Oscillation Theory of First Order Functional Differential Equations with Deviating Arguments (*).

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Summary. – *New oscillation criteria are established for the first order functional differential equation* (*) $y'(t) + p(t)y(g(t)) = 0$ *and its nonlinear analogue. The results are presented so that a remarkable duality existing between the case where* (*) *is retarded* $(g(t) < t)$ *and the case where* (*) *is advanced* $(g(t) > t)$ *is apparent. Possible extension of the results for* (*) *to equations with several deviating arguments is attempted. Finally, it is shown that there exists a class of autonomous equations for which the oscillation situation can be completely characterized.*

Introduction.

In this paper we are concerned with the differential equation

$$y'(t) + p(t)y(g(t)) = 0 \tag{1}$$

and related functional differential equations with deviating arguments. Without further mention we assume that $p(t)$ and $g(t)$ are continuous on $[a, \infty)$, $g(t)$ is nondecreasing and $\lim_{t\to\infty} g(t) = \infty$. The deviating argument $g(t)$ (or equation (1)) is said to be retarded or advanced according to whether $g(t) \leqq t$ or $g(t) \geqq t$ for $t \geqq a$; $g(t)$ is said to be of mixed type if $g(t) - t$ changes sign infinitely often as $t \to \infty$.

We restrict our attention to those solutions $y(t)$ of equation (1) which exist on some half-line $[T_y, \infty)$ and satisfy $\sup\{|y(t)|: t \geqq T\} > 0$ for any $T \geqq T_y$. Such a solution is called a proper solution of (1). We make the standing hypothesis that (1) does possess proper solutions. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is termed oscillatory (resp. nonoscillatory) if all of its solutions are oscillatory (resp. nonoscillatory).

One of the striking features of the functional differential equation (1) is that it may be oscillatory, though the corresponding ordinary differential equation $y' + p(t)y = 0$ is always nonoscillatory. For example, it is known ([8], [9]) that

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the equations

$$y'(t) + py(t-\tau) = 0\,, \tag{2}$$

$$y'(t) - py(t+\tau) = 0\,, \tag{3}$$

where p and τ are positive constants, are both oscillatory if and only if $p\tau > 1/e$. The oscillation of first order functional differential equations, which is generated by the deviating arguments involved, has been studied by numerous authors; see e.g. the papers [1-24].

Now, the examples (2) and (3) given above suggest a remarkable « duality » existing between equations with retarded arguments and the corresponding (or companion) equations with advanced arguments. This kind of duality has been observed by KOPLATADZE and ČANTURIJA [6] and KUSANO [8]. The objective of this paper is to establish some new oscillation criteria for equation (1) and a nonlinear analogue of it, laying particular emphasis on the duality between the retarded and advanced cases. We also show that a similar duality holds for differential equations with deviating arguments of mixed type. Retarded and advanced equations of the form (1) are discussed in the first two sections; Section 1 and Section 2 concern, respectively, the case where the coefficient $p(t)$ is of constant sign and the case where $p(t)$ is of variable sign. Equation (1) in which the deviating argument $g(t)$ is of mixed type is studied in Section 3. In Section 4 the results of Section 1 are extended to nonlinear retarded and advanced equations of the form

$$y'(t) + p(t)f\big(y(g(t))\big) = 0\,. \tag{4}$$

Possible extension of the results regarding (1) and (4) to equations with several arguments is attempted in the final Section 5. There it is also shown that there is a class of autonomous functional differential equations for which a necessary and sufficient condition for oscillation can be established. Our results supplement, improve, extend and unify the previous results obtained in the papers [7, 9, 11, 13, 14, 17, 19, 20, 22, 23].

1. – Retarded and advanced equations with one-signed coefficients.

We begin by considering equation (1) in which the coefficient $p(t)$ is of constant sign.

THEOREM 1. – (i) *Suppose that $p(t) \geqq 0$ and $g(t) \leqq t$ for $t \geqq a$. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} p(s)\,ds > \frac{1}{e}\,, \tag{5}$$

then equation (1) *is oscillatory. If, on the other hand,*

$$\int_{g(t)}^{t} p(s)\,ds \leqq \frac{1}{e} \qquad \textit{for all sufficiently large } t\,, \tag{6}$$

then equation (1) *has a nonoscillatory solution.*

(ii) *Suppose that* $p(t) \leqq 0$ *and* $g(t) \geqq t$ *for* $t \geqq a$. *If*

$$\liminf_{t\to\infty} \int_{t}^{g(t)} [-p(s)]\,ds > \frac{1}{e}\,, \tag{7}$$

then equation (1) *is oscillatory. If, on the other hand,*

$$\int_{t}^{g(t)} [-p(s)]\,ds \leqq \frac{1}{e} \qquad \textit{for all sufficiently large } t\,, \tag{8}$$

then equation (1) *has a nonoscillatory solution.*

PROOF. – The first part of this theorem has recently been proved by KOPLATADZE and ČANTURIJA [7]. We present a proof of the second part which is dual to the one given in [7].

Let $p(t) \leqq 0$ and $g(t) \geqq t$ and suppose that (1) has a nonoscillatory solution $y(t)$. We may assume with no loss of generality that $y(t) > 0$ for $t \geqq t_0$. From (1), $y'(t) = -p(t)\,y(g(t)) \geqq 0$, $t \geqq t_0$, so that $y(t)$ is nondecreasing for $t \geqq t_0$. In particular, $y(g(t)) \geqq y(t)$, $t \geqq t_0$, and hence we have $y'(t) = -p(t)\,y(g(t)) \geqq -p(t)\,y(t)$, $t \geqq t_0$, or

$$y'(t)/y(t) \geqq -p(t) \qquad \text{for } t \geqq t_0\,. \tag{9}$$

In view of (7) there exist constants $t_1 > t_0$ and c such that

$$\int_{t}^{g(t)} [-p(s)]\,ds > c > 1/e \qquad \text{for } t \geqq t_1\,. \tag{10}$$

Integrating (9) over $[t, g(t)]$ and using (10), we obtain

$$\begin{aligned} y(g(t)) &\geqq y(t)\exp\Big[\int_{t}^{g(t)} [-p(s)]\,ds\Big] \\ &\geqq e^{c} y(t) \geqq ecy(t)\,, \qquad t \geqq t_1\,, \end{aligned} \tag{11}$$

where we have used the inequality $e^{x} \geqq ex$ for $x \geqq 0$. From (1) and (11) we have

$$y'(t) \geqq ec[-p(t)]\,y(t)\,, \qquad t \geqq t_1\,.$$

Dividing this inequality by $y(t)$ and integrating over $[t, g(t)]$ yields

$$y(g(t)) \geqq y(t) \exp\left[ec\int_t^{g(t)}[-p(s)]\,ds\right]$$
$$\geqq e^{ec^2}y(t) \geqq (ec)^2 y(t)\,, \qquad t \geqq t_1\,.$$

Now, combine this with (1), divide the resulting inequality by $y(t)$ and integrate from t to $g(t)$. Continuing this process, we conclude that, for any integer $\nu > 0$,

$$y(g(t)) \geqq (ec)^\nu y(t)\,, \qquad t \geqq t_1\,.$$

Since $ec > 1$ and ν is arbitrary, it follows that $y(g(t)) = \infty$ for any $t \geqq t_1$, a contradiction. Thus, equation (1) cannot have a nonoscillatory solution if (7) is satisfied.

Next, suppose that (8) holds for $t \geqq t_0$. Let Y denote the set of all continuous functions $y(t)$ which are continuous and nondecreasing on $[t_0, \infty)$ and satisfy the following inequalities:

$$1 \leqq y(t) \leqq \exp\left[e\int_{t_0}^{t}[-p(s)]\,ds\right],$$
$$y(g(t)) \leqq ey(t)\,, \qquad t \geqq t_0\,.$$

Y is a non-empty, closed and convex subset of the locally convex space $C[t_0, \infty)$ of continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Define the operator Φ: $Y \to C[t_0, \infty)$ by

$$\Phi y(t) = \exp\left(\int_{t_0}^{t}[-p(s)]\frac{y(g(s))}{y(s)}\,ds\right). \tag{12}$$

It is a matter of simple computation to show that Φ is a continuous operator mapping Y into a compact subset of Y. Therefore, by the Schauder-Tychonoff fixed point theorem, Φ has a fixed point y in Y, and we see that this fixed point $y = y(t)$ is a nonoscillatory solution of equation (1) on $[t_0, \infty)$. This completes the proof.

EXAMPLE 1. – Consider the equations

$$y'(t) + at^\alpha y(\log t) = 0\,, \tag{13}$$

$$y'(t) - at^\alpha y(e^t) \quad = 0\,, \tag{14}$$

where $a > 0$ and α are constant. If we put $p(t) = at^\alpha$ and $g(t) = \log t$, then we have

$$\lim_{t\to\infty}\int_{g(t)}^{t} p(s)\,ds = \begin{cases} \infty & (\alpha \geqq -1)\,, \\ 0 & (\alpha < -1)\,, \end{cases}$$

and if we put $p(t) = -at^{\alpha}$ and $g(t) = e^t$, then we have

$$\lim_{t\to\infty}\int_t^{g(t)}[-p(s)]\,ds = \begin{cases} \infty & (\alpha \geqq -1)\,, \\ 0 & (\alpha < -1)\,. \end{cases}$$

From Theorem 1 it follows that equations (13) and (14) are oscillatory for any $a > 0$ if and only if $\alpha \geqq -1$.

EXAMPLE 2. – Consider the equations

$$y'(t) + aty\left(t - \frac{1}{t}\right) = 0\,, \tag{15}$$

$$y'(t) - aty\left(t + \frac{1}{t}\right) = 0\,, \tag{16}$$

where $a > 0$ is a constant. Since

$$\lim_{t\to\infty}\int_{t-(1/t)}^{t} as\,ds = \lim_{t\to\infty}\int_{t}^{t+(1/t)} as\,ds = a\,,$$

(15) and (16) are oscillatory if $a > 1/e$, and have nonoscillatory solutions if $a < 1/e$.

2. – Retarded and advanced equations with oscillating coefficients.

Very recently, LADAS, SFICAS and STAVROULAKIS [14] have established criteria for oscillation of differential equations of the forms

$$y'(t) + p(t)\,t(t-\tau) = 0\,,$$

$$y'(t) + p(t)y(t+\tau) = 0\,,$$

where $\tau > 0$ is a constant and $p(t)$ may change sign as $t \to \infty$. To the best of the authors' knowledge, they seem to be the first who discovered effective oscillation criteria for functional differential equations with oscillating coefficients. Our purpose here is to extend their results to the case where $g(t)$ is a general retarded or advanced argument.

THEOREM 2. – (i) *Suppose that $g(t) \leqq t$ for $t \geqq a$. Equation* (1) *is oscillatory if there exists a sequence of numbers $\{t_n\}_{n=1}^{\infty}$ with the following properties: $t_n \to \infty$ as $n \to \infty$; the intervals $\{[g(g(t_n)), t_n]\}_{n=1}^{\infty}$ are disjoint;*

$$p(t) \geqq 0 \quad on \quad \bigcup_{n=1}^{\infty}[g(g(t_n)), t_n]\,; \tag{17}$$

$$\int_{g(t_n)}^{t_n} p(s)\,ds \geqq 1 \quad for\ n = 1, 2, \ldots\,. \tag{18}$$

(ii) *Suppose that* $g(t) \geqq t$ *for* $t \geqq a$. *Equation* (1) *is oscillatory if there exists a sequence of numbers* $\{t_n\}_{n=1}^{\infty}$ *with the following properties*: $t_n \to \infty$ *as* $n \to \infty$; *the intervals* $\{[t_n, g(g(t_n))]\}_{n=1}^{\infty}$ *are disjoint*;

$$-p(t) \geqq 0 \quad on \quad \bigcup_{n=1}^{\infty} [t_n, g(g(t_n))]\,; \tag{19}$$

$$\int_{t_n}^{g(t_n)} [-p(s)]\, ds \geqq 1 \quad for\ n = 1, 2, \ldots . \tag{20}$$

PROOF. – We only prove the first part. The proof of the second part proceeds by duality. Suppose to the contrary that (1) with $p(t) \geqq 0$ and $g(t) \leqq t$ has a non-oscillatory solution $y(t)$. We may assume that $y(t) > 0$ for $t \geqq t_0$. Since, by (17),

$$y'(t) = -p(t)y(g(t)) \leqq 0 \quad \text{on} \quad \bigcup_{n=N}^{\infty} [g(g(t_n)), t_n]$$

provided N is sufficiently large, $y(t)$ is nonincreasing on $\bigcup_{n=N}^{\infty}[g(g(t_n)), t_n]$, and so $y(g(t))$ is nonincreasing on $\bigcup_{n=N}^{\infty}[g(t_n), t_n]$. Integrating (1) from $g(t_n)$ and t_n and using the nonincreasing nature of $y(g(t))$, we get

$$\begin{aligned} 0 &= y(t_n) - y(g(t_n)) + \int_{g(t_n)}^{t_n} p(s)y(g(s))\, ds \\ &\geqq y(t_n) - y(g(t_n)) + y(g(t_n)) \int_{g(t_n)}^{t_n} p(s)\, ds \end{aligned}$$

for $n \geqq N$, which leads to

$$y(t_n) + y(g(t_n))\left[\int_{g(t_n)}^{t_n} p(s)\, ds - 1\right] \leqq 0\,, \qquad n \geqq N\,.$$

This, however, is a contradiction, since the left-hand side is positive by virtue of (18). The proof is complete.

In the following theorem, g^n denotes the n-times iteration of g:

$$g^0(t) = t\,, \qquad g^n(t) = g(g^{n-1}(t))\,, \qquad n = 1, 2, \ldots .$$

THEOREM 3. – (i) *Suppose that* $g(t) \leqq t$ *for* $t \geqq a$. *Suppose that there is a sequence of numbers* $\{t_n\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$ *as* $n \to \infty$, *the intervals* $\{[g^n(t_n), t_n]\}_{n=1}^{\infty}$ *are disjoint and*

$$p(t) \geqq 0 \quad on \quad \bigcup_{n=1}^{\infty} [g^n(t_n), t_n]\,. \tag{21}$$

If there is a constant c such that

$$\int_{g(t)}^{t} p(s)\,ds > c > \frac{1}{e} \qquad \textit{for} \qquad t \in \bigcup_{n=1}^{\infty} [g^{n-1}(t_n), t_n]\,, \tag{22}$$

then equation (1) *is oscillatory.*

(ii) *Suppose that* $g(t) \geqq t$ *for* $t \geqq a$. *Suppose that there is a sequence of numbers* $\{t_n\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$ *as* $n \to \infty$, *the intervals* $\{[t_n, g^n(t_n)]\}_{n=1}^{\infty}$ *are disjoint and*

$$-p(t) \geqq 0 \qquad \textit{on} \qquad \bigcup_{n=1}^{\infty} [t_n, g^n(t_n)]\,. \tag{23}$$

If there is a constant c such that

$$\int_{t}^{g(t)} [-p(s)]\,ds > c > \frac{1}{e} \qquad \textit{for} \qquad t \in \bigcup_{n=1}^{\infty} [t_n, g^{n-1}(t_n)]\,, \tag{24}$$

then equation (1) *is oscillatory.*

PROOF. - We only prove the second part. Assume the existence of a positive solution $y(t)$ on $[t_0, \infty)$. Then

$$y'(t) = -p(t)y(g(t)) \geqq 0 \qquad \text{on} \qquad \bigcup_{n=N}^{\infty} [t_n, g^n(t_n)]\,,$$

provided N is sufficiently large, so that $y(g(t))$ is nondecreasing on $\bigcup_{n=N}^{\infty} [t_n, g^{n-1}(t_n)]$.

Let $t \in \bigcup_{n=N}^{\infty} [g(t_n), g^{n-2}(t_n)]$ be fixed and choose t^* so that

$$t_n \leqq t^* < t < g(t^*) \leqq g^{n-1}(t_n)$$

and

$$\int_{t^*}^{t} [-p(s)]\,ds > \frac{c}{2} \qquad \text{and} \qquad \int_{t}^{g(t^*)} [-p(s)]\,ds > \frac{c}{2}\,. \tag{25}$$

Integrating (1) over $[t^*, t]$ and $[t, g(t^*)]$, respectively, and taking (23) and (25) into account, we obtain

$$y(t) - y(t^*) = \int_{t^*}^{t} [-p(s)]\,y(g(s))\,ds > \frac{c}{2}\,y(g(t^*))\,,$$

and

$$y(g(t^*)) - y(t) = \int_{t}^{g(t^*)} [-p(s)]\,y(g(s))\,ds > \frac{c}{2}\,y(g(t))\,.$$

From the above inequalities it follows that

$$\frac{y(g(t))}{y(t)} < \frac{4}{c^2} \quad \text{on} \quad \bigcup_{n=N}^{\infty} [g(t_n), g^{n-2}(t_n)]. \tag{26}$$

On the other hand, we have

$$y'(t) = -p(t)y(g(t)) \geqq -p(t)y(t), \quad t \in \bigcup_{n=N_1}^{\infty} [t_n, g^{n-2}(t_n)],$$

for some $N_1 > N$. Divide the above by $y(t)$ and integrate it over $[t, g(t)]$. We then have

$$y(g(t)) \geqq y(t) \exp\left[\int_t^{g(t)} [-p(s)]\, ds\right]$$

$$\geqq e^c y(t) \geqq ecy(t), \quad t \in \bigcup_{n=N_1}^{\infty} [t_n, g^{n-2}(t_n)].$$

This combined with (1) yields

$$y'(t) \geqq ec[-p(t)]y(t), \quad t \in \bigcup_{n=N_1}^{\infty} [t_n, g^{n-2}(t_n)],$$

from which we can derive the following inequality

$$y(g(t)) \geqq y(t) \exp\left[ec\int_t^{g(t)} [-p(s)]\, ds\right]$$

$$\geqq e^{ec^2} y(t) \geqq (ec)^2 y(t), \quad t \in \bigcup_{n=N_2}^{\infty} [t_n, g^{n-3}(t_n)],$$

for some $N_2 > N_1$. Continuing in this manner, we conclude that, for any integer $\nu > 0$, there is an integer $N_\nu > 0$ such that $N_1 < N_2 < \ldots < N_\nu$ and

$$y(g(t)) \geqq (ec)^\nu y(t), \quad t \in \bigcup_{n=N_\nu}^{\infty} [t_n, g^{n-\nu-1}(t_n)].$$

Since ν is arbitrary, it follows that

$$\limsup_{t\to\infty} \frac{y(g(t))}{y(t)} = \infty,$$

which contradicts (26). This completes the proof of the second part of Theorem 3. The first part can be proved similarly.

EXAMPLE 3. – Consider the equations

$$y'(t) + \frac{2}{t}\sin(\log t)\, y\left(\frac{2}{t}\right) = 0\,, \tag{27}$$

$$y'(t) - \frac{2}{t}\sin(\log t)\, y(2t) = 0\,. \tag{28}$$

Put $p(t) = (2/t)\sin(\log t)$ and $g(t) = t/2$. If $t_n = \sqrt{2}\exp[(2n + \frac{1}{2})\pi]$, $n = 1, 2, \ldots$, then $g(g(t_n)) > \exp[2n\pi]$, $p(t) > 0$ on $[g(g(t_n)), t_n]$ and

$$\int_{g(t_n)}^{t_n} p(s)\, ds = 4\sin\big((\log 2)/2\big) > 1\,.$$

So, by (i) of Theorem 2, equation (27) is oscillatory. Similarly, via (ii) of Theorem 2, it can be shown that equation (28) is also oscillatory.

EXAMPLE 4. – If $|c| > 1/e$, then the following equations are oscillatory:

$$y'(t) + ct\sin t\, y\left(t - \frac{1}{t}\right) = 0\,, \tag{29}$$

$$y'(t) + ct\sin t\, y\left(t + \frac{1}{t}\right) = 0\,, \tag{30}$$

We first suppose $c > 0$. Put $p(t) = ct\sin t$, $g(t) = t - (1/t)$ and $t_n = [2n^2 + \frac{1}{2}]\pi$, $n = 1, 2, \ldots$. Let α_n denote the smaller root of the quadratic equation in α

$$n\alpha^2 - t_n^2\alpha + t_n^2 = 0\,.$$

It is easy to verify that $\alpha_n > 1$ and $\alpha_n \to 1$ as $n \to \infty$. Since

$$k\alpha_n^2 - t_n^2\alpha_n + t_n^2 < 0 \qquad \text{for } 1 \leqq k < n\,,$$

we have

$$\frac{\alpha_n}{t_n} - \frac{1}{t_n - k\alpha_n/t_n} = \frac{-k\alpha_n^2 + t_n^2\alpha_n - t_n^2}{t_n(t_n^2 - k\alpha_n)} > 0 \tag{31}$$

for $1 \leqq k < n$, provided n is large enough. We claim that

$$g^k(t_n) > t_n - \frac{k\alpha_n}{t_n} \qquad \text{for} \qquad 1 \leqq k \leqq n\,. \tag{32}$$

Since $\alpha_n > 1$, (32) holds for $k = 1$. Assuming that (32) is true for some k, $1 \leqq k < n$, we see with the help of (31) that it is also true for $k+1$ as follows:

$$g^{k+1}(t_n) = g^k(t_n) - \frac{1}{g^k(t_n)}$$
$$> t_n - \frac{k\alpha_n}{t_n} - \frac{1}{t_n - k\alpha_n/t_n}$$
$$> t_n - \frac{k\alpha_n}{t_n} - \frac{\alpha_n}{t_n} = t_n - \frac{(k+1)\alpha_n}{t_n} .$$

Thus (32) holds, and in particular we have

$$t_n - \frac{n\alpha_n}{t_n} < g^n(t_n) < t_n ,$$

which implies that

$$\lim_{n\to\infty} [t_n - g^n(t_n)] = 0 . \tag{33}$$

Since $\sin t_n = 1$, we see that $p(t) \geqq 0$ on $\bigcup_{n=N}^{\infty} [g^n(t_n), t_n]$, provided N is sufficiently large. On the other hand, by the mean value theorem, we have

$$\int_{g(t)}^{t} p(s)\, ds = \frac{ct^* \sin t^*}{t} \qquad \text{for some } t^* \in [g(t), t] . \tag{34}$$

Note that if $t \in [g^n(t_n), t_n]$, then

$$g^n(t_n) - \frac{1}{g^n(t_n)} \leqq g(t) \leqq t^* \leqq t \leqq t_n . \tag{35}$$

From (33), (34) and (35) we conclude that there exist a constant c' and an integer N' such that $\int_{g(t)}^{t} p(s)\, ds > c' > 1/e$ for $t \in \bigcup_{n=N'}^{\infty} [g^n(t_n), t_n]$. Thus the hypotheses of (i) of Theorem 3 are satisfied, and so equation (29) is oscillatory if $c > 1/e$. If $c < 0$, then the same argument applies by taking $t_n = [2n^2 + \frac{3}{2}]\pi$. Equation (30) can be analyzed analogously. We remark that if we apply Theorem 2, then we have the weaker conclusion that (29) and (30) are oscillatory for $|c| > 1$.

3. – Equations with deviating arguments of mixed type.

Let us now consider equation (1) in which $g(t)$ may be of mixed type. IVANOV and ŠEVELO [3] have recently given oscillation criteria for such an equation with one-signed coefficients. The purpose of this section is to proceed further to discuss the case where $p(t)$ is oscillatory.

We introduce the following notation:

$$\mathcal{A} = \{t \in (a, \infty) : g(t) > t\},$$
$$\mathcal{R} = \{t \in (a, \infty) : g(t) < t\}.$$

The sets $\mathcal{A}$ and $\mathcal{R}$ are called the advanced and the retarded part of $g(t)$, respectively. If $g(t)$ is of mixed type, then $\mathcal{A}$ and $\mathcal{R}$ are unbounded and countable unions of disjoint open intervals.

THEOREM 4. – (i) *Suppose that R is unbounded and is a countable union of disjoint open intervals: $\mathcal{R} = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$. If there is an infinite number of intervals $\{(\alpha_{n_k}, \beta_{n_k})\}_{k=1}^{\infty}$ such that*

$$p(t) \geqq 0 \quad \text{on} \quad \bigcup_{k=1}^{\infty} [\alpha_{n_k}, \beta_{n_k}] \tag{36}$$

and if

$$\int_{g(t_k)}^{t_k} p(s)\, ds \geqq 1 \quad \text{for some } t_k \in (\alpha_{n_k}, \beta_{n_k}),\ k = 1, 2, \ldots, \tag{37}$$

then equation (1) *is oscillatory.*

(ii) *Suppose that $\mathcal{A}$ is unbounded and is a countable union of disjoint open intervals: $\mathcal{A} = \bigcup_{n=1}^{\infty} (\gamma_n, \delta_n)$. If there is an infinite number of intervals $\{(\gamma_{n_k}, \delta_{n_k})\}_{k=1}^{\infty}$ such that*

$$-p(t) \leqq 0 \quad \text{on} \quad \bigcup_{k=1}^{\infty} [\gamma_{n_k}, \delta_{n_k}] \tag{38}$$

and if

$$\int_{t_k}^{g(t_k)} [-p(s)]\, ds \geqq 1 \quad \text{for some } t_k \in (\gamma_{n_k}, \delta_{n_k}),\ k = 1, 2, \ldots, \tag{39}$$

then equation (1) *is oscillatory.*

PROOF. We prove the first part (i). Let $y(t)$ be a nonoscillatory solution of (1) which may be assumed to be eventually positive without loss of generality. We have for sufficiently large k

$$y'(t) = -p(t)\, y(g(t)) \leqq 0 \quad \text{for} \quad t \in [\alpha_{n_k}, \beta_{n_k}],$$

so that $y(t)$ is nonincreasing on $[\alpha_{n_k}, \beta_{n_k}]$. Since $g(t)$ is nondecreasing and $g(t) = t$ at the endpoints α_{n_k} and β_{n_k}, we have $\alpha_{n_k} \leqq g(t) \leqq \beta_{n_k}$ for $\alpha_{n_k} \leqq t \leqq \beta_{n_k}$, and so and so $y(g(t))$ is nonincreasing on $[\alpha_{n_k}, \beta_{n_k}]$. An integration of (1) over $[g(t_k), t_k]$ yields, as in the proof of (i) of Theorem 2,

$$y(t_k) + y(g(t_k))\Big[\int_{g(t_k)}^{t_k} p(s)\,ds - 1\Big] \leqq 0\,,$$

which is clearly impossible. Thus the proof of (i) of Theorem 4 is complete. A parallel argument applies to the proof of the second part (ii).

EXAMPLE 5. – Consider the equation

$$y'(t) - c\sin t y(t + \sin t) = 0\,, \tag{40}$$

where $c > 0$ is a constant. Here $p(t) = -c\sin t$, $g(t) = t + \sin t$ and

$$\mathcal{A} = \bigcup_{n=0}^{\infty} (2n\pi, (2n+1)\pi)\,, \qquad \mathcal{R} = \bigcup_{n=1}^{\infty} ((2n-1)\pi, 2n\pi)\,.$$

Since $p(t) > 0$ on $\mathcal{R}$ and

$$\int_{g(t_n)}^{t_n} p(s)\,ds = c\sin 1$$

for $t_n = (2n - \frac{1}{2})\pi \in ((2n-1)\pi, 2n\pi)$, $n = 1, 2, \ldots$, by (i) of Theorem 4, equation (40) is oscillatory if $c > 1/\sin 1$. One may also apply (ii) of Theorem 4 by taking $t_n = (2n + \frac{1}{2})\pi$.

4. – Nonlinear equations with deviating arguments.

We are interested in extending the previous results to nonlinear equations of the form

$$y'(t) + p(t)f(y(g(t))) = 0 \tag{4}$$

where $f(y)$ is continuous on R and satisfies $yf(y) > 0$ for $y \neq 0$. Here we restrict ourselves to the case where $p(t)$ is of one sign and extend part of Theorem 1 as follows.

THEOREM 5. – (i) *Suppose that* $p(t) \geqq 0$ *and* $g(t) \leqq t$ *for* $t \geqq a$. *Suppose moreover that*

$$\lambda = \limsup_{y\to 0} \frac{|y|}{|f(y)|} < \infty\,. \tag{41}$$

If

$$\liminf_{t\to\infty}\int_{g(t)}^{t} p(s)\,ds > \frac{\lambda}{e}, \tag{42}$$

then equation (4) *is oscillatory.*

(ii) *Suppose that* $p(t)\leqq 0$ *and* $g(t)\geqq t$ *for* $t\geqq a$. *Suppose moreover that*

$$\mu = \limsup_{|y|\to\infty}\frac{|y|}{|f(y)|}<\infty\,. \tag{43}$$

If

$$\liminf_{t\to\infty}\int_{t}^{g(t)} [-p(s)]\,ds > \frac{\mu}{e}, \tag{44}$$

then equation (4) *is oscillatory.*

PROOF. – We only prove the second part. Let $y(t)$ be an eventually positive solution of (4). Condition (44) implies that

$$\int_a^\infty [-p(t)]\,dt = \infty\,.$$

Since $\int_a^\infty [-p(t)]\,dt<\infty$ is a necessary and sufficient condition for (4) to have a bounded nonoscillatory solution, it follows that $\lim_{t\to\infty} y(t)=\infty$. Suppose $\mu>0$. Then, in view of (43) we can choose $T>a$ so large that

$$f(y(t))\geqq\frac{1}{2\mu}\,y(t)\qquad\text{for}\qquad t\geqq T\,. \tag{45}$$

For each t sufficiently large there exists a t^* such that $t^*<t<g(t^*)$,

$$\int_{t^*}^{t}[-p(s)]\,ds\geqq\frac{\mu}{2e}\qquad\text{and}\qquad\int_{t}^{g(t^*)}[-p(s)]\,ds\geqq\frac{\mu}{2e}\,. \tag{46}$$

We let t be large enough so that $t^*\geqq T$. Integrating (4) over $[t^*, t]$ and $[t, g(t^*)]$ and using (45) and (46), we see that

$$y(t)-y(t^*)=\int_{t^*}^{t}[-p(s)]f\big(y(g(s))\big)\,ds\geqq\frac{1}{2\mu}\int_{t^*}^{t}[-p(s)]y(g(s))\,ds\geqq\frac{1}{2e}\,y(g(t^*))$$

$$y(g(t^*))-y(t)=\int_{t}^{g(t^*)}[-p(s)]f\big(y(g(s))\big)\,ds\geqq\frac{1}{2\mu}\int_{t}^{g(t^*)}[-p(s)]y(g(s))\,ds\geqq\frac{1}{4e}\,y(g(t))\,.$$

Combining the above inequalities yields

$$y(t) \geqq \frac{1}{(4e)^2}\, y(g(t)) \quad \text{for} \quad t \geqq T_1\,,$$

provided $T_1 > T$ is sufficiently large. Let

$$\omega = \liminf_{t\to\infty} \frac{y(g(t))}{y(t)}\,. \tag{47}$$

Then ω is finite: $1 \leqq \omega \leqq (4e)^2$.

We now divide (4) by $y(t)$ and integrate it over $[t, g(t)]$, obtaining

$$\log \frac{y(g(t))}{y(t)} = \int\limits_t^{g(t)} [-p(s)] \frac{f\big(y(g(s))\big)}{y(s)}\, ds = \int\limits_t^{g(t)} [-p(s)] \frac{f\big(y(g)s))\big)}{y(g(s))}\, \frac{y(g(s))}{y(s)}\, ds\,. \tag{48}$$

Taking lower limits on both sides of (48) and using (47), (44) and (43), we obtain $\log \omega > \omega/e$. But this is impossible since $\log x \leqq x/e$ for all $x > 0$. The case where $\mu = 0$ can be discussed similarly. Thus equation (4) cannot have an eventually positive solution. Likewise, (4) has no eventually negative solution.

EXAMPLE 6. – Consider the advanced equation

$$y'(t) - \frac{t^m}{2\log(1+2t)}\, y(2t) \log\left[1 + |y(2t)|\right] = 0\,, \tag{49}$$

where m is a constant. If we put $p(t) = -t^m/2\log(1+2t)$, $g(t) = 2t$ and $f(y) = y\log\big(1+|y|\big)$, then $\lim_{|y|\to\infty} y/f(y) = 0$ and

$$\lim_{t\to\infty} \int\limits_t^{g(t)} [-p(s)]\, ds = \begin{cases} \infty & (m > -1)\,, \\ 0 & (m \leqq -1)\,. \end{cases}$$

Hence, by (ii) of Theorem 5, equation (49) is oscillatory if $m > -1$. Note that $y(t) = t$ is a nonoscillatory solution of (49) with $m = -1$. If $m < -1$, then (49) has a bounded nonoscillatory solution $y(t)$, since $\int^{\infty} [-p(t)]\, dt < \infty$.

REMARK 1. – Equations of the form (4) with different nonlinearities have been studied in IVANOV and ŠEVELO [3], KITAMURA and KUSANO [4] and ŠEVELO and IVANOV [18]. In these papers a duality between the advanced and retarded cases is clearly described.

REMARK 2. – It would be of interest to obtain oscillation criteria for nonlinear equations of the form (4) in which the coefficient $p(t)$ changes sign infinitely often as $t \to \infty$.

5. – Equations with several deviating arguments.

In this section we consider equations with several deviating arguments of the types

$$y'(t) + \sum_{i=1}^{N} p_i(t) y(g_i(t)) = 0 , \tag{50}$$

$$y'(t) + p(t) f\big(y(g_1(t)), \ldots, y(g_N(t))\big) = 0 , \tag{51}$$

where $p_i(t)$ and $g_i(t)$, $1 \leqq i \leqq N$, are continuous on $[a, \infty)$ and $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leqq i \leqq N$. The previous results, except Theorem 4, for equations (1) and (4) allow natural extensions to the above equations (50) and (51), respectively. Below we state the extended versions of Theorems 1, 3 and 5.

THEOREM 1′. – (i) *Suppose that* $p_i(t) \geqq 0$ *and* $g_i(t) \leqq t$ *for* $t \geqq a$, $1 \leqq i \leqq N$. *Suppose moreover that there exists a continuous nondecreasing function* $g^*(t)$ *such that* $g_i(t) \leqq g^*(t) \leqq t$ *for* $t \geqq a$, $1 \leqq i \leqq N$. *If*

$$\liminf_{t\to\infty} \int\limits_{g^*(t)}^{t} \sum_{i=1}^{N} p_i(s)\, ds > \frac{1}{e} , \tag{52}$$

then equation (50) *is oscillatory. If, on the other hand, there exists a continuous nondecreasing function* $g_*(t)$ *such that* $g_*(t) \leqq g_i(t)$ *for* $t \geqq a$, $1 \leqq i \leqq N$, $\lim_{t\to\infty} g_*(t) = \infty$ *and*

$$\int\limits_{g_*(t)}^{t} \sum_{i=1}^{N} p_i(s)\, ds \leqq \frac{1}{e} \qquad \textit{for all sufficiently large } t , \tag{53}$$

then (50) *has a nonoscillatory solution.*

(ii) *Suppose that* $p_i(t) \leqq 0$ *and* $g_i(t) \geqq t$ *for* $t \geqq a$, $1 \leqq i \leqq N$. *Suppose moreover that there exists a continuous nondecreasing function* $g_*(t)$ *such that* $t \leqq g_*(t) \leqq$ $\leqq g_i(t)$ *for* $t \geqq a$, $1 \leqq i \leqq N$. *If*

$$\liminf_{t\to\infty} \int\limits_{t}^{g_*(t)} \sum_{i=1}^{N} [-p_i(s)]\, ds > \frac{1}{e} , \tag{54}$$

then equation (50) *is oscillatory. If, on the other hand, there exists a continuous nondecreasing function* $g^*(t)$ *such that* $g_i(t) \leqq g^*(t)$ *for* $t \geqq a$, $1 \leqq i \leqq N$, *and*

$$\int\limits_{t}^{g^*(t)} \sum_{i=1}^{N} [-p_i(s)]\, ds \leqq \frac{1}{e} \qquad \textit{for all sufficiently large } t , \tag{55}$$

then equation (50) *has a nonoscillatory solution.*

THEOREM 3′. – (i) *Suppose that there is a continuous nondecreasing function $g^*(t)$ such that $g_i(t) \leqq g^*(t) \leqq t$ for $t \geqq a$, $1 \leqq i \leqq N$. Suppose moreover that there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ as $n \to \infty$, the intervals $\{[(g^*)^n(t_n), t_n]\}_{n=1}^{\infty}$ are disjoint and*

$$p_i(t) \geqq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} [(g^*)^n(t_n), t_n]\,, \; 1 \leqq i \leqq N\,. \tag{56}$$

If there is a constant c such that

$$\int_{g^*(t)}^{t} \sum_{i=1}^{N} p_i(s)\, ds > c > \frac{1}{e} \qquad \text{for} \qquad t \in \bigcup_{n=1}^{\infty} [(g^*)^{n-1}(t_n), t_n]\,, \tag{57}$$

then equation (50) *is oscillatory.*

(ii) *Suppose that there is a continuous nondecreasing function $g_*(t)$ such that $t \leqq g_*(t) \leqq g_i(t)$ for $t \geqq a$, $1 \leqq i \leqq N$. Suppose moreover that there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ as $n \to \infty$, the intervals $\{[t_n, (g_*)^n(t_n)]\}_{n=1}^{\infty}$ are disjoint and*

$$-p_i(t) \geqq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} [t_n, (g_*)^n(t_n)]\,, \; 1 \leqq i \leqq N\,. \tag{58}$$

If there is a constant c such that

$$\int_{t}^{g_*(t)} \sum_{i=1}^{N} [-p_i(s)]\, ds > c > \frac{1}{e} \qquad \text{for} \qquad t \in \bigcup_{n=1}^{\infty} [t_n, (g_*)^{n-1}(t_n)]\,, \tag{59}$$

then equation (50) *is oscillatory.*

THEOREM 5′. – (i) *Suppose that $p(t) \geqq 0$ and $g_i(t) \leqq t$ for $t \geqq a$, $1 \leqq i \leqq N$. Suppose moreover that $f(y_1, \ldots, y_N)$ is a continuous function on R^N such that*

$$y_1 f(y_1, \ldots, y_N) > 0 \qquad \text{for} \qquad y_1 y_i > 0\,, \; 1 \leqq i \leqq N\,, \tag{60}$$

and

$$\lambda = \limsup_{\substack{y_i \to 0 \\ 1 \leqq i \leqq N}} \frac{|y_1|^{\alpha_1} \ldots |y_N|^{\alpha_N}}{|f(y_1, \ldots, y_N)|} < \infty \tag{61}$$

for some nonnegative constants α_i, $1 \leqq i \leqq N$, with $\sum_{i=1}^{N} \alpha_i = 1$. If there is a continuous nondecreasing function $g^*(t)$ such that $g_i(t) \leqq g^*(t) \leqq t$ for $t \geqq a$, $1 \leqq i \leqq N$, and

$$\liminf_{t \to \infty} \int_{g^*(t)}^{t} p(s)\, ds > \frac{\lambda}{e}\,, \tag{62}$$

then equation (51) *is oscillatory.*

(ii) *Suppose that* $p(t) \leqq 0$ *and* $g_i(t) \geqq t$ *for* $t \geqq a$, $1 \leqq i \leqq N$. *Suppose moreover that* $f(y_1, \ldots, y_N)$ *is a continuous function on* R^N *satisfying* (60) *and*

$$\mu = \limsup_{\substack{|y_i| \to \infty \\ 1 \leqq i \leqq N}} \frac{|y_1|^{\beta_1} \ldots |y_N|^{\beta_N}}{|f(y_1, \ldots, y_N)|} < \infty \tag{63}$$

for some nonnegative constants β_i, $1 \leqq i \leqq N$, *with* $\sum_{i=1}^{N} \beta_i = 1$. *If there is a continuous nondecreasing function* $g_*(t)$ *such that* $t \leqq g_*(t) \leqq g_i(t)$ *for* $t \geqq a$, $1 \leqq i \leqq N$, *and*

$$\liminf_{t \to \infty} \int_t^{g_*(t)} [-p(s)]\, ds > \frac{\mu}{e}, \tag{64}$$

then equation (51) *is oscillatory.*

The first part of Theorem 1′ has recently been obtained by KOPLATADZE and ČANTURIJA [7]. Theorem 2′ could easily be formulated. These theorems can be proved by proceeding, with slight modifications, as in the proof of the corresponding theorems (without « primes »). The details are left to the reader.

EXAMPLE 7. – Consider the retarded equation

$$y'(t) + \frac{1}{et} y\left(\frac{t}{e}\right) + \frac{1}{2et} y\left(\frac{t}{e^2}\right) = 0, \tag{65}$$

which is a special case of (50) in which $p_1(t) = 1/et$, $p_2(t) = 1/2et$, $g_1(t) = t/e$ and $g_2(t) = t/e^2$. One can take $g^*(t) = t/e$. Since

$$\int_{g^*(t)}^{t} [p_1(s) + p_2(s)]\, ds = \frac{3}{2e} > \frac{1}{e}$$

for all $t > 0$, equation (65) is oscillatory by (i) of Theorem 1′. Note that each of the equations

$$y'(t) + \frac{1}{et} y\left(\frac{t}{e}\right) = 0 \quad \text{and} \quad y'(t) + \frac{1}{2et} y\left(\frac{t}{e^2}\right) = 0$$

has a nonoscillatory solution.

EXAMPLE 8. – Consider the equations

$$y'(t) + at \sin t\, y\left(t - \frac{1}{t}\right) + bt \cos t\, y\left(t - \frac{2}{t}\right) = 0, \tag{66}$$

$$y'(t) + at \sin t\, y\left(t + \frac{1}{t}\right) + bt \cos t\, y\left(t + \frac{2}{t}\right) = 0, \tag{67}$$

where a and b are nonzero constants. With regard to (66) we can take $p_1(t) = at \sin t$, $p_2(t) = bt \cos t$, $g_1(t) = t - (1/t)$, $g_2(t) = t - (2/t)$ and $g^*(t) = t - (1/t)$. Noting that

$$\int_{g^*(t)}^{t} p(s)\, ds = (a^2 + b^2)^{1/2} \int_{t-(1/t)}^{t} s \cos (s - \theta)\, ds$$

for some θ, $0 < \theta < 2\pi$, and arguing as in Example 4 by taking $t_n = 2n^2\pi + \theta$, we conclude from (i) of Theorem 3′ that (66) is oscillatory if $(a^2 + b^2)^{1/2} > 1/e$. The same is true of equation (67).

EXAMPLE 9. – Consider the advanced equation

$$y'(t) - at^m [y(2t)]^{1/3} [y(4t)]^{2/3} = 0\,, \tag{68}$$

where $a > 0$ and m are constants. This is a special case of (51) in which $f(y_1, y_2) = y_1^{1/3} y_2^{2/3}$, $g_1(t) = 2t$, $g_2(t) = 4t$ and $p(t) = -at^m$. One can take $g_*(t) = 2t$. Condition (63) is satisfied with $\beta_1 = 1/3$, $\beta_2 = 2/3$ and $\mu = 1$. Since

$$\int_{t}^{g_*(t)} [-p(s)]\, ds = \begin{cases} \infty & (m > -1)\,, \\ a \log 2 & (m = -1)\,, \end{cases}$$

according to (ii) of Theorem 5′, equation (68) is oscillatory for any $a > 0$ if $m > -1$ or if $m = -1$ and $a > 1/(e \log 2)$. Note that when $m = -1$ and $a = 2^{-7/3}$ equation (68) has a nonoscillatory solution $y(t) = t^2$. It is not had to see that if $m < -1$, then (68) has a bounded nonoscillatory solution.

REMARK 3. – Theorems 1′, 3′ and 5′ can further be extended to equations of the forms

$$y'(t) + a(t) y(t) + \sum_{i=1}^{N} p_i(t) y(g_i(t)) = 0\,,$$

$$y'(t) + a(t) y(t) + p(t) f\big(y(g_1(t)), \ldots, y(g_N(t))\big) = 0\,,$$

so that some of the previous oscillation criteria presented in the papers [3, 11, 17, 20-23] are covered.

Finally we show that there is a class of functional differential equations for which the oscillation situation can be completely characterized.

THEOREM 6. – *Consider the equations*

$$y'(t) + f\big(y(t - \tau_1), \ldots, y(t - \tau_N)\big) = 0\,, \tag{69}$$

$$y'(t) - f\big(y(t + \tau_1), \ldots, y(t + \tau_N)\big) = 0\,, \tag{70}$$

where τ_i, $1 \leqq i \leqq N$, are positive constants with $\tau_1 \leqq \ldots \leqq \tau_N$. Suppose that $f(y_1, \ldots, y_N)$ is continuous on R^N and increasing in each y_i, $1 \leqq i \leqq N$, and satisfies

$$f(\alpha y_1, \ldots, \alpha y_N) = \alpha f(y_1, \ldots, y_N) \quad \text{for all } \alpha \in R . \tag{71}$$

Then a necessary and sufficient condition for equations (69) *and* (70) *to be oscillatory is that*

$$\min_{\lambda \geqq 0} [-\lambda + f(\exp[\lambda\tau_1], \ldots, \exp[\lambda\tau_N])] > 0 . \tag{72}$$

PROOF. – Suppose that (70) has a nonoscillatory solution $y(t)$. We may suppose that $y(t) > 0$ for $t \geqq t_0$. Since, by (70),

$$y'(t) > f(y(t+\tau_1), \ldots, y(t+\tau_N)) > 0 , \qquad t \geqq t_0 ,$$

$y(t)$ is increasing for $t \geqq t_0$, and so, using (71) and the increasing nature of f, we have

$$f(y(t+\tau_1), \ldots, y(t+\tau_N)) > f(y(t), \ldots, y(t)) = f(1, \ldots, 1) y(t) , \tag{73}$$

$$f(y(t+\tau_1), \ldots, y(t+\tau_N)) \geqq f(y(t+\tau_1), \ldots, y(t+\tau_1)) = f(1, \ldots, 1) y(t+\tau_1) , \tag{74}$$

and

$$f(y(t+\tau_1), \ldots, y(t+\tau_N)) \leqq f(y(t+\tau_N), \ldots, y(t+\tau_N)) = f(1, \ldots, 1) y(t+\tau_N) . \tag{75}$$

Let Λ denote the following set of positive numbers:

$$\Lambda = \{\lambda > 0 \colon y'(t) - \lambda y(t) > 0 \text{ for all sufficiently large } t\} .$$

In view of (73) we have

$$\begin{aligned} 0 &= y'(t) - f(y(t+\tau_1), \ldots, y(t+\tau_N)) \\ &< y'(t) - f(1, \ldots, 1) y(t) , \qquad r \geqq t_0 , \end{aligned}$$

so that $f(1, \ldots, 1) \in \Lambda$, that is, Λ is not empty. It can be shown that Λ is bounded from above. In fact, from (70) and (74) it follows that

$$y'(t) - f(1, \ldots, 1) y(t+\tau_1) \geqq 0 , \qquad t \geqq t_0 . \tag{76}$$

Integrating (76) over the intervals $[t - (\tau_1/2), t]$ and $[t, t + (\tau_1/2)]$, we get

$$y(t) - y\left(t - \frac{\tau_1}{2}\right) \geqq f(1, \ldots, 1) \int_{t-(\tau_1/2)}^{t} y(s+\tau_1)\, ds \geqq \frac{\tau_1}{2} f(1, \ldots, 1)\, y\left(t + \frac{\tau_1}{2}\right),$$

and

$$y\left(t+\frac{\tau_1}{2}\right)-y(t) \geqq f(1, \ldots, 1) \int_t^{t+(\tau_1/2)} y(s+\tau_1)\, ds \geqq \frac{\tau_1}{2} f(1, \ldots, 1) y(t+\tau_1) .$$

Combining these inequalities yields

$$y(t) \geqq \tfrac{1}{4}[\tau_1 f(1, \ldots, 1)]^2 y(t+\tau_1) ,$$

which implies that the function

$$\varphi(t)=\frac{y(t+\tau_1)}{y(t)}$$

is bounded for $t \geqq t_0+(\tau_1/2)$. If $n>0$ is an integer such that $\tau_N \leqq n\tau_1$, then

$$\frac{y(t+\tau_N)}{y(t)} \leqq \frac{y(t+n\tau_1)}{y(t)}=\varphi(t)\varphi(t+\tau_1) \ldots \varphi(t+(n-1)\tau_1) ,$$

and so there is a constant $M>0$ such that

$$\frac{y(t+\tau_N)}{y(t)} \leqq M , \quad t \geqq t_0+\frac{\tau_1}{2} . \tag{77}$$

From (70), (75) and (77) it follows that

$$\begin{aligned} 0 &= y'(t)-f(y(t+\tau_1), \ldots, y(t+\tau_N)) \\ &\geqq y'(t)-f(1, \ldots, 1) y(t+\tau_N) \\ &= y'(t)-f(1, \ldots, 1) \frac{y(t+\tau_N)}{y(t)} y(t) \\ &\geqq y'(t)-M f(1, \ldots, 1) y(t) , \quad t \geqq t_0+\frac{\tau_1}{2} . \end{aligned}$$

This shows that $Mf(1, \ldots, 1) \notin \Lambda$. Thus Λ is bounded from above. Consequently, there is a $\lambda_0>0$ such that $\lambda_0 \in \Lambda$ but $\lambda_0+m \notin \Lambda$, where m denotes the left-hand side of (72). Put

$$z(t)=\exp[-\lambda_0 t] y(t) .$$

Since $z'(t)=\exp[-\lambda_0 t][y'(t)-\lambda_0 y(t)]>0$, $z(t)$ is increasing. Using this fact, the increasing nature of f and (64), we see that

$$\begin{aligned} 0 &= y'(t)-f(y(t+\tau_1), \ldots, y(t+\tau_N)) \\ &= \exp[\lambda_0 t][z'(t)+\lambda_0 z(t)]-f(\exp[\lambda_0(t+\tau_1)] z(t+\tau_1), \ldots, \exp[\lambda_0(t+\tau_N)] z(t+\tau_N)) \end{aligned}$$

$$
\begin{aligned}
&= \exp[\lambda_0 t][z'(t) + \lambda_0 z(t) - f(\exp[\lambda_0 \tau_1] z(t+\tau_1), \ldots, \exp[\lambda_0 \tau_N] z(t+\tau_N))] \\
&< \exp[\lambda_0 t]\{z'(t) - [-\lambda_0 + f(\exp[\lambda_0 \tau_1], \ldots, \exp[\lambda_0 \tau_N])] z(t)\} \\
&\leqq \exp[\lambda_0 t][z'(t) - m z(t)] \\
&= y'(t) - (\lambda_0 + m) y(t), \qquad t \geqq t_0 .
\end{aligned}
$$

This implies that $\lambda_0 + m \in \Lambda$, a contradiction. We therefore conclude that (70) must be oscillatory provided (72) is satisfied.

If (72) is violated, then there is a $\lambda^* > 0$ such that

$$-\lambda^* + f(\exp[\lambda^* \tau_1], \ldots, \exp[\lambda^* \tau_N]) = 0 .$$

The function $y(t) = \exp[\lambda^* t]$ is a nonoscillatory solution of (70), since

$$
\begin{aligned}
y'(t) - f(y(t+\tau_1), \ldots, y(t+\tau_N)) \\
&= \lambda^* \exp[\lambda^* t] - f(\exp[\lambda^*(t+\tau_1)], \ldots, \exp[\lambda^*(t+\tau_N)]) \\
&= \exp[\lambda^* t][\lambda^* - f(\exp[\lambda^* \tau_1], \ldots, \exp[\lambda^* \tau_N])] = 0 .
\end{aligned}
$$

Equation (69) can be discussed analogously. This finishes the proof.

Remark 4. - The linear equation with constant coefficients and constant deviations

$$y'(t) + \sum_{i=1}^{N} p_i y(t - \tau_i) = 0 \tag{78}$$

is a special case of (69) (resp. (70)) satisfying the hypotheses of Theorem 6 if $p_i > 0$ and $\tau_i > 0$, $1 \leqq i \leqq N$, (resp. if $p_i < 0$, $\tau_i < 0$, $1 \leqq i \leqq N$). Condition (72) then reduces to

$$\min_{\lambda \geqq 0}\left[-\lambda + \sum_{i=1}^{N} p_i \exp[\lambda \tau_i]\right] > 0 \qquad \left[\text{resp. } \min_{\lambda \geqq 0}\left[-\lambda - \sum_{i=1}^{N} p_i \exp[-\lambda \tau_i]\right] > 0\right].$$

A characterization for the oscillation of (78) in the retarded case was obtained by Tramov [24]. A different proof has recently be given by Ladas, Sficas and Stavroulakis [13]. Our proof presented above is an adaptation by duality of their method to the advanced case. We note that equations (69) and (70) may not be linear as the following example shows.

Example 10. - Consider the equations

$$y'(t) + p[y(t-3)]^{1/3}[y(t-6)]^{2/3} = 0 , \tag{79}$$

$$y'(t) - p[y(t+3)]^{1/3}[y(t+6)]^{2/3} = 0 , \tag{80}$$

where p is a positive constant. The function $f(y_1, y_2) = p y_1^{1/3} y_2^{2/3}$ satisfies the hypotheses of Theorem 6, and so these equations are oscillatory if and only if

$$\min_{\lambda \geq 0} (-\lambda + p \exp[5\lambda]) > 0 ,$$

that is, $p > 1/5e$.

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