On Approximation by Operator Semigroups of a General Type (*).

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Summary. – *For operator semigroups of class* (C_0) *on a Banach space* X *it is well known that the saturation class can be characterized as the relative completion with respect to* X *of the domain of the infinitesimal generator. This remains true for strongly measurable semigroups* $\{T(t),\ t>0\}$ *having a closed infinitesimal operator* A_0, *but it becomes false if* A_0 *is non-closed. We prove that a characterization is given by* $\|A_0T(t)f\| = \mathcal{O}(1),\ t\to 0+$ *for a fairly general class of semigroups, including certain particular semigroups which belong to Oharu's class* $(C_{(1)})$, *or are of growth order less than one.*

0. – Introduction.

The purpose of this paper is to investigate saturation classes of semigroups of a general type, including semigroups of Oharu's classes $(C_{(k)})$ ([10; p. 250]), which are not necessarily Abel summable. Semigroups of class $(C_{(k)})$ are met e.g. in connection with generalizations of continuous and discrete Trotter type theorems (see [13], [7]).

In the most simple case of a (C_0)-semigroup on a Banach space X with norm $\|\cdot\|$, it is well-known that the saturation order is always $\mathcal{O}(t)$ (see [2; Thm. 2.1.2]) and the saturation class $S(\{T(t)\})$, defined by

$$S(\{T(t)\}) = \{f\in X;\ \|T(t)f-f\| = \mathcal{O}(t),\ t\to 0+\}\,,$$

can be characterized by (see [1])

$$S(\{T(t)\}) = \widetilde{D(A)}^X\,. \tag{0.1}$$

Here A denotes the infinitesimal generator of the semigroup, $D(A)$ its domain, equipped with the graph norm $\|\cdot\|_{D(A)}$, and $\widetilde{D(A)}^X$ the relative completion of $D(A)$ with respect to X, defined by

$$\widetilde{D(A)}^X = \{f\in X;\ \exists\{f_n\}\subset D(A) \text{ with } \|f_n\|_{D(A)} = \mathcal{O}(1),\ \|f_n - f\| = o(1),\ n\to\infty\}\,. \tag{0.2}$$

In more general classes of semigroups, which will be investigated here, the infinitesimal operator A_0 is no longer closed, i.e. A_0 does not coincide with the infi-

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nitesimal generator A, the closure of A_0 (cf. [8; pp. 306, 344] for the basic definitions). So $\widetilde{D(A_0)}^X$ and $\widetilde{D(A)}^X$ can be considered as the first two candidates for a characterization of $S(\{T(t)\})$ then. Here we use the above definition of the relative completion of $D(A_0)$, though $D(A_0)$ is not necessarily a Banach subspace of X, i.e.

$$(0.3)\qquad \widetilde{D(A_0)}^X = \{f \in X;\ \exists\{f_n\} \subset D(A_0) \text{ with } \|f_n\|_{D(A_0)} = \mathcal{O}(1),\ \|f_n - f\| = o(1),\ n \to \infty\}\,,$$

where $\|f\|_{D(A_0)} = \|f\| + \|Af\| = \|f\|_{D(A)}$ for $f \in D(A_0)$. Our first objective is to show that neither of them is suited for this purpose (Theorem 1), a fact which makes a study of such general semigroups interesting. In particular, a precise description of when (0.1) holds or not can be given in terms of the closedness of A_0. For counterexamples we use two particular semigroups, which belong to Oharu's class $(C_{(1)})$, and have been considered in [7]. This will be section 1.

In section 2 we investigate a third candidate for characterizing $S(\{T(t)\})$, i.e. the condition

$$(0.4)\qquad \|A_0 T(t) f\| = \mathcal{O}(1)\,, \qquad t \to 0+\,.$$

For (C_0)-semigroups, (0.4) is a more or less trivial equivalence to $f \in S(\{T(t)\})$, but for more general semigroups, including semigroups of class $(C_{(k)})$, the fact that this equivalence remains valid appears to be new and harder to prove. For reflexive spaces X, we first prove Theorem 2, which reduces the proof of the equivalence to the verification of several conditions, the crucial one of which is (*b*). For non-reflexive spaces we have a less complete result (Theorem 3), which, however, suffices to treat the periodic example mentioned in section 1.

Applications of these results to the particular semigroups of section 1 will be considered in section 3.

1. – Saturation, $D(A_0)$, $D(A)$, and their relative completions.

Let X be a (complex) Banach space with norm $\|\cdot\|$, $[X]$ the space of bounded linear operators on X into X, and $\{T(t), t > 0\}$ a strongly measurable semigroup in X. Moreover, let $X_0 = \bigcup_{t>0} T(t)(X)$, let ω_0 be the type of the semigroup and $\Sigma = \{f \in X;\ \|T(t)f - f\| = o(1),\ t \to 0+\}$ its continuity set. By $R_0(\lambda)$ we denote the Laplace transform of $T(t)$, i.e., for $\lambda \in \mathbf{C}$ and $f \in X$

$$(1.1)\qquad R_0(\lambda) f = \int_0^\infty e^{-\lambda t}\, T(t) f\, dt\,,$$

whenever the integral exists as a Bochner integral. If $\operatorname{Re}(\lambda) > \omega_0$ then $\Sigma \subset D(R_0(\lambda))$.

The semigroup is supposed to satisfy the following conditions ([10; p. 249])

(C.1) $\bar{X}_0 = X$,

(C.2) there is an $\omega_1 > \omega_0$ such that there exists an operator $R(\lambda) \in [X]$ for all λ with $\operatorname{Re}(\lambda) > \omega_1$ and $R(\lambda)|_{X_0} = R_0(\lambda)|_{X_0}$,

(C.3) if $R(\lambda) f = 0$ for $\lambda > \omega_1$, then $f = 0$.

Denoting by A_0 the infinitesimal operator of the semigroup, conditions (C.1)-(C.3) imply the closability of A_0 and hence the existence of the infinitesimal generator A. If $\lambda \in \boldsymbol{C}$, $\operatorname{Re}(\lambda) > \omega_1$, λ belongs to the resolvent set $\varrho(A)$, and, denoting by $R(\lambda, A)$ the resolvent operator of A at λ, one has $R(\lambda) = R(\lambda, A)$ ([10; Lemma 6.2]). Moreover, property $(\mathrm{i})_0$ will be used, i.e. ([8; p. 322])

$$(\mathrm{i})_0 \quad \int_0^1 \|T(t) f\| \, dt \leqslant M_f < \infty, \quad \forall f \in X.$$

The class of semigroups with properties (C.1)-(C.3) and $(\mathrm{i})_0$ contains, for example, the classes of semigroups of growth order α for $\alpha \in [0, 1)$ (see [11], [15], [14], [6]).

For the proof of Theorem 1 we need two simple lemmas, the proofs of which will be omitted (concerning Lemma 2 cf. [14; Thm. 3]).

LEMMA 1. – *Let $\{T(t), t > 0\}$ be a strongly measurable semigroup satisfying* (C.1)-(C.3) *and* $(\mathrm{i})_0$. *Then $R_0(\lambda) = R(\lambda, A)$ for all λ with* $\operatorname{Re}(\lambda) > \omega_1$.

Defining $C(t)$ by

$$C(t) f = \frac{1}{t} \int_0^t T(u) f \, du \tag{1.2}$$

for $t > 0$, $f \in X$, one has $C(t) \in [X]$ under the assumptions of Lemma 1. Moreover,

LEMMA 2. – *Let $\{T(t), t > 0\}$ be as in Lemma 1. Then A_0 is closed if and only if $\|C(t) f - f\| = o(1)$, $t \to 0+$ for all $f \in X$.*

THEOREM 1. – *Let $\{T(t), t > 0\}$ be a strongly measurable semigroup satisfying* (C.1)-(C.3) *and* $(\mathrm{i})_0$. *Then the following assertions hold:*

(*a*) *If A_0 is closed,, i.e. $A_0 = A$,*

$$S(\{T(t)\}) = \widetilde{D(A)}^X .$$

In particular, if X is reflexive,

$$S(\{T(t)\}) = D(A) .$$

(*b*) *If* A_0 *is non-closed*

$$D(A_0) \begin{matrix} \subset S(\{T(t)\}) \underset{\neq}{\subseteq} \\ \underset{\neq}{\subseteq} \quad \not\supset \quad \\ D(A) \subset \end{matrix} \widetilde{D(A_0)}^X \subset \widetilde{D(A)}^X .$$

In particular, if X *is reflexive,*

$$D(A_0) \subset S(\{T(t)\}) \underset{\neq}{\subseteq} D(A) .$$

(*c*) *Moreover, there exist semigroups* $\{T(t), t > 0\}$ *and spaces* X *for which*

$$D(A_0) \underset{\neq}{\subseteq} S(\{T(t)\}) \underset{\not\supset}{\not\subset} D(A) ,$$

and, in case X *is reflexive, there still exist semigroups for which* $D(A_0) \underset{\neq}{\subseteq} S(\{T(t)\})$.

REMARK 1. – Part (*a*) of the theorem is well-known in case $T(t)$ forms a (C_0)-semigroup. Then (C.1)-(C.3) and $(\mathrm{i})_0$ are obviously satisfied, and the result is due to BERENS [1].

PROOF. – (*a*) Let $\|T(t)f - f\| = \mathcal{O}(t)$, $t \to 0+$. Then $f \in \Sigma$ and thus $C(t)f \in D(A)$ and $AC(t)f = t^{-1}(T(t)f - f)$ for all $t > 0$. Choosing $f_n = C(n^{-1})f$ in (0.2), it follows that $\|f_n - f\| = o(1)$ and $\|f_n\|_{D(A)} = \mathcal{O}(1)$, $n \to \infty$, thus $f \in \widetilde{D(A)}^X$.

Conversely, assume that there is a sequence $\{f_n\} \subset D(A)$ such that $\|f_n\|_{D(A)} = \mathcal{O}(1)$ and $\|f_n - f\| = o(1)$ as $n \to \infty$. Since

$$D(A_0) = D(A) \quad \text{and} \quad \lim_{t\to 0+} t^{-1}(T(t)g - g) = Ag$$

for each $g \in D(A)$ the uniform boundedness principle yields constants M and t_0 such that $\|t^{-1}(T(t)g - g)\| \leq M\|g\|_{D(A)}$ for each $g \in D(A)$ and $0 < t \leq t_0$. Setting $g = f_n$, there is a constant M' such that $t^{-1}\|T(t)f_n - f_n\| \leq M'$ uniformly in $n \in \mathbf{N}$ and $t \in (0, t_0]$. Letting $n \to \infty$ it follows that $\|T(t)f - f\| = \mathcal{O}(t)$, $t \to 0+$.

If X is reflexive, $D(A)$ is reflexive, too, and $\widetilde{D(A)}^X = D(A)$.

(*b*) (i) The inclusions $D(A_0) \underset{\neq}{\subseteq} D(A)$, $D(A_0) \subset S(\{T(t)\})$, and $\widetilde{D(A_0)}^X \subset \widetilde{D(A)}^X$ are trivial.

(ii) $D(A) \subset \widetilde{D(A_0)}^X$. Let $f \in D(A)$. Since A is the closure of A_0 there is a sequence $\{f_n\} \subset D(A_0)$ such that $f_n \to f$ and $A_0 f_n \to Af$ as $n \to \infty$. The existence of the latter limit implies $\|A_0 f_n\| = \mathcal{O}(1)$, $n \to \infty$, and thus $f \in \widetilde{D(A_0)}^X$ by (0.3).

(iii) $D(A) \not\subset S(\{T(t)\})$: We show that there exists an $f_0 \in D(A) \setminus S(\{T(t)\})$. Since A_0 is non-closed, Lemma 2 furnishes an $h_0 \in X$ such that

$$\limsup_{t\to 0+} \|C(t)h_0 - h_0\| > 0 .$$

But for elements f of the dense set X_0 we have $\|C(t)f - f\| \to 0$, $t \to 0+$, and thus the Banach-Steinhaus theorem and the uniform boundedness principle yield a $g_0 \in X$ such that $\limsup_{t\to 0+} \|C(t)g_0\| = +\infty$. If $0 < t \leqslant 1$ and $\lambda > 0$, it follows by (i)$_0$ that

$$\left\|t^{-1}\int_0^t e^{-\lambda s}T(s)g_0\,ds\right\| \geqslant \|C(t)g_0\| - \left\|t^{-1}\int_0^t (1-e^{-\lambda s})T(s)g_0\,ds\right\| \geqslant$$
$$\geqslant \|C(t)g_0\| - \lambda\int_0^1 \|T(s)g_0\|\,ds \geqslant \|C(t)g_0\| - \lambda M_{g_0}\,.$$

Choosing $\lambda > \max\{0, \omega_1\}$, λ belongs to $\varrho(A)$, so that $f_0 = R(\lambda, A)g_0$ is defined and belongs to $D(A)$. Lemma 1 implies $f_0 = R_0(\lambda)g_0$, thus

$$\limsup_{t\to 0+} t^{-1}\|T(t)f_0 - f_0\| \geqslant \limsup_{t\to 0+}\left\{t^{-1}e^{\lambda t}\int_0^t e^{-\lambda s}T(s)g_0\,ds - \frac{e^{\lambda t}-1}{t}f_0\right\} = +\infty\,,$$

which yields the assertion.

(iv) $S(\{T(t)\}) \subsetneqq \widetilde{D(A_0)}^X$: The inclusion $S(\{T(t)\}) \subset \widetilde{D(A_0)}^X$ follows as in the proof of part (*a*). Assuming $S(\{T(t)\}) = \widetilde{D(A_0)}^X$, one obtains by (ii) $D(A) \subset S(\{T(t)\})$, a contradiction to (iii). In order to prove (*c*) we use two examples of [7] and S. G. KREIN [9] (cf. also SUNOUCHI [12]).

EXAMPLE I. - We choose $X = C_{2\pi}$, the space of continuous 2π-periodic functions with maximum-norm, and $T(t) = T_\varphi(t)$, where

$$T_\varphi(t)(f;x) = \sum_{k\in\mathbf{Z}} e^{-t\varphi(|k|)} f^\wedge(k)\, e^{ikx} \tag{1.3}$$

for $f \in C_{2\pi}$ and $t > 0$. Here $\mathbf{Z}$ denotes the set of integers, $f^\wedge(k)$ the k-th Fourier coefficient of the function f, and φ a function of class Ω_1 defined as follows. Denoting by $C^r(0,\infty)$ the space of functions with continuous r-th derivative on $(0,\infty)$ we set (cf. [4], [5])

$$\tag{1.4} \begin{cases} \Omega_0 = \{\varphi;\ \varphi\colon [0,\infty) \to \mathbf{R},\ \varphi(0) = 1,\ \varphi \in C^1(0,\infty),\ \varphi'(x) > 0 \\ \qquad\qquad \forall x > 0 \text{ and } \lim_{x\to+\infty} \varphi(x) = +\infty\}\,, \\ \Omega_1 = \{\varphi \in \Omega_0;\ \varphi(x) = e^{\gamma(x)},\ \gamma \in C^3(0,\infty),\ \exists x_0 > 0 \\ \qquad \text{with } \gamma''(x) \leqslant 0,\ \gamma'''(x) \geqslant 0\ \forall x \geqslant x_0 \text{ and } \limsup_{x\to+\infty} |\gamma''(x)|(\gamma'(x))^{-2} = 0\}\,. \end{cases}$$

For each $\varphi \in \Omega_1$, the semigroup $\{T_\varphi(t), t > 0\}$ satisfies (C.1)-(C.3) and (i)$_0$. Moreover we want to show that

$$D(A_{\varphi,0}) \subsetneqq S(\{T_\varphi(t)\}) \not\subset D(A_\varphi)\,, \tag{1.5}$$

where $A_{\varphi,0}$ denotes the infinitesimal operator and A_φ the infinitesimal generator with domain $D(A_\varphi) = \{f \in C_{2\pi};\ \exists g \in C_{2\pi}$ with $\varphi(|k|) f^\wedge(k) = g^\wedge(k),\ \forall k \in \mathbf{Z}\}$. To prove (1.5) we take

$$f_0(x) = \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{i}{2k\varphi(|k|)} e^{ikx}.$$

Since $\varphi(|k|) \hat{f_0}(k) = i(2k)^{-1}$, $\forall k \in \mathbf{Z} \setminus \{0\}$, f_0 does not belong to $D(A_\varphi)$, nor to $D(A_{\varphi,0})$. In view of part (*b*), (1.5) follows if we show that $\|T_\varphi(t) f_0 - f_0\| = \mathcal{O}(t)$ as $t \to 0+$. For this purpose we define

$$R_{\varphi,\varrho}(f; x) = \sum_{|k| \leq \varrho} \left(1 - \frac{\varphi(|k|)}{\varphi(\varrho + 1)}\right) f^\wedge(k)\, e^{ikx} \tag{1.6}$$

for $f \in C_{2\pi}$ and $\varrho > 0$, and obtain by [16; p. 61] and [4; Lemma 2]

$$\begin{aligned}\|\varphi(\varrho+1)(R_{\varphi,\varrho} f_0 - f_0)\| &\leq \left\| \sum_{0<|k|\leq\varrho} \frac{i e^{ikx}}{2k} \right\| + \sum_{k>\varrho} \frac{\varphi(\varrho+1)}{k\varphi(k)} \leq \\ &\leq \left\| \sum_{0<k\leq\varrho} \frac{\sin kx}{k} \right\| + \frac{\varphi(\varrho+1)}{\varrho} \sum_{k>\varrho} \frac{1}{\varphi(k)} = \mathcal{O}(1), \qquad \varrho \to \infty.\end{aligned} \tag{1.7}$$

On the other hand, one has

$$\begin{aligned}&\left\| R_{\varphi,\varrho} f_0 - T_\varphi\left(\frac{1}{\varphi(\varrho+1)}\right) f_0 \right\| \leq \\ &\leq \left\| \sum_{0<|k|\leq\varrho} \left\{ \exp\left(-\frac{\varphi(|k|)}{\varphi(\varrho+1)}\right) - 1 + \frac{\varphi(|k|)}{\varphi(\varrho+1)} \right\} \frac{i e^{ikx}}{2k\varphi(|k|)} \right\| + \\ &+ \left\| \sum_{|k|>\varrho} \exp\left(-\frac{\varphi(|k|)}{\varphi(\varrho+1)}\right) \frac{i e^{ikx}}{2k\varphi(|k|)} \right\| = I_1 + I_2,\end{aligned} \tag{1.8}$$

say. Since the function $x^{-1}\varphi(x)$ is increasing for x large enough, we have

$$I_1 \leq \sum_{0<|k|\leq\varrho} \frac{\varphi^2(|k|)}{2\varphi^2(\varrho+1)} \frac{1}{2|k|\varphi(|k|)} = \frac{1}{2\varphi^2(\varrho+1)} \sum_{0<k\leq\varrho} \frac{\varphi(k)}{k} = \mathcal{O}\left(\frac{1}{\varphi(\varrho+1)}\right), \qquad \varrho \to \infty.$$

By [4; Lemma 2],

$$I_2 \leq 2 \sum_{k>\varrho} \frac{1}{2k\varphi(k)} \leq \frac{1}{\varrho} \sum_{k>\varrho} \frac{1}{\varphi(k)} = \mathcal{O}\left(\frac{1}{\varphi(\varrho+1)}\right), \qquad \varrho \to \infty.$$

Hence, one has by (1.7) and (1.8)

$$\begin{aligned}\left\| T_\varphi\left(\frac{1}{\varphi(\varrho+1)}\right) f_0 - f_0 \right\| &\leq \left\| T_\varphi\left(\frac{1}{\varphi(\varrho+1)}\right) f_0 - R_{\varphi,\varrho} f_0 \right\| + \|R_{\varphi,\varrho} f_0 - f_0\| = \\ &= \mathcal{O}\left(\frac{1}{\varphi(\varrho+1)}\right), \qquad \varrho \to \infty.\end{aligned}$$

Setting $t^{-1} = \varphi(\varrho + 1)$, the proof of (1.5) is complete.

EXAMPLE II. – We choose $X = \boldsymbol{L}^2 = L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ with elements $f = (f_1, f_2)$ and norm $\|f\| = (\|f_1\|_2^2 + \|f_2\|_2^2)^{1/2}$, where $L^2(\boldsymbol{R})$ has the usual norm

$$\|f_1\|_2 = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f_1(x)|^2 \, dx\right)^{1/2}.$$

As a semigroup $T(t)$ we take $T_q(t)$ with $2 < q < 4$, which is defined by

$$[T_q(t) f]^\wedge(v) = \begin{pmatrix} e^{-tv^2} & tv^q e^{-tv^2} \\ 0 & e^{-tv^2} \end{pmatrix} \begin{pmatrix} f_1^\wedge(v) \\ f_2^\wedge(v) \end{pmatrix}, \tag{1.9}$$

$f = (f_1, f_2) \in \boldsymbol{L}^2$, $v \in \boldsymbol{R}$, where $f^\wedge(v)$ denotes the Fourier transform of f (see S. G. KREIN [9], [12], and [7] for integer values of q). The infinitesimal generator exists and is given by

$$[P_q f]^\wedge(v) = \begin{pmatrix} -v^2 f_1^\wedge(v) + v^q f_2^\wedge(v) \\ -v^2 f_2^\wedge(v) \end{pmatrix}; \tag{1.10}$$

$f \in D(P_q)$, $v \in \boldsymbol{R}$, $D(P_q)$ consisting of those $f \in \boldsymbol{L}^2$ for which the right-hand side belongs to $\boldsymbol{L}^2$. Then (C.1)-(C.3) and $(\mathrm{i})_0$ are satisfied, and we claim that

$$D(P_{q,0}) \subsetneqq S(\{T_q(t)\}) \subsetneqq D(P_q), \tag{1.11}$$

where $P_{q,0}$ is the infinitesimal operator. In view of (*b*) we only have to prove that the first inclusion is proper. As a counterexample we take $f_\varepsilon \in \boldsymbol{L}^2$ defined by

$$f_\varepsilon^\wedge(v) = \begin{pmatrix} v^{-(2+\varepsilon)} \\ v^{-(q+\varepsilon)} \end{pmatrix} \quad \text{if } v \geqslant 1, \; f_\varepsilon^\wedge(v) = 0 \quad \text{if } v < 1, \; \varepsilon > -3/2 \tag{1.12}$$

and show that

$$f_{1/2} \notin D(P_{q,0}) \quad \text{and} \quad f_{1/2} \in S(\{T_q(t)\}). \tag{1.13}$$

Indeed, assuming $f_{1/2} \in D(P_{q,0})$, i.e. $\|t^{-1}(T_q(t) f_{1/2} - f_{1/2}) - P_{q,0} f_{1/2}\|$ tends to zero as $t \to 0+$, implies

$$0 = \lim_{t \to 0+} \int_1^\infty \left| \frac{e^{-tv^2} - 1}{tv^2} \frac{1}{\sqrt{v}} + e^{-tv^2} \frac{1}{\sqrt{v}} + v^2 \frac{1}{v^{5/2}} - v^q \frac{1}{v^{q+1/2}} \right|^2 dv$$

$$= \lim_{t \to 0+} \frac{1}{2} \int_t^\infty \left(\frac{1 - e^{-x}}{x} - e^{-x} \right)^2 \frac{dx}{x} \geqslant M > 0,$$

a contradiction. Thus $f_{1/2} \notin D(P_{q,0})$. Moreover, there is a constant M such that

$$\left\|\frac{T_q(t)f_{1/2}-f_{1/2}}{t}\right\|^2 \leqslant M+\frac{1}{\sqrt{2\pi}}\int_t^\infty\left|\frac{e^{-tv^2}-1}{tv^2}+e^{-tv^2}\right|^2\frac{dv}{v}$$

$$\leqslant M+\frac{1}{2\sqrt{2\pi}}\int_t^\infty\left(\frac{1-e^{-x}}{x}+e^{-x}\right)^2\frac{dx}{x}=\mathcal{O}(1)\,,\qquad t\to 0+\,.$$

This completes the proof of Theorem 1.

In connection with Example II we further note that $f_\varepsilon \in D(P_q)$ iff $\varepsilon > 5/2 - q$. Moreover, $\{T_q(t), t>0\}$ is of growth order $\alpha = q/2 - 1$.

2. – A characterization of $S(\{T(t)\})$.

For (C_0)-semigroups with $T(t)(X) \subset D(A_0)$ for each $t>0$ the characterization

$$f \in S(\{T(t)\}) \Leftrightarrow \|A_0 T(t) f\| = \mathcal{O}(1)\,,\quad t\to 0+ \tag{2.1}$$

holds (see [2; Prop. 2.3.1]). In comparison with the condition $f\in \widetilde{D(A_0)}^X$ (cf. (0.3)), the condition on the right of (2.1) means that instead of an arbitrary sequence $\{f_n\}\subset D(A_0)$ only the special sequences $\{T(t_n)f\}$ with $t_n\to 0$ are admitted. Our purpose here is to show that this characterization remains valid in a more general context.

If X is reflexive the following theorem generalizes (2.1) to semigroups with non-closed infinitesimal operator A_0.

THEOREM 2. – *Let X be a reflexive Banach space and $\{T(t), t>0\}$ a strongly measurable semigroup with infinitesimal operator A_0 and $T(t)(X)\subset D(A_0)$ for each $t>0$. Let $\{T(t), t>0\}$ satisfy* (i)$_0$ *as well as the following conditions:*

(*a*) *the infinitesimal generator A exists,*

(*b*) $\|C(t)g\| = \mathcal{O}(1)$, $t\to 0+$ *for $g\in X$ implies* $\|T(t)g\| = \mathcal{O}(1)$, $t\to 0+$.

For $f\in\Sigma$ the following are equivalent:

$$\|T(t)f-f\| = \mathcal{O}(t)\,,\quad t\to 0+\,, \tag{2.2}$$

$$\|A_0T(t)f\| = \mathcal{O}(1)\,,\quad t\to 0+\,. \tag{2.3}$$

PROOF. – Let $\|A_0T(t)f\| = \mathcal{O}(1)$, $t\to 0+$. By the strong continuity of the semigroup one has for each $s>0$

$$\|A_0T(s+h)f - A_0T(s)f\| = \left\|\left\{T\left(\frac{s}{2}+h\right)-T\left(\frac{s}{2}\right)\right\}A_0T\left(\frac{s}{2}\right)f\right\|\to 0\,,\qquad h\to 0+\,. \tag{2.4}$$

Thus also $A_0T(\cdot)f$ is strongly continuous on $(0, \infty)$, and, since $\int_0^t \|A_0T(u)f\|\,du = \mathcal{O}(t)$ as $t \to 0+$, $A_0T(\cdot)f$ is also Bochner integrable on $(0, t)$ for $t > 0$. Using this, (a), and [8; Thm. 3.7.12], (2.2) follows by

$$\|t^{-1}(T(t)f - f)\| = \|A_0C(t)f\| = \|AC(t)f\| = \left\|t^{-1}\int_0^t AT(u)f\,du\right\| = \\ = \left\|t^{-1}\int_0^t A_0T(u)f\,du\right\| \leqslant t^{-1}\int_0^t \|A_0T(u)f\|\,du = \mathcal{O}(1)\,, \qquad t \to 0+\,.$$

Conversely, let $\|T(t)f - f\| = \mathcal{O}(t)$, $t \to 0+$. Since X is reflexive the weak compactness theorem furnishes weak convergence for a positive null sequence $\{t_n\}$, i.e. there is a $g \in X$ such that

$$f^*\left(\frac{T(t_n)f - f}{t_n}\right) \to f^*(g)\,, \qquad n \to \infty \tag{2.5}$$

for each $f^* \in X^*$, or,

$$[f^* \circ T(t)]\left(\frac{T(t_n)f - f}{t_n}\right) \to [f^* \circ T(t)](g)\,, \qquad n \to \infty\,,$$

so that we also have weak convergence of the sequence $\{T(t)\,t_n^{-1}(T(t_n)f - f)\}$ to $T(t)g$. Since $T(t)f \in D(A_0)$ for $t > 0$ we obtain

$$f^*(T(t)g) = \lim_{n\to\infty} f^*\left(\frac{T(t_n) - I}{t_n}\right)T(t)f = f^*(A_0T(t)f)$$

for each $f^* \in X^*$, and it follows that

$$T(t)g = A_0T(t)f\,, \qquad \forall t > 0\,. \tag{2.6}$$

The strong measurability of $A_0T(\cdot)f$ follows as in (2.4), and $\|A_0T(\cdot)f\|$ is Lebesgue integrable on $(0, t)$ by (2.6) and $(\mathrm{i})_0$, thus $A_0T(\cdot)f$ is Bochner integrable on $(0, t)$. As above one has

$$t^{-1}(T(t)f - f) = A_0C(t)f = t^{-1}\int_0^t A_0T(t)f\,du = t^{-1}\int_0^t T(u)g\,du = C(t)g\,.$$

Thus $\|C(t)g\| = \mathcal{O}(1)$, $t \to 0+$ and by (b)

$$\|A_0T(t)f\| = \|T(t)g\| = \mathcal{O}(1)\,, \qquad t \to 0+\,,$$

and the proof is complete.

REMARK 2. – If, in particular, the semigroup belongs to (C_0), conditions $(i)_0$, (a), and (b) are satisfied and Theorem 2 reduces to [2; Prop. 2.3.1] in the case of a reflexive space X.

For non-reflexive spaces we can show that characterization (2.1) remains valid, provided that the space X can be continuously embedded in some space Y and the semigroup $\{T(t), t > 0\}$ can be extended in the following way. We say the Banach space X satisfies condition

(E_1) if there are a Banach space Y and a separable normed linear space Z such that

(*a*) $X \subset Y$ and $\|g\|_X = \|g\|_Y$ for all $g \in X$,

(*b*) there exists an isometric and isomorphic mapping K from Y to Z^*, the dual of Z.

A semigroup $\{T(t), t > 0\}$ on a Banach space satisfying (E_1) is said to satisfy condition

(E_2) if (*a*) $T(t) \in [X]$ can be extended to some $\tilde{T}(t) \in [Y]$ for $t > 0$,

(*b*) for each $t > 0$ there is an $S(t) \in [Z]$ whose dual operator $S^*(t)$ satisfies $S^*(t)K = K\tilde{T}(t)$.

THEOREM 3. – *Let X be a Banach space satisfying* (E_1) *and let* $\{T(t), t > 0\}$ *be a strongly measurable semigroup with infinitesimal operator* A_0 *and* $T(t)(X) \subset D(A_0)$, $\forall t > 0$ *which satisfies* (E_2).

Moreover, we suppose

(*a*) *there exists the infinitesimal generator A of* $\{T(t), t > 0\}$ *on* X,

(*b*) $\{\tilde{T}(t), t > 0\}$ *is strongly measurable and* $(i)_0$ *is valid for* $\tilde{T}(t)$ *on* Y, *i.e.* $\tilde{C}(t)g = t^{-1}\int_0^t \tilde{T}(u)g\,du$ *is well defined for* $t > 0$, $g \in Y$,

(*c*) $\|\tilde{C}(t)g\| = \mathcal{O}(1)$, $t \to 0+$ *for* $g \in Y$ *implies* $\|\tilde{T}(t)g\| = \mathcal{O}(1)$, $t \to 0+$.

Then (2.2) *and* (2.3) *are equivalent for each* $f \in \Sigma$.

PROOF. – The proof of the implication (2.3) $\Rightarrow$ (2.2) is the same as in Theorem 2. Conversely, let $\tilde{T}_K(t)$ be the operator from Z^* to Z^* associated to $\tilde{T}(t)$ by

$$\tilde{T}_K(t) := K\tilde{T}(t)K^{-1}, \quad t > 0. \tag{2.7}$$

By (E_1) (*b*), $\|\tilde{T}_K(t)\|_{[Z^*]} = \|\tilde{T}(t)\|_{[Y]}$ and, by (E_2), there is an $S(t) \in [Z]$ such that

$$S^*(t) = \tilde{T}_K(t), \quad t > 0. \tag{2.8}$$

By (2.2) and (E_1) one has $\|K(t^{-1}(T(t)f-f))\|_{Z^*}=\mathcal{O}(1)$, $t\to 0+$. The weak * compactness theorem yields a positive null sequence $\{t_n\}$ and a $G^*\in Z^*$ such that

$$\lim_{n\to\infty}\left[K\left(\frac{T(t_n)f-f}{t_n}\right)\right](h)=G^*(h)\qquad \forall h\in Z\,. \tag{2.9}$$

Defining $g:=K^{-1}G^*\in Y$ we obtain for $t>0$ and $h\in Z$

$$\begin{aligned}[K(\tilde T(t)g)](h)=[S^*(t)G^*](h)=G^*(S(t)h)&=\lim_{n\to\infty}\left[K\left(\frac{T(t_n)f-f}{t_n}\right)\right](S(t)h)\\ &=\lim_{n\to\infty}\left[S^*(t)K\left(\frac{T(t_n)f-f}{t_n}\right)\right](h)=\lim_{n\to\infty}\left[K\left(\tilde T(t)\frac{T(t_n)f-f}{t_n}\right)\right](h)\\ &=\lim_{n\to\infty}\left[K\left(\frac{T(t_n)f-f}{t_n}T(t)f\right)\right](h)=[K(A_0T(t)f)](h)\end{aligned}$$

and thus

$$\tilde T(t)g=A_0T(t)f\,,\qquad t>0\,. \tag{2.10}$$

In particular, $\tilde T(t)g\in X$ for $t>0$. Moreover, $\tilde T(\cdot)g$ is strongly measurable by (*b*) and, in view of $(\mathrm{i})_0$, Bochner integrable on $(0,t)$, $t\leqslant 1$. By (2.10) one has

$$t^{-1}(T(t)f-f)=A_0C(t)f=t^{-1}\int_0^t A_0T(u)f\,du=t^{-1}\int_0^t\tilde T(u)g\,du=\tilde C(t)g$$

for $t>0$. Condition (2.2) implies $\|\tilde C(t)g\|=\mathcal{O}(1)$, $t\to 0+$ and (*c*) yields $\|\tilde T(t)g\|=\mathcal{O}(1)$, $t\to 0+$, so that, by (2.10),

$$\|A_0T(t)f\|=\|\tilde T(t)g\|=\mathcal{O}(1)\,,\qquad t\to 0+\,.$$

REMARK 3. – If, in addition, X is reflexive and separable the assumptions of Theorem 3 reduce to those of Theorem 2, for one may choose $Y=X$, $Z=X^*$ and $\tilde T(t)=T(t)$, $S(t)=T^*(t)$ in order to verify (E_1) and (E_2). Indeed, X^{**} is separable and so is X^*, and for K the canonical mapping can be chosen.

3. – Two applications.

In this section we show that the results of section 2 can be applied to the examples of section 1.

EXAMPLE I. – We shall use Theorem 3 to prove that (2.1) remains true for $X=C_{2\pi}$ and $T(t)=T_\varphi(t)$, cf. (1.3), where φ is an arbitrary element of Ω_1 (cf. (1.4)).

In order to verify (E_1) we choose $Y = L^\infty_{2\pi}$ and $Z = L^1_{2\pi}$ with norms

$$\|f\|_\infty = \operatorname{ess\,sup} |f(x)| \quad \text{and} \quad \|f\|_1 = (2\pi)^{-1}\int_{-\pi}^{\pi} |f(x)|\, dx$$

respectively.

Indeed, (*a*) holds trivially and the congruence of $L^\infty_{2\pi}$ with the dual of $L^1_{2\pi}$ is also clear in view of the Riesz representation theorem, where

$$(Kg)h = (2\pi)^{-1}\int_{-\pi}^{\pi} g(u)h(u)\, du \quad \text{for } g \in L^\infty_{2\pi},\ h \in L^1_{2\pi}.$$

Hence (E_1) is satisfied.

Let $\varphi \in \Omega_1$. Since

$$\sum_{k\in\mathbf{Z}} h^{-1}(e^{-h\varphi(|k|)} - 1)\, e^{-t\varphi(|k|)} f^\wedge(k)\, e^{ikx}$$

tends to

$$\sum_{k\in\mathbf{Z}} \big(-\varphi(|k|)\big)\, e^{-t\varphi(|k|)} f^\wedge(k)\, e^{-ikx}$$

as $h \to 0+$ uniformly in $x \in [-\pi, \pi)$ for $t > 0$, $f \in C_{2\pi}$, one has $T_\varphi(t)(C_{2\pi}) \subset D(A_{\varphi,0})$, $\forall t > 0$.

Let $\tilde{T}_\varphi(t)$ denote the trivial extension of $T_\varphi(t)$ from $C_{2\pi}$ to $L^\infty_{2\pi}$. Choosing

$$S_\varphi(t)(h; x) = \sum_{k\in\mathbf{Z}} e^{-t\varphi(|k|)} h^\wedge(k)\, e^{ikx} \tag{2.11}$$

for $h \in L^1_{2\pi}$, $t > 0$, one has $K\tilde{T}_\varphi(t)K^{-1} = S^*_\varphi(t)$. Hence $\{T_\varphi(t), t > 0\}$ satisfies (E_2) and, (*a*) and (*b*) being trivial, it remains to verify condition (*c*).

Setting

$$U_{\varphi,\varrho}(g; x) = \sum_{|k|\le\varrho} g^\wedge(k)\, e^{ikx} + \sum_{|k|>\varrho} \frac{\varphi(\varrho+1)}{\varphi(|k|)}\, g^\wedge(k)\, e^{ikx}$$

for $g \in L^\infty_{2\pi}$, $\varrho > 0$, and $t^{-1} = \varphi(\varrho + 1)$, and, denoting by $\tilde{C}_\varphi(t)$ and $\tilde{R}_{\varphi,\varrho}$ the trivial extensions of $C_\varphi(t)$ and $R_{\varphi,\varrho}$, we have

$$\|\tilde{C}_\varphi(t) - \tilde{T}_\varphi(t)\|_{[L^\infty_{2\pi}]} \le \|\tilde{C}_\varphi(t) - U_{\varphi,\varrho}\|_{[L^\infty_{2\pi}]} + \|U_{\varphi,\varrho} - \tilde{R}_{\varphi,\varrho}\|_{[L^\infty_{2\pi}]} + \|\tilde{R}_{\varphi,\varrho} - \tilde{T}_\varphi(t)\|_{[L^\infty_{2\pi}]}. \tag{2.12}$$

In order to prove that $\|U_{\varphi,\varrho} - \tilde{R}_{\varphi,\varrho}\|_{[L^\infty_{2\pi}]} = \mathcal{O}(1)$, $\varrho \to \infty$ we consider the associated kernels and set $\gamma(\varrho) = \log \varphi(\varrho)$:

$$\Big\| \sum_{|k|\le\varrho} \frac{\varphi(|k|)}{\varphi(\varrho+1)} e^{ikx} + \sum_{|k|>\varrho} \frac{\varphi(\varrho+1)}{\varphi(|k|)} e^{ikx} \Big\|_1$$

$$\le \frac{1}{2\pi} \int_{-\gamma'(\varrho)}^{\gamma'(\varrho)} \Big\{ \Big| \sum_{|k|\le\varrho} \frac{\varphi(|k|)}{\varphi(\varrho+1)} e^{ikx} \Big| + \Big| \sum_{|k|>\varrho} \frac{\varphi(\varrho+1)}{\varphi(|k|)} e^{ikx} \Big| \Big\} dx$$

$$+ \frac{1}{\pi} \int_{\gamma'(\varrho)}^{\pi} \Big\{ \Big| \sum_{|k|\le\varrho} \Big(1 - \frac{\varphi(|k|)}{\varphi(\varrho+1)}\Big) e^{ikx} \Big| + \Big| \sum_{|k|>\varrho} \frac{\varphi(\varrho+1)}{\varphi(|k|)} e^{ikx} + \sum_{|k|\le\varrho} e^{ikx} \Big| \Big\} dx = I_1 + I_2 ,$$

say. By an application of [3; Lemma 2.3(*b*)] the uniform boundedness of I_1 follows immediately. A repeated application of Abel's transformation yields the same for I_2. Similarly, the uniform boundedness of the other terms of (2.12) follows. Thus the hypotheses of Theorem 3 are satisfied, so that, together with (1.5) and Theorem 1, one obtains

$$D(A_{\varphi,0}) \subsetneq S(\{T_\varphi(t)\}) = \{f \in C_{2\pi};\ \|A_{\varphi,0} T_\varphi(t) f\| = \mathcal{O}(1),\ t \to 0+\}$$
$$\subsetneq \widetilde{D(A_\varphi)}^{C_{2\pi}} = \{f \in C_{2\pi};\ \exists g \in L^\infty_{2\pi} \text{ with } \varphi(|k|) f^\wedge(k) = g^\wedge(k)\ \forall k \in \mathbf{Z}\} .$$

Analogous results are valid for $\{T_\varphi(t),\, t > 0\}$ in the space $X = L^1_{2\pi}$.

EXAMPLE II. - Here we apply Theorem 2. Since

$$\Big[\frac{T_q(h) - I}{h} T_q(t) f\Big]^\wedge (v) = \begin{pmatrix} \dfrac{e^{-hv^2} - 1}{h} \{e^{-tv^2} f_1^\wedge(v) + tv^q e^{-tv^2} f_2^\wedge(v)\} + v^q e^{-hv^2} e^{-tv^2} f_2^\wedge(v) \\ \dfrac{e^{-hv^2} - 1}{h} e^{-tv^2} f_2^\wedge(v) \end{pmatrix}$$

for $f \in \mathbf{L}^2$, $t > 0$, (cf. (1.9)), one has $T_q(t)(\mathbf{L}^2) \subset D(P_{q,0})$ for each $t > 0$. Moreover, $\{T_q(t),\, t > 0\}$ is strongly measurable and $(\mathrm{i})_0$ is satisfied. For $g \in \mathbf{L}^2$ we have, for $2 < q < 4$,

$$\|T_q(t) g\| = \mathcal{O}(1 + \|tv^q e^{-tv^2} g_2^\wedge(v)\|_2) , \quad t \to 0+ ,$$
$$\Big\| v^{q-2} \Big(\frac{1 - e^{-tv^2}}{tv^2} - e^{-tv^2}\Big) g_2^\wedge(v) \Big\|_2 = \mathcal{O}(1 + \|C_q(t) g\|) , \quad t \to 0+ ,$$

where $C_q(t) g := t^{-1} \int_0^t T_q(u) g\, du$ for $t > 0$. Defining

$$[E_q(t) h]^\wedge(v) = \frac{tv^2 e^{-tv^2}}{(tv^2)^{-1}(1 - e^{-tv^2}) - e^{-tv^2}} h^\wedge(v)$$

for $h \in L^2(\boldsymbol{R})$, $t > 0$, $v \in \boldsymbol{R}$ and observing that $\|E_q(t)\|_{[L^2(\boldsymbol{R})]} \leq 2$, $\forall t > 0$, we obtain

$$\|T_q(t)g\| = \mathcal{O}\left(1 + 2\left\|\left(\frac{1-e^{-tv^2}}{tv^2} - e^{-tv^2}\right)v^{q-2}\hat{g_2}(v)\right\|_2\right) = \mathcal{O}(1 + \|C_q(t)g\|)$$

as $t \to 0+$, which is condition (*b*) of Theorem 2. Thus, by Theorem 2 and (1.11), we obtain for $2 < q < 4$

$$D(P_{q,0}) \subsetneqq S(\{T_q(t)\}) = \{f \in \boldsymbol{L}^2;\ \|P_{q,0}\, T_q(t) f\| = \mathcal{O}(1),\ t \to 0+\} \subsetneqq D(P_q)\,.$$

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REFERENCES

[1] H. BERENS, *Interpolationsmethoden zur Behandlung von Approximationsprozessen auf Banachräumen* (Lecture Notes in Mathematics 64), Springer, Berlin - Heidelberg - New York, 1968.

[2] P. L. BUTZER - H. BERENS, *Semi-Groups of Operators and Approximation*, Springer, Berlin - Heidelberg - New York, 1967.

[3] W. DAHMEN, *Trigonometric approximation with exponential error orders*, I: *Construction of asymptotically optimal processes; generalized de la Vallée Poussin sums*, Math. Ann., **230** (1977), pp. 57-74.

[4] W. DAHMEN - E. GÖRLICH, *Best approximation with exponential error orders and intermediate spaces*, Math. Z., **148** (1976), pp. 7-21.

[5] W. DAHMEN - E. GÖRLICH, *The characterization problem for best approximation with exponential error orders and evaluation of entropy*, Math. Nachr., **76** (1977), pp. 163-179.

[6] G. DA PRATO, *Semigruppi di crescenza n*, Ann. Scuola Norm. Sup. Pisa, **20** (1966), pp. 753-782.

[7] E. GÖRLICH - D. PONTZEN, *Approximation of operator semigroups of Oharu's class* $(C_{(k)})$, Tôhoku Math. J., **34** (1982), pp. 539-552.

[8] E. HILLE - R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., Vol. **31**, 1957.

[9] S. G. KREIN, *Linear Differential Equations in Banach Space*, Transl. Math. Monographs, Amer. Math. Soc., Vol. **29**, 1971.

[10] S. OHARU, *Semigroups of linear operators in a Banach space*, Publ. RIMS, Kyoto Univ., **7** (1971-72), pp. 205-260.

[11] P. E. SOBOLEVSKII, *On semigroups of growth order* α, Soviet Math. Dokl., **12** (1971), pp. 202-205.

[12] H. SUNOUCHI, *Convergence of semi-discrete difference schemes of Cauchy problems*, Tôhoku Math. J., **22** (1970), pp. 394-408.

[13] T. TAKAHASHI - S. OHARU, *Approximation of operator semigroups in a Banach space*, Tôhoku Math. J., **24** (1972), pp. 505-528.

[14] P. P. ZABREIKO - A. V. ZAFIEVSKII, *On a certain class of semigroups*, Soviet Math. Dokl., **10** (1969), pp. 1523-1526.

[15] A. V. ZAFIEVSKII, *On semigroups with singularities summable with a power weight at zero*, Soviet Math. Dokl., **11** (1970), pp. 1408-1411.

[16] A. ZYGMUND, *Trigonometric Series I*, Cambridge University Press, 1959.