Nonoscillatory Solutions of Linear Differential Equations with Deviating Arguments (*).

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Summary. – *The equation to be considered is of the form* (1) $x^{(n)}(t) + \sigma p(t)x(g(t)) = 0$ $(t > a)$, *where* $\sigma = \pm 1$, $p(t) > 0$ *for* $t \geqq a$ *and* $g(t) \to \infty$ *as* $t \to \infty$. *It is well-known that a nonoscillatory solution* $x(t)$ *of* (1) *satisfies* (2) $x(t)x^{(i)}(t) > 0$ $(0 \leqq i \leqq l)$, $(-1)^{i-l}x(t)x^{(i)}(t) > 0$ $(l \leqq i \leqq n)$ *for some integer* l, $0 \leqq l \leqq n$, $(-1)^{n-l-1}\sigma = 1$. *In this paper, for a given* l *such that* $0 < l < n$, $(-1)^{n-l-1}\sigma = 1$, *necessary conditions and sufficient conditions are found for* (1) *to have a solution* $x(t)$ *which satisfies* (2), *and a necessary and sufficient condition is established in order that for every* $\lambda > 0$ *the equation* $x^{(n)}(t) + \lambda\sigma p(t)x(g(t)) = 0$ $(t > a)$ *has a solution* $x(t)$ *which satisfies* (2). *Related results are also contained.*

1. – Introduction.

In this paper we examine the oscillatory and nonoscillatory behavior of solutions of linear differential equations with deviating arguments of the form

$$x^{(n)}(t) + \sigma p(t)x(g(t)) = 0 , \qquad t > a , \tag{1}$$

where the following conditions are always assumed:

(i) $n \geqq 2$, $\sigma = +1$ or -1;

(ii) $p(t)$ is continuous and positive on $[a, \infty)$;

(iii) $g(t)$ is continuous on $[a, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of Eq. (1) we mean a function $x(t)$ which is defined on some half-line $[T_x, \infty)$ and satisfies (1) for $t \geqq T_x$ and $\sup\{|x(t)|: t \geqq T\} > 0$ for any $T \geqq T_x$. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

If $x(t)$ is a nonoscillatory solution of (1), then $\sigma x(t)x^{(n)}(t) < 0$ for all sufficiently large t, and so by a lemma of KIGURADZE [5] there exists an integer $l \in \{0, 1, ..., n\}$, $(-1)^{n-l-1}\sigma = 1$, and a $t_0 > T_x$ such that

$$\begin{cases} x(t)x^{(i)}(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for } 0 \leqq i \leqq l \\ (-1)^{i-l}x(t)x^{(i)}(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for } l \leqq i \leqq n . \end{cases} \tag{2}$$

(*) Entrata in Redazione il 2 marzo 1983.

A function satisfying (2) is said to be a (nonoscillatory) function of degree l. We use the symbol $\mathcal{N}_l$ to denote the totality of nonoscillatory solutions of degree l of (1). If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then we have

$$
(3) \qquad \begin{aligned}
&\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_{n-1} && \text{for } n \text{ even}, \ \sigma = 1\\
&\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_{n-1} && \text{for } n \text{ odd}, \ \sigma = 1\\
&\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_{n-2} \cup \mathcal{N}_n && \text{for } n \text{ even}, \ \sigma = -1\\
&\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_{n-2} \cup \mathcal{N}_n && \text{for } n \text{ odd}, \ \sigma = -1 .
\end{aligned}
$$

In the case where $g(t) \equiv t$, that is, (1) is an ordinary differential equation, it is known that the classes $\mathcal{N}_0$ and $\mathcal{N}_n$ in (3) are always nonempty. That $\mathcal{N}_0 \neq \emptyset$ follows from a classical theorem of HARTMAN and WINTNER [4] (see also HARTMAN [3, p. 508, Cor. 2.2]), while $\mathcal{N}_n \neq \emptyset$ is clear since the solution $x(t)$ of (1) with $\sigma = -1$ such that $x^{(i)}(a) > 0$ $(i = 0, 1, \ldots, n-1)$ satisfies $x^{(i)}(t) > 0$ for $t > a$ $(i = 0, 1, \ldots, n)$.

Each of the classes $\mathcal{N}_l$ $(0 < l < n)$ may further be minutely classified according to the possible asymptotic behavior of its members as $t \to \infty$. Let $x(t) \in \mathcal{N}_l$ $(0 < < l < n)$. Since $|x^{(l)}(t)|$ is decreasing, $|x^{(l)}(t)|$ has a nonnegative limit as $t \to \infty$. It is clear that if $\lim_{t\to\infty} |x^{(l)}(t)| > 0$, then $\lim_{t\to\infty} |x^{(l-1)}(t)| = \infty$. On the other hand, since $|x^{(l-1)}(t)|$ is increasing, either $|x^{(l-1)}(t)|$ has a finite limit as $t \to \infty$ or $|x^{(l-1)}(t)|$ tends to ∞ as $t \to \infty$. Consequently we have the following three possibilities:

$$
(4) \qquad \lim_{t\to\infty} |x^{(l-1)}(t)| = \infty , \qquad \lim_{t\to\infty} |x^{(l)}(t)| = \text{const} \neq 0;
$$

$$
(5) \qquad \lim_{t\to\infty} |x^{(l-1)}(t)| = \infty , \qquad \lim_{t\to\infty} x^{(l)}(t) = 0;
$$

$$
(6) \qquad \lim_{t\to\infty} |x^{(l-1)}(t)| = \text{const} \neq 0 , \qquad \lim_{t\to\infty} x^{(l)}(t) = 0 .
$$

A solution $x(t)$ satisfying (4) [resp. (6)] can be regarded as a «maximal» [resp. «minimal»] element in $\mathcal{N}_l$; a solution $x(t)$ satisfying (5) may be referred to as an «intermediate» element in $\mathcal{N}_l$. We use the notation $\mathcal{N}_l[\max]$, $\mathcal{N}_l[\text{int}]$ and $\mathcal{N}_l[\min]$ to denote the set of all nonoscillatory solutions $x(t)$ in $\mathcal{N}_l$ satisfying (4), (5) and (6), respectively. Thus we have

$$
\mathcal{N}_l = \mathcal{N}_l[\max] \cup \mathcal{N}_l[\text{int}] \cup \mathcal{N}_l[\min] .
$$

It is not difficult to give necessary and sufficient conditions for (1) to have solutions of classes $\mathcal{N}_l[\max]$ and $\mathcal{N}_l[\min]$ (Lemma 5 below). Of particular interest, therefore, is to find necessary and/or sufficient conditions for (1) to have solutions of classes $\mathcal{N}_l$ and $\mathcal{N}_l[\text{int}]$.

In this paper we obtain first necessary conditions and sufficient conditions for (1) to have a solution of class $\mathcal{N}_l$ $(0 < l < n)$, and then combine them to establish a necessary and sufficient condition in order that the associated differential equation with a parameter λ

$$(7_\lambda) \qquad x^{(n)}(t) + \lambda\sigma p(t)x(g(t)) = 0\,, \qquad t > a\,,$$

has a solution of class $\mathcal{N}_l$ $(0 < l < n)$ for every $\lambda > 0$. As a consequence we can find a characterization for the situation in which (7_λ) has a solution of class $\mathcal{N}_l[\text{int}]$ $(0 < l < n)$ for every $\lambda > 0$. We can also give a characterization for the situation in which (7_λ) is almost oscillatory for every $\lambda > 0$. Here and hereafter we say that Eq. (1) is *almost oscillatory* if the extreme case occurs for (1) in which all classes $\mathcal{N}_l$ $(0 < l < n)$ in (3) are empty. (Of course, the notion of almost oscillation makes sense only when either $n > 3$, $\sigma = \pm 1$ or $n = 2$, $\sigma = 1$.)

This work is strongly motivated by the papers of KUSANO [7] and NAITO [17]. Related results are contained in ČANTURIJA [1, 2], KIGURADZE [5], KOPLATADZE and ČANTURIJA [6], LOVELADY [11-15] and TRENCH [18].

Throughout the paper we assume that l is an integer such that $0 < l < n$, $(-1)^{n-l-1}\sigma = 1$.

2. – Preparatory results.

The following lemmas will be needed in proving our results.

LEMMA 1 (KIGURADZE [5, the proof of Lemma 2]). – *Let $x(t)$ be a positive function of degree l. Then $x(t)$ satisfies the inequalities*

$$x^{(l-j)}(t) \geqq \frac{1}{j}(t - t_0)x^{(l-j+1)}(t) \quad \text{for } t \geqq t_0 \qquad (1 \leqq j \leqq l)\,,$$

and in particular

$$x(t) \geqq \frac{1}{l!}(t - t_0)^{l-1}x^{(l-1)}(t) \quad \text{for } t \geqq t_0\,, \qquad \frac{x(t)}{(t - t_0)^l} \text{ is nonincreasing for } t > t_0\,.$$

LEMMA 2 (KUSANO and NAITO [10]). – *If there is a positive function $v(t)$ of degree l satisfying the inequality*

$$\sigma\{v^{(n)}(t) + \sigma p(t)v(g(t))\} \leqq 0$$

for all sufficiently large t, then the equation

$$u^{(n)}(t) + \sigma p(t)u(g(t)) = 0$$

has an eventually positive solution of degree l.

LEMMA 3 (MAHFOUD [16]). – *Suppose that there is a function* $G(t) \in C^1[a, \infty)$ *such that*

$$G(t) \leqq \min\{g(t), t\}, \quad G'(t) > 0, \quad G(t) \to \infty \quad (t \to \infty) \tag{8}$$

and let $G^{-1}(t)$ *be the inverse function of* $G(t)$. *If the ordinary differential equation*

$$v''(t) + \frac{p\big(G^{-1}(t)\big)}{G'\big(G^{-1}(t)\big)}\, v(t) = 0$$

is oscillatory, then all solutions of the equation

$$u''(t) + p(t)u\big(g(t)\big) = 0$$

is oscillatory.

LEMMA 4 (KUSANO and NAITO [10]). – *Suppose that there is a function* $H(t) \in C^1[a, \infty)$ *such that*

$$H(t) \geqq \max\{g(t), t\}, \quad H'(t) > 0 \tag{9}$$

and let $H^{-1}(t)$ *be the inverse function of* $H(t)$. *If the ordinary differential equation*

$$v''(t) + \frac{p\big(H^{-1}(t)\big)}{H'\big(H^{-1}(t)\big)}\, v(t) = 0$$

is nonoscillatory, then the equation

$$u''(t) + p(t)u\big(g(t)\big) = 0$$

has a nonoscillatory solution.

LEMMA 5. – (i) *Eq.* (1) *has a solution of class* $\mathcal{N}_l[\max]$ *if and only if*

$$\int^{\infty} t^{n-l-1}\big(g(t)\big)^l p(t)\, dt < \infty .$$

(ii) *Eq.* (1) *has a solution of class* $\mathcal{N}_l[\min]$ *if and only if*

$$\int^{\infty} t^{n-l}\big(g(t)\big)^{l-1} p(t)\, dt < \infty .$$

In fact, it is easily verified that $x \in \mathcal{N}_l[\max]$ if and only if $x(t)$ satisfies $\lim_{t\to\infty} x(t)/t^l = \text{const} \neq 0$, and $x \in \mathcal{N}_l[\min]$ if and only if $x(t)$ satisfies $\lim_{t\to\infty} x(t)/t^{l-1} = \text{const} \neq 0$. On the other hand, it is known that for an integer k, $0 \leqq k \leqq n-1$, there exists a solution $x(t)$ of (1) such that $\lim_{t\to\infty} x(t)/t^k = \text{const} \neq 0$ if and only if

$$\int^{\infty} t^{n-k-1}(g(t))^k p(t)\, dt < \infty .$$

Thus Lemma 5 is immediate.

3. – Main results.

THEOREM 1. – *If Eq.* (1) *has a nonoscillatory solution of class* $\mathcal{N}_l$, *then for all sufficiently large* T *the equation*

$$u''(t) + \frac{(t-T)^{n-l-1}(g(t)-T)^{l-1}}{(n-l)!\,l!}\, p(t)u(g(t)) = 0 \tag{10}$$

has a nonoscillatory solution.

PROOF. – Suppose that (1) has a nonoscillatory solution $x(t)$ of class $\mathcal{N}_l$. Without loss of generality we may assume that $x(t)$ is eventually positive, and from (2) it follows that

$$x^{(i)}(t) > 0 \quad (0 \leqq i \leqq l) \quad \text{and} \quad (-1)^{i-l}x^{(i)}(t) > 0 \quad (l \leqq i \leqq n) \tag{11}$$

for $t \geqq T$, where $T > t_0$ is such that $x(g(t)) > 0$ for $t \geqq T$. By Taylor's formula with remainder, we have

$$x^{(l)}(t) = \sum_{j=0}^{n-l-1} \frac{x^{(l+j)}(\tau)}{j!}(t-\tau)^j + \frac{1}{(n-l-1)!}\int_{\tau}^{t}(t-s)^{n-l-1}x^{(n)}(s)\, ds =$$
$$= \sum_{j=0}^{n-l-1} \frac{(-1)^j x^{(l+j)}(\tau)}{j!}(\tau-t)^j + \frac{1}{(n-l-1)!}\int_{t}^{\tau}(s-t)^{n-l-1}p(s)x(g(s))\, ds .$$

Now using (11), we obtain

$$x^{(l)}(t) \geqq \frac{1}{(n-l-1)!}\int_{t}^{\tau}(s-t)^{n-l-1}p(s)x(g(s))\, ds$$

for $T \leqq t \leqq \tau$. Letting $\tau \to \infty$, we have

$$x^{(l)}(t) \geqq \frac{1}{(n-l-1)!}\int_{t}^{\infty}(s-t)^{n-l-1}p(s)x(g(s))\, ds \tag{12}$$

for $t \geqq T$. Integrating (12) yields

$$(13)\qquad x^{(l-1)}(t) \geqq x^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_T^t \int_s^\infty (r-s)^{n-l-1} p(r) x(g(r))\, dr\, ds$$

$$= x^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_T^t \left(\int_T^r (r-s)^{n-l-1}\, ds\right) p(r) x(g(r))\, dr +$$

$$+ \frac{1}{(n-l-1)!}\int_t^\infty \left(\int_T^t (r-s)^{n-l-1}\, ds\right) p(r) x(g(r))\, dr$$

for $t \geqq T$. Using the inequality

$$\int_T^t (r-s)^{n-l-1}\, ds \geqq \frac{1}{n-l}(t-T)(r-T)^{n-l-1} \qquad (T \leqq t \leqq r)$$

in (13), we get

$$(14)\qquad x^{(l-1)}(t) \geqq x^{(l-1)}(T) + \frac{1}{(n-l)!}\int_T^t (r-T)^{n-l} p(r) x(g(r))\, dr +$$

$$+ \frac{t-T}{(n-l)!}\int_t^\infty (r-T)^{n-l-1} p(r) x(g(r))\, dr$$

for $t \geqq T$. Denote the right hand side of (14) by $y(t)$. As easily verified, $y(t)$ is positive and satisfies the equality

$$(15)\qquad y''(t) + \frac{1}{(n-l)!}(t-T)^{n-l-1} p(t) x(g(t)) = 0$$

for $t \geqq T$. From Lemma 1 it follows that

$$(16)\qquad x(g(t)) \geqq \frac{1}{l!}(g(t)-T)^{l-1} x^{(l-1)}(g(t)) \geqq \frac{1}{l!}(g(t)-T)^{l-1} y(g(t))$$

for all large t. Combining (15) with (16) yields

$$y''(t) + \frac{(t-T)^{n-l-1}(g(t)-T)^{l-1}}{(n-l)!\, l!} p(t) y(g(t)) \leqq 0$$

for all large t. Applying now Lemma 2 with $n = 2$, $\sigma = 1$, we conclude that Eq. (10) has an eventually positive solution. The proof of Theorem 1 is complete.

Let $G(t) \in C^1$ be a function satisfying (8). Then, according to Lemma 3, all solutions of Eq. (10) are oscillatory if the ordinary differential equation

$$(17)\qquad v''(t) + \frac{(G^{-1}(t)-T)^{n-l-1}\left(g(G^{-1}(t))-T\right)^{l-1} p(G^{-1}(t))}{(n-l)!\, l!\, G'(G^{-1}(t))} v(t) = 0$$

is oscillatory. If we apply the well-known oscillation criteria of Hille and Fite to (17), we have the following result.

COROLLARY 1. – *If Eq.* (1) *has a nonoscillatory solution of class* $\mathcal{N}_l$, *then*

$$\limsup_{t\to\infty} G(t)\int_t^\infty s^{n-l-1}(g(s))^{l-1}p(s)\,ds \leqq (n-l)!\,l!$$

and

$$\liminf_{t\to\infty} G(t)\int_t^\infty s^{n-l-1}(g(s))^{l-1}p(s)\,ds \leqq \frac{(n-l)!\,l!}{4}$$

hold for every $G(t)\in C^1$ *satisfying* (8).

We now look for sufficient conditions under which (1) has a nonoscillatory solution of class $\mathcal{N}_l$.

THEOREM 2. – *If for some* $T\geqq a$ *the equation*

$$u''(t)+\frac{(t-T)^{n-l-1}(g(t)-T)^{l-1}}{(n-l-1)!(l-1)!}p(t)u(g(t))=0 \tag{18}$$

has a nonoscillatory solution, then Eq. (1) *has a nonoscillatory solution of class* $\mathcal{N}_l$.

PROOF. – Let $u(t)$ be a positive nonoscillatory solution of (18). There is a number $T_1\geqq T$ such that $u(t)>0$, $u(g(t))>0$ and $g(t)\geqq T$ for $t\geqq T_1$. Noting that $u'(t)>0$ for $t\geqq T_1$ and integrating (18) from t to ∞, we get

$$u'(t)\geqq\int_t^\infty \frac{(s-T)^{n-l-1}(g(s)-T)^{l-1}}{(n-l-1)!(l-1)!}p(s)u(g(s))\,ds$$

for $t\geqq T_1$, whence it follows that

$$u(t)\geqq\int_{T_1}^t\int_s^\infty \frac{(r-T)^{n-l-1}(g(r)-T)^{l-1}}{(n-l-1)!(l-1)!}p(r)u(g(r))\,dr\,ds \tag{19}$$

for $t\geqq T_1$. Define the function $v(t)$ by

$$v(t)=\int_{T_1}^t \frac{(t-s)^{l-1}}{(l-1)!}\int_s^\infty \frac{(r-s)^{n-l-1}(g(r)-T)^{l-1}}{(n-l-1)!(l-1)!}p(r)u(g(r))\,dr\,ds \tag{20}$$

for $t\geqq T_1$. From (19) and (20) we easily see that

$$v(t)\leqq\frac{(t-T_1)^{l-1}}{(l-1)!}u(t) \tag{21}$$

for $t \geqq T_1$. It is also easy to see that $v(t)$ is of degree l and satisfies

$$-\sigma v^{(n)}(t) = \frac{(g(t) - T)^{l-1}}{(l-1)!} p(t) u(g(t)) \tag{22}$$

for $t \geqq T_1$. From (21) and (22) it follows that

$$\sigma\{v^{(n)}(t) + \sigma p(t) v(g(t))\} \leqq 0$$

for all large t. Applying Lemma 2, we conclude that Eq. (1) has an eventually positive solution of degree l. This completes the proof of Theorem 2.

Let $H(t) \in C^1$ be a function satisfying (9). Then Lemma 4 implies that (18) has a nonoscillatory solution if the ordinary differential equation

$$v''(t) + \frac{(H^{-1}(t) - T)^{n-l-1}\big(g(H^{-1}(t)) - T\big)^{l-1} p(H^{-1}(t))}{(n-l-1)!(l-1)!H'(H^{-1}(t))} v(t) = 0 \tag{23}$$

is nonoscillatory. Applying Hille's nonoscillation criterion to (23), we have the following result.

COROLLARY 2. – *If*

$$\limsup_{t\to\infty} H(t) \int_t^\infty s^{n-l-1}(g(s))^{l-1} p(s)\, ds < \frac{(n-l-1)!(l-1)!}{4}$$

holds for some $H(t) \in C^1$ *satisfying* (9), *then Eq.* (1) *has a nonoscillatory solution of class* $\mathcal{N}_l$.

If the deviating argument $g(t)$ satisfies

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t} \leqq \limsup_{t\to\infty} \frac{g(t)}{t} < \infty\,, \tag{24}$$

then necessary and sufficient conditions can be given for (7_λ) to have solutions of classes $\mathcal{N}_l$ and $\mathcal{N}_l[\text{int}]$ for all $\lambda > 0$.

THEOREM 3. – *Suppose that* (24) *holds. Let l be fixed. Then, Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$ *for all* $\lambda > 0$ *if and only if*

$$\lim_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\, ds = 0\,. \tag{25}$$

THEOREM 4. – *Suppose that* (24) *holds. Let l be fixed. Then, Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$[int] *for all* $\lambda > 0$ *if and only if*

$$\lim_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\, ds = 0 \tag{25}$$

and

$$\int^\infty s^{n-1} p(s)\, ds = \infty\,. \tag{26}$$

In fact, we have the following stronger results.

THEOREM 5. – *Suppose that* (24) *holds. Then the statements* (i)-(iv) *below are equivalent*:

(i) *for any l and for all* $\lambda > 0$ *Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$;

(ii) *there is an l such that for all* $\lambda > 0$ *Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$;

(iii) *for all* $\lambda > 0$ *there is an l such that Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$ (*that is, for all* $\lambda > 0$ *Eq.* (7_λ) *is never almost oscillatory*);

(iv) *condition* (25) *is satisfied.*

THEOREM 6. – *Suppose that* (24) *holds. Then the statements* (i)-(iv) *below are equivalent*:

(i) *for any l and for all* $\lambda > 0$ *Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$[int];

(ii) *there is an l such that for all* $\lambda > 0$ *Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$[int];

(iii) *for all* $\lambda > 0$ *there is an l such that Eq.* (7_λ) *has a nonoscillatory solution of class* $\mathcal{N}_l$[int];

(iv) *conditions* (25) *and* (26) *are satisfied.*

PROOF OF THEOREM 5. – Clearly (i) implies (ii), and (ii) implies (iii). It follows from (24) that there are positive constants $c_1 < 1$ and $c_2 > 1$ such that $c_1 t < g(t) < c_2 t$ for all large t. Suppose that (iii) holds, i.e., suppose that for all $\lambda > 0$ there is an $l = l(\lambda)$ (in general, depending on λ) such that (7_λ) has a solution of class $\mathcal{N}_l$. By Corollary 1 we deduce that

$$\limsup_{t\to\infty} G(t) \int_t^\infty s^{n-l-1}(g(s))^{l-1}[\lambda p(s)]\, ds \leqq (n-l)!\, l!$$

for all $\lambda > 0$, where $G(t) = c_1 t$. But this can be satisfied only when (25) is satisfied. Thus (iii) implies (iv). Suppose that (iv) holds. Then

$$\limsup_{t\to\infty} H(t)\int_t^\infty s^{n-l-1}(g(s))^{l-1}[\lambda p(s)]\, ds = 0 < \frac{(n-l-1)!(l-1)!}{4}$$

for any l and for all $\lambda > 0$, where $H(t) = c_2 t$. By Corollary 2 Eq. (7_λ) possesses a solution of class $\mathcal{N}_l$ for any l and for all $\lambda > 0$. The proof of Theorem 5 is complete.

PROOF OF THEOREM 6. – Clearly (i) implies (ii), and (ii) implies (iii). Suppose that (iii) holds and let $x_\lambda(t)$ be an eventually positive solution of class $\mathcal{N}_l$[int] of (7_λ), where l may depend on λ. By Lemma 1, $x_\lambda(t)/(t-t_0)^l$ is nonincreasing for all large t. Consider then the linear ordinary differential equation

$$(27) \qquad y^{(n)}(t) + \sigma P_\lambda(t) y(t) = 0\ ,$$

where $P_\lambda(t) = \lambda p(t) x_\lambda(g(t))/x_\lambda(t)$ (>0). With the aid of (24) it can be shown that $P_\lambda(t) \leqq k_\lambda p(t)$ for all large t, where k_λ is a positive constant (depending on λ). If (26) is not satisfied, then

$$\int^\infty s^{n-1} P_\lambda(s)\, ds < \infty\ ,$$

and so that (27) has a fundamental system of solutions $\{y_1(t), \ldots, y_n(t)\}$ such that

$$\lim_{t\to\infty} \frac{y_j(t)}{t^{j-1}} = \text{const} \neq 0 \qquad (j = 1, \ldots, n)\ .$$

This, however, is a contradiction to the fact that $x_\lambda(t)$ is a solution of class $\mathcal{N}_l$[int] of (27). Thus (26) is satisfied. That (25) is satisfied follows from Theorem 5.

Finally suppose that (iv) holds. Since (25) is satisfied, it follows from Theorem 5 that Eq. (7_λ) has a solution $x_{l\lambda}(t)$ of class $\mathcal{N}_l$ for any l and for all $\lambda > 0$. Since (24) and (26) are satisfied, this solution $x_{l\lambda}(t)$ is neither of class $\mathcal{N}_l$[max] nor of class $\mathcal{N}_l$[min], so that $x_{l\lambda}(t)$ must be of class $\mathcal{N}_l$[int]. Thus (iv) implies (i). The proof of Theorem 6 is complete.

We can also obtain the following result.

THEOREM 7. – *Suppose that* (24) *holds. Then the statements* (i)-(iv) *below are equivalent*:

(i) *for any l and for all $\lambda > 0$ Eq.* (7_λ) *has no nonoscillatory solution of class $\mathcal{N}_l$ (that is, for all $\lambda > 0$ Eq.* (7_λ) *is almost oscillatory)*;

(ii) *there is an l such that for all $\lambda > 0$ Eq.* (7_λ) *has no nonoscillatory solution of class $\mathcal{N}_l$*;

(iii) *for all* $\lambda > 0$ *there is an* l *such that Eq.* (7_λ) *has no nonoscillatory solution of class* $\mathcal{N}_l$;

(iv) *the condition*

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)\,ds = \infty \tag{28}$$

is satisfied.

PROOF. – Clearly (i) implies (ii), and (ii) implies (iii). It follows from (24) that there are positive constants $c_1 < 1$ and $c_2 > 1$ such that $c_1 t < g(t) < c_2 t$ for all large t. Suppose that (iii) holds, i.e., suppose that for all $\lambda > 0$ there is an $l = l(\lambda)$ (depending on λ) such that (7_λ) does not have a nonoscillatory solution of class $\mathcal{N}_l$. By Corollary 2 we must have

$$\limsup_{t\to\infty} H(t)\int_t^\infty s^{n-l-1}\big(g(s)\big)^{l-1}[\lambda p(s)]\,ds \geqq \frac{(n-l-1)!(l-1)!}{4}$$

for all $\lambda > 0$, where $H(t) = c_2 t$. But this is possible only if (28) is satisfied. Thus (iii) implies (iv). If (28) is satisfied, then clearly

$$\limsup_{t\to\infty} G(t)\int_t^\infty s^{n-l-1}\big(g(s)\big)^{l-1}[\lambda p(s)]\,ds = \infty > (n-l)!\,l!$$

for any l and for all $\lambda > 0$, where $G(t) = c_1 t$, so that by Corollary 1 Eq. (7_λ) has no nonoscillatory solution of class $\mathcal{N}_l$ for any l and for all $\lambda > 0$. This completes the proof of Theorem 7.

It should be noted that in the equations

$$x^{(n)}(t) + \lambda p(t)x\big(g(t)\big) = 0\,, \tag{7_λ^+}$$

$$x^{(n)}(t) - \lambda p(t)x\big(g(t)\big) = 0\,, \tag{7_λ^-}$$

if $g(t)$ satisfies (24), then

(i) for all $\lambda > 0$ (7_λ^+) is never almost oscillatory if and only if for all $\lambda > 0$ (7_λ^-) is never almost oscillatory;

(ii) for all $\lambda > 0$ (7_λ^+) has a nonoscillatory solution of an «intermediate» class if and only if for all $\lambda > 0$ (7_λ^-) has a nonoscillatory solution of an «intermediate» class, and

(iii) for all $\lambda > 0$ (7_λ^+) is almost oscillatory if and only if for all $\lambda > 0$ (7_λ^-) is almost oscillatory.

As examples consider the equations

(29) $$x^{(n)}(t) \pm \lambda t^{\alpha}(\log t)^{\beta} x(t+\tau) = 0 ,$$

(30) $$x^{(n)}(t) \pm \lambda t^{\alpha}(\log t)^{\beta} x(\gamma t) = 0 ,$$

(31) $$x^{(n)}(t) \pm \lambda t^{\alpha}(\log t)^{\beta} x(t + c \sin t) = 0 ,$$

where $\lambda, \alpha, \beta, \tau, \gamma$ and c are constants with $\lambda > 0, \gamma > 0$. Eq. (29) is retarded or advanced according as $\tau < 0$ or $\tau > 0$; (30) is retarded or advanced according as $\gamma < 1$ or $\gamma > 1$; (31) is an equation with deviating argument of mixed type if $c \neq 0$. All the deviating arguments in these equations satisfy (24). By Theorems 5-7, in particular, they have nonoscillatory solutions of every class $\mathcal{N}_l$ for all $\lambda > 0$ if and only if either $\alpha < -n$ or $\alpha = -n, \beta < 0$; they have nonoscillatory solutions of every class $\mathcal{N}_l[\text{int}]$ for all $\lambda > 0$ if and only if $\alpha = -n, -1 \leqq \beta < 0$; they are almost oscillatory for all $\lambda > 0$ if and only if either $\alpha > -n$ or $\alpha = -n, \beta > 0$.

As mentioned in the introduction, if $g(t) \equiv t$, then $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_n \neq \emptyset$ in (3). Several authors have observed that it may happen that $\mathcal{N}_0$ or $\mathcal{N}_n$ or both are empty, and more strongly that all $\mathcal{N}_l$ disappear if $g(t) \not\equiv t$ and the deviation $|t - g(t)|$ is sufficiently large. For example, KUSANO [9] has shown that every solution of the equation

$$x^{(n)}(t) - px(t + \sin t) = 0$$

is oscillatory provided $p > 0$ is sufficiently large. For related results the reader is referred to KOPLATADZE and ČANTURIJA [6] and KUSANO [8].

Acknowledgment. – The author would like to thank Professor TAKAŜI KUSANO for his helpful suggestions concerning this work.

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