}$ (resp. of $\bar{g}$).

The suspension degrees depend only on l but not on m or X. As a result we can deduce

9.4. LEMMA. – For any $\bar{g} \in \overline{\mathcal{K}}(S^{p+l}, E_l \overline{\wedge} X^+)$ we have in $\overline{\mathcal{K}}$ a homotopy

$$\Sigma^{|p|} \psi \gamma_u \bar{g} \simeq \bar{g} . \tag{14}$$

This completes the proof of the assertion that γ_u in (12) is injective, because it ensures that ψ is stably an inverse to γ_u.

Moreover we have established:

9.5. LEMMA. – Let $\lambda\colon \mathcal{E} \to \tilde{\mathcal{E}}$ be a $\mathcal{F}$-module-homomorphism of spectra, then we have induced continuous mappings $F_i(\lambda)$ for the related function spaces (for fixed X) turning $F_i(X) = F_i(X;\ )$ into a functor from the category of CW-$\mathcal{F}$-module spectra into the category of based topological spaces. The assignment $\psi = \psi_X$ (for fixed X!) reveals itself in this situation as a natural tranformation of functors.

PROOF. – The assertion follows immediately from the construction of ψ.

10. – Appendix.

In this appendix we are 1) proving propositions A1 and A2 2) recollecting the necessary material on 2-homotopies, 2-categories and 2-functors in order to be able 3) to perform the explicit construction of a shape mapping $\bar{f} \in \overline{\mathcal{K}}(X, Y)$ (resp. $\bar{f} \in \overline{\mathcal{K}}(S, X \overline{\wedge} Z)$) and 4) we provide proofs of proposition A9, A10, which ensure that every inclusion of compact metric spaces is a cofibration in $\overline{\mathcal{K}}$ and that this property of the inclusion $i\colon A \subset B$ is equivalent to the fact that $A \times I \cup B \times 0$ is, in $\overline{\mathcal{K}}$, a strong deformation retract of $B \times I$. The last fact is not needed in this paper but well-known within the framework of ordinary homotopy theory ([6], 4.1.7. *a*), p. 155).

10.1. *Proofs of propositions A*1, *A*2.

The proof of the following proposition is due to J. DUGUNDJI. It represents the essential part of the proof of lemma 5.2. (« Sitnikov's lemma »):

A1 PROPOSITION. – Let M be any metric space, $X \subset M$ be any subspace and Y

an ANE (cf. § 1); let $f\colon X \to Y$ be continuous, then there exists a $g \simeq f$ which extends over a neighbourhood $U \supset X$ in M.

PROOF. – Consider the subspace $Q = X\times 0 \cup M\times[0,1] \subset M\times I$ and identify $X\times 0$ with X. Because $X\times 0 \subset Q$ is a closed subset, there exists an extension $F\colon W \to Y$ of f over a neighbourhood W of $X\times 0 \subset Q$. We find an open set $V \subset M\times I$ such that $W = V \cap Q$ and for each $x \in X$ a neighbourhod U_x of x in M such that $U_x\times[0, t] \subset V$ for suitable $t_x > 0$. Assume that $\{U_x\}$ is a locally finite family of open sets and define $U = \cup U_x$. Because M is paracompact we find a real valued continuous function $\varphi\colon U \to \mathbb{R}$ with $\varphi(u) > 0$ for all $u \in U$ and $\varphi(u) \leqslant \sup\{t_x | u \in U_x\}$. In W we have the space $X' = \{(x, \varphi(x) | x \in X)\}$ homeomorphic to X. The mapping $\phi(x, t) = F(x, \varphi(x)t)$ provides us with a homotopy $\phi\colon f \simeq F|X' = g\colon X \to Y$. However $F|\{(u, \varphi(u))|u \in U\} = g'\colon U \to Y$ is the desired extension (up to homotopy) of g over U.

This completes the proof of A1.

To accomplish a proof of lemma 5.2. we need a second assertion which easily follows from A1:

A2 PROPOSITION. – Let M be a metric space such that the cone $CM \supset CX$ is again metrizable (which is true for example if M is a compact manifold), let U be a neighbourhood of $X \subset M$ and let $F\colon U \to Y$ be continuous, such that $F|X = f \simeq 0$. Then we find a neighbourhood V of X in M, $V \subset U$ such that $F|V \simeq 0$.

PROOF. – We have a map $\phi\colon CX \to Y$ reflecting the fact that $f \simeq 0$. This together with F provides us with a map $\psi\colon U\times 0 \cup CX \to Y$. We have $D = U\times \times 0 \cup CX \subset CM$ and can therefore apply A1 to the effect that we obtain an extension $\Lambda\colon W \to Y$ of a $\psi' \simeq \psi$ over an open neighbourhood V of X in M such that $CV \subset W$. Take $x \in X$ and find a neighbourhood V_x of x in M such that $CV_x \subset W$. This V_x exists because the unit interval is compact. Now $V = \cup V_x$ has the desired property. This completes the proof of proposition A2.

10.2. *Miscellaneous results concerning the construction of a strong shape category.*

A shape mapping $\bar{f} \in \overline{\mathfrak{K}}(X, Y)$ is a 2-functor $\bar{f}\colon \mathfrak{T}_Y \to \mathfrak{T}_X$ having special properties. The explicit construction of such a map is rather involved unless one provides a manageable subcategory $\mathfrak{T}_Y' \subset \mathfrak{T}_Y$ having the property that every suitable functor $T\colon \mathfrak{T}_Y' \to \mathfrak{T}_X$ admits a natural extension to such a $\bar{f}$.

In order to settle this (independently of [5] in particular § 4) we take advantage to say a little more about homotopies between homotopies (so-called 2-homotopies) about 2-categories and finally to make precise what we understand by a 2-functor. Although these concepts are treated by many different authors in different ways (e.g. CH. EHRESMANN, J. BENABOU etc.) it turns out to be more convenient to give the necessary definitions explicitely:

10.2.1. 2-*Homotopies*. – A 2-homotopy between two 1-homotopies $\xi\colon \omega_0 \simeq \omega_m$ is an equivalence class of mappings $A\colon X\times I_n\times I_m \to Y$ having the properties

$$\begin{aligned} A(x, t_1, 0) &= \omega_0(x, t_1) \\ A(x, t_1, m) &= \omega_m(x, t_1) \end{aligned} \qquad t_1 \in I_n$$

$$\begin{aligned} A(x, 0, t_2) &= f_0(x) \\ A(x, n, t_2) &= f_n(x) \end{aligned} \qquad t_2 \in I_m\,.$$

Here, by an abuse of notation, we write ω_0, ω_m for suitable representatives of the corresponding classes $\omega_0, \omega_m\colon f_0 \simeq f_n$.

The equivalence relation between these mapping A is generated by the following relations:

1) We define $B\sim A$, $B\colon X\times I_{n-1}\times I_m \to Y$ whenever $A|X\times[n-1, n]\times I_m$ is 2-stationary (i.e. $A(x, t_1, t_2) = f_n(x)$, $n-1 \leqslant t_1 \leqslant n$, $t_2 \in I_m$) and $A|X\times I_{n-1}\times I_m = B$.

This condition allows us to define 2-homotopies between 1-homotopies of eventually different lengths.

Now let $S\colon X\times I_n\times I_m\to Y$ resp. $T\colon X\times I_n\times I_m \to Y$ be two mappings such that

$$S|X\times I_{k-1}\times I_m \cup X\times[k+1, n]\times I_m\,, \qquad 0 \leqslant k < n$$

is 2-stationary (resp. T ... is 2-stationary) while

$$S|X\times[k-1, k+1]\times I_m$$

(resp. $T|$...) maps as indicated in fig. 1 (resp. fig. 2).

Then we set $S\sim 1\colon \omega\simeq\omega$ ($T\sim 1\colon \omega\simeq\omega$) where we denote by 1 as usual the identical 2-homotopy represented by

$$E\colon X\times I_n\times I_0 \to Y\,, \qquad I_0 = [0, 0]\,.$$

Given a homotopy $\omega = \varepsilon_1 \dots \varepsilon_k$ with elementary or stationary ε_i, then 2-homotopies of the kind T allow us to go over to an equivalent and simultaneously 2-homotopic $\omega' = \varepsilon_1' \dots \varepsilon_k'$ (again of length k) without non-stationary parts of the kind $\varepsilon_i' = \varepsilon_{i+1}'^{-1}$.

In a next step we can apply homotopies of the kind S to the effect that all possible stationary homotopies ε_i' somewhere in the middle or at the end are transported to the first place. Finally, the given ω is transformed into a homotopy $\omega'' = \varepsilon_1'' \dots \varepsilon_k''$ of length k such that $\varepsilon_1'' = \dots = \varepsilon_j''$ are stationary and that $\varepsilon_{j+1}'' \dots \varepsilon_k''$ forms a reduced word in the sense of lemma 1.1.

This allows us to compose two 2-homotopies $\xi\colon \omega_0 \simeq \omega_1$, $\zeta\colon \omega_1' \simeq \omega_2$ with $\omega_1 = \varepsilon_1 \dots \varepsilon_k$, $\omega_1' = \varepsilon_1' \dots \varepsilon_k'$ being equivalent: According to the just given argument

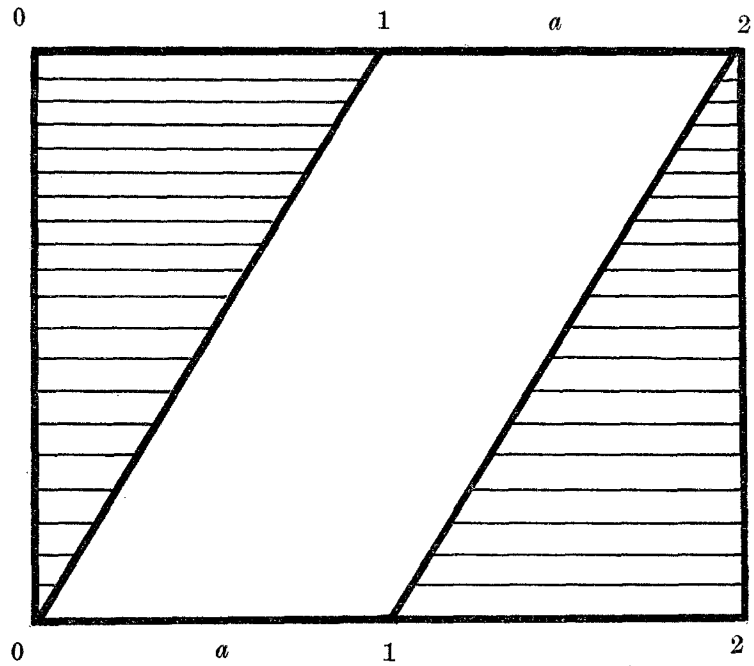


Fig. 1.

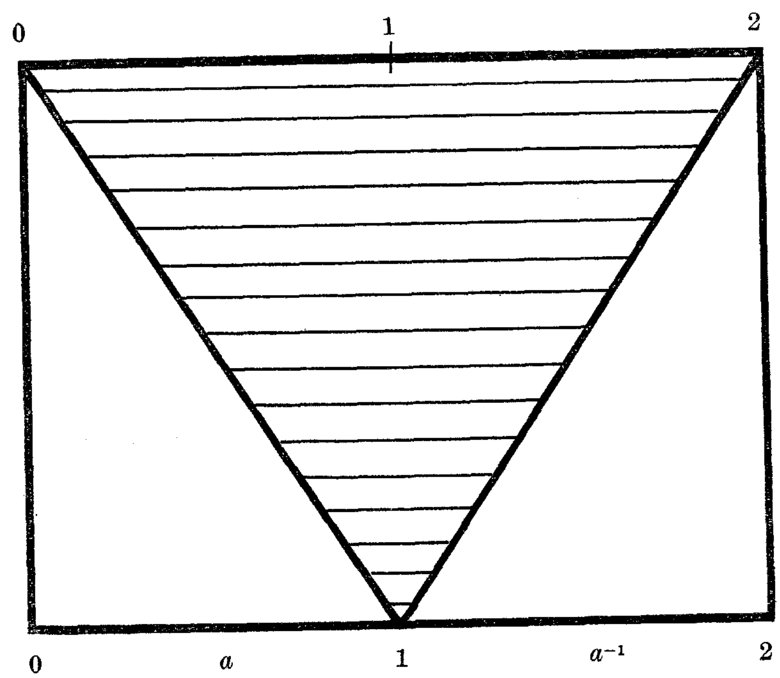


Fig. 2.

we find a 2-homotopy $\pi\colon \omega_1 \simeq \omega_1'$ which is a product of 2-homotopies of the form S and T so that one is enabled to define $\zeta\circ\pi\circ\xi\colon \omega_0 \simeq \omega_2$ (with $\circ$-product defined in an obvious way).

The equivalence 1) finally allows us to drop the assumption that ω_1 and ω_1' are given by words of equal length k, because this can be enforced by starting with sufficiently many stationary homotopies.

Finally we define, analogously to the case of 1-homotopies

3) $(A\colon X\times I_n \times I_m \to Y) \sim 1$ whenever A is stationary in the third variable and

4) for any $A\colon X\times I_n\times I_m \to Y$

$$A\circ A^{-1} \sim 1$$

with $A^{-1}(x, t_1, t_2) = A(x, t_1, m - t_2)$.

So a 2-homotopy is an equivalence class of mappings $A\colon X\times I_n\times I_m \to Y$ with an equivalence relation which is defined by 1) - 4).

By an abuse of notation we will, analogous to the case of 1-homotopies not distinguish between the class ξ and a representing map which we also denote by ξ.

The proof of the following facts are omitted:

α) $\mathcal{K}(f, f')$ $\big(f, f' \in \mathcal{K}(X, Y)$ fixed$\big)$ is a category having classes of 2-homotopies as morphisms such that a homotopy

$$\nu\colon f_1 \simeq f \ \ (\mu\colon f' \simeq f_1') \qquad \text{operates as a functor}$$

$$\nu^*\colon \mathcal{K}(f, f') \to \mathcal{K}(f_1, f') \ \big(\mu_*\colon \mathcal{K}(f, f') \to \mathcal{K}(f, f_1')\big)\,.$$

Moreover for $\nu_1\colon f_2 \simeq f_1$, $\nu_2\colon f_1 \simeq f$ one has

$$(\nu_2\nu_1)^* = \nu_1^*\nu_2^* \text{ resp. for } \mu\,.$$

β) Any $a \in \mathcal{K}(X', X)$ $\big($resp. $b \in \mathcal{K}(Y, Y')\big)$ induces a functor

$$a^*\colon \mathcal{K}(X, Y) \to \mathcal{K}(X', Y) \ \big(\text{resp. } b_*\colon \mathcal{K}(X, Y) \to \mathcal{K}(X, Y')\big)\,.$$

γ) Let $\omega\colon f_0 \simeq f_1$ be any homotopy, $\omega\colon X\times I_n \to Y$, $\omega'\colon X\times I_n \to Y$ a second homotopy such that $\omega'|X\times[0, 1]$ behaves like ω (after a linear contraction of I_n onto I) and where $\omega'|X\times[1, n]$ is stationary. Then there exists an elementary 2-homotopy (i.e. one with $m - 1$) $\xi\colon \omega \simeq \omega'$.

Since we are doing 3-*stage strong shape theory*, we are not dealing with these individual 2-homotopies itself but with (3-)homotopy classes of 2-homotopies: Two mappings $A_0, A_1\colon X\times I_n\times I_m \to Y$ (representing 2-homotopies ξ_0, ξ_1) are defined to belong to the same homotopy class ($A_0 \simeq A_1$) whenever there exists a continuous family A_t, $0 \leqslant t \leqslant 1$ of mappings, all representing 2-homotopies ξ_t between the *same*

1-homotopies $\xi_t\colon \omega_0 \simeq \omega_1$. So we start with 2-homotopies as defined above, being equivalence classes of mappings A, B, S, T, and allow these mappings to vary within its 3-homotopy classes.

One could of course define a concept of a 3-homotopy (in complete analogy to that of a 2-homotopy by using mappings $X \times I_n \times I_m \times X_k \to Y$ satisfying certain conditions on the boundary) and call A_0 3-homotopic to A_1 ($A_0 \simeq A_1$) whenever there exists such a 3-homotopy between A_0, A_1. However, since we are only interested in the related 3-*homotopy classes* rather than in the 3-homotopies itself, we can avoid this concept.

10.2.2. 2-*Categories and* 2-*functors*. – The only 2-*categories* which appear in this paper are 1) categories of topological spaces with homotopies as 2-morphisms and 2) the categories $\mathcal{T}_Y$ resp. $\mathcal{T}_{Y_1} \overline{\wedge} \ldots \overline{\wedge} \mathcal{T}_{Y_n}$ (cf. § 1).

They have the following properties which in turn can be used as a definition of an abstract 2-category $\mathcal{K}$: We assume $\mathcal{K}$ to be a category such that all $\mathcal{K}(X, Y)$, $X, Y \in \operatorname{Ob} \mathcal{K}$ are again categories (whose morphisms are called 2-morphisms). All $f \in \mathcal{K}(X', X)$ $\big(g \in \mathcal{K}(Y, Y')\big)$ induce functors $f^*\colon \mathcal{K}(X, Y) \to \mathcal{K}(X', Y)$ $\big($resp. $g_*\colon \mathcal{K}(X, Y) \to \mathcal{K}(X, Y')\big)$, in such a way that 1) $(f_2 f_1)^* = f_1^* f_2^*$ $\big((g_2 g_1)_* = g_{2*} g_{1*}\big)$ whenever the compositions are defined; 2) the identities 1_X induce identity functors and 3) the diagrams

$$\begin{array}{ccc} \mathcal{K}(X, Y) & \xrightarrow{g_*} & \mathcal{K}(X, Y') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{K}(X', Y) & \xrightarrow[g_*]{} & \mathcal{K}(X', Y') \end{array}$$

are all commutative. Moreover we have that all 2 morphisms (coming from 1- or 2-homotopies) are always isomorphisms.

In [5] we did not explicitely require 3) in the definition of a 2-category because we were able to get along with the weaker concept.

Unlike the situation with 2-categories we need abstract 2-*functors* (they are defining our shape morphisms).

So we include the explicit definition:

Let $\mathcal{K}, \mathfrak{L}$ be two 2-categories, then a 2-*functor* $T\colon \mathcal{K} \to \mathfrak{L}$ is

1) an assignment $T\colon \operatorname{Ob} \mathcal{K} \to \operatorname{Ob} \mathfrak{L}$ $\big(X \to T(X)\big)$;

2) for fixed $X, Y \in \mathcal{K}$ an assignment $\mathcal{K}(X, Y) \to \mathfrak{L}\big(T(X), T(Y)\big)$;

3) for fixed $f_1, f_2 \in \mathcal{K}(X, Y)$ an assignment

$$\mathcal{K}(X, Y)(f_1, f_2) = \mathcal{K}(f_1, f_2) \to \mathfrak{L}\big(T(X), T(Y)\big)(Tf_1, Tf_2) = \mathfrak{L}(Tf_1, Tf_2)$$

such that the following conditions hold:

a) We assume

$$T(1_X) = 1_{T(X)} \quad \text{for all } X \in \mathcal{K} .$$

b) Moreover for $f_1 \in \mathcal{K}(X, Y)$, $f_2 \in \mathcal{K}(Y, Z)$ there is given a 2-morphism

$$\omega\colon T(f_2 f_1) \to T(f_2)\, T(f_1) ,$$

satisfying the following compatibility conditions:

Let $f_i \in \mathcal{K}$ be morphisms, $i = 1, 2, 3$, such that $f_1(f_2 f_3)$ (hence also $(f_1 f_2) f_3$) are defined, then we have a commutative diagram

$$\begin{array}{ccc} T\big((f_1 f_2) f_3\big) & \xrightarrow{\ \omega\ } & T(f_1 f_2)\, T(f_3) \\ \omega \big\downarrow & & \big\downarrow \omega T(f_3) \\ T(f_1)\, T(f_2 f_3) & \xrightarrow[\omega T(f_1)]{} & T(f_1)\, T(f_2)\, T(f_3) \end{array}$$

with the corresponding 2-morphisms ω.

c) For fixed X, Y the assignment 2) is an ordinary functor, such that for $f \in \mathcal{K}(X', X)$, $g \in \mathcal{K}(Y, Y')$ the 2-morphisms ω in *a*) induce natural transformations ω^f, ω_g fitting into the diagrams

$$\begin{array}{ccc} \mathcal{K}(X, Y) & \longrightarrow & \mathfrak{L}\big(T(X), T(Y)\big) \\ \big\downarrow f^* & & Tf^* \big\downarrow \\ \mathcal{K}(X', Y) & \longrightarrow & \mathfrak{L}\big(T(X'), T(Y)\big) \end{array} \qquad \begin{array}{ccc} \mathcal{K}(X, Y) & \longrightarrow & \mathfrak{L}\big(T(X), T(Y)\big) \\ g_* \big\downarrow & & Tg_* \big\downarrow \\ \mathcal{K}(X, Y') & \longrightarrow & \mathfrak{L}\big(T(X), T(Y')\big) \end{array}$$

In other words: For a $\alpha \in \mathcal{K}(X, Y)(f_1, f_2)$ we have

$$(Tf)^*\, T(\alpha)\, \omega^f = \omega^f\, T\big(f^*(\alpha)\big)$$

resp.

$$(Tg)_*\, T(\alpha)\, \omega_g = \omega_g\, T\big(g_*(\alpha)\big) .$$

REMARKS. – 1) in the applications the 2-morphisms of the category $\mathfrak{T}_Y$ (cf. § 1) are of the form (ν, ξ) where ξ is a 2-homotopy *up to homotopy* in the sense mentioned at the end of 10.2.1. (i.e. two 2-homotopies ξ_0, ξ_1 are homotopic whenever there exists a continuous family ξ_t of 2-homotopies, all between the same, fixed 1-homotopies).

2) We have repeated the definition of a 2-functor in full detail, because in [5], p. 28 1° we required $T\colon \mathcal{K} \to \mathfrak{L}$ to be an ordinary functor rather than a functor *up*

to 2-morphisms (which is the kind of definition needed for the purpose of strong shape theory). For a category which is ordered (like for example $\mathfrak{T}'_Y$ in [5], p. 20) we can always achieve strict functoriality (because we can confine ourselves to the definition of $T(f)$ for *indecomposable* morphisms).

In the general case however the usage of the stronger notion of a 2-functor (requiring that all 2-morphisms in a) are identities) is not adequate for strong shape theory and leads into trouble.

3) The present definition of a strong shape category could be refined by working with 3-, 4-, ..., n-categories and functors instead of 2-categories and 2-functors. In general this leads to a different kind of strong shape theory. However there is some evidence that for compact metric spaces the n-shape categories, $n \geqslant 3$, are all equivalent.

4) In the applications all 2-morphisms are always isomorphisms, so we do not have to take care about the direction of the ω's in a).

10.2.3. *Explicit construction of shape morphisms.* – Let Y be any metric space then we establish a subcategory $\mathfrak{T}''_Y \subset \mathfrak{T}_Y$ in the following way. It is well-known that every metric space Y can be embedded in an ANR M as a closed subspace (see [8], 5.2., p. 21).

The objects of $\mathfrak{T}''_Y$ are inclusions $i\colon Y \subset C(\subset M)$, where C denotes an open neighbourhood (hence an ANR). The 1-morphisms are of the form $(r, 1)\colon i_1 \to i_2$ where $i_k\colon Y \subset C_k$, $r\colon C_1 \subset C_2$ are inclusions. The 2-morphisms are identities.

We have the following assertion:

A3 PROPOSITION. – Let $T\colon \mathfrak{T}''_Y \to \mathfrak{T}_X$ ($X \in \mathcal{K}$ arbitrary) be an ordinary functor, having the following properties:

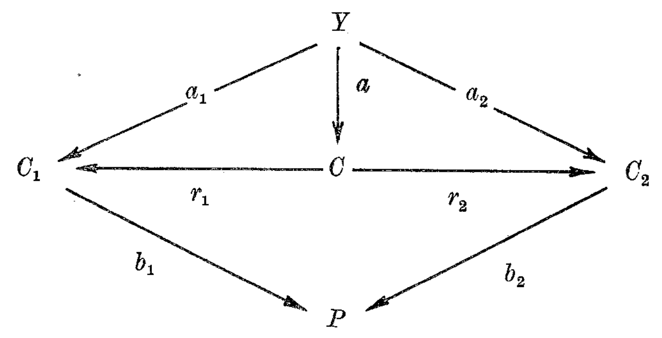
$T\,1)$ $g \in \mathfrak{T}''_Y$, $g\colon Y \to P \Rightarrow T(g)\colon X \to P$;

$T\,2)$ $(r, 1)\colon g_1 \to g_2 \Rightarrow T(r, 1) = (r, \omega)$ for suitable ω. Then there exists a $\bar{f} \in \overline{\mathcal{K}}(X, Y)$ such that $\bar{f} | \mathfrak{T}''_Y = T$.

The main tool for proving this is embodied in the following

A4 LEMMA. – Let $\omega\colon b_1 a_1 \simeq b_2 a_2$, $a_i \in \mathfrak{T}''_Y$ be given then there exists a diagram

(1)



$a \in \mathfrak{S}''_Y$, $(r_k, 1) \in \mathfrak{S}''_Y(a, a_k)$ with commutative triangles and a homotopy

$$\omega': b_1 r_1 \simeq b_2 r_2$$

such that

$$\omega' a = \omega .$$

PROOF. – Assume that ω is an elementary homotopy (see § 1), then we find an extension F of $\omega: Y \times I \to P$ over a $C \times I$, $C \subset Y$ open in M.

This follows because $Y \times I$ is a closed subset of the ANR $M \times I$. By eventually shrinking this C, we can assume that $C \subset C_1 \cap C_2$. As a result we obtain $\omega': C \times \times I \to P$ such that $\omega = \omega'(a \times 1)$ (here the homotopy is considered as a continuous mapping!).

$$\omega' i_0 = b_1 r_1 , \qquad \omega' = b_2 r_2 .$$

This completes the proof A4 for elementary homotopies. The general case follows similarly.

As an immediate consequence we conclude:

α) Let $g: Y \to P \in \mathfrak{S}$ be continuous and $g = b_1 a_1 = b_2 a_2$ any two factorizations with $a_k \in \mathfrak{S}''_Y$, then there exists a diagram (1) and a homotopy ω' (which is not necessarily the identity).

β) Let $\omega'': b_1 r_1 \simeq b_2 r_2$ be a second elementary homotopy fitting into (1), then there exists a further factorization $a = pa'$, $a' \in \mathfrak{S}''_Y$ and a homotopy class of 2-homotopies $[\xi]: \omega' p \simeq \omega'' p$ such that $\xi a'$ is the identity.

This follows by the same argument as α), after having replaced $Y \times I$ by $Y \times I \times \times I \cup C \times 0 \times I \cup C \times 1 \times I \subset M \times I \times I$. The second and the third summand enter, to make sure that ξ is really a 2-homotopy (i.e. stationary in the third variable s for $t = 0, 1$, cf. § 1).

There is of course an analogous assertion available for higher homotopies.

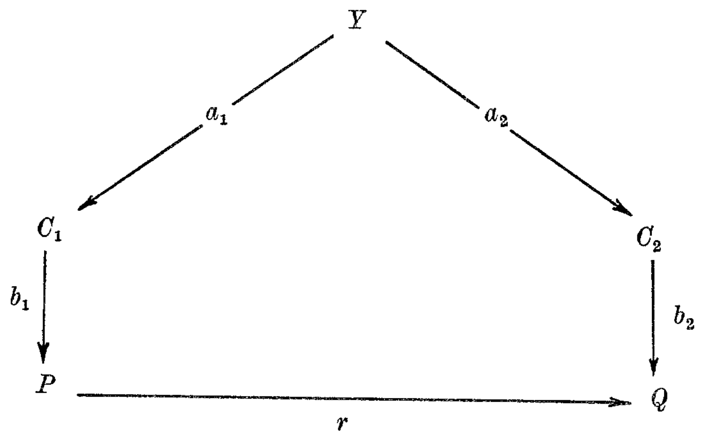
PROOF OF A3. – Let $T: \mathfrak{S}''_Y \to \mathfrak{S}_X$ be any functor such that $T\,1)$, $T\,2)$ holds. Then we define $\bar{f}: \mathfrak{S}_Y \to \mathfrak{S}_X$ on the objects:

Let $g \in \mathfrak{S}_Y$ be any object. We choose a fixed decomposition $g = ba$, $a \in \mathfrak{S}''_X$ (which exists without being uniquely determined) and define

$$\bar{f}(g) = gT(a) .$$

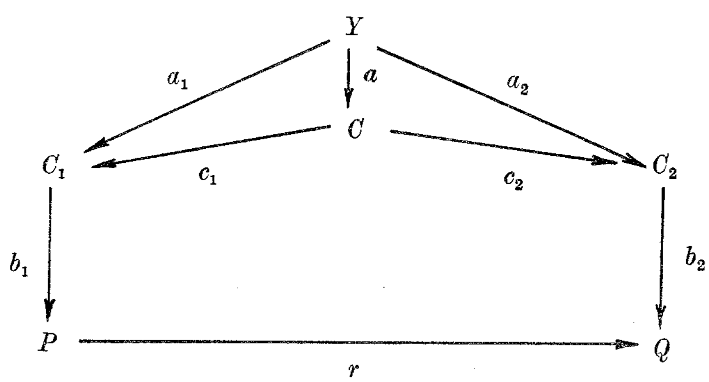
Let $(r, \omega): g_1 \to g_2$ be any morphisms in $\mathfrak{S}_Y$ (assuming for the moment that ω

is elementary), $g_1 = b_1 a_1$, $g_2 = b_2 a_2$, then we apply lemma A4 to the diagram



which enables us to make a choice of a homotopy $\mu\colon r b_1 c_1 \simeq b_2 c_2$ in

(2)



$a \in \mathfrak{T}''_Y$, $(c_i, 1) \in \mathfrak{T}''_Y(a, a_i)$, such that $\mu a = \omega$. Now we can apply the functor T to the upper triangles to the effect that we obtain a (in general non-elementary!) homotopy

$$\omega'\colon r\bar{f}(g_1) = r b_1 T(a_1) \simeq r b_1 c_1 T(a) \simeq b_2 c_2 T(a) \simeq b_2\, T(a_2) = \bar{f}(g_2)\,.$$

This ω' has the following explicit form:

$$\omega' = (b_2\gamma_2)\circ\mu T(a)\circ(b_1\gamma_1)$$

where we set

$$T(c_j, 1) = (c_j, \gamma_j)\,.$$

For non-elementary homotopies we proceed analogously.

As a result we define

$$\bar{f}(r, \omega) = (r, \omega') .$$

Observe that according to β), with a different choice of μ in (2) we would end up with a new (r, ω') which is connected to the given one by a 2-morphism $(1, \xi)$ in $\mathfrak{T}_Y$. Our definition of ω' can be arranged such that the relations 11), 12) in § 1 for homotopies are respected.

In particular we define $\bar{f}(1_Y, 1)$: $\bar{f}(g) = \bar{f}(g)$ by $(1_X, 1)$. All this can be used to establish $\bar{f}$ on the 1-morphisms as a 2-functor.

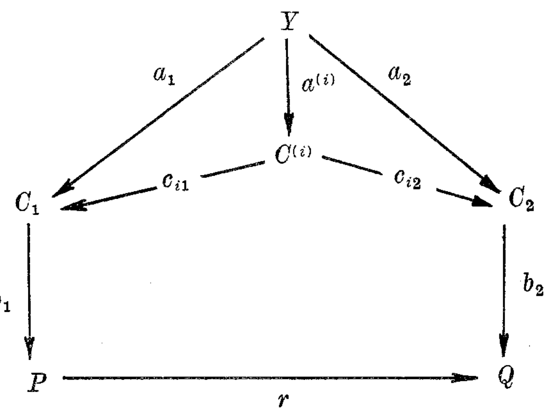
We still have to construct $\bar{f}(\nu, \xi) = (\nu, \xi')$ for a 2-morphism (ν, ξ): $(r_1, \omega_1) \approx \approx (r_2, \omega_2)$ in $\mathfrak{T}_Y$.

In a first step this will be accomplished for elementary 2-homotopies ξ:

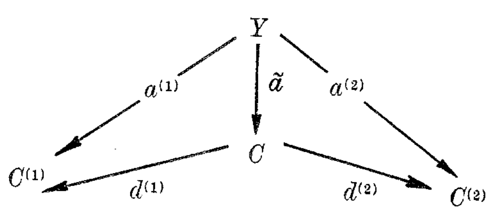
We transform the identity 2-morphism into the identity.

Let $(\nu, \xi) \neq (1, 1)$ be given. To ω_i: $r_0 b_1 a_1 \simeq b_2 a_2$ we have the chosen homotopies μ_i: $b_i c_{i1} \simeq b_2 c_{i2}$ fitting into the diagram

(3)



$i = 1, 2$ $\mu_i a^{(i)} = \omega_i$. There exists a common refinement of the $a^{(i)}$, i.e. a diagram



$(d^{(i)}, 1) \in \mathfrak{T}''_Y(\tilde{a}, a^{(i)})$, $\tilde{a} \in \mathfrak{T}''_Y$. Now we apply A4 to $Y \times I_n \times I \cup C \times 0 \times I \cup C \times \{n\} \times \times I \subset M \times I_n \times I$ (for the same reason as in β)) obtaining an elementary 2-homotopy

$$\eta: \mu_2 d^{(2)} e \circ \nu(b_1 c_{11} d^{(1)} e) \simeq \mu_1 d^{(1)} e$$

where $\tilde{a} = ea'$, $a' \in \mathfrak{T}''_Y$, $(e, 1) \in \mathfrak{T}''_Y(a', \tilde{a})$ are suitably chosen, such that $\eta a' = \xi$. Here ξ is a 2-homotopy

$$\xi\colon \omega_2 \circ \nu(b_1 a_1) = \mu_2 a^{(2)} \circ \nu(b_1 a_1) = \big(\mu_2 d^{(2)} e \circ \nu(b_1 c_{11} d^{(1)} e)\big) a' \simeq \omega_1 = \mu_1 a^{(1)} = \mu_1 d^{(1)} e a' .$$

This enables us to define the required ξ' by

$$\xi' = \eta T(a') .$$

It can be easily checked that ξ' is a 2-homotopy

$$\xi'\colon \omega_2' \circ \nu \bar{f}(b_1 a_1) \simeq \omega_1'$$

where we set $\bar{f}(r_i, \omega_i) = (r_i, \omega_i')$. This can be done for each (ν, ξ) separately in such a way that $\bar{f}(\nu, \xi)^{-1} = \bar{f}(\nu^{-1}, \xi^{-1})$. For non-elementary homotopies we establish ξ' by composing the elementary factors. Moreover β) ensures that the homotopy class of ξ' is independent of the choices involved. This can be easily verified.

In the present case $\bar{f}$ is automatically becoming functorial, because every morphism can be represented as a composition of indecomposable ones (which was *not* true in the case of 1-morphisms!).

This completes the construction of $\bar{f}$ which is easily recognized as a 2-functor, fulfilling the requirements on a strong shape morphism. The missing details are immediate and, as well as the fact that $\bar{f}|\mathfrak{T}''_Y = T$ left to the reader.

Let Y be a compact metric space, then we have a still simpler category $\mathfrak{T}'_Y \subset \mathfrak{T}_Y$ available ([5], § 4):

Take any embedding of Y into a Hilbert cube Q. Then we have a sequence of compact ANRs C_n, $C_1 \supset C_2 \supset \ldots$, such that $\cap\, C_n = Y$. The objects of $\mathfrak{T}'_Y$ are again the inclusions $i_n\colon Y \subset C_n$ and the 2-morphisms are of the form $(r_p^m\colon C_m \subset C_n, 1)$, $m \geqslant n$. Every morphism appears in a unique way as the product of indecomposable ones. Therefore it is sometimes easier to describe a specific functor $T\colon \mathfrak{T}'_Y \to \mathfrak{T}_X$.

As a corollary to A3 we have:

A5 PROPOSITION. – Assertion A3 holds for compacta with $\mathfrak{T}'_Y$ replacing $\mathfrak{T}''_Y$.

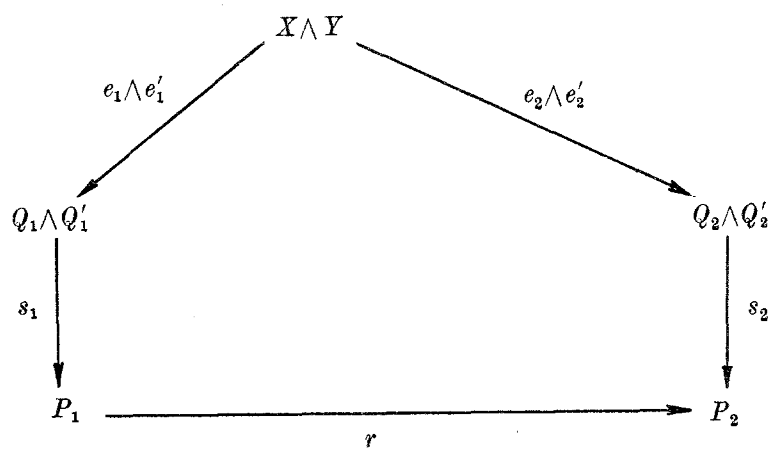
REMARKS. – 1) Proposition A5 has already been discussed in [5], § 4 although the present proof is independent of [5].

2) Assume in A3 that $X = Y$ and that T is the identity onto $\mathfrak{T}'_X \subset \mathfrak{T}_X$, then the constructed $\bar{f}$ is easily seen to be also the identity.

3) For a $Y \in \mathfrak{T}$ (which is not necessarily metric) we can use instead of $\mathfrak{T}''_Y$ the category consisting of the single object $1\colon Z \to Z$ with the identity morphism solely.

The proof of proposition 1.4. requires:

A6 LEMMA. – Assume X, Y being compact metric spaces and consider the following diagram



$Q_i, Q_i', P_i \in \mathfrak{T}$, together with a homotopy $\omega: rs_1(e_1 \wedge e_1') \simeq s_2(e_2 \wedge e_2')$.

Then there exist mappings $e \wedge e': X \wedge Y \to Q \wedge Q'$, $Q, Q' \in \mathfrak{T}$, $t_i \wedge t_i': Q \wedge Q' \to Q_i \wedge Q_i'$ such that $(t_i \wedge t_i')(e \wedge e') = e_i \wedge e_i'$ $(i = 1, 2)$ as well as a homotopy $\delta: rs_1(t_1 \wedge t_1') \simeq$ $\simeq s_2(t_2 \wedge t_2')$ which satisfies

$$\delta(e \wedge e') = \omega \, .$$

PROOF. – The proof follows the pattern of the proof of A4: We find immediately mappings $\tilde{e}: X \wedge Y \to \tilde{Q} \in \mathfrak{T}$, $\tilde{t}_i: \tilde{Q} \to Q_i \wedge Q_i'$ such that $\tilde{t}_i \tilde{e} = e_i \wedge e_i'$ and a homotopy δ' fitting into the corresponding diagram. However the compactness of X and Y guarantees that we can require $\tilde{e}$ to be of the form $\tilde{e} = e \wedge e'$, where $e: X \subset Q$, $e': Y \subset Q'$ are compact ANR neighbourhoods of X resp. Y in some Hilbert cube (cf. 1.5.). By eventually taking finer approximations Q, Q' of X resp. Y, we obtain mappings $\tilde{t}_i = t_i \wedge t_i'$. The final homotopy δ is now immediately established.

Let X be metric and Z be either an ANE or a metric space, then we are trying to develop a subcategory $\mathfrak{T}' \subset \mathfrak{T}_X \overline{\wedge} \mathfrak{T}_Z$ such that again an assertion like A3 ensures the existence of a map $\bar{f} \in \overline{\mathfrak{K}}(S, X \overline{\wedge} Z)$ whenever a special kind of functor $T: \mathfrak{T}' \to \mathfrak{T}_S$ (for a given space S) is given. This has to be settled for the proof of theorem 7.1.

In what follows, we mean by $\mathfrak{T}_X''(\mathfrak{T}_Z'')$ either the category which we have just constructed or $\mathfrak{T}_X'(\mathfrak{T}_Z')$ whenever X (or Z) is compact metric.

The objects of $\mathfrak{T}'$ are decompositions of continuous mappings of the form $i \wedge a$, $i \in \mathfrak{T}_Y''$, $a \in \mathfrak{T}_Z''$ while the morphisms are of the form defined in § 1 (1) with s_1, $s_2 =$ identity $r = r_1 \wedge r_2$ and all the homotopies being trivial. By an abuse of notation we will simply write $(r_1 \wedge r_1, 1): (i_1 \wedge a_1) \to (i_2 \wedge a_2)$ for such a morphism.

A7 PROPOSITION. – Let $S \in \mathcal{K}$ be any space. A functor $T\colon \mathcal{T}' \to \mathcal{T}_S$ with the properties

$T\,1)$ $(i \wedge a) \in \mathcal{T}'$, $i\colon X \subset C$, $a\colon Z \subset U \Rightarrow T(i \wedge a)\colon S \to C \wedge U$;

$T\,2)$ $(r_1 \wedge r_2, 1) \in \mathcal{T}'\big((i_1 \wedge a_1), (i_2 \wedge a_2)\big) \Rightarrow T(r_1 \wedge r_2, 1) = (r_1 \wedge r_2, \omega)$ for suitable ω determines a $\bar{f} \in \overline{\mathcal{K}}(S, \overline{X \wedge Z})$ such that $\bar{f}|\mathcal{T}' = T$.

PROOF. – One immediately recognize that we have merely to apply lemma A4 to each factor in the $\wedge$-product separately, in order to get an assertion which allows us to argue as in the proof of A3.

In order to find a specific functor T with properties $T\,1)$, $T\,2)$, the following assertion will be helpful:

A8 PROPOSITION. – Let $T\colon \mathcal{T}' \to \mathcal{T}_S$ be an assignment with the following properties:

a) For fixed $i \in \mathcal{T}''_X$, $T'(\ ) = T(i \wedge\ )$ behaves functorially.

b) For fixed $a \in \mathcal{T}''_Z$, $T''(\ ) = T(\ \wedge a)$ behaves functorially.

c) T fulfills $T\,1)$ and $T\,2)$ in proposition A7.

Then T is a functor.

PROOF. – Each morphism $(r_1 \wedge r_2, 1)\colon i_1 \wedge a_1 \to i_2 \wedge a_2$ in $\mathcal{T}'$ allows a decomposition

$$(r_1 \wedge r_2, 1) = (1 \wedge r_2, 1)(r_1 \wedge 1, 1)$$

with suitable identities 1. From this observation A8 follows easily.

10.3. *Cofibrations in the strong shape category.*

In order to keep this paper independent of other articles on shape theory and because we want to present an application of A3 we will include proofs of the following assertions A9, A10:

A9 PROPOSITION. – Let $i\colon A \subset B$ be an inclusion of metric spaces and A compact. Then $h(i)$ is a cofibration in $\overline{\mathcal{K}}$.

PROOF. – Let $d\colon A \times 0 \approx A \subset B$ be the inclusion, then $A \times I \cup_d B$ with the identification topology is homeomorphic to $A \times I \cup B \times 0$ (equipped with the topology of a subspace of $B \times I$). By a well-known argument the assertion is equivalent to the existence of a mapping $\bar{\varrho} \in \overline{\mathcal{K}}(B \times I, A \times I \cup B \times 0)$ having the property that $\bar{\varrho}k = 1$, where $k\colon A \times I \cup B \times 0 \subset B \times I$ denotes the inclusion.

We embed B in an ANR M, hence we have

$$A\times I\cup B\times 0\subset B\times I\subset M\times I\,.$$

Let $j\colon A\times I\cup B\times 0\subset C$ be an object of $\mathfrak{T}''_{A\times I\cup B\times 0}$, then we find a (rel. M) open $W\supset B$ and an open $U\supset A$, such that $W\times 0\cup U\times I\subset C$ and that both are maximal with this property (i.e. $W=(M\times 0)\cap C$). Moreover since A is compact we find a decreasing sequence $U_1\supset U_2\supset\ldots$, having the property that every U is contained in some U_i and therefore in a $U_{i(C)}$ with maximal index $i=i(C)$. All spaces are metric, hence normal, therefore we find to each C a continuous $\varphi_C\colon B\to I$ such that $\varphi_C|A=1$, $\varphi_C|B-B\cap U_{i(C)}=0$ (φ_C should only depend on the index $i(C)$). Now we set

$$\bar{\varrho}(j)(b,t)=\big(b,\varphi_C(b)t\big)\,,$$

which is a continuous function

$$\bar{\varrho}(j)\colon B\times I\to W\times 0\cup U_{i(C)}\times I\subset C\,.$$

Let $(r\colon C_1\subset C_2, 1)$ be a morphism in $\mathfrak{T}''_{A\times I\cup B\times 0}$, then we define $\bar{\varrho}(r,1)=(r,\omega)$ in the following way:

$$\omega=1\ldots i(C_1)=i(C_2)$$

$$\omega(b,t,s)=\big(b,(1-s)t\varphi_{C_1}(b)+s\varphi_{C_2}(b)t\big)\in C_2$$

for $i(C_2)=i(C_1)+1$ and as the composition of such homotopies for $i(C_2)=i(C_1)+n$.

This yields a functor $\bar{\varrho}|\mathfrak{T}''_{A\times I\cup B\times 0}=T$ fulfilling T1), T2) in A3. So we obtain a $\bar{\varrho}\in\overline{\mathfrak{K}}(B\times I, A\times I\cup B\times 0)$. Since we have $\bar{\varrho}(g)|A\times I\cup B\times 0=g$ for any $g\in\mathfrak{T}_{A\times I\cup B\times 0}$ resp. for the 1- and 2-morphisms according to remark 2) (following A5), we conclude that

$$\bar{\varrho}h(k)=\bar{\varrho}k=1\,.$$

Although the following assertion is not explicitely needed in the present paper, we include it, because it has its well-known counterpart in ordinary homotopy theory:

A10 PROPOSITION. – Let $i\colon A\subset B$ be an inclusion of compact metric spaces, then $A\times I\cup B\times 0$ is a strong deformation retract of $B\times I$ in $\overline{\mathfrak{K}}$. More precisely: There exists a $\overline{F}\in\overline{\mathfrak{K}}(B\times I\times I, B\times I)$ such that $\overline{F}|B\times I\times 0=k\bar{\varrho}$, $\overline{F}|B\times I\times 1=1$, $\overline{F}|(A\times I\cup B\times 0)\times I=p$ (= projection).

PROOF. – The proof is analogous to that of A9: Set $X=B\times I\times 0\cup B\times 0\times I\cup\cup A\times I\times I\cup B\times I\times 1\subset B\times I\times I$, then by glueing together the prescribed shape mappings on the specific subspaces (which is easily seen to be possible) we get a $\bar{G} \in \bar{\mathfrak{K}}(X, B \times I)$. Now set $B' = B \times 0 \times I$, $A' = A \times 0 \times I \cup B \times 0 \times 0 \cup B \times 0 \times 1$ and apply A9 to the inclusion $i' \colon A' \subset B'$, then we obtain a retraction $\bar{R} \in \bar{\mathfrak{K}}(B \times I \times I, X)$ because $X = A' \times I \cup B' \times 0$. Hence we can form

$$\bar{F} = \bar{G}\,\bar{R}\,,$$

which has the required properties.

This completes the proof of A10.

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