Bounds for the Nonhomogeneous GASPT Equation (*).

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Summary. - *Weighted a priori bounds for the equation* $\Delta u + (\mu/y)u_y = f(\mu > 0)$, *in the half-plane* $y > 0$, *are proved. If* $p > 1$, $0 < \alpha + p^{-1} < 1 + \mu$, u *has bounded support and* $y^{\mu} u_y \to 0$ *(as* $y \to 0_+$*), then the* L^p *norms of* $y^{\alpha} u$ *and* $y^{\alpha}|D^2 u|$ *are bounded by the* L^p *norm of* $y^{\alpha} f$. *A boundary value problem in a rectangle is also studied in the appropriate weighted Sobolev class.*

0. - Introduction.

The main goal of this paper is to prove a priori bounds for solutions of the equation

$$\mathcal{W}u = u_{xx} + u_{yy} + (\mu/y)u_y = f\,, \qquad \mu \text{ a positive constant}\,, \tag{0.1}$$

in the half-plane $y > 0$. We will write $\mathcal{W}_{\mu}$ when we need to emphasize the dependence on μ.

The operator $\mathcal{W}$ is a model operator for elliptic operators singular on a line; also, $y\mathcal{W}$ is an elliptic operator degenerating on $y = 0$. Many different questions are connected with this operator; let us recall some of them.

(*a*) Assume $\mu = m - 2 \in \mathcal{N}$; then u is solution of $\mathcal{W}u = 0$ if and only if $u_0(x_1, \dots, x_m) = \big(x_1, (x_2^2 + \dots + x_m^2)^{\frac12}\big)$ is a solution of $\Delta u_0 = 0$. For this fact the equation $\mathcal{W}u = 0$ has been called GASPT (generalized axially symmetric potential theory) equation. A natural extension of the above remark is in TALENTI [37] (here recalled in a slightly different form): given $\mu > 0$, fix an integer $m \geqslant \mu + 2$ and define $\alpha(\mu) = \mu/((m-2) + (m+1)\mu)$; α is an increasing function of μ and $\alpha(0) = 0$, $\alpha(m-2) = 1/m$; define: $u_0(x_1, \dots, x_m) = u\big((1/\alpha - (m-1))^{\frac12} x_1, (x_2^2 + \dots + x_m^2)^{\frac12}\big)$ and

$$\mathcal{U}u = \alpha\, \Delta u + (1 - m\alpha) \sum_{ij=2}^{m} x_i x_j (x_2^2 + \dots + x_m^2)^{-\frac12} u_{x_i x_j} \tag{0.2}$$

then

$$\mathcal{U}u_0(x_1, \dots, x_m) = \big(1 - (m-1)\alpha\big)\Big((\mathcal{W}u)\big((1/\alpha - (m-1))^{\frac12} x_1\,,\ (x_2^2 + \dots + x_m^2)^{\frac12}\big)\Big)\,.$$

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The operator $\mathcal{U}$ is a uniformly elliptic second order operator in $\mathcal{R}^m$, with (lower) ellipticity constant α, trace $\equiv 1$ and discontinuous coefficients on the axis $x_2 = \ldots = x_m = 0$; it has been introduced by URAL'TSEVA [38] as counterexample operator to existence theorems in the Sobolev space $W^{2,m}$; in other words the Ural'tseva operator $\mathcal{U}$ on axially symmetric functions is the same as the operator $\mathcal{W}$.

(*b*) If u, v are solutions of the generalized Cauchy-Riemann system:

$$y^{\mu} u_x = v_y \tag{0.3}$$

$$y^{\mu} u_y = -v_x \tag{0.4}$$

then $\mathcal{W}u = 0$ and $\Delta v - (\mu/y)v_y = 0$. This system was studied, for positive μ, by BERS and GELBARG [3] and it is connected with problems of mechanics of continua (BERS and GELBARG [3], PAYNE [33]).

The equation $\mathcal{W}u = 0$ can be written in complex form as:

$$u_{z\bar{z}} - (\mu/2)(u_{\bar{z}} - u_z)/(z - \bar{z}) = 0 \tag{0.5}$$

with $z = x + iy$, $\bar{z} = x - iy$; the equation (0.5) is a Euler-Poisson–Darboux equation and the Riemann function can be explicitly written for it (see VEKUA [39], GILBERT [12]) and used to construct solutions of $\mathcal{W}u = 0$.

(*c*) The equation $\mathcal{W}u = f$ can be written in variational form as:

$$(y^{\mu} u_x)_x + (y^{\mu} u_y)_y = y^{\mu} f$$

and variational techniques can be used to study it (see e.g. NIKOL'SKII-LIZORKIN [30], BOLLEY-CAMUS [4]).

(*d*) The operator $\mathcal{W}$ is connected with various classes of special functions and integral transforms. Its fundamental solutions can be written by using hypergeometric functions (OLEVSKII [32], in particular one can use Legendre functions of second kind in 3. below), Bessel functions (WEINSTEIN [40], 1. below); solving the equation $\mathcal{W}u = 0$ by separation of variables leads to Bessel functions again, Gegenbauer functions (see GILBERT [12]). Good tools to study the operator $\mathcal{W}$ are the Hankel in y or the Fourier-Hankel transform (Fourier in x, Hankel in y transform, see KIPRIJANOV [18], [19], [20]). WEINSTEIN [40] used results for $\mathcal{W}u = 0$ to give a proof of the Weber-Schafheitlin theorem. In the book of GILBERT [12] function theoretic methods, related to the complex equation (0.5) and to expansions in special functions, are used to study solutions of $\mathcal{W}u = 0$ looking in particular at the analyticity and to the singular points of them.

(*e*) The problem of the analyticity near $y = 0$, was earlier studied by HENRICI [13] and KRIVENKOV [24], [25]. The first author found that the problem:

$\mathcal{W}u = 0$, $u(x, 0) = u_0(x)$, with u_0 given analytic function, has the solution:

$$u(x, y) = \text{const}\int_0^1 [t(1-t)]^{\mu/2-1} u_0(x + iy(1-2t))\, dt .$$

Krivenkov found that, if $\mu \geqslant 1$, then a solution of $\mathcal{W}u = 0$, continuous up to $y = 0$, can be extended across $y = 0$ as a even analytic function; if $0 < \mu < 1$, to get the same result, one needs the extra condition:

$$\lim_{y\to 0} y^{\mu} u_y = 0 .$$

Henrici's representation formula above was used by Radjabov (see L. G. MIKHAILOV [29]) to solve the Dirichlet problem for $\mathcal{W}u = 0$ (with suitable boundary conditions); the problem was changed into a Abel type integral equation; the unknown is u at $y = 0$.

The operator $\mathcal{W}$ is connected with fractional integrals and derivatives (see HENRICI [13], LEVITAN [26], ERDÉLYI [7], [8]); as an example, if $\mathcal{W}u = 0$ then:

$$u = I(h) , \qquad \Delta h = 0$$

where:

$$I(h)(x, y) = 2y^{1-\mu}(\Gamma(\mu/2))^{-1}\int_0^y (y^2 - t^2)^{\mu/2-1} h(x, t)\, dt;$$

a similar formula (with suitable change of variables) was used by WEINSTEIN [40] to write the fundamental solution of the operator $\mathcal{W}$.

(*f*) Changes of variables connect the operator $\mathcal{W}$ with other special operators. The change of variables:

$$x' = ax , \quad y' = y^b , \quad u(x, y) = v(ax, y^b) , \quad a > 0,\ b > 0$$

gives:

$$\mathcal{W}u(x, y) = \left(a^2 v_{x'x'} + b^2 (y')^{2-2/b} v_{y'y'} + b(b - 1 + \mu)(y')^{1-2/b} v_{y'}\right)(ax, y^b) .$$

Choose $a = b = 2$; then:

$$\mathcal{W}u(x, y) = 4\left(v_{x'x'} + y' v_{y'y'} + ((1 + \mu)/2)\, v_{y'}\right)(2x, y^2) .$$

The operator on the right hand side is a particular case of an operator introduced by M. V. KEL'DISCH [17]; in $y' > 0$ it is an elliptic-parabolic operator. Boundary value problems for this class of operators have been considered by M. V. KEL'DISCH and G. FICHERA [10] (see the OLEJNIK-RADKEVIČ book [31]).

Choose $a = b = \frac{2}{3}$; then:

$$\mathcal{W}u(x, y) = (\tfrac{2}{3})^2 y'[y' v_{x'x'} + v_{y'y'} + (3\mu/2 - \tfrac{1}{2}) v_{y'}/y'^2]\,(2x/3, y^{\frac{2}{3}})\,.$$

The operator in square brackets (when $\mu = \frac{1}{3}$) is the Tricomi operator, in $y' > 0$ (on this operator see e.g. the M. M. SMIRNOV book [36] and the references therein).

The above remarks just sketch the many connections of the operator $\mathcal{W}$. Surveys on results on $\mathcal{W}$ and generalizations are in HUBER [15], GILBERT [12], TALENTI [37], MIKHAILOV [29]. An extensive work on $\mathcal{W}$ and many extensions has been made by I. A. KIPRIJANOV and coworkers; let us quote [18]-[23]. Also, many results for $\mathcal{W}$ can be obtained as particular cases of results on general singular or degenerate elliptic equations (see e.g. AVANTAGGIATI [2], TALENTI [37], ALESSANDRINI [1], OLEJNIK-RADKEVIC [31], M. M. SMIRNOV [36], BOLLEY-CAMUS [4], DUNNINGER-LEVINE [7], KIPRIJANOV [18]-[23], CA'C [5], LO [27], [28], SCHECHTER [35].

The main result of this paper is the following (theorems 3.4 and 2.1 below):

Assume $\mu > 0$, $p > 1$, $0 < \alpha + 1/p < 1 + \mu$. There exists a constant c such that, for every u of class C^∞ in $y > 0$ which satisfies

$$\lim_{y \to 0} y^\mu \int_{\mathcal{R}} |u_y(x, y)|\, dx = 0$$

and has bounded support, the inequality:

$$\iint_{y>0} (y^\alpha |D^2 u|)^p\, dx\, dy \leqslant c \iint_{y>0} |y^\alpha \mathcal{W}u|^p\, dx\, dy$$

holds.

Moreover, for every open rectangle R in $y > 0$, there exists a constant c_1 such that, for every u as above, with support in R, the bound:

$$\iint_{y>0} |y^\alpha u|^p\, dx\, dy \leqslant c_1 \iint_{y>0} |y^\alpha \mathcal{W}u|^p\, dx\, dy$$

holds.

These results have been proved, for some choices of α and p, by Kiprijanov. One can prove that the bounds above are sharp: if $(1/p, \alpha)$ is outside of the domain $0 < 1/p < 1$, $0 < 1/p + \alpha < 1 + \mu$, then the above inequalities do not hold (remark 4.2 of section 4.).

In section 4. a boundary value problem in a rectangle for the nonhomogeneous equation (0.1) is solved within a suitable weighted Sobolev class.

Boundary value problems in different domains will be discussed in a forthcoming paper.

Let us introduce a few notations. We define: $\mathcal{R}^2_+ = \{(x, y) \in \mathcal{R}^2 : y > 0\}$ and write $\boldsymbol{x} = (x, y)$; we will also write:

$$\|u\|_{\alpha,p} = \Big(\iint_{\mathcal{R}^2_+} |y^\alpha u(\boldsymbol{x})|^p \, dx \, dy\Big)^{1/p}, \qquad 1 < p < \infty ;$$

$$|u(\cdot, y)|_s = \Big(\int_{\mathcal{R}} |u(\boldsymbol{x})|^s \, dx\Big)^{1/s}, \qquad 1 \leqslant s < \infty .$$

More notations: $p' = p/(p-1)$; Du will be u_x or u_y, D^2u will be u_{xx}, u_{xy} or u_{yy}.

1. - The equation $Wu = 0$ in a rectangle.

The separation of variables technique will be used here to study the equation $\mathcal{W}u = 0$ in a open rectangle R, in $y > 0$, with one side on $y = 0$, to get L^p a priori bounds.

A change of variables of the form:

$$x' = ax + b , \qquad y' = ay , \qquad (a > 0)$$

will change the operator by a positive, multiplicative constant. Thus, in the statements of the theorems, R will be a rectangle of the form:

$$R = R_{y_0} = (x_1, x_2) \times (0, y_0) = A \times (0, y_0) , \qquad 0 < y_0 < \infty ;$$

however in the proofs, for simplicity, R will be of the form $(0, \pi) \times (0, y_0)$; we will write R_{y_0} when we need to emphasize the dependence on y_0; we also define:

$$T = \{\boldsymbol{x} \colon 0 < y < y_0, \ x = x_1, \text{ or } x = x_2\}$$

$$E = \{(x, y_0) \colon x \in A\}$$

$$S = \{\boldsymbol{x} \colon x \in A, \ -y_0 < y < y_0\} .$$

Let us look for solutions of $\mathcal{W}u = 0$ in $y > 0$, of the form:

$$u_n(\boldsymbol{x}) = v_n(y) \sin nx ; \tag{1.1}$$

the functions v_n are solutions of the differential equations:

$$v_n'' + (\mu/y) v_n' - n^2 v_n = 0 ; \tag{1.2}$$

two linearly independent solutions of (1.2) are:

$$(ny)^{(1-\mu)/2} I_{(\mu-1)/2}(ny) , \qquad (ny)^{(1-\mu)/2} K_{(\mu-1)/2}(ny)$$

(I_r, K_r modified Bessel functions of first and third type). Let us define:

$$h(t) = (t/2)^{(1-\mu)/2} \Gamma((\mu+1)/2) I_{(\mu-1)/2}(t) ;$$

clearly $h(ny)$ is a solution of (1.2); moreover, since:

$$I_r(z) = \sum_{m=0}^{\infty} (z/2)^{2m+r} / (m! \Gamma(m+r+1)) \qquad (z \in \mathbb{C})$$

one sees that $\mathcal{R} \ni t \to h(t)$ is a real, positive, increasing convex function; in the complex plane h is holomorphic and entire; it satisfies the equation: $h'' + (\mu/y)h' = h$. The asymptotic expansion of I_r: $I_r(y) = \text{const} \cdot e^y y^{-\frac{1}{2}}(1 + 0(1/y))$ $(y \gg 1)$ (Erdelyi and others II, p. 86) gives:

$$h(y)/h(y_0) \leqslant \text{const}\,[(1+y)/(1+y_0)]^{-\mu/2} \exp[y - y_0] \qquad 0 \leqslant y \leqslant y_0 . \tag{1.3}$$

The differential equation above and the properties $h' > 0$, $h'' > 0$ give the inequalities:

$$h'(y) \leqslant y h(y)/\mu \tag{1.4}$$

$$h''(y) \leqslant h(y) . \tag{1.5}$$

REMARK 1.1. - *The problem:*

$$v'' + (\mu/y)v' - n^2 v = 0 \quad \text{in } (0, y_0), \qquad v(y_0) = 0 ,$$

with the extra condition

$$\lim_{y \to 0} y^{\mu} v' = 0 ,$$

has the unique solution $v \equiv 0$.

For a more general and abstract version of this remark see D. R. DUNNINGER and H. A. LEVINE [7].

Let $s \geqslant 1$, $u_0 \in L^s(A)$, u defined in R, such that:

$$\lim_{y \to y_0^-} \int_A |u(x) - u_0(x)|^s \, dx = 0 ;$$

we will write $u|_E = u_0$, for short.

THEOREM 1.1. – *Let* $s \geq 1$. *The problem*:

$$\mathcal{W}u = 0 \quad in\ R, \tag{1.7}$$

$$u|_T = 0\,, \qquad \lim_{y\to 0^+}\int_A y^\mu |u_y(\boldsymbol{x})|\,dx = 0\,, \qquad u|_E = u_0 \in L^s(A) \tag{1.8}$$

has a unique solution $u \in C^0(R \cup T) \cap C^2(R)$. *Moreover* $u \in C^2(\bar{R}\setminus\bar{E})$ *and can be extended to an analytic, even function in* S, *such that*:

$$\Big(\int_A |u(\boldsymbol{x})|^s\,dx\Big)^{1/s} \leq |u_0|_{L^s(A)}\,, \qquad 0 < y \leq y_0; \tag{1.9}$$

for every $y' \in (0, y_0)$ *there exists* $c_{y'}$, *not depending on* u, *such that*:

$$\|u\|_{C^2(\bar{R}_{y'})} \leq c_{y'} |u_0|_{L^s(A)}\,. \tag{1.10}$$

PROOF. – Let us prove the uniqueness [1]. Let $\bar{u} \in C^2(R) \cap C^0(R \cup T)$ be a solution of (1.7), (1.8) with $u_0 = 0$. By classical regularity theorems, $\bar{u}$ is in $C^\infty(R \cup T)$.

Let us expand $\bar{u}$ in Fourier sine series in x. We get:

$$u(\boldsymbol{x}) = \sum_{n=1}^{\infty} u_n(y) \sin nx$$

with

$$u_n(y) = (2/\pi)\int_0^\pi \bar{u}(\boldsymbol{x}) \sin nx\,dx\,.$$

The equation (1.7) and the boundary conditions (1.8) give:

$$u_n'' + (\mu/y)u_n' - n^2 u_n = 0 \quad \text{in } (0, y_0)$$

$$\lim_{y\to y_0} u_n(y) = 0\,, \quad \lim_{y\to 0} y^\mu u_n'(y) = 0$$

and thus $u_n \equiv 0$ (Remark 1); then $\bar{u} \equiv 0$ in R.

To prove the existence we just need to write down a solution $u \in C^2(R) \cap C^0(R \cup T)$ of (1.7), satisfying the boundary conditions (1.8). By using the boundedness of the sequence $\Big\{2\pi^{-1}\int_0^\pi u_0(x)\sin nx\,dx\Big\}$ and the bounds (1.3)-(1.5), one sees that the series:

$$\sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{h(ny)}{h(ny_0)} \sin nx \int_0^\pi u_0(t) \sin nt\,dt \tag{1.11}$$

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[1] The uniqueness part of this theorem could be also deduced from the results of D. R. GUNNINGER and H. A. LEVINE [7].

is uniformly convergent with first and second derivatives in every $\bar{R}y'$, with $0 < y' < y_0$, to a function $\bar{u} = g(u_0) \in C^2(\bar{R}y_0 \setminus \bar{E})$ solution of (1.7) with $\bar{u}|_T = 0$. It remains to show that $\bar{u}$ satisfies the boundary conditions on E, to prove the a priori bounds and the regularity results.

Assume, for a moment, $u_0 = f_0 \in C_0^\infty(E)$; then the corresponding $g(f_0) \in C^2(\bar{R}y_0)$, and $g(f_0)|_E = f_0$. By the maximum principle (WEINSTEIN [40], KAROL [16]),

$$\max_{\bar{R}y \setminus \bar{E}} |g(f_0)| \leqslant \max_{E} |f_0| .$$

Let us define: $\boldsymbol{w} = (x_0, y_0) \in E$ and

$$F(\boldsymbol{x}, \boldsymbol{w}) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{h(ny)}{h(ny_0)} \sin nx \sin nx_0$$

$(0 \leqslant y < y_0;\ x, y_0 \in [0, \pi])$. By (1.3), (1.4), (1.5) F, as a function of $\boldsymbol{x}$, is in $C^2(\bar{R} \setminus \bar{E})$ and is a solution of $\mathcal{W}_\mu F(\cdot, \boldsymbol{w}) = 0$ in $R(\boldsymbol{w} \in E)$; moreover, the function $g(f_0)$ above can be written as:

$$g(f_0)(\boldsymbol{x}) = \int_0^\pi F(\boldsymbol{x}, \boldsymbol{w}) f_0(x_0)\, dx_0 . \tag{1.12}$$

The following properties for F can be proved:

(i) $F(\boldsymbol{x}, \boldsymbol{w}) \geqslant 0$ (consequence of the maximum principle for $\mathcal{W}$);

(ii) $F(x, y, x_0, y_0) = F(x_0, y, x, y_0)$ $\quad (x, x_0 \in [0, \pi]$ and $0 \leqslant y < y_0)$;

(iii) $0 \leqslant \int_0^\pi F(\boldsymbol{x}, \boldsymbol{w})\, dx_0 \leqslant 1 \quad (\boldsymbol{x} \in \bar{R}y_0 \setminus \bar{E})$.

Let $f_n \in C_0^\infty(E)$, $0 \leqslant f_n \leqslant 1$, $f_n \to 1$ in $L^1(A)$; the maximum principle and (1.12) give:

$$0 \leqslant \int_0^\pi F(\boldsymbol{x}, \boldsymbol{w}) f_n(x_0)\, dx_0 \leqslant 1 , \qquad \boldsymbol{x} \in \bar{R}y_0 \setminus \bar{E} ;$$

if we let $n \to \infty$, on these inequalities, we get (iii);

(iv) for every $y' \in (0, y_0)$, there exists $C_{y'}$, such that:

$$\|F(\cdot, \boldsymbol{w})\|_{C^2(\bar{R}_{y'})} < C_{y'} , \qquad \boldsymbol{w} \in E .$$

Assume, now, $f_0 \in L^s(A)$. The expansion (1.11) can be written as (1.12). Thus, the properties (i)-(iv) of F and standard techniques (see e.g. ZYGMUND [41], III-IV) give the boundary conditions $u|_E = f_0 \in L^s(E)$, and the bounds (1.9), (1.10).

The regularity of the solution can be deduced from expansion (1.11), which defines a $C^2(S)$ even function. The analyticity follows from KRIVENKOV [24], [25] results. |||

REMARK 1.2. – The condition:

$$\lim_{y\to 0} y^{\mu}\int_A |u_y(x, y)|\, dx = 0 \tag{1.13}$$

was used to prove uniqueness only of the theorem above; more precisely, if u satisfies (1.13), then the Fourier (in x) coefficients $u_n(y)$ of u satisfy condition (1.6) of Remark 1.1. Thus (1.13) may be replaced by any other assumption on u implying for u_n, at $y = 0$, a condition which in turn would give uniqueness for the problem in Remark 1.1. As an example, if $\mu \geqslant 1$, (1.13) may be replaced by:

$$\max_{y\to 0}\lim \int_A |u(x, y)|\, dx < \infty;$$

this condition matches with KEL'DISCH [17] problem E, when one changes (as in (*f*) above) $\mathcal{W}$ into a Kel'disch operator (see KEL'DISCH [17]); also it matches with FICHERA [10] approach to boundary value problems for elliptic-parabolic operators.

2. – The a priori bound for u.

Let us prove a representation formula for solutions of $\mathcal{W}_\mu u = f$. We will keep the notations of 1.; moreover, if v is the solution of the problem: $\mathcal{W}_\mu v = 0$ in $R = A \times (0, y_0)$, with boundary conditions $v|_T = 0$, $v|_E = f_0 \in L^s(A)$ and:

$$\lim_{y\to 0^+} y^{\mu}\int_A |v_y(\boldsymbol{x})|\, dx = 0\ ,$$

we will write $v(\cdot, y) = \mathcal{G}^A_{y,y_0} f_0$.

The following lemma holds.

LEMMA 2.1. – *Let* $u \in C^\infty(\mathcal{R}^2_+)$, *such that* $\operatorname{supp} u$ *is bounded and*

$$\lim_{y\to 0^+} y^{\mu}\int_{\mathcal{R}} |u_y(\boldsymbol{x})|\, dx = 0\ . \tag{2.1}$$

Let us define $f = \mathcal{W}u$ *and assume moreover that* $\operatorname{supp} u \subset R = A \times (0, y_0)$ *and* $y^\mu f \in L^1(\mathcal{R}^2_+)$. *Then*:

$$u(\cdot, y) = \int_{y_0}^{y} dt \int_0^t \left(\frac{t'}{t}\right)^{\mu} \mathcal{G}^A_{y,t}\, \mathcal{G}^A_{t',t}\, f(\cdot, t')\, dt' \qquad (0 \leqslant y < y_0)\ . \tag{2.2}$$

PROOF. – Let:

$$u(x, y) = \sum_{n=1}^{\infty} u_n(y) \sin nx ,$$

$$f(x, y) = \sum_{n=1}^{\infty} f_n(y) \sin nx$$

$(0 < y < \infty)$ the Fourier expansions of u, f in sine series in x. The equation $\mathcal{W}u = f$ gives:

$$u_n'' + (\mu/y)u_n' - n^2 u_n = f_n$$
$$\lim_{y\to 0^+} y^\mu u_n'(y) = u_n(y_0) = 0 .$$

It is easily seen that $y^\mu f_n \in L^1(0, +\infty)$ and that the unique solution of this problem is:

$$u_n(y) = \int_{y_0}^{y} \frac{h(ny)}{h^2(nt)}\, dt \int_0^t \left(\frac{t'}{t}\right)^\mu h(nt') f_n(t')\, dt' .$$

And thus:

$$u(x, y) = \sum_{n=1}^{\infty} \sin nx \int_{y_0}^{y} \frac{h(ny)}{h^2(nt)}\, dt \int_0^t \left(\frac{t'}{t}\right)^\mu h(nt') f_n(t')\, dt' .$$

It is not difficult to see that one can exchange series and integrals and get:

$$u(\mathbf{X}) = \int_{y_0}^{y} dt \int_0^t \left(\frac{t'}{t}\right)^\mu \left[\sum_{n=1}^{\infty} \frac{h(ny)h(nt')}{h^2(nt)} f_n(t') \sin nx\right] dt' .$$

By recalling (1.11) and the definition of $\mathcal{G}$ we have the theorem. |||

THEOREM 2.2. – *Let* $p > 1$, $\alpha \in \mathcal{R}$, $\mu > 0$ *such that* $0 < \alpha + p^{-1} < 1 + \mu$. *Let* $u \in C^\infty(\mathcal{R}^2_+)$ *such that* supp u *is bounded and* (2.1) *holds; assume also* $y^\alpha \mathcal{W}u \in L^p(\mathcal{R}^2_+)$. *Then*:

$$\lim_{y\to 0^+} y^\mu |u(\cdot, y)|_1 = 0 . \tag{2.3}$$

Moreover, for every $R = A \times (0, y_0)$ *there exists* $C_1(\mu, \alpha, p, R)$ *such that for* u *as above, with support in* R, *the inequality*:

$$\|u\|_{\alpha,p} \leq C_1(\mu, \alpha, p, R) \|\mathcal{W}u\|_{\alpha,p} \tag{2.4}$$

holds.

PROOF. – First of all, let us notice that, by Hölder inequality, $y^\mu \mathcal{W} u \in L^1(\mathcal{R}^2_+)$. Let $R = A \times (0, y_0)$ a rectangle containing supp u. By previous lemma:

$$\int_{\mathcal{R}} |u(x, y)|\, dx = \int_y^{y_0} dt \int_0^t \left(\frac{t'}{t}\right)^\mu \int_{\mathcal{R}} |\{\mathcal{G}^A_{y,t}\mathcal{G}^A_{t',t} f(\cdot, t')\}(x)|\, dx\,.$$

Formula (1.9), with $s = 1$, gives:

$$\int_{\mathcal{R}} |u(\boldsymbol{x})| px \leqslant \int_y^{y_0} dt \int_0^t \left(\frac{t'}{t}\right)^\mu \int_{\mathcal{R}} |f(x, t')|\, dx\, dt' = \int_{[}^{y_0} t^{-\mu}\, dt \int_{A\times[0,t]} (t')^{\mu-\alpha}(t')^\alpha |f(x, t')|\, dx\, dt'\,.$$

By Hölder inequality:

$$(2.5)\qquad \int_{\mathcal{R}} |u(\boldsymbol{x})|\, dx \leqslant \int_y^{y_0} t^{-\mu}\, dt \left(\iint_{A\times[0,t]} (t')^{(\mu-\alpha)p'}\, dt'\right)^{1/p'} \cdot \left(\iint_{A\times[0,t[} |t'^\alpha f(x, t')|^p\, dx\, dt'\right)^{1/p} \leqslant$$
$$\leqslant \left(\frac{\text{length of } A}{1 + (\mu - \alpha)p'}\right)^{1/p'} \left(\frac{(y_0)^{1-\alpha+1/p'} - y^{1-\alpha+1/p'}}{1 - \alpha + 1/p'}\right) \left(\iint_{\mathcal{R}^2_+} |y^\alpha \mathcal{W} u|^p\, dx\, dy\right)^{1/p}$$

(if $1 - \alpha + 1/p' = 0$, the second factor in the right hand side should be read $\log (y_0/y)$). From (2.5) and the inequality $\mu + 1 - \alpha - 1/p' > 1$, (2.3) follows.

Let us prove (2.4). We have:

$$\|u\|_{\alpha,p} = \left(\int_0^{y_0} y^{\alpha p} |u(\cdot, y)|_p^p\, dy\right)^{1/p}.$$

By the previous lemma, Minkowsky integral inequality and (1.9) (with $s = p$) we get:

$$|u(\cdot, y)|_p \leqslant \int_y^{y_0} dt \int_0^t \left(\frac{t'}{t}\right)^\mu |f(\cdot, t')|_p\, dt'\,, \qquad 0 \leqslant y < y_0\,.$$

It follows:

$$\|u\|^{\alpha,p} \leqslant \left\{\int_0^{y_0} y^{\alpha p} \left[\int_y^{y_0} dt \int_0^t \left(\frac{t'}{t}\right)^\mu |f(\cdot, t')|_p\, dt'\right]^p dy\right\}^{1/p}.$$

Multiplying the last integral by $1 = t^{-\alpha} \cdot t^\alpha$, using Hölder inequality in $\int_y^{y_0}$ and the fact $\alpha + p^{-1} > 0$, one can show that there exists a finite constant C (depending on

α, p, y_0 only), such that:

$$\|u\|_{\alpha,p} \leqslant C\left\{\int_0^{y_0} t^{\alpha p}\left[\int_0^t \left(\frac{t'}{t}\right)^{\mu} |f(\cdot, t')|_p \, dt'\right]^p dt\right\}^{1/p}.$$

By the condition $\alpha + p^{-1} < 1 + \mu$ and Hardy inequality, we have:

$$\|u\|_{\alpha,p} \leqslant c y_0 (1 + \mu - (\alpha + p^{-1}))^{-1}\left(\int_0^{y_0} t^{\alpha p} |f(\cdot, t)|_p^p \, dt\right)^{1/p};$$

the thesis follows. |||

REMARK 2.1. – In the previous proof we have used a discrete Fourier in x, Hankel in y, transform. These techniques have been used extensively by KIPRIJANOV [18], [19], [20] (see also the bibliography therein).

REMARK 2.2. – In the above theorem we have actually proved a sharper result than (2.3). In fact we have proved (see (2.5)) that there exist C_1, C_2 depending on μ, α, p, diam supp u, such that:

$$|u(\cdot, y)|_1 \leqslant (C_1 + C_2 y^{1-\alpha+1/p'}) \|\mathfrak{W}u\|_{\alpha,p}$$

$(0 < y,\ 1 - \alpha + 1/p' \neq 0$; if $1 - \alpha + 1/p' = 0$, there will be logs in the bracket).

REMARK 2.3. – Let us explicitly notice that estimate (2.4) holds also for functions $u \in C^2(R \cup T \cup E)$, $u|_{T \cup E} = 0$, $y^{\mu}\int_A |u_y(x, y)|\, dx \to 0$ as $y \to 0^+$, such that

$$\iint_R y^{\alpha p}(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2)^{p/2}\, dx\, dy < +\infty,$$

R being the rectangle $A \times (0, y_0)$.

Indeed, functions in the class above can be approximated by functions $v = v(x, y)$ of class C^∞ in the strip $0 < y < y_0$, odd, 2π periodic and vanishing at $y = y_0$. For functions in the latter class the proof of (2.4) goes without changes.

3. – The a priori bounds for the derivatives.

Let us prove a representation formula for solutions of $\mathfrak{W}u = f$ in $\mathcal{R}_+^2$. We will write $\boldsymbol{w} = (w, t) \in \mathcal{R}_+^2$.

In [40] WEINSTEIN proved that for $\boldsymbol{x} \in \mathcal{R}_+^2$, $\boldsymbol{w} \in \mathcal{R}_+^2$, $\boldsymbol{x} \neq \boldsymbol{w}$, the function:

$$H(\boldsymbol{x}, \boldsymbol{w}) = H(\boldsymbol{w}, \boldsymbol{x}) = \int_0^{\pi} ((x - w)^2 + y^2 + t^2 + 2yt \cos h)^{-\mu/2} \sin^{\mu-1} h \, dh \tag{3.1}$$

is regular and $\mathfrak{W}H(\cdot, \boldsymbol{w}) = 0$ in $\mathcal{R}^2_+ \setminus \{\boldsymbol{w}\}$; moreover:

$$H(\boldsymbol{x}, \boldsymbol{w}) = -(yt)^{-\mu/2} \log |\boldsymbol{x} - \boldsymbol{w}| + H_1(\boldsymbol{x}, \boldsymbol{w}) , \tag{3.2}$$

where H_1 is regular at $\boldsymbol{x} = \boldsymbol{w}$; the function H is a fundamental solution for the operator $\mathfrak{W}$.

Another fundamental solution (actually the Green function for the Dirichlet problem in the half disk) was constructed by OLEVSKII [32] by using hypergeometric functions.

Here we will use a modified form of Weinstein fundamental solution:

$$G(\boldsymbol{x}, \boldsymbol{w}) = \frac{1}{\pi} t^\mu H(\boldsymbol{x}, \boldsymbol{w})$$

($\boldsymbol{x}, \boldsymbol{w} \in \mathcal{R}^2_+$, $\boldsymbol{x} \neq \boldsymbol{w}$). G is a regular function if $\boldsymbol{x} \neq \boldsymbol{w}$; moreover:

(*a*) if $\boldsymbol{w} \in \mathcal{R}^2_+$, then: $\mathfrak{W}G(\cdot, \boldsymbol{w}) = 0$ $(\boldsymbol{x} \in \mathcal{R}^2_+ \setminus \{\boldsymbol{w}\})$;

(*b*) if $\boldsymbol{x} \in \mathcal{R}^2_+$, then: $\mathfrak{W}^* G(\boldsymbol{x}, \cdot) = (\Delta - (\mu/t)\, \partial/\partial t + \mu/t^2)\, G(\boldsymbol{x}, \cdot) = 0$

$$(\boldsymbol{w} \in \mathcal{R}^2_+ \setminus \{\boldsymbol{x}\}) .$$

(*c*) if $\boldsymbol{x} \in \mathcal{R}^2_+$, $\boldsymbol{w} \in \mathcal{R}^2_+$, $\boldsymbol{x} \neq \boldsymbol{w}$, by (3.2), it follows:

$$G(\boldsymbol{x}, \boldsymbol{w}) = -\frac{1}{\pi} (t/y)^{\mu/2} \cdot \log |\boldsymbol{x} - \boldsymbol{w}| + \psi_1(\boldsymbol{x}, \boldsymbol{w}) \tag{3.3}$$

with ψ_1 regular at $\boldsymbol{x} = \boldsymbol{w}$;

(*d*) assume $\boldsymbol{x} \in \mathcal{R}^2_+$; there exists k (depending on y only) such that, if $\boldsymbol{w} \in (-\infty, +\infty) \times [0, y/2]$:

$$|G(\boldsymbol{x}, \boldsymbol{w})| \leqslant kt^\mu , \qquad |(\partial/\partial t) G(\boldsymbol{x}, \boldsymbol{w}) - \mu G(\boldsymbol{x}, \boldsymbol{w})/t| \leqslant kt^\mu .$$

By using (*a*)-(*d*), the following representation formula can be proved.

LEMMA 3.1. – *Let* $u \in C^\infty(\mathcal{R}^2_+)$ *such that*: (i) *u has bounded support*; (ii)$y^\mu |u(\cdot, y)|_1$, $y^\mu |u_y(\cdot, y)|_1$ *tend to zero as* $y \to 0^+$; (iii) $y^\mu \mathfrak{W}u \in L^1(\mathcal{R}^2_+)$; *then*:

$$u(\boldsymbol{x}) = -\iint_{\mathcal{R}^2_+} G(\boldsymbol{x}, \boldsymbol{w})\, \mathfrak{W}u(\boldsymbol{w})\, d\boldsymbol{w} .$$

PROOF. – Let $\boldsymbol{x} = (x, y) \in \mathcal{R}^2_+$, $h > 0$ a small parameter ($h < y/2$ will do), and $0_h = \{\boldsymbol{w} = (w, t) \in \mathcal{R}^2_+ : 0 < h < t\}$; by (3.2) and (*b*) above, we have the Stokes formula:

$$u(\boldsymbol{x}) + \int_{t=h} [G(\boldsymbol{x}, \boldsymbol{w})\, \partial u/\partial t - u(\partial/\partial t - \mu/t) G(\boldsymbol{x}, \boldsymbol{w})]\, dw = -\iint_{0_h} G(\boldsymbol{x}, \boldsymbol{w})\, \mathfrak{W}u\, d\boldsymbol{w} \tag{3.4}$$

($\boldsymbol{x} \in 0_h$). As $h \to 0^+$, by (d) above, we have:

$$\iint_{0_h} G(\boldsymbol{x}, \boldsymbol{w})\, \mathcal{W}u(\boldsymbol{w})\, d\boldsymbol{w} \to \iint_{\mathcal{R}^2_+} G(\boldsymbol{x}, \boldsymbol{w})\, \mathcal{W}u(\boldsymbol{w})\, d\boldsymbol{w}$$

$$\int_{t=h} [G(\boldsymbol{x}, \boldsymbol{w})\, \partial u/\partial t(\boldsymbol{w}) - u(\boldsymbol{w})(\partial/\partial t - \mu/t) G(\boldsymbol{x}, \boldsymbol{w})]\, dw \to 0$$

and the thesis follows. |||

REMARK 3.1. – If $u \in C_0^\infty(\mathcal{R}^2)$, and u is even in y, then u satisfies the hypothesis of previous lemma. Also, if u satisfies the hypotesis of theorem 2.2, the lemma above applies.

For later pourposes we need sharp evaluations of the derivatives of $G(\boldsymbol{x}, \boldsymbol{w})$. Let $b(\boldsymbol{x}, \boldsymbol{w})$ a positive function in $\boldsymbol{x} \in \mathcal{R}^2_+$, $\boldsymbol{w} \in \mathcal{R}^2_+$, $\boldsymbol{x} \neq \boldsymbol{w}$, defined by:

$$b(\boldsymbol{x}, \boldsymbol{w}) = |\boldsymbol{x} - \boldsymbol{w}|^2/(2yt)\,.$$

LEMMA 3.2. – *There exists a positive constant H such that, if*

$$\boldsymbol{x} \in \mathcal{R}^2_+\,, \qquad \boldsymbol{w} \in \mathcal{R}^2_+\,, \qquad \boldsymbol{x} \neq \boldsymbol{w}\,, \qquad b(\boldsymbol{x}, \boldsymbol{w}) < 1\,,$$

then:

$$D^2_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{w}) = -\frac{1}{2\pi} D^2_{\boldsymbol{x}} \log |\boldsymbol{x} - \boldsymbol{w}| + S(\boldsymbol{x}, \boldsymbol{w}) \tag{3.5}$$

where

$$|S(\boldsymbol{x}, \boldsymbol{w})| \leqslant H\big(b(\boldsymbol{x}, \boldsymbol{w})\big)^{-\frac{1}{2}} y^{-2}\,.$$

PROOF. – As a consequence of the inequality $b(\boldsymbol{x}, \boldsymbol{w}) < 1$ we have: (a) $\big((x-w)^2 + y^2 + t^2\big)/(2yt) = 1 + b(\boldsymbol{x}, \boldsymbol{w}) < 2$; ($b$) $\frac{1}{4} < y/t < 4$, $|x - w|/y < 4$; (c) $|1 - t/y| < \sqrt{8b(\boldsymbol{x}, \boldsymbol{w})}$.

Let us evaluate the derivatives of b:

$$b_x = (x - w)/(yt)\,, \qquad b_y = \big(y^2 - (x-w)^2 - t^2\big)/(2ty^2)$$
$$b_{xx} = 1/(yt)\,, \qquad b_{xy} = -(x - w)/(y^2 t)\,,$$
$$b_{yy} = \big((x - w)^2 + t^2\big)/(y^3 t)\,.$$

In $b < 1$ by (a), (b), (c), we have:

$$|b_x| \leqslant c y^{-1} \sqrt{b}\,, \qquad |b_y| \leqslant c y^{-1} \sqrt{b}\,, \tag{3.6}$$

$$|b_{xx}| \leqslant c/y^2\,, \qquad |b_{xy}| \leqslant c/y^2\,, \qquad |b_{yy}| \leqslant c/y^2 \tag{3.7}$$

(in the lemma c will be any constant not depending on $\boldsymbol{x}, \boldsymbol{w}$).

Let us write now $G(\boldsymbol{x}, \boldsymbol{w})$ in a different way. Let $P_{\mu/2-1}(z)$ the Legendre function of the first kind (which is holomorphic in $|1-z|<2$, and $P_{\mu/2-1}(1)=1$), and $Q_{\mu/2-1}(z)$ the Legendre function of second kind (holomorphic in the complex plane cut along the real axis from $-\infty$ to 1).

An integral representation of $Q_{\mu/2-1}$ is:

$$Q_{\mu/2-1}(z) = 2^{-\mu/2}\int_0^\pi (z-\cos t')^{-\mu/2}(\sin t')^{\mu-1}\,dt'$$

(see ERDÉLYI and others [9], I, p. 155, formula (3.5) and substitution $t'=\pi-t$) This formula and (3.1) give:

$$G(\boldsymbol{x}, \boldsymbol{w}) = \frac{1}{\pi}(t/y)^{\mu/2}Q_{\mu/2-1}(1+b(\boldsymbol{x}, \boldsymbol{w})) \qquad (\boldsymbol{x}\neq\boldsymbol{w})\,. \tag{3.8}$$

The function $Q_{\mu/2-1}$ can be written as:

$$\begin{aligned}Q_{\mu/2-1}(z) &= 2^{-1}P_{\mu/2-1}(z)\{-\log[(z-1)/(z+1)]-2\gamma-2\psi(\mu/2)\}+\\ &+\pi^{-1}\sin(\pi\mu/2)\sum_{l=1}^{\infty}(l!)^{-2}\Gamma(l+\mu/2)\Gamma(l+1-\mu/2)\cdot[\psi(l+1)-\psi(1)]((1-z)/2)^l\end{aligned} \tag{3.9}$$

where γ is the Euler-Mascheroni constant ($\gamma^{\sim}\cdot 577$), $\psi(z)=\Gamma'(z)/\Gamma(z)$ and the last power series is holomorphic in $|1-z|/2<1$ (ERDÉLYI and others [9], I, p. 149).

It follows that $G(\boldsymbol{x}, \boldsymbol{w})$ $(\boldsymbol{x}\neq\boldsymbol{w})$ can be written as:

$$\begin{aligned}G(\boldsymbol{x}, \boldsymbol{w}) &= (t/y)^{\mu/2}[R_1(b(\boldsymbol{x}, \boldsymbol{w}))\log b(\boldsymbol{x}, \boldsymbol{w}) + R_2(b(\boldsymbol{x}, \boldsymbol{w}))] =\\ &= S_1(\boldsymbol{x}, \boldsymbol{w})\log b(\boldsymbol{x}, \boldsymbol{w}) + S_2(\boldsymbol{x}, \boldsymbol{w})\end{aligned} \tag{3.10}$$

where the functions $R_i(b)$ are regular in $|b|<2$ and $R_1(0)=1/(2\pi)$.

Let us prove (3.5), by using (*a*), (*b*), (*c*) and (3.6), (3.7), (3.10). We have, in $b(\boldsymbol{x}, \boldsymbol{w})<1$:

$$|D_x(t/y)^{\mu/2}|\leqslant c/y\,, \qquad |D_x^2(t/y)^{\mu/2}|\leqslant c/y^2 \tag{3.11}$$

$$|D_xS_i|\leqslant c/y\,, \qquad |D_x^2S_i|\leqslant c/y^2 \qquad (i=1,2); \tag{3.12}$$

$$|S_1+1/(2\pi)|\leqslant c\sqrt{b}$$

and:

$$|D_x\log b(\boldsymbol{x}, \boldsymbol{w})|\leqslant c(y\sqrt{b})^{-1}$$

$$|D_x^2\log|\boldsymbol{x}-\boldsymbol{w}|^2|\leqslant\frac{c}{y^2b}\,.$$

By using these inequalities, we get

$$|D_x^2 G(\boldsymbol{x}, \boldsymbol{w}) + (2\pi)^{-1} D_x^2 \log|\boldsymbol{x}-\boldsymbol{w}|^2| \leqslant c(|(D_x^2 S_1)\lg b| + |D_x S_1||D_x \log b| + \\ + |S_1 + 1/(2\pi)||D_x^2 \log|\boldsymbol{x}-\boldsymbol{w}|^2| + |S_1 D_x^2 \lg(yt)| + |D_x^2 S_2|) \leqslant c y^{-2}/\sqrt{b}$$

and, by this inequality, the theorem follows. |||

LEMMA 3.3. – *Let* $\varepsilon > 0$; *there exists* k *such that, if* $\boldsymbol{x} \in \mathcal{R}_+^2$, $\boldsymbol{w} \in \mathcal{R}_+^2$, $b(\boldsymbol{x}, \boldsymbol{w}) > \varepsilon$, *then*:

$$|D_x^2 G(\boldsymbol{x}, \boldsymbol{w})| \leqslant k t^\mu [(x-w)^2 + t^2 + y^2]^{-1-\mu/2}. \tag{3.13}$$

PROOF. – In the region of the complex plane, given by $|z| > 1$ cut along the real axis from $-\infty$ *to* $+1$, the Legendre function of second kind $Q_{\mu/2-1}$ can be written as:

$$Q_{\mu/2-1}(z) = 2^{-\mu/2}\pi^{\frac{1}{2}}\Gamma(\mu/2) z^{-\mu/2}\Gamma(\mu/2)^{-1}\cdot F(\mu/4 + \tfrac{1}{2}, \mu/4, \mu/2 + \tfrac{1}{2}; z^{-2})$$

(ERDÉLYI and others [9], I, p. 122; $F(a, b, c; z)$ is Gauss hypergeometric function, which is holomorphic in $|z| < 1$).

This formula and (3.8) give, in $b(\boldsymbol{x}, \boldsymbol{w}) > 0$:

$$G(\boldsymbol{x}, \boldsymbol{w}) = c t^\mu [(x-w)^2 + y^2 + t^2]^{-\mu/2}\cdot \\ \cdot F(\mu/4 + \tfrac{1}{4}, \mu/4, \mu/2 + \tfrac{1}{2}, \{2yt/[(x-w)^2 + y^2 + t^2]^2\})$$

(c will be any constant not depending on $\boldsymbol{x}, \boldsymbol{w}$).

Let us consider the function:

$$g(x, y, t) = (x^2 + y^2 + t^2)^{-\mu/2}\cdot F(\mu/4 + \tfrac{1}{2}, \mu/4, \mu/2 + \tfrac{1}{2}; \{2yt/[x^2 + y^2 + t^2]^2\})$$

in the cone $\{(x, y, t) \in \mathcal{R}^3: 2yt/[x^2 + y^2 + t^2] \leqslant 1/(1 + \varepsilon/2)\}$; the function g is homogeneous of degree-μ and smooth in the cone; thus the second derivatives of g are homogeneous of degree -2-μ and bounded on the intersection of the cone and the unit sphere. This is equivalent to say that, in $b(\boldsymbol{x}, \boldsymbol{w}) > \varepsilon$:

$$|D_x^2 G(\boldsymbol{x}, \boldsymbol{w})| \leqslant c t^\mu [(x-w)^2 + y^2 + t^2]^{-\mu/2-1}. \quad |||$$

THEOREM 3.4. – *Let* $p > 1$, $\mu > 0$, $0 < \alpha + 1/p < 1 + \mu$. *There exists* K *such that, for every* $u \in C^\infty(\mathcal{R}_+^2)$ *satisfying*:

(i) u has bounded support;

(ii) $\lim_{y\to 0^+} y^\mu |u_y(\cdot, y)|_1 = 0$;

(iii) $y^\alpha \mathcal{W} u \in L^p(\mathcal{R}_+^2)$, *the inequality*

$$\|D^2u\|_{\alpha,p} \leqslant K\|\mathcal{W}u\|_{\alpha,p} \tag{3.14}$$

holds.

PROOF.

Step 1. – First of all, we need to write the second derivatives of u in terms of $\mathcal{W}u$.

Notice that, by theorem 2.2, u satisfies (i) and (ii) of lemma 3.1; (iii) of lemma 3.1 is a consequence of (iii) above and Hölder inequality, as in theorem 2.2.

Thus, lemma 3.1 gives:

$$u(\boldsymbol{x}) = -\iint_{\mathcal{R}^2_+} G(\boldsymbol{x}, \boldsymbol{w})\,\mathcal{W}u(\boldsymbol{w})\, d\boldsymbol{w}\,;$$

using formula (3.3) above, we have that:

$$D^2u(\boldsymbol{x}) = c\,\mathcal{W}u(\boldsymbol{x}) + {}^*\!\!\iint_{\mathcal{R}^2_+} - D^2_{\boldsymbol{x}}G(\boldsymbol{x}, \boldsymbol{w})\,\mathcal{W}u(\boldsymbol{w})\, d\boldsymbol{w}\,,$$

where c depends on the choice of the derivative D^2 and last integral is in principal value.

Step 2. – Define $E_{\boldsymbol{x}} = \{\boldsymbol{w} \in \mathcal{R}^2_+ : |\boldsymbol{x} - \boldsymbol{w}| < (\sqrt{3} - 1)y\}$,

$$\boldsymbol{w} = (w, t)\,, \qquad g(\boldsymbol{w}) = t^\alpha\,\mathcal{W}u(\boldsymbol{w})\,,$$

and:

$$\Phi_1(\boldsymbol{x}) = {}^*\!\!\iint_{E_{\boldsymbol{x}}} (y/t)^\alpha D^2_{\boldsymbol{X}}G(\boldsymbol{x}, \boldsymbol{w})g(\boldsymbol{w})\, d\boldsymbol{w}\,,$$

$$\Phi_2(\boldsymbol{x}) = \iint_{\mathcal{R}^2_+ \setminus E_{\boldsymbol{x}}} (y/t)^\alpha D^2_{\boldsymbol{x}}G(\boldsymbol{x}, \boldsymbol{w})g(\boldsymbol{w})\, d\boldsymbol{w}\,.$$

The thesis will be proved if it exists k (not depending on g), such that:

$$\|\Phi_1\|_{0,p} \leqslant k\|g\|_{0,p}\,, \tag{3.15}$$

$$\|\Phi_2\|_{0,p} \leqslant k\|g\|_{0,p}\,. \tag{3.16}$$

Step 3₁. – Proof of (3.15).

Notice that $E_{\boldsymbol{x}} \subset \{\boldsymbol{w} \in \mathcal{R}^2_+ : b(\boldsymbol{x}, \boldsymbol{w}) < 1\}$. By using lemma 3.2, we can write

$\Phi_1 = \Psi_1 + \Psi_2 + \Psi_3$, where:

$$\Psi_1(\boldsymbol{x}) = -\frac{1}{2\pi} {}^*\!\!\iint_{E_{\boldsymbol{x}}} D_X^2 \log |\boldsymbol{x} - \boldsymbol{w}| g(\boldsymbol{w})\, d\boldsymbol{w}$$

$$\Psi_2(\boldsymbol{x}) = \iint_{E_{\boldsymbol{x}}} (y/t)^\alpha S(\boldsymbol{x}, \boldsymbol{w}) g(\boldsymbol{w})\, d\boldsymbol{w}$$

$$\Psi_3(\boldsymbol{x}) = -\frac{1}{2\pi} \iint_{E_{\boldsymbol{x}}} [(y/t)^\alpha - 1] D_x^2 \log |\boldsymbol{x} - \boldsymbol{w}| g(\boldsymbol{w})\, d\boldsymbol{w}\,.$$

Let us extend $g \equiv 0$ in $y \leqslant 0$ and Ψ_1 accordingly; we can write Ψ_1 as:

$$\Psi_1(\boldsymbol{x}) = \frac{1}{2\pi}\Big(- {}^*\!\!\iint_{\mathfrak{R}^2} + \iint_{\mathfrak{R}^2 \setminus E_{\boldsymbol{x}}}\Big) D_x^2 \log |\boldsymbol{x} - \boldsymbol{w}| g(\boldsymbol{w})\, d\boldsymbol{w}\,.$$

The first integral in the right hand side is a standard singular integral; its L^p norm can be bounded by $c\|g\|_{0,p}$, for all $p > 1$; the second integral can be bounded by using thm. 1, chapter 2 of CALDERON-ZYGMUND [6]:

$$\iint_{\mathfrak{R}^2} \Big| \iint_{|\boldsymbol{x}-\boldsymbol{w}| > (\sqrt{3}-1)y} D_x^2 \log |\boldsymbol{x} - \boldsymbol{w}| g(\boldsymbol{w})\, d\boldsymbol{w} \Big|^p d\boldsymbol{x} \leqslant$$

$$\leqslant \iint_{\mathfrak{R}^2} \sup_{\lambda > 0} \Big| \iint_{|\boldsymbol{x}-\boldsymbol{w}| > 1/\lambda} D_x^2 \log |\boldsymbol{x} - \boldsymbol{w}| g(\boldsymbol{w})\, d\boldsymbol{w} \Big|^p d\boldsymbol{x} \leqslant c \iint_{\mathfrak{R}^2} |g(\boldsymbol{x})|^p\, d\boldsymbol{x}\,.$$

Thus:

$$\|\Psi_1\|_{0,p} \leqslant c \|g\|_{0,p} \tag{3.17}$$

(c will be any constant not depending on g).

In $b(\boldsymbol{x}, \boldsymbol{w}) < 1$, we have ((b) lemma 3.2) $\frac{1}{4} < y/t < 4$. Thus, by (3.5) and Hölder inequality:

$$\iint_{\mathfrak{R}^2_+} |\Psi_2(\boldsymbol{x})|^p\, d\boldsymbol{x} \leqslant c \iint_{\mathfrak{R}^2_+} y^{-2p} \Big| \iint_{E_{\boldsymbol{x}}} (b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}} g(\boldsymbol{w})\, d\boldsymbol{w} \Big|^p d\boldsymbol{x} \leqslant$$

$$\leqslant c \iint_{\mathfrak{R}^2_+} y^{-2p} \Big[\Big(\iint_{E_{\boldsymbol{x}}} (b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}}\, d\boldsymbol{w} \Big)^{1/p'} \Big(\iint_{E_{\boldsymbol{x}}} (b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}} |g(\boldsymbol{w})|^p\, d\boldsymbol{w} \Big)^{1/p} \Big]^p d\boldsymbol{x}\,.$$

It is not difficult to show that:

$$\iint_{E_{\boldsymbol{x}}} (b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}}\, d\boldsymbol{w} \leqslant c y^2\,.$$

Thus:

$$\iint_{\mathfrak{R}^2_+} |\Psi_2(\boldsymbol{x})|^p\, d\boldsymbol{x} \leqslant c \iint_{\mathfrak{R}^2_+} y^{-2}\, d\boldsymbol{x} \iint_{E_{\boldsymbol{x}'}} (b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}} |g(\boldsymbol{w})|^p\, d\boldsymbol{w}\,.$$

Exchanging the integrals in the right hand side, we get:

$$\iint_{\mathcal{R}^2_+} |\Psi_2(\boldsymbol{x})|^p \, d\boldsymbol{x} \leqslant c \iint_{\mathcal{R}^2_+} |g(\boldsymbol{w})|^p \, d\boldsymbol{w} \iint_{|\boldsymbol{x}-\boldsymbol{w}|<(\sqrt{3}-1)y} y^{-2}(b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}} \, d\boldsymbol{w} \, .$$

It is not difficult to see that there exists a constant $c > 0$ such that

$$\int_{|\boldsymbol{x}-\boldsymbol{w}|<(\sqrt{3}-1)y} y^{-2}(b(\boldsymbol{x}, \boldsymbol{w}))^{-\frac{1}{2}} \, d\boldsymbol{x} \leqslant c$$

($c = 2^q$ will do); it follows:

$$\|\Psi_2\|_{L^p(\mathcal{R}^2_+)} \leqslant c \|g\|_{L^p(\mathcal{R}^2_+)} \, . \tag{3.18}$$

As in lemma 3.2, one can prove that

$$|1 - (y/t)^\alpha| \cdot |D^2 \log |\boldsymbol{x} - \boldsymbol{w}|| \leqslant c y^{-2}[b(\boldsymbol{x}, \boldsymbol{w})]^{-\frac{1}{2}};$$

with the same proof above, we have:

$$\|\Psi_3\|_{L^p(\mathcal{R}^2_+)} \leqslant c \|g\|_{L^p(\mathcal{R}^2_+)} \, . \tag{3.19}$$

The inequalities (3.17), (3.18), (3.19) give (3.15).

Step 3_2. – Proof of (3.16).
Let

$$k_0 = (\sqrt{3} - 1)/\sqrt{2} \, , \quad k_1 = 1 - k_0 (> 0) \, , \quad k_2 = 1 + k_0 \, ,$$
$$T_{\boldsymbol{x}} = \{\boldsymbol{w} \in \mathcal{R}^2_+ : |x - w| < k_0 y, \ |y - t| < k_0 y\};$$

let us define:

$$F_{y,t}(x) = (y/t)^\alpha y^\mu |x^2 + y^2 + t^2|^{-1-\mu/2} \, , \quad x \in \mathcal{R}, \ y, t > 0 \, .$$

Notice that, for $\boldsymbol{x} \in \mathcal{R}^2_+$:

$$\{\boldsymbol{w} \in \mathcal{R}^2_+ : b(\boldsymbol{x}, \boldsymbol{w}) \leqslant \tfrac{1}{4}\} \subset T_{\boldsymbol{x}} \subset E_{\boldsymbol{x}};$$

by lemma 3.3, we get:

$$|D^2_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{w})| \leqslant k t^\mu [(x - w)^2 + t^2 + y^2]^{-1-\mu/2}$$

$(\boldsymbol{x} \in \mathfrak{R}^2_+,\ \boldsymbol{w} \notin T_{\boldsymbol{x}})$; thus:

$$|\Phi_2(\boldsymbol{x})| \leqslant c \iint_{\mathfrak{R}^2_+ \setminus T_{\boldsymbol{x}}} F_{y,t}(x-w) g(\boldsymbol{w})\, d\boldsymbol{w}\,.$$

Let us write $g_t = |g(\cdot, t)|$, $t > 0$; then:

$$|\Phi_2(\boldsymbol{x})| \leqslant c\left\{\int_0^{k_1 y} (F_{y,t} * g_t)(x)\, dt + \int_{k_2 y}^{+\infty} (F_{y,t} * g_t)(x)\, dt + \int_{k_1 y}^{k_2 y} dt \int_{|x-w|>k_0 y} F_{y,t}(x-w) g_t(w)\, dw\right\}$$

($*$ is the convolution in $\mathfrak{R}$).

Let J_1, J_2, J_3 the $L^p(\mathfrak{R}^2_+)$ norms of the last three terms; (3.16) will be proved, if:

$$J_\nu \leqslant c\|g\|_{L^p(\mathfrak{R}^2_+)}\,, \qquad \nu = 1, 2, 3\,. \tag{3.20}$$

By Minkowski integral inequality:

$$J_1 = \left(\int_0^{+\infty}\left[\left|\int_0^{ky} (F_{yt} * g_t)\, dt\right|_p\right]^p dy\right)^{1/p} \leqslant \left(\int_0^{+\infty}\left[\int_0^{ky} |F_{yt} * g_t|_p\, dt\right]^p dy\right)^{1/p}.$$

By convolution theorem:

$$J_1 \leqslant \left(\int_0^{+\infty}\left[\int_0^{ky} |F_{yt}|_1 \cdot |g_t|_p\, dt\right]^p dy\right)^{1/p},$$

we have:

$$|F_{yt}|_1 = \int_{\mathfrak{R}} (y/t)^\alpha t^\mu [x^2 + y^2 + t^2]^{1+\mu/2}\, dx = (y/t)^\alpha t^\mu (y^2 + t^2)^{-\frac{1}{2}-\mu/2} \int_{\mathfrak{R}} (1 + t^2)^{-1-\mu/2}\, dt \tag{3.21}$$

and:

$$(y^2 + t^2)^{-\frac{1}{2}-\mu/2} \leqslant y^{-1-\mu};$$

thus:

$$J_1 \leqslant c\left(\int_0^{+\infty}\left[y^{-1+\alpha-\mu}\int_0^{k_1 y} t^{\mu-\alpha}|g_t|_p\, dt\right]^p dy\right)^{1/p}.$$

Making the change of variable $\theta = k_1 y$, and using Hardy inequality, we get

$$J_1 \leqslant c(p/(p\mu - p - \alpha p - 1))\left(\int_0^{+\infty} |g_t|_p^p\, dt\right)^{1/p} \leqslant c\|g\|_{L^p(\mathfrak{R}^2_+)};$$

notice that $1 + p(\alpha - \mu - 1) < 0$ is equivalent to $\alpha + 1/p < 1 + \mu$, and the first of (3.20) is proved.

Similarly:

$$J_2 \leqslant \left(\int_0^{+\infty} \left[\int_{k_2 y}^{+\infty} |F_{yt}|_1 |g_t|_p \, dt \right]^p dy \right)^{1/p};$$

from (3.21) and $(y^2 + t^2)^{-\frac{1}{2}-\mu/2} \leqslant t^{-\mu-1}$, we get:

$$J_2 \leqslant c \left(\int_0^{+\infty} \left[y^\alpha \int_{k_2 y}^{+\infty} t^{-\alpha-1} |g_t|_p \, dt \right]^p dy \right)^{1/p}.$$

By scaling y, and using Hardy inequality, we get:

$$J_2 \leqslant c \|g\|_{L^p(\mathcal{R}^2_+)} .$$

Let us prove the last of the (3.20)'s. Let χ_y be the characteristic function of $\mathcal{R} \setminus [-k_0 y, k_0 y]$; J_3 can be written as:

$$J_3 = \left(\int_0^{+\infty} \left| \int_{k_1 y}^{k_2 y} (\chi_y \cdot F_{yt}) * g_t \, dt \right|_p^p dy \right)^{1/p}.$$

Thus:

$$J_3 \leqslant c \left(\int_0^{+\infty} \left[\int_{k_1 y}^{k_2 y} |\chi_y F_{yt}|_1 \cdot |g_t|_p \, dt \right]^p dy \right)^{1/p}.$$

If $0 < k_1 y \leqslant t \leqslant k_2 y$:

$$|\chi_y F_{yt}|_1 \leqslant c \int_{|x| > k_0 y} t^\mu (x^2 + y^2 + t^2)^{-1-\mu/2} \, dx \leqslant c/y .$$

It follows:

$$J_3 \leqslant c \left(\int_0^{+\infty} \left[y^{-1} \int_0^{k_2 y} |g_t|_p \, dt \right]^p dx \right)^{1/p} \leqslant c \|g\|_{L^p(\mathcal{R}^2_+)} .$$

Thus estimate (3.16) is established. The proof of the theorem is so complete. |||

REMARK 3.2. – It is worthy noting explicitly that, under the assumptions of theorem 3.4, an estimate similar to (3.14) holds also for $(1/y) u_y$.

4. – A boundary value problem for the nonhomogeneous equation.

In this section we deal with a boundary value problem for the non homogeneous equation (0.1) in a rectangle R, within a proper weighted Sobolev class.

Precisely, in the rectangle

$$R = (x_1, x_2) \times (0, y_0) = A \times (0, y_0)\,, \qquad 0 < y_0 < +\infty$$

we look for solutions of the equation

$$\mathcal{W}_\mu u = f \tag{4.1}$$

satisfying the boundary conditions

$$u|_{T \cup E} = 0 \tag{4.2}$$

and

$$\lim_{y \to 0^+} y^\mu \int_A |u_y(x, y)|\, dx = 0\,. \tag{4.3}$$

The condition (4.2) in the appropriate trace sense; as in section 1., T stands for the union of vertical sides of R while E denotes the horizontal side $y = y_0$.

The natural function spaces for the problem we are going to study are as follows.

Let $p > 1$, $\alpha \in \mathcal{R}$. To begin with, let $L^p_\alpha(R)$ the class of all measurable functions $u = u(x, t)$ defined in R such that

$$\|u\|^p_{L^p_\alpha(R)} = \iint_R y^{\alpha p} |u|^p\, dx\, dy < +\infty\,.$$

Next let $X^{2,p}_\alpha(R)$ denote the completion of the space

$$\{u \in C^2(R) \colon \|u\|_{X^{2,p}_\alpha(R)} < +\infty\}$$

where

$$\|u\|^p_{X^{2,p}_\alpha(R)} = \sum_{0 \leq |\beta| \leq 2} \|D^\beta u\|^p_{L^p_\alpha(R)} + \left\|\frac{u_y}{y}\right\|^p_{L^p_\alpha(R)}\,.$$

The following properties hold.

LEMMA 4.1. – *For any $u \in X^{2,p}_\alpha(R)$, $0 < \alpha + 1/p < 1 + \mu$, $\mu > 0$, the following estimates hold:*

$$y^\mu \int_A |u_y(x, y)|\, dx \leq (x_2 - x_1)^{1/p'} (\mu p' - \alpha p' + 1)^{-1/p'} \cdot y^{\mu - \alpha + 1/p'} \left\| u_{yy} + \frac{\mu}{y} u_y \right\|_{L^p_\alpha(R)} \tag{4.4}$$

where $1/p + 1/p' = 1$, $0 < y < y_0$, *and*

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} \leqslant \frac{p}{p(1-\alpha+\mu)-1}\left\|u_{yy}+\frac{\mu}{y}u_y\right\|_{L^p_\alpha(R)}. \tag{4.5}$$

PROOF. – It is enough to take $u \in C^2(R)$ such that $\|u\|_{X^{2,p}_\alpha(R)} < +\infty$. For $0 < \varepsilon < y < y_0$, one has:

$$y^\mu u_y(x, y) - \varepsilon^\mu u_y(x, \varepsilon) = \int_\varepsilon^y \frac{\partial}{\partial y}\left(y^\mu u_y(x, y)\right) dy = \int_\varepsilon^y y^\mu \left\{u_{yy} + \frac{\mu}{y} u_y\right\} dy . \tag{4.6}$$

On the other hand

$$\iint_R y^\mu \left|\frac{u_y}{y}\right| dx\, dy \leqslant (\text{const})\left(\iint_R y^{\alpha p}\left|\frac{u_y}{y}\right|^p dx\, dy\right)^{1/p} < +\infty \tag{4.7}$$

$$\iint_R y^\mu |u_{yy}|\, dx\, dy \leqslant (\text{const})\left(\iint_R y^{\alpha p}|u_{yy}|^p\, dx\, dy\right)^{1/p} < +\infty, \tag{4.8}$$

where $\text{const} = (x_2 - x_1)^{1/p'}(\mu p' - \alpha p' + 1)^{-1/p'} y_0^{\mu-\alpha+1/p'}$.

It follows that, for almost every $x \in A$, the integral in (4.6) is convergent as $\varepsilon \to 0^+$ and therefore $\lim_{\varepsilon\to 0^+} \varepsilon^\mu u_y(x, \varepsilon)$ is finite. By (4.7) clearly

$$\lim_{\varepsilon\to 0^+} \varepsilon^\mu u_y(x, \varepsilon) = 0 .$$

Thus, for $0 < y < y_0$, one has:

$$y^\mu u_y(x, y) = \int_0^y y^\mu \left\{u_{yy} + \frac{\mu}{y} u_y\right\} dy . \tag{4.9}$$

Taking the absolute value, integrating over $A = (x_1, x_2)$ and using Hölder inequality we easily get the stated estimate (4.4).

To get then estimate (4.5) we take into account (4.9) and make use of Hardy's inequality. |||

LEMMA 4.2. – *Let* $u \in X^{2,p}_\alpha(R)$ *and assume* $0 < \alpha + 1/p < 1$. *Then*

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} \leqslant \frac{p}{p(1-\alpha)-1}\|u_{yy}\|_{L^p_\alpha(R)} .$$

PROOF. – As in lemma 4.1, take $u \in C^2(R)$ such that $\|u\|_{X^{2,p}_\alpha(R)} < +\infty$.

It turns out that

$$u_y(x, y) = \int_0^y u_{yy}(x, \eta)\, d\eta \,.$$

Indeed, because of the fact that $u_y/y \in L^p_\alpha(R)$ and $0 < \alpha + 1/p < 1$, one has

$$\lim_{y \to 0^+} u_y(x, y) = 0 \,.$$

Therefore, via Hardy's inequality, the stated estimate easily follows. |||

Now in what follows $X^{2,p}_{\alpha,\gamma_0}(R)$ will stand for the closure, with respect to $X^{2,p}_\alpha(R)$' of the space

$$\{u \in C^2(R \cup T \cup E) \colon u|_{T \cup E} = 0, \ \|u\|_{X^{2,p}_\alpha(R)} < +\infty\} \,.$$

We define

$$\|u\|_{X^{2,p}_{\alpha,\gamma_0}(R)} = \left(\iint y^{\alpha p}\left(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2 + \left|\frac{u_y}{y}\right|^2\right)^{p/2} dx\, dy\right)^{1/p} . \tag{4.10}$$

It is not hard to see that the imbedding of $X^{2,p}_{\alpha,\gamma_0}(R)$ in $X^{2,p}_\alpha(R)$ is continuous; namely there exists a constant H, depending only on R, such that for any $u \in X^{2,p}_{\alpha,\gamma_0}(R)$

$$\|u\|_{X^{2,p}_\alpha(R)} \leqslant H \|u\|_{X^{2,p}_{\alpha,\gamma_0}(R)} \,.$$

Thus in the space $X^{2,p}_{\alpha,\gamma_0}(R)$ the norms

$$\|\cdot\|_{X^{2,p}_{\alpha,\gamma_0}(R)} \quad \text{and} \quad \|\cdot\|_{X^{2,p}_\alpha(R)}$$

are equivalent.

Moreover, if $0 < \alpha + 1/p < 1$ then, by lemma 4.2, in the space $X^{2,p}_{\alpha,\gamma_0}(R)$ an equivalent norm is the following one

$$\|u\|_{X^{2,p}_{\alpha,\gamma_0}(R)} = \left(\iint_R y^{\alpha p}(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2)^{p/2}\, dx\, dy\right)^{1/p} . \tag{4.11}$$

Finally, denote by $\overset{0}{X}{}^{2,p}_\alpha(R)$ the closure in the $X^{2,p}_\alpha$-topology of the space

$$\{u \in C^\infty(\mathcal{R}^2_+) \colon \operatorname{supp} u \subset R, \ \|u\|_{X^{2,p}_\alpha(R)} < +\infty\} \,.$$

Let $\mu > 0$, $0 < \alpha + 1/p < 1 + \mu$. Owing to theorem 3.2 and recalling remark 3.2, *for any* $u \in \overset{0}{X}{}^{2,p}_\alpha(R)$ *the following inequality holds*:

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} + \|D^2 u\|_{L^p_\alpha(R)} \leqslant C \|\mathcal{W} u\|_{L^p_\alpha(R)} ,$$

where C is a constant depending on μ, α, p, R.

REMARK 4.1. – We stress the fact that functions $u \in X^{2,p}_{\alpha,\gamma_0}(R)$ satisfy condition (4.3); for take into account estimate (4.4).

Moreover, we are able to prove a similar estimate for functions $u \in X^{2,p}_{\alpha,\gamma_0}(R)$.

LEMMA 4.3. – *For any* $u \in X^{2,p}_{\alpha,\gamma_0}(R)$, $p > 1$, $0 < \alpha + 1/p < 1 + \mu$, $\mu > 0$, *the following a priori estimate holds*:

$$\|u\|_{X^{2,p}_{\alpha,\gamma_0}(R)} \leqslant C \|\mathcal{W} u\|_{L^p_\alpha(R)}, \tag{4.12}$$

C being a constant depending on μ, α, p, R.

PROOF. – We start by making the following remark: for functions $v \in C^\infty(\{\boldsymbol{x} \in \mathcal{R}^2_+ : x \geqslant k\})$ with bounded support, satisfying conditions

$$v|_{x=k} = 0 \quad \text{and} \quad \lim_{y\to 0^+} y^\mu \int |v_y|\, dx = 0 ,$$

an inequality of the type

$$\left\|\frac{v_y}{y}\right\|_{\alpha,p} + \|D^2 v\|_{\alpha,p} \leqslant (\text{const}) \|\mathcal{W} v\|_{\alpha,p}$$

holds, the norms being taken in the region $\{\boldsymbol{x} \in \mathcal{R}^2_+ : x > k\}$.

For, it is enough to take the reflection $\tilde{v}$ of v through $x = k$ and apply to it the results of previous section. Indeed $\tilde{v} \in C^{1,1}_0(\mathcal{R}^2_+)$ and so $\tilde{v} \in \overset{0}{X}{}^{2,p}_\alpha(\mathcal{R}^2_+)$; moreover $\mathcal{W}\tilde{v} = \widetilde{\mathcal{W} v}$. An analogous inequality holds for functions v as above but with support in $\{\boldsymbol{x} \in \mathcal{R}^2_+ : x \leqslant k\}$.

Now to prove the lemma we suitably make a partition of unity in $\bar{R}$: $1 = \varphi_1 + \varphi_2 + \varphi_3$, $\varphi_i \in C^\infty_0(\bar{R})$, $\varphi_i \geqslant 0$ $(i = 1, 2, 3)$. More precisely, let $\psi(t)$ a $C^\infty([0, +\infty))$ function such that $0 \leqslant \psi \leqslant 1$

$$\psi(t) = 0 \quad \text{if} \quad 0 \leqslant t \leqslant \eta/2$$

$$\psi(t) = 1 \quad \text{if} \quad \eta \leqslant t$$

with $0 < \eta < \min\{x_2 - x_1, y_0\}$.

Define in $\bar{R}$:

$$\varphi_1(x, y) = \psi(y)$$

$$\varphi_2(x, y) = \psi(x - x_1)[1 - \psi(y)]$$

$$\varphi_3(x, y) = [1 - \psi(x - x_1)][1 - \psi(y)] .$$

Assume u smooth in $R \cup T \cup E$, $u|_{T\cup E} = 0$ and such that $\|u\|_{X^{2,p}_\alpha(R)} < +\infty$. Define $u_i = u\varphi_i$, $i = 1, 2, 3$.

Notice that on the support of the function u_1 the operator $\mathcal{W}$ is smooth and on the other hand the functions u_2 and u_3 fall in the remark we started with. Therefore in R we have

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} + \|D^2u\|_{L^p_\alpha(R)} \leqslant (\text{const}) \sum_1^3 \|\mathcal{W}u_i\|_{L^p_\alpha(R)} .$$

By explicit evaluation

$$\mathcal{W}u_i = \varphi_i \mathcal{W}u + u\mathcal{W}\varphi_i + 2(\nabla u)\cdot(\nabla\varphi_i) .$$

Thus, because of the above proper choice of φ_i and by means of interpolation inequalities, we get

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} + \|D^2u\|_{L^p_\alpha(R)} \leqslant (\text{const})[\|\mathcal{W}u\|_{L^p_\alpha(R)} + \|u\|_{L^p_\alpha(R)}] .$$

Recall now that estimate (2.4) holds (see remark 2.3). We infer that

$$\left\|\frac{u_y}{y}\right\|_{L^p_\alpha(R)} + \|D^2u\|_{L^p_\alpha(R)} \leqslant (\text{const})\|\mathcal{W}u\|_{L^p_\alpha(R)} .$$

An approximation argument finally allows us to deduce the claimed estimate (4.12) for any $u \in X^{2,p}_{\alpha,y_0}(R)$. |||

The a priori estimate (4.12) yields at once *uniqueness* of solution to the equation (4.1) in the class $X^{2,p}_{\alpha,y_0}(R)$. Uniqueness for the problem (4.1)-(4.3) has been also guaranteed in theorem 1.1.

It is not difficult then to get the following existence (uniqueness) result.

THEOREM 4.1. – *Assume that* $f \in L^p_\alpha(R)$, $p > 1$, $0 < \alpha + 1/p < 1 + \mu$, $\mu > 0$. *Then there exists a (unique) solution* $u \in X^{2,p}_{\alpha,y_0}(R)$ *of equation* (4.1).

PROOF. – Suppose first $f \in C^\infty_0(R)$. Then the function $u = u(x, y)$ given by (2.2) turns out to be a C^∞ solution of equation (4.1) in R. Moreover such a function u vanishes on the sides $x = x_1$, $x = x_2$, $y = y_0$ of R; the condition on $y = 0$ is also satisfied. Owing to the estimate (4.12) the function u belongs to the space $X^{2,p}_{\alpha,y_0}(R)$ and

$$\|u\|_{X^{2,p}_{\alpha,y_0}(R)} \leqslant C\|f\|_{L^p_\alpha(R)}$$

with constant C depending on μ, α, p, R. Take now $f \in L^p_\alpha(R)$. There exists a sequence of functions $f_n \in C^\infty_0(R)$ converging to f in $L^p_\alpha(R)$. Thus, as we have shown, for each integer n there exists a function u_n such that

$$\begin{cases} u_n \in X^{2,p}_{\alpha,y_0}(R) , \\ \mathcal{W}_\mu u_n = f_n . \end{cases}$$

Since $\mathcal{W}_\mu(u_m - u_n) = f_m - f_n$ we have

$$\|u_m - u_n\|_{X^{2,p}_{\alpha,\gamma_0}(R)} \leqslant C\|f_m - f_n\|_{L^p_\alpha(R)} \to 0\,, \qquad (m, n \to \infty)\,.$$

Therefore, by the completeness of $X^{2,p}_{\alpha,\gamma_0}(R)$, there exists a function $u \in X^{2,p}_{\alpha,\gamma_0}(R)$ such that $\|u_n - u\|_{X^{2,p}_{\alpha,\gamma_0}(R)} \to 0$ $(n \to \infty)$.

It is an easy matter to see that $\mathcal{W}_\mu u = f$ a.e. in R.

The proof of the theorem is so complete. |||

REMARK 4.2. – We notice that in our study the condition $0 < \alpha + 1/p < 1 + \mu$ is sharp.

A) If $p > 1$, $\alpha + 1/p = 1 + \mu$, $\mu > 0$, the a priori bound (4.12) does not hold. For, let $\psi \in C^\infty(\mathcal{R})$, $0 \leqslant \psi \leqslant 1$, $\psi(t) = 0$ if $t \leqslant 0$, $\psi(t) = 1$ if $t \geqslant 1$.

In the rectangle $R = (0, 2) \times (0, 2)$ we define

$$v_n(x, y) = \psi(x)\psi(2 - x)\psi(ny)\psi(2 - y)\,, \qquad n \in \mathcal{N}\,.$$

We have $v_n \in C_0^\infty(R)$. Moreover:

$$(4.13) \qquad \begin{cases} \displaystyle\iint_R \left|y^{1-\mu+\alpha}\frac{\partial^2 v_n}{\partial x^2}\right|^p dx\,dy \leqslant C_1 \\ \displaystyle\iint_R \left|y^{1-\mu+\alpha}\frac{\partial^2 v_n}{\partial y^2}\right|^p dx\,dy \leqslant C_2 \\ \displaystyle\iint_R \left|y^{\alpha-\mu}\frac{\partial v_n}{\partial y}\right|^p dx\,dy \leqslant C_3 \end{cases}$$

and

$$(4.14) \qquad \iint_R |y^{\alpha-\mu-1}v_n|^p\,dx\,dy \geqslant C_4 \log n$$

where C_1, C_2, C_3, C_4 are constants independent of n.

Let us define $u_n = y^{1-\mu}v_n$, $n \in \mathcal{N}$. By (4.13) and (4.14) we get

$$\iint_R |y^\alpha \mathcal{W} u_n|^p\,dx\,dy \leqslant C_5$$

and

$$\iint_R \left|y^\alpha \frac{\partial^2 u_n}{\partial y^2}\right|^p dx\,dy \geqslant C_6(\log n - 1)$$

where C_5 and C_6 are constants independent of n.

The above two inequalities show that the a priori estimate (4.12) fails.

B) If $\mu > 0$, $p > 1$, $\alpha + 1/p \leqslant 0$ the existence theorem 4.1 does not hold.

Indeed, let $R = (0, \pi) \times (0, 2)$ and ψ the function considered above. We define

$$\bar{u}(x, y) = \sin x h(y) \psi(2 - y)$$

where h is the function of section 2.

Notice that $\bar{u} \in C^\infty(\mathcal{R}^2)$, $\bar{u} = 0$ on $T \cup E$, $\bar{u}_y(x, 0) = 0$ and moreover $\mathcal{W}\bar{u} \in C^\infty(\mathcal{R}^2) \cap L^p(R)$.

Let us show that there is no function $u \in X^{2,p}_{\alpha,\gamma_0}(R)$, $\alpha + 1/p \leqslant 0$, such that $\mathcal{W}u = \mathcal{W}\bar{u}$. In fact, if there were such a function u we would have that

$$v = u - \bar{u} \in W^{2,p}_{\mathrm{loc}}(R \cup T \cup E)\,, \qquad y^\mu \int_0^\pi |v_y|\, dx \to 0 \qquad \text{as } y \to 0$$

and $\mathcal{W}v = 0$. Thus $v \in C^2(R) \cap C^0(R \cup T \cup E)$ and further, by the uniqueness theorem 1.1, we would have $u = \bar{u}$. On the other hand, we have

$$\iint_R |y^\alpha \bar{u}_{xx}|^p\, dx\, dy = +\infty\,,$$

so that $\bar{u}$ does not belong to $X^{2,p}_{\alpha,\gamma_0}(R)$. Contradiction. |||

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