On the Monodromy of Weierstrass Points (*).

GIUSEPPE CANUTO (Pavia) (**)

Summary. – *In this paper we consider the family of curves of genus $g = 2m$ with a g_3^1 lying on a particular rational normal scroll S in $\mathbf{P}^{g-1}(\mathbf{C})$. We define a covering of this family representing the Weierstrass points and we study the monodromy. Applying the techniques of* [3] *we prove that if $g = 4$ the monodromy is the full symmetric group and for general $g = 2m$ it is transitive. We show also that the generic curve of the family has only normal Weierstrass points generalizing a classical result. We work always over the complex numbers.*

1. – Monodromy groups and Weierstrass points.

Canonical curves and rational normal scrolls

Let $\mathbf{P}^k, \mathbf{P}^l$ be complementary subspaces in $\mathbf{P}^n$, $n = k + l + 1$. The rational normal scroll $S_{n,l} = S$ is the surface consisting of the straight lines joining corresponding points of the rational normal curves in $\mathbf{P}^k$ and $\mathbf{P}^l$ [1].

It's a surface of degree $k + l$ and $\operatorname{Pic}(S) = Z\cdot H \oplus Z\cdot L$ where H is the hyperplane section and L is a line of the ruling. The canonical bundle K_C is equal to $-2H + (n-3)L$. If $C \subset S$ is a canonical curve, it's easy to see that $C = 3H + (3-n)L$.

In particular C has g_3^1.

On the other hand it's well known [6] that if C is a canonical curve of genus g in $\mathbf{P}^{g-1}$ which is trigonal, then C lies on a rational normal scroll $S_{k,l}$.

If we fix g the possible values for k are given by

$$\frac{g-4}{3} \leqslant k \leqslant \frac{g-2}{2}$$

and among the canonical curves of genus g with a g_3^1 the generic one lies on the scroll such that $l-k$ is minimum (we can always suppose $k \leqslant l$) [4].

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Weierstrass points.

Let C be a Riemann surface of genus g and p a point of C. According to Riemann-Roch we have the following behaviour for the divisors $k\cdot p$ where $k = 0, 1, 2, \dots$

$$\begin{array}{lcccccc} k & 0 & 1 & \dots & 2g-1 & 2g & \dots \\ h^0(k\cdot p) & 1 & 1 & \dots & g & g+1 & \dots \end{array}$$

so there are integers $1 = a_1 < a_2 < \dots < a_g$ such that

$$h^0(a_i\cdot p) = h^0\big((a_i - 1)\cdot p\big)$$

for all i.

These numbers are called *gap values* of p: notice that the first gap value is always 1, [2].

A point p is called regular if the gap sequence is $1, 2, \dots, g$ otherwise a Weierstrass point.

The weight of such a point is defined to be

$$W(p) = \Sigma_i(a_i - i)\,.$$

A Weierstrass point is hyperelliptic if the gap sequence is $a_i = 2i - 1$ and C is hyperelliptic if and only if it contains a hyperelliptic Weierstrass point.

If the gap sequence at p is $1, 2, \dots, g-1, g+1$ then p is called a normal Weierstrass point.

These are the only points with weight one. The total weight of C is the sum of the weights of its points and this number is

$$W = (g-1)g(g+1)\,.$$

In particular the number of Weierstrass points is finite and it's equal to W exactly when all the Weierstrass points are normal. We have the following classical

THEOREM. – *The generic Riemann surface of genus $g \geqslant 3$ has only normal Weierstrass points.*

(See [1], chapter 2 or [5].)

It's clear that b is not a gap if and only if there exists a meromorphic function on C whose only singularity is a pole of order exactly b at p.

It follows that the set of non-gap values at p is closed under addition. Using Riemann-Roch we see that p is a Weierstrass point if and only if $h^0(K_C - g\cdot p)$ is different from zero, where K_C is the canonical divisor of C.

Galois groups and monodromy groups.

We recall here some facts about Galois and monodromy groups. We refer to [3] for complete proofs. Suppose X and Y are irreducible varieties of the same dimension over the complex numbers and $\pi\colon Y\to X$ a map of degree d. Let p be a generic point of X and $\Gamma=\pi^{-1}(p)=\{q_1, q_2, \ldots, q_d\}$ the fiber of π at p.

We can consider two subgroups of the full symmetric group on d elements Σ_d:

(1) If we normalize the function fields extension $K(Y)/K(X)$ we get a Galois extension $L/K(X)$ and we can consider the Galois group $\mathrm{Gal}\,(L/K(X))$ which we denote also $\mathrm{Gal}\,(\pi)$.

(2) The second group is the monodromy group $\mathrm{Mon}\,(\pi)$ defined as follows: if U is a Zariski open of X over which π is unramified, there is a natural action of $\pi_1(U,p)$ on Γ defined as usual by lifting loops.

In other words we have a morphism

$$\pi_1(U,p)\to\Sigma_d$$

and the image is by definition $\mathrm{Mon}\,(\pi)$.

We have ([3]).

THEOREM. – *For* $Y\xrightarrow{\pi}X$ *as above, the monodromy group* $\mathrm{Mon}\,(\pi)$ *is isomorphic to the Galois group* $\mathrm{Gal}\,(\pi)$.

In this paper we consider the scroll $S=S_{m-1,m-1}\subset \boldsymbol{P}^{g-1}$ where $g=2m\geqslant 4$ which is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1=Q$.

If L_1, L_2 are lines of the two rulings of Q we have

$$L=L_2$$

say, and

$$H=L_1+(m-1)L_2$$

and S is the image of Q under the embedding given by the complete linear system $|H|$.

The complete linear system

$$|C|=|3L_1+(m+1)L_2|$$

of curves of type $(3, m+1)$ on Q has projective dimension $4m+7=2g+7$ and the generic element C is a canonical curve of genus $g=2m$ in $\boldsymbol{P}^{g-1}$.

Let's put $X=|C|\simeq\boldsymbol{P}^{2g+7}$.

If $\boldsymbol{P}^{g-1*}$ is the set of hyperplanes of $\boldsymbol{P}^{g-1}$ let Z be the subvariety of $S\times\boldsymbol{P}^{g-1*}$ consisting of the couples (z,H) such that $z\in H\cap S$.

The variety Z is smooth, irreducible, of dimension g since it's fibered in $\boldsymbol{P}^{g-2}$ over S.

We define then $I \subseteq X \times Z$ as follows

$$I = \{(C, z, H) \text{ s.t. } m_z(C \cdot H) \geqslant g\}$$

where $m_z(C \cdot H)$ is the intersection multiplicity of C with H at z.

If $\pi\colon I \to X$ is the projection and C is a smooth curve of X, a couple (z, H) belongs to $\pi^{-1}(C)$ exactly when z is a Weierstrass point of C and H is a section of $(K_C - g \cdot z)$.

In what follows a curve D on $S \approx Q$ is said of type (a, b) if D is linearly equivalent to $aL_1 + bL_2$ i.e. we refer to the basis L_1, L_2 of Pic (S).

We recall also the following elementary fact: a curve of type $(1, b)$ is smooth if and only if it is irreducible.

2. – Transitivity.

In this section we want to look at the irreducible components of I by studying the projection map

$$\varphi\colon I \to Z\,.$$

We have the following

PROPOSITION 1. – *I has only one component of maximal dimension $2g + 7$ mapping over X.*

More precisely if X_1 is the subvariety of X consisting of curves C which are singular and X_2 the subvariety consisting of curves having a point of total ramification for the g_3^1 and if $J = \pi^{-1}(X_1 \cup X_2)$, we want to prove that among the components of I of dimension greater or equal to $2g + 7$ only one is not contained in J.

Denoting by $L_2(z)$ the line of the second ruling of $Q \approx S$ through the point z, for any couple k, l such that $k + l \leqslant m - 1$, we can introduce the subvariety $Z_{k,l}$ of Z consisting of couples (z, H) such that if $E = H \cap S$ we can write

$$E = E_1 + E_2 + E_3$$

where

E_1 is a curve of type $(1, k)$ containing z;

E_2 is equal to $lL_2(z)$;

E_3 consists of $(m-1) - (k + l)$ lines L_2.

The variety parametrizing couples (z, E_1), where E_1 is a curve of type $(1, k)$ containing z, is smooth of dimension $2 + 2k$.

Since the complete linear system $|E_3|$ has dimension $(m-1)-(k+l)$, we see that $Z_{k,l}$ is closed irreducible variety of dimension $m+k+1-l$. When one of the lines of E_3 coincides with $L_2(z)$ we get a point (z, H) in $Z_{k,l+1}$ and when E_1 can be written as $E_1' + L_2$ where E_1' is a curve of type $(1, k-1)$ containing z we get a point of $Z_{k-1,l}$.

In other words we have the inclusions

$$Z_{k,l} \supseteq Z_{k,l+1}$$

$$Z_{k,l} \supseteq Z_{k-1,l}\,.$$

For $l = 1, 2, \ldots, m-1$ we introduce also the subvariety Z_l of Z consisting of couples (z, H) s.t. if $E = H \cap S$ then

$$E = E_2 + R$$

where R is any curve of type $(1, m-1-l)$ and $E_2 = lL_2(z)$.

It's easy to see that Z_l is closed, irreducible of dimension $2m - 2l + 1$ and

$$Z_l \supseteq Z_{l+1}$$

$$Z_l \supseteq Z_{k,l}$$

for all k s.t. $k + l \leqslant m - 1$.

The set $U_{k,l} \subseteq Z_{k,l}$ consisting of couples (z, H) such that the irreducible components of $E = H \cap S$ passing through z are exactly a curve of type $(1, k)$ and the line $L_2(z)$ with multiplicity l is Zariski open in $Z_{k,l}$ since its complement in $Z_{k-1,l} \cup \cup (Z_{k,l} \cap Z_{l+1})$.

In the same way the set $U_l \subseteq Z_l$ consisting of couples (z, H) s.t. the only component of E through z is $L_2(z)$ with multiplicity l is Zariski open in Z_l since its complement consists of the union of Z_{l+1} and all the $Z_{k,l}$'s for $k = 0, \ldots, m-1-l$.

Take now a point $p = (z, H)$ in $Z_{k,l} - U_{k,l}$. We have the following possibilities for $E = H \cap S$:

(1) E contains an irreducible component E_1 of type $(1, k)$ through z, and $L_2(z)$ with multiplicity $l' > l$.
Then $p \in Z_{k,l+1}$.

(2) E contains an irreducible component E_1' of type $(1, k')$ through z with $k' < k$ z with $k' < k$ and $L_2(z)$ with multiplicity $l' \geqslant l$.
Then $p \in Z_{k-1,l}$.

(3) E can be written as

$$E = E_1' + E_2 + E_3'$$

where

E_1' is irreducible of type $(1, k')$ with $k' < k$ and it does not contain z;

E_2 is $lL_2(z)$;

E_3' contains $L_2(z)$.

If $k \geqslant k' + 2$ we can write

$$E = E_1 + E_2 + E_3$$

where

$$E_1 = E_1' + L_2(z)$$
$$E_3 = E_3' - L_2(z)$$

and $p \in Z_{k'+1,l} \subseteq Z_{k-1,l}$.
If $k = k' + 1$ and E_3' contains $L_2(z)$ at least twice we can write

$$E = E_1 + E_2' + E_3$$

where

$$E_1 = E_1' + L_2(z)$$
$$E_2' = E_2 + L_2(z)$$
$$E_3 = E_3' - 2L_2(z)$$

and $p \in Z_{k,l+1}$.
Finally if $k = k' + 1$ and E_3' contains $L_2(z)$ with multiplicity one then $p \in \in Z_{k,l} \cap U_{l+1}$.

From the previous analysis it follows that if Z^α is the closed subvariety of Z consisting of the union of the $Z_{k,l}$'s and Z_n's of dimension smaller or equal than α the set $Z^\alpha - Z^{\alpha-1} = V^\alpha$ is open in Z^α and it consists of the union of the following sets

$$U_{k,l} \quad \text{for } k, l \quad \text{s.t.} \quad \begin{cases} k + l \leqslant m - 1 \\ \alpha = m + 1 + k - l \end{cases}$$

$$U_n \quad \text{for } n \quad \text{s.t.}$$

$$\alpha = 2m - 2n + 1$$

$$Z_{k,l} \cap U_{l+1} \quad \text{for all } k, l \quad \text{s.t.}$$

$$\begin{cases} k + l \leqslant m - 1 \\ \alpha = m + 1 + k - l \\ \dim U_{l+1} > \alpha \end{cases}$$

$$\begin{cases} \text{i.e. for all } k, l \text{ s.t.} \\ \alpha = m + 1 + k - l \\ k + l < m - 1 \,. \end{cases}$$

We can look now at the fibers of φ over V^α.

If $p = (z, H)$ belongs to U_n (for some n) a curve C of type $(3, m+1)$ has

$$m_z(C \cdot H) = n \cdot m_z\big(C \cdot L_2(z)\big)\,.$$

If C is non-singular, it does not contain $L_2(z)$ and we can have

$$m_z\big(C \cdot L_2(z)\big) = 3 \quad \text{in which case } C \in X_2$$

or

$$m_z\big(C \cdot L_2(z)\big) \leqslant 2$$

and then

$$m_z(C \cdot H) \leqslant 2n < g$$

because $n \leqslant m - 1$.

If follows that if $p \in U_n$ then $\varphi^{-1}(p) \subseteq J$.

Take now a point $p = (z, H)$ in $U_{k,l}$ where $\alpha = m + 1 + k - l$ and call E_1 the component of type $(1, k)$ through z of $E = H \cap S$.

The curve E_1 is smooth and rational and the linear system $|C|$ of curves of type $(3, m+1)$ cuts on E_1 the complete system of degree $m + 1 + 3k$. Let's consider the curves C such that $m_z(E_1 \cdot C) \geqslant g - l$.

If

$$g - l \leqslant 3k + 1 + m \quad \text{i.e. if} \quad m \leqslant l + 3k + 1$$

they form a linear system of projective dimension

$$(2g + 7) - (g - l) = g + l + 7$$

and they satisfy also the condition $m_z(C \cdot H) > g$ since they intersect $L_2(z)$ at z.

In case $g-l>3k+1+m$ a curve C can have $m_z(C\cdot E_1)\geqslant g-l$ only when it contains E_1 as a component.

On the other hand if $m_z(C\cdot H)\geqslant g$ and $m_z(C\cdot E_1)<g-l$ we must have $m_z(C\cdot\cdot L_2(z))\geqslant 2$.

Now if

$$m_z(C\cdot L_2(z))\geqslant 3 \quad \text{then } C\in X_1\cup X_2$$

and if C is non-singular and $m_z(C\cdot L_2(z))=2$ then

$$m_z(C\cdot H)=2l+1\leqslant 2(m-1)+1<g\,.$$

In fact in the last case the intersection multiplicity at z of C with E_1 is one because E_1 is transverse to $L_2(z)$.

It follows that if $p\in V^g=U_{m-1,0}$ the fiber $\varphi^{-1}(p)$ is a $\boldsymbol{P}^{g+7}$ and if $p\in V^\alpha$, $\alpha<g$,

$$\varphi^{-1}(p)\subseteq J$$

or

$$\varphi^{-1}(p)=\boldsymbol{P}^\beta\cup J_p$$

where $J_p\subseteq J$ and $\alpha+\beta<2g+7$.

In particular the only component of dimension $2g+7$ mapping over X is $H=$ = closure of $\varphi^{-1}(V_g)$. Q.E.D.

As immediate corollary we get

THEOREM 1. – *The monodromy group is transitive.*

PROOF (see [3]). – We can find a Zariski open set $A\subseteq X$ such that $B=\pi^{-1}(A)$ is contained in Y and $\pi\colon B\to A$ is unramified. Since Y is irreducible, B is connected and for a generic $C\in A$ we can join two points of $\pi^{-1}(C)$ with an arc γ contained in B. The action on $\pi^{-1}(C)$ associated to $\pi(\gamma)$ carries the initial point of γ to the end point. Q.E.D.

3. – Normal Weierstrass points.

A Weierstrass point z on a smooth curve C of genus g is not normal when one of the following occurs

$$h^0(K_C-(g-1)z)\geqslant 2 \tag{1}$$

$$h^0(K_C-(g+1)z)>0\,. \tag{2}$$

(Notice that each of them automatically implies that z is a Weierstrass point.)

The two possibilities are distinct: the gap sequence $1, 2, \ldots, g-2, g, g+1$ satisfies (1) but not (2) and the sequence $1, 2, \ldots, g-1, g+2$ satisfies (2) but not (1).

It's easy to see that the generic curve $C \in X$ doesn't have points of type (2).

If we define in fact $\tilde{I} \subset I$ as

$$\tilde{I} = \{(C, z, H) \text{ s.t. } m_z(C \cdot H) \geqslant g+1\}$$

for any $p = (z. H) \in V^g$, $\varphi^{-1}(z, H) \cap \tilde{I}$ is isomorphic to a projective space of dimension $g+6$. It follows that $Y \cap \tilde{I}$ is a proper subevariety of Y.

We want to see now that the generic $C \in X$ does not have points of type (1).

For this define $I' \subseteq X \times Z$ as

$$I' = \{(C, p, H) \text{ s.t. } m_z(C \cdot H) \geqslant g-1\}$$

and call π' and φ' the projection maps over X and Z.

If C_0 is a smooth curve of X and z_0 a point of C_0, the set of H such that $(C_0, z_0, H) \in I'$ is isomorphic to the projective space $\boldsymbol{P}H^0(K_{C_0} - (g-1)\cdot z_0)$.

It particular it will have positive dimension exactly when z_0 is a point of type (1) for C_0. There is an obvious component of I' defined as

$$T = \text{closure of } \{(C, z, H) \in I' \text{ s.t. } C \text{ is smooth and } h^0(C - (g-1)\cdot z) = 1\}.$$

If every curve C of X has a point of type (1), then in I' we could find at least two components of dimension greater or equal to $2g+8$ mapping over X: the second one from the union of $\boldsymbol{P}H^0(K_C - (g-1)\cdot z)$, $C \in X$, $z \in C$.

As in the previous Theorem it is enough to prove that in I' there is only one component of dimension greater or equal to $2g+8$ which is not contained in

$$J' = \pi'^{-1}(X_1 \cup X_2).$$

Repeating the arguments of the previous Theorem we get easily

(i) if $p = (z, H) \in U_n$, $n = 1, \ldots, m-1$

$$\varphi'^{-1}(p) \subseteq J';$$

(ii) if $p = (z, H) \in U_{k,l}$, $k + l \leqslant m-1$ and if $l < m-1$

$$\varphi'^{-1}(p) = \boldsymbol{P}^{\beta'} \cup J'_p$$

where $J'_p \subseteq J'$ and $\beta' = g + l + 8$;

(iii) if $p = (z, H) \in U_{0,m-1}$ then $\varphi'^{-1}(p)$ consists of the union of a projective space $\boldsymbol{P}^{3m+7}$, representing the curves of type $(3, m+1)$ with a contact of order greater or equal to m with $E_1 = L_1(z)$ at z, a projective space $\boldsymbol{P}^{2g+5}$, representing curves of type $(3, m+1)$ with a contact of order greater or equal to 2 with $L_2(z)$ at z, and $J'_p \subseteq J$.

We conclude, again by dimension count, that the only component of dimension $2g+8$ (or greater) mapping over X comes from the inverse image of $U_{m-1,0}$ in the map φ'.

In particular

THEOREM 2. – *The generic element C of X contains only normal Weierstrass points.*

4. – Curves of genus four.

In this section we consider the case $g = 4$ and we prove that the monodromy is the full symmetric group.

We conjecture in fact that this is true for any $g = 2m$ but unfortunately we are not able at this moment to give a proof for the general case.

In the case under consideration $S = Q$ is the quadric in $\boldsymbol{P}^3$, H is the hyperplane section and the curves C are the curves of type $(3, 3)$ so that $X \simeq \boldsymbol{P}^{15}$.

We also have:

$U = U_{1,0}$ is subset of Z consisting of couples (z, H) where $z \in H \cap Q$ and $H \cap Q$ is smooth. It has dimension 4;

$U_1 = \{(z, H)$ s.t. $H \cap Q = L_1 + L_2$ and $z \in L_2$, $z \notin L_1\}$;

$U_{0,0} = \{(z, H)$ s.t. $H \cap Q = L_1 + L_2$ and $z \in L_1$, $z \notin L_2\}$.
These two substes of Z have both dimension 3.

$U_{0,1} = \{(z, H)$ s.t. $H \cap Q = L_1(z) + L_2(z)$ i.e. H is the tangent plane to Q at $z\}$.
It has dimension 2.

According to the results of the previous sections a generic $C \in X$ is smooth with only normal Weierstrass points and doesn't have points of total ramification for either one of the two g_3^1's determined by the rulings of Q. Let's fix such a curve C_0 and let (C_0, z_0, H_0) be a point of $\pi^{-1}(C_0)$ where we can suppose $(z_0, H_0) \in U = U_{1,0}$.

Let's also call J the subset of I consisting of triples (C, z, H) where C doesn't satisfy one of the previous conditions. We observe that if $p = (z, H)$ is a point of U_1 (resp. $U_{0,0}$) the set $\varphi^{-1}(p)$ is contained in J because a curve C has $m_z(C \cdot H) \geqslant 4$ only when it contains $L_2(z)$ (resp. $L_1(z)$).

On the other hand if $p = (z, H)$ is a point of $U_{0,1}$ a smooth curve C has $m_z(C \cdot H) \geqslant 4$ only if z is a point of total ramification for one of the two g_3^1 so that also in this case $\varphi^{-1}(p) \subset J$.

Fallowing the technique explained in [3] we prove

PROPOSITION 2. – *The stabilizer of* (C_0, z_0, H_0) *in the monodromy group acts transitively on the remaining points of* $\Gamma = \pi^{-1}(C_0)$.

PROOF. – Let's define

$$X_0 = \{C \in X \text{ s.t. } m_{z_0}(C \cdot H_0) \geqslant 4\}:$$

it's a subspace of X isomorphic to $\boldsymbol{P}^{11}$.

The inverse image of X_0, $\pi^{-1}(X_0)$ contains a component consisting of triples (C, z_0, H_0) where $C \in X_0$.

We call $I_0 \subset X_0 \times Z$ the closure in $\pi^{-1}(X_0)$ of the complement of that component.

By the same argument we used to prove simple transitivity it will be enough to prove that I_0 contains only one component of dimension 11 mapping over X_0. This will be done as usual by looking at the projection $\varphi_0 \colon I_0 \to Z$.

We notice that $\varphi_0^{-1}(z, H) \subset J$ if $z = z_0$, $H \neq H_0$ or $z \neq z_0$ but $H = H_0$.

In the first case in fact a smooth curve $C \in X_0$ having $m_z(C \cdot H) \geqslant 4$ has z_0 as a Weierstrass point of weight > 1 because $h^0(K_C - 4 \cdot z_0)$ is at least 2.

In the second case any $C \in X_0$ xith $m_z(C \cdot H_0) \geqslant 4$ must contain $E_0 = H_0 \cap Q$.

It follows that it's enough to look at the fibers of φ_0 over points $p = (z, H) \in U$ where $z \neq z_0$, $H \neq H_0$.

We claim that for any such p, $\varphi_0^{-1}(p) \simeq \boldsymbol{P}^7$.

Suppose that $z_0 \notin H$ and call E the intersection $H \cap Q$.

The curves of the form $E + R$ where R is any curve of type $(2, 2)$ form a linear system of dimension 8 and they certainly have a 4-fold intersection with H at z.

A curve of this type will have 4 intersections with H_0 at z_0 when $m_{z_0}(R \cdot H_0) \geqslant 4$. It follows easily that the requirement of having a 4-fold intersection with H_0 at z_0 imposes 4 conditions on this linear system.

In other words the conditions we are imposing at (z_0, H_0) and at (z, H) are independent so that $\varphi_0^{-1}(p) = \boldsymbol{P}^7$.

Obviously the same is true is $z \notin H_0$. If z, z_0 are the points of intersection of H and H_0 on Q we choose two points r_1 and r_2 on $E_0 = H_0 \cap Q$ different from z, z_0 and two other points s_1, s_2 on $E = H \cap Q$ different from z, z_0.

In the space of curves of type $(3, 3)$ having 4-fold intersection with H at z and with H_0 at z_0, those passing through r_1, r_2, s_1, s_2 have codimension at most 4.

On the other hand they contain both E_0 and E_1 because they have at least 7 intersections with each.

It follows that also in this case the conditions at (z_0, H_0) and (z, H) are independent and $\varphi_0^{-1}(p) = \boldsymbol{P}^7$. Q.E.D.

Since we know that for any even $g \geqslant 4$ the monodromy group is transitive, from the previous proposition we immediately get

COROLLARY. – *If* $g = 4$, *the monodromy group is twice transitive.*

We come now to the last step which consists in proving that for $g = 4$ the monodromy group contains a simple transposition.

We have the following

LEMMA (see [3], page 698). – *Let $\pi\colon Y \to X$ be as before. Call d the degree of π. If there is a point C of X such that $\pi^{-1}(C)$ consists exactly of $d-1$ distinct points—i.e. $d-2$ simple points $q_1, \ldots, q_{d-2}$ and one double point q_{d-1}—and if Y is locally irreducible at q_{d-1}, then the monodromy group contains a simple transposition.*

In our situation we have to prove that there is a curve $C \in X$ with one Weierstrass point of weight two and all the others of weight one. There are two types of Weierstrass points of weight two corresponding to the sequences 1236 and 1245.

Let's consider first a curve C (smooth) having a point z with gap sequence 1245.

We denote by L the tangent line to C at z and L_1, L_2 the two lines of Q through z.

The tangent plane to Q at z, say H, intersects Q in $L_1 + L_2$. Since $h^0(K_C - 3\cdot z) = 2$, all the planes containing L have three intersections with C at z.

In particular this is true for H and L_1 or L_2 must be the tangent line to C at z, since one of them has a double intersection with C at z (at least). Suppose $L = L_1$.

Since a plane containing L and different from H cuts Q along a residual line not containing z, we see that $m_z(C\cdot L) = 3$ and the only plane with a contact of order four is H.

It's also clear that there are no planes with contact af order five.

This obviously checks with Riemann-Roch. Viceversa if C is a smooth curve of type (3, 3) s.t. $m_z(C\cdot L_1) = 3$ then any plane containing L_1 has three intersections with C at z so that $h^0(K_C - 3\cdot z) > 2$. In fact since we can always find a plane transversal to C at z we must have

$$h^0(K_C - z) = 3$$

$$h^0(K_C - 2\cdot z) = h^0(K_C - 3\cdot z) = 2\,.$$

The tangent plane to Q at z has then one more intersection with C at z and no plane can have five intersections with C at z. The gap sequence at z is 1245.

Suppose now we fix z and $L_1 = L_1(z)$ and consider the set of curves C of type (3, 3) such that $m_z(C\cdot L_1) \geqslant 3$. It's immediate to check that a generic curve of this set is smooth, $C\cdot L_1 = 3\cdot z$ and $m_z\big(C\cdot L_2(z)\big) = 1$.

The curves $2L_1(z) + 2L_2(z) + L_1' + L_2'$ where $L_1' \neq L_1(z)$, $L_2' \neq L_2(z)$ intersect any line of one of the two rulings different from L_1', L_2', $L_1(z)$, $L_2(z)$ only in one point.

Moving L_1', L_2' we conclude that the generic C of the set defined before does not have points of total ramification for either g_3^1 outside z.

We proved the following

PROPOSITION 3. – (*a*) *on a smooth curve C of type* (3, 3) *on the quadric Q the Weier-*

strass points with gap sequence 1245 *are exactly the points of total ramification for one sof the two* g_3^1*'s given by the rulings of* Q.

(*b*) *If we fix* $z \in Q$ *and one of the two lines through* z, *say* $L_1 = L_1(z)$, *the generic curve* C *of type* (3, 3) *s.t.* $m_z(C \cdot L_1) \geqslant 3$ *is smooth and* z *is the unique Weierstrass point with gap sequence* 1245 *on* C.

Consider now a smooth curve C of type (3, 3) with a point z having gap sequence 1236.

From Riemann-Roch

$$h^0(K_C - 4\cdot z) = h^0(K_C - 5\cdot z) = 1 .$$

Furthermore the points t such that $h^0(K_C - 5\cdot t) = 1$ are exactly the points with weight > 1 and a sequence different from 1245. If we take $p_0 = (z_0, H_0) \in U$ let's call E_0 the intersection $H_0 \cap Q$ and

$$W = \{C \in X \text{ s.t. } m_{z_0}(C\cdot H_0) \geqslant 5\} .$$

PROPOSITION 4. – *W is isomorphic to* $\mathbf{P}^{10}$, *the generic* $C \in W$ *is smooth and* z_0, *as a point of* C, *has the gap sequence* 1236.

PROOF. – The proof of the first statement is trivial. It's easy to see that the generic C is smooth at z_0.

By Bertini's Theorem we get that the generic C is smooth everywhere. Checking all the possible sequences at z_0, we have by Riemann-Roch $h^0(K_C - 5\cdot z_0) = 1$ so that H_0 is the only plane with a contact of order greater or equal to five at z_0.

The point z_0 could have a sequence different from 1236 only if $h^0(K_C - 6\cdot z_0) = 1$.

Since the curves of type (3, 3) cut on E_0 the complete system of degree six, this does not happen for the generic $C \in W$. Q.E.D.

The next proposition tells us that the generic $C \in W$ has only one Weierstrass point of weight > 1.

PROPOSITION 5. – *The Weierstrass points different from* z_0 *of the generic* $C \in W$ *are normal.*

PROOF. – We will show that for a generic $C \in W$

(*a*) there is no point $z \neq z_0$ such that the weight of z is $\geqslant 3$ or the weight is 2 and the gap sequence is 1236;

(*b*) C does not contain any point with gap sequence 1245.

Like in Proposition 2, for (*a*) it's enough to prove that the subset M of $W \times Z$ given by

$$M = \{(C, z, H) \text{ s.t. } m_z(C \cdot H) \geqslant 5,\ z \neq z_0,\ H \neq H_0\}$$

cannot map onto W.

As usual we look at the map $\psi: M \to Z$. Following the same arguments as in Proposition 2 we see that it's enough to look at the fibers of φ over points $p = (z, H) \in U$ and that if $z_0 \notin H$ (or $z \notin H_0$) the conditions of contact of order $\geqslant 5$ at z and z_0 are independent.

In case $z \in H_0$ and $z_0 \in H$, if we take $r \in H$, $s \in H_0$ different from z, z_0 among the curves with the required contact at z, z_0 those containing the points r and s form a subspace of codimension $\leqslant 2$.

On the other hand they form a $\boldsymbol{P}^3$ since they must contain E_0 and $E = H \cap Q$. This implies that $\varphi^{-1}(p) \simeq \boldsymbol{P}^5$ for any $p = (z, H) \in U$, $z \neq z_0$, $H \neq H_0$, and by dimension count M cannot map onto W.

For the point (*b*) let's consider a couple (z, L) where $z \neq z_0$ and L contains z and belongs to a fixed ruling of Q.

Let t be the point $L \cap E_0$. If t is different from z, the curves with 3 intersections with L at z have codimension 3 among the curves of type $(3, 3)$ containing E_0.

In fact they must contain (E_0 and) L. So they form a $\boldsymbol{P}^5$ and the requirement of having a contact or order 3 with L at z imposes 3 conditions on W.

If $t = E_0 \cap L = z$ let r (resp. s) be a point of L (resp. E_0) different from z (resp. z, z_0).

The curves of type $(3, 3)$ having 5 intersections with H_0 at z_0, 3 intersections with L at z and passing through r and s form a $\boldsymbol{P}^5$ because they contain again E_0 and L.

Since they have codimension $\leqslant 2$ among those satisfying the first two requirements by dimension count we conclude that the generic $C \in W$ does not contain points with sequence 1245. Q.E.D.

In order to conclude that the monodromy group contains the transposition we check, according to the Lemma, that Y is irreducible at (C, z_0, H_0) where C is a smooth curve of W.

Since (z_0, H_0) is in U, there is a neighborhood B of (z_0, H_0) in Z such that $Y|_B = \dot{B} \times \boldsymbol{P}^{11}$.

In particular Y is irreducible at (C, z_0, H_0) and we proved

THEOREM. – *If $g = 4$ the monodromy group is the full symmetric group on* $60 = (g-1)g(g+1)$ *elements.* Q.E.D.

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