Multiplicity Results for Asymptotically Homogeneous Semilinear Boundary Value Problems (*).

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Summary. – *This paper treats nonlinear elliptic boundary value problems of the form*

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

in the space $L^2(\Omega)$ by degree theoretic methods. Emphasis is placed on existence of multiple solutions in the case, where the nonlinearity f crosses several eigenvalues of the corresponding eigenvalue problem $\Delta\theta + \lambda\theta = 0$ with zero boundary values. No differentiability conditions (but Lipschitz type conditions) on f are assumed. A main tool is a new a priori bound for solutions (Theorem 1). The method is not confined to the selfadjoint case. It applies also to some time-periodic parabolic and hyperbolic problems.

1. – Introduction.

This paper is concerned with proving existence and multiplicity results for the equation

$$\Delta u + f(x, u) = h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

in a bounded region Ω of $\boldsymbol{R}^n$, with various assumptions on the existence of the limits

$$a(x) = \lim_{u\to-\infty} f(x, u)/u, \qquad b(x) = \lim_{u\to+\infty} f(x, u)/u.$$

Problems of this sort arose first in a classical paper by Dolph, where he showed that if a, b satisfy $\lambda_n + \varepsilon < a, b < \lambda_{n+1} - \varepsilon$, where λ_n are the eigenvalues of the Laplacian with Dirichlet boundary conditions, then (1) has a solution for all h. There has been considerable work done on weakening this assumption on a and b in the case where $n > 1$. Dancer [5] treated the case $f(x, u) = f(u)$, $a = \lambda_n$ and $b = \lambda_{n+1}$. This was later extended by Berestycki and De Figueiredo [2], who gave a certain sufficient condition that (1) should have a solution for all h. This

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condition allowed the possibility that $\lambda_n + \varepsilon < a(x) \leqslant \lambda_{n+1}$ and $\lambda_n \leqslant b(x) < \lambda_{n+1} - \varepsilon$. Further improvements have been given in this direction by MAWHIN [8] and by MAWHIN and WARD [10], to situations where only lim sup's and lim inf's exist and to problems where the resolvant is non-compact.

It is obvious that if $a \equiv \lambda_n$, $b \equiv \lambda_{n+1}$ gives solutions for all h by degree theoretic methods, one could perturb to allow $a \equiv \lambda_n - \varepsilon$ and $b \equiv \lambda_{n+1} + \varepsilon$ for sufficiently small ε, thereby allowing the nonlinearity to cross at least two eigenvalues. Multiplicity results in this context have been the subject of [6], [7] and [12]. In section 2, we give a constructive a priori estimate which for a wide range of functions $a(x)$ and $b(x)$ allows us to decide whether (1) still has solutions for all h. We give some applications of this a priori bound.

Another direction of research has been to take $h(x) = 0$ in (1) and to assume that $a(x)$ and $b(x)$ are contained in the interval $[\lambda_n + \varepsilon, \lambda_{n+1} - \varepsilon]$ and that $(\partial f/\partial u)(x, 0)$ is in the interval $[\lambda_{n+k} + \varepsilon, \lambda_{n+k+1} - \varepsilon]$ and that $f(x, 0) \equiv 0$. In this case CASTRO and LAZER [3] for $f(x, u) = f(u)$ and CHANG [4] showed that there exists a nontrivial solution for any $k \geqslant 1$. CHANG also considers the case of more general linear operators, some with non-compact resolvant. Using degree theory, in the case of odd k, we show that with considerably weaker conditions on the behavior of f at zero and infinity, we can obtain the existence of nontrivial solutions. In the case that $k = 1$, we show the existence of at least two nontrivial solutions. This represents an improvement of existing theorems in [1] where stronger restrictions on the nonlinear function f are imposed. These restrictions tend to require f to be, in some sense, « odd-like ».

Since many of the previous methods used are variational, they do not apply to problems where the linear operator is non-selfadjoint, as in the case of a parabolic problem with periodic-Dirichlet boundary conditions. At the end of this paper, we make some remarks showing how our methods apply to this problem.

Throughout the paper, Ω is a bounded region in $\boldsymbol{R}^n$, sufficiently smooth that the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

has eigenvalues $\lambda_1 < \lambda_2 \leqslant \ldots$ with eigenfuctions $\theta_1, \theta_2, \ldots$ which are an orthogonal basis in the Hilbert space $H = L^2(\Omega)$ and satisfy $\theta_1(x) > 0$ in Ω. The function $f(x, u)$ will always satisfy Carathéodory hypotheses (measurable in $x \in \Omega$, continuous in $u \in \boldsymbol{R}$).

2. – The a priori estimate.

We consider problem (1) in the Hilbert space $H = L^2(\Omega)$. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\} = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{\lambda_p, \ldots, \lambda_q\}$ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. Let $H_1 = \operatorname{span}\{\theta_i\colon p \leqslant i \leqslant q\}$ and $H_2 = H_1^\perp$. Let P denote orthogonal projection on H_1.

Let $c \notin \Lambda$ and $\lambda_p < c < \lambda_q$. Then

$$\left\{\begin{aligned} &\|(\Delta + c)^{-1}\|_{H_1} = \frac{1}{\varrho_1}, \qquad \text{where } \varrho_1 = \operatorname{dist}(c, \Lambda_1)\,, \\ &\|(\Delta + c)^{-1}\|_{H_2} = \frac{1}{\varrho_2}, \qquad \text{where } \varrho_2 = \operatorname{dist}(c, \Lambda_2) > \varrho_1\,. \end{aligned}\right. \tag{3}$$

Let us assume that f satisfies the estimate

$$\alpha(x)|u| \leqslant f(x, u) - cu \leqslant \beta(x)|u|\,, \tag{4}$$

where

$$M = \sup_{\Omega} \big(|\alpha(x)|, |\beta(x)|\big) < \infty\,. \tag{5}$$

For $u \in H$, the function $g(x) = f(x, u(x)) - cu(x)$ is in H, and $\|g\| \leqslant M\|u\|$. Decompose g as $g = v + w$, $v = Pg$, $w = (I - P)g$. Then there exists δ, $0 \leqslant \delta \leqslant 1$, such that

$$\|v\|^2 = \delta^2 M^2 \|u\|^2\,, \qquad \|w\|^2 \leqslant (1 - \delta^2) M^2 \|u\|^2\,,$$

and thus

$$\begin{aligned} \|v\|^2 = \int gv\,dx = \int gv^+\,dx - \int gv^-\,dx \leqslant \\ \leqslant \int \beta|u|\,v^+\,dx - \int \alpha|u|\,v^-\,dx \leqslant \\ \leqslant \|u\|\big(\|\beta v^+\| + \|\alpha v^-\|\big)\,. \end{aligned}$$

Let $S = \{v \in H_1 \colon \|v\| = 1\}$ be the unit sphere in H_1 and write

$$\gamma^+ = \frac{1}{M} \max_{v \in S} \|\beta v^+\|\,, \qquad \gamma^- = \frac{1}{M} \max_{v \in S} \|\alpha v^-\|\,. \tag{6}$$

Note that $\gamma^+ < 1$, $\gamma^- < 1$ and $(\gamma^+)^2 + (\gamma^-)^2 \leqslant 1$ if $p \geqslant 2$, because all functions in H_1 must change sign, and they are all orthogonal to θ_1. It follows that $\|v\|^2 \leqslant \|u\|\,\|v\|\,M\gamma$, where $\gamma = \gamma^+ + \gamma^-$, which implies $\delta \leqslant \gamma$.

Summing up, we get

$$\begin{aligned} \|(\Delta + c)^{-1} g(x)\|^2 \leqslant \|u\|^2 M^2 \left(\frac{\delta^2}{\varrho_1^2} + \frac{(1 - \delta^2)}{\varrho_2^2}\right) \leqslant \\ \leqslant M^2 \|u\|^2 \left\{\frac{\gamma^2}{\varrho_1^2} + \frac{1 - \gamma^2}{\varrho_2^2}\right\}, \end{aligned}$$

since $\varrho_1 < \varrho_2$. Thus we have proved

THEOREM 1. – Under the assumptions (4) the estimate

$$\|(\Delta + c)^{-1}(f(x, u) - cu)\| \leqslant k_0\|u\| , \qquad k_0^2 = \frac{M^2\gamma^2}{\varrho_1^2} + M^2\frac{(1-\gamma^2)}{\varrho_2^2} \tag{7}$$

holds, where ϱ, M and $\gamma = \gamma^+ + \gamma^-$ are defined by (3), (5), (6), and $p \geqslant 2$.

Naturally this theorem is of interest mainly when $k_0 < 1$.

Let $f = f_1 = b(x)u^+ - a(x)u^-$ and assume that

$$\lambda_n - \varepsilon \leqslant a(x) \leqslant c \leqslant b(x) \leqslant \lambda_{n+1} + \varepsilon , \tag{8}$$

where $c = \frac{1}{2}(\lambda_n + \lambda_{n+1})$. Let $\Lambda_1 = \{\lambda_p, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_q\}$, where $\lambda_p = \lambda_n$ and $\lambda_q = \lambda_{n+1}$; in other words, H_1 is the span of *all* the eigenfunctions corresponding to the eigenvalues λ_n or λ_{n+1}. Then $\varrho_1 = \frac{1}{2}(\lambda_{n+1} - \lambda_n)$, $\varrho_2 = \min(\lambda_{q+1} - c, c - \lambda_{p-1}) > \varrho_1$, $\alpha(x) = 0$, $M = \varrho_1 + \varepsilon$, $\gamma = \gamma^+ < 1$, $\gamma^- = 0$, and

$$k_0^2 = \left(\frac{\varrho_1 + \varepsilon}{\varrho_1}\right)^2 \gamma^2 + \left(\frac{\varrho_1 + \varepsilon}{\varrho_2}\right)^2 (1 - \gamma^2) .$$

Now assume that ε_0 is the positive solution of $k_0^2 = 1$.

THEOREM 2. – Let f satisfy

$$\lim_{u \to -\infty} \frac{f(x, u)}{u} = a(x) , \qquad \lim_{u \to +\infty} \frac{f(x, u)}{u} = b(x) \qquad \text{uniformly in } x \in \Omega , \tag{9}$$

and assume that (8) holds, where $n \geqslant 2$ and $\varepsilon < \varepsilon_0$. Then equation (1) has a solution for all $h(x) \in L^2(\Omega)$.

PROOF. – One has merely to observe that for any fixed h and for R sufficiently large $\|(\Delta + c)^{-1}(h(x) - f(x, u))\| < R$ for all $\|u\| \leqslant R$. This is a direct consequence of the estimate (7) for $f_1(x, u) = b(x)u^+ - a(x)u^-$ (one uses a decomposition $f = f_1 + f_2 + f_3$ with f_2 bounded and $|f_3(x, u)| \leqslant \delta|u|$, where $k_0 + \delta/\varrho_1 < 1$). From this and the compactness of $(\Delta + c)^{-1}$, we can conclude that the map

$$u \to (\Delta + c)^{-1}(h(x) - f(x, u))$$

has a fixed point in the ball of radius R in H. This proves the theorem. $\square$

REMARKS. – Notice that theorem 2 allows that $a(x) = \lambda_n$ and $b(x) = \lambda_{n+1}$, but also allows $a(x)$ and $b(x)$ to take values in the intervals $[\lambda_n - \varepsilon, c]$, $[c, \lambda_{n+1} + \varepsilon]$ respectively $(c = \frac{1}{2}(\lambda_n + \lambda_{n+1}))$, where ε can be estimated. To make this clear, we consider

The one-dimensional case. – Let $\Omega = [0, \pi] \subset \mathbf{R}^1$, $\lambda_n = n^2$, $\theta_n = \sqrt{2/\pi}\,\sin nx$ ($n = 1, 2, 3, \ldots$). Then the estimate

$$\eta_k^2 = \max\left(\|\theta_k^+\|^2, \|\theta_k^-\|^2\right) = \begin{cases} \dfrac{1}{2} & \text{for } k \text{ even}, \\[2ex] \dfrac{1}{2} + \dfrac{1}{2k} & \text{for } k \text{ odd} \end{cases} \tag{10}$$

holds.

Under the assumptions of theorem 2, we have $\varrho_1 = n + \frac{1}{2}$, $\varrho_2 = 3n - \frac{1}{2}$ and $\gamma^2 = \frac{1}{2} + 1/2n$, n odd, or $\gamma^2 = \frac{1}{2} + 1/(2n+2)$, n even, and

$$k_0^2 \leqslant \left(\frac{2n+1+2\varepsilon}{2n+1}\right)^2 \frac{n+1}{2n} + \left(\frac{2n+1+2\varepsilon}{6n-1}\right)^2 \frac{n-1}{2n}\,.$$

For n large and $\varepsilon = xn$, this gives

$$k_0^2 \approx (1+x)^2 \frac{1}{2} + \frac{1}{2\cdot 9}(1+x)^2 = \frac{5}{9}(1+x)^2\,,$$

and equating the right hand side to one gives $x = -1 + 3/\sqrt{5} = 0.3416 > 1/3$. Thus in the one-dimensional case, the theorem states that $\varepsilon = 1/3n$ is admissible for large n, or that the values $a(x)$ and $b(x)$ need only lie in the intervals $[\lambda_n - \varepsilon, c]$ and $[c, \lambda_{n+1} + \varepsilon]$, where ε is approximately $1/6$ of the length of the adjoining intervals $(\lambda_{n-1}, \lambda_n)$ or $(\lambda_{n+1}, \lambda_{n+2})$. It is worth noticing that these results do not depend on the distribution of $a(x)$ and $b(x)$ in their respective intervals.

Assume now that $p \geqslant 2$,

$$\lambda_{p-1} < \lambda_p \leqslant \lambda_n < c < \lambda_{n+1} \leqslant \lambda_q < \lambda_{q+1}$$

and

$$\lambda_{p-1} + \varepsilon \leqslant a(x) \leqslant c \leqslant b(x) \leqslant \lambda_{q+1} - \varepsilon\,,$$

which implies $\varrho_1 = \min(\lambda_{n+1} - c, c - \lambda_n)$, $M \leqslant \max(c - \varepsilon - \lambda_{p-1}, \lambda_{q+1} - \varepsilon - c) < \varrho_2$. In this case, $\gamma = \gamma^+ < 1$, $M/\varrho_2 < 1$, but M/ϱ_1 may be large.

If $\beta(x) = \max(b(x) - c, c - a(x))$ is for the most part small and only allowed to be close to $M = \max \beta(x)$ on a small set, then γ would be small and $k_0^2 < 1$. Thus the nonlinearity may asymptotically cross as many eigenvalues as we please, so long as it only does so on small sets. (It would be easy to give explicit conditions in the one-dimensional case, using (10) and the fact that $|\theta_n(x)| \leqslant \sqrt{2/\pi}$.)

3. – Multiplicity results.

In this section, we shall consider some improvements on theorems concerning the existence of non-trivial solutions of

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega \tag{11}$$

under the assumptions that $f(x, 0) = 0$ and that $f(x, u)/u$ behaves differently at zero and at infinity.

We first prove a degree theoretic lemma which is, in essence, a restatement of the main a priori estimate. The Leray-Schauder degree is denoted by d.

Lemma 1. - Assume that f satisfies the hypotheses of theorem 1 with $k_0 < 1$, where $\lambda_k < c < \lambda_{k+1}$. Then

$$d(u - (-\Delta)^{-1} f(x, u), \eta B, 0) = (-1)^k \quad \text{for all } \eta > 0 ,$$

where B is the unit ball in H.

Proof. - According to (7),

$$\|(-\Delta - c)^{-1} \lambda(f(x, u) - cu)\| < \|u\|$$

for all $u \neq 0$ and all $\lambda \in [0, 1]$. Thus the problem

$$-\Delta u - cu = \lambda(f(x, u) - cu) \quad \text{in } \Omega , \qquad u = 0 \quad \text{on } \partial\Omega$$

has only the zero solution for $0 \leqslant \lambda \leqslant 1$. This shows that

$$d(u - (-\Delta)^{-1} f(x, u), \eta B, 0) = d(u - (-\Delta)^{-1} cu, \eta B, 0)$$

by the homotopy invariance of degree. But by a familiar computation (see, for example, [11]) this last degree is $(-1)^k$. □

Now assume that f is globally Lipschitz continuous in u, uniformly for $x \in \bar{\Omega}$, and that

$$\text{(12)} \qquad b_0(x) = \lim_{u \to 0+} \frac{f(x, u)}{u} , \quad a_0(x) = \lim_{u \to 0-} \frac{f(x, u)}{u} , \quad \text{uniformly in } x .$$

We write $f = f_0 + g$ with $f_0(x, u) = b_0(x) u^+ - a_0(x) u^-$. Assume that f_0 satisfies the hypotheses of theorem 1 with $k_0 < 1$, where $\lambda_k < c < \lambda_{k+1}$. It follows that there exists $L > 0$ such that

$$|f_0(x, u) - cu| + |g(x, u)| \leqslant L|u| \quad \text{for all } x \text{ and } u .$$

Let $R = (-\Delta - c)^{-1}$ and $K = \overline{R(B)}$. If u is a solution of (11) with $\|u\| \leqslant \varepsilon$, then $u = R(f - cu)$, and we get $u \in \varepsilon LK$.

Choose $\gamma > 0$ with $2\gamma\|R\| + k_0 < 1$. Then there is a $\varrho > 0$ such that

$$|g(x, u)| \leqslant \gamma |u| \quad \text{for all } x \in \bar{\Omega},\ |u| \leqslant \varrho .$$

Let

$$u^{\varrho} = \max(-\varrho, \min(u, \varrho)) .$$

Then

$$|g(x, u)| \leqslant |g(x, u) - g(x, u^{\varrho})| + |g(x, u^{\varrho})| \leqslant$$
$$\leqslant L|(u-\varrho)^{+}| + L|(u+\varrho)^{-}| + \gamma|u^{\varrho}| .$$

Since K is compact, there exists a modulus of continuity $\delta(s)$ ($\delta(s) \to 0$ as $s \to 0+$) such that

$$\|(u-\varrho)^{+}\| \quad \text{and} \quad \|(u+\varrho)^{-}\| \leqslant \varepsilon L \delta(\varepsilon L) ;$$

cf. [11; Lemma 1]. This implies

$$\|g(x, u)\| \leqslant 2L^2 \varepsilon \delta(\varepsilon L) + \gamma\varepsilon \leqslant 2\gamma\varepsilon ,$$

if $\varepsilon > 0$ is chosen sufficiently small. Hence, if $u = R(f - cu)$, $\|u\| \leqslant \varepsilon$, then, by the choice of γ,

$$\|u\| \leqslant \|Rg\| + \|R(f_0 - cu)\| \leqslant$$
$$\leqslant 2\gamma\|R\|\varepsilon + k_0\varepsilon < \varepsilon .$$

Since the above reasoning remains valid if g is replaced by λg with $0 \leqslant \lambda \leqslant 1$, it follows that

$$d(u - (-\Delta)^{-1} f(x, u), \varepsilon B, 0) = d(u - (-\Delta)^{-1} f_0(x, u), \varepsilon B, 0) = (-1)^k$$

by lemma 1. Thus we have proved

THEOREM 3. – Let f be globally Lipschitz in u (uniformly in x) with $f(x, 0) = 0$. Assume that (12) holds and that the function $f_0(x, u) = b_0(x)u^{+} - a_0(x)u^{-}$ satisfies the a priori estimate (7) with $k_0 < 1$ and $c \in (\lambda_k, \lambda_{k+1})$. Then there exists $\varepsilon_0 > 0$ such that

$$d(u - (-\Delta)^{-1} f(x, u), \varepsilon B, 0) = (-1)^k \quad \text{for } 0 < \varepsilon < \varepsilon_0 .$$

Now we make some assumptions on the behavior of f at infinity. We assume that $a(x)$ and $b(x)$ exist such that

$$(13) \qquad a(x) = \lim_{u \to -\infty} \frac{f(x, u)}{u} , \quad b(x) = \lim_{u \to +\infty} \frac{f(x, u)}{u} , \quad \text{uniformly in } x ,$$

where a and b are such that the function $f_1(x, u) = b(x)u^{+} - a(x)u^{-}$ satisfies the estimate of theorem 1 with $k_0 < 1$ and $\lambda_n < c < \lambda_{n+1}$.

Theorem 4. – With the above assumptions on $f(x, u)$, there exists N_0 so that

$$d(u - (-\Delta)^{-1} f(x, u), NB, 0) = (-1)^n \quad \text{for all } N \geqslant N_0 .$$

Proof. – Writing $f = f_1 + f_2 + f_3$ with $|f_2(x, u)| \leqslant \delta |u|$ for all u and f_3 bounded, the assertion follows by homotopy. Note that δ can be chosen arbitrarily small. □

We are now able to prove the existence of non-trivial solutions, when $n - k$ is odd.

Theorem 5. – Assume that the hypotheses of theorems 3 and 4 are satisfied with $n - k$ odd. Then the equation

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega , \qquad u = 0 \quad \text{on } \partial\Omega$$

has at least one non-trivial solution.

Proof. – The proof is by excision. If, for example, $k < n$ and k is even, n odd, we have for large N and small ε that

$$d(u - (-\Delta)^{-1} f(x, u), \varepsilon B, 0) \;= +1 ,$$
$$d(u - (-\Delta)^{-1} f(x, u), NB, 0) = -1$$

and thus

$$d(u - (-\Delta)^{-1} f(x, u), NB \setminus \varepsilon B, 0) = -2 .$$

Therefore there exists at least one solution in $NB \setminus \varepsilon B$. □

Remarks. – We observe that these hypotheses allow $f^+ = (\partial f/\partial u)(x, 0+)$ and $f^- = (\partial f/\partial u)(x, 0-)$ to take values outside the interval $(\lambda_k, \lambda_{k+1})$; see the remark at the end of section 2.

In particular, in the one-dimensional case, for large n, $a(x)$ could lie entirely in the upper sixth of $(n^2, (n+1)^2)$, $b(x)$ entirely in the lower sixth of $((n+2)^2, (n+3)^2)$, and $a_0(x)$ and $b_0(x)$ have values in the interval $(n^2, (n+1)^2)$.

4. – The existence of three solutions.

One can obtain additional information in the case where one eigenvalue is crossed, and more stringent restrictions are imposed on f. We assume:

(A) $n \geqslant 2$, λ_n is a simple eigenvalue.

(B) $f(x, 0) = 0$ and

$$\lambda_{n-1} + \varepsilon' \leqslant \frac{f(x, u) - f(x, v)}{u - v} \leqslant \lambda_{n+1} - \varepsilon' \qquad (u \neq v)$$

for some $\varepsilon' > 0$.

(*C*) The limits in (9) and (12) exist (uniformly in x), and the functions $a_0(x)$, $b_0(x)$, $a(x)$, $b(x)$ satisfy the hypotheses of theorems 3 and 4, with $k = n - 1$.

(For example, assumption (*C*) is satisfied if there exist $c_0 \in (\lambda_{n-1}, \lambda_n)$, $c_1 \in (\lambda_n, \lambda_{n+1})$ such that $\lambda_{n-1} + \varepsilon' \leqslant a_0(x) \leqslant c_0 \leqslant b_0(x) \leqslant \lambda_n + \varepsilon_0$, $\lambda_n - \varepsilon_1 \leqslant a(x) \leqslant c_1 \leqslant b(x) \leqslant \lambda_{n+1} - \varepsilon'$, where ε_0, ε_1 are positive constants depending on c_0, c_1.)

Let P be the orthogonal projection onto the space spanned by θ_n, the eigenfunction associated with λ_n, and let $c = \frac{1}{2}(\lambda_{n-1} + \lambda_{n+1})$. We assume for the moment that $c \neq \lambda_n$. In view of the remarks of section 2, the operator $(I - P)(-\Delta - c)^{-1}$ has norm $1/\varrho_2$, where $\varrho_2 = \frac{1}{2}(\lambda_{n+1} - \lambda_{n-1})$. By (*B*), the function $f(x, u) - cu$ is Lipschitz continuous in u with Lipschitz constant $\varrho_2 - \varepsilon'$. Hence, the nonlinear operator

$$u \to (I - P)(-\Delta - c)^{-1}(f(x, u) - cu)$$

is Lipschitz continuous, with Lipschitz constant $k_0 < 1$. In the case where $\lambda_n = \frac{1}{2}(\lambda_{n+1} + \lambda_{n-1})$ we choose c close to λ_n and arrive at the same conclusion $k_0 < 1$. Define $T\colon H \to H$ by $Tu = (-\Delta - c)^{-1}(f(x, u) - cu)$. Then for fixed $v \in PH$, there is a unique $w = w(v)$ which solves

$$w = (I - P)T(v + w)\,.$$

If $v = t\theta_n$, we write $w(t) = w(t\theta_n)$. (This is, of course, the usual Ljapunov-Schmidt method.) The function $w(t)$ is Lipschitz continuous with Lipschitz constant $k_0/(1 - k_0)$.

The equation $u = Tu$ is equivalent to

$$v = PT(v + w)\,, \qquad w = (I - P)T(v + w)\,,$$

and thus, the problem of solving $u = Tu$ is reduced to that of finding zeros of the one-dimensional function $\eta(t) \equiv t - \langle \theta_n, PT(t\theta_n + w(t))\rangle$. The following is a prism lemma which occurs in [6], [7].

Lemma 2. – Let $t_1 < t_2$, with $\eta(t_1)\cdot\eta(t_2) \neq 0$. Choose $R > 0$ so large that for all $t \in [t_1, t_2]$

$$\|w(t)\| < R \quad \text{and} \quad (1 - k_0)^{-1}\|(I - P)T(t\theta_n)\| < R\,,$$

and define

$$D = D(t_1, t_2, R) = \{u \in H\colon \|(I - P)u\| < R, Pu = t\theta_n, t_1 < t < t_2\}\,.$$

Then $d(I - T, D, 0)$ is defined, and

$$d(I - T, D, 0) = 1 \quad \Leftrightarrow \eta(t_1) < 0 < \eta(t_2)\,,$$

$$d(I - T, D, 0) = -1 \Leftrightarrow \eta(t_1) > 0 > \eta(t_2)\,.$$

We are now in a position to prove the following

THEOREM 6. – Assume that f satisfies the hypotheses (A), (B), (C). Then the equation

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega , \qquad u = 0 \quad \text{on } \partial\Omega$$

possesses at least three solutions.

PROOF. – By the reasoning of theorems 3 and 4, it is easy to show that there exist $N_0 > \tau_0 > 0$ such that for all $0 < \tau < \tau_0$ and $N > N_0$ the degrees $d_1 = d(I - T, NB, 0)$ and $d_2 = d(I - T, \tau B, 0)$ are defined, and $d_1 d_2 = -1$.

Using the notation of the prism lemma, define

$$D_1 = D(-N, N, R) , \qquad D_2 = D(-\tau, \tau, R) .$$

If R is chosen sufficiently large, the lemma is applicable. But since there are no solutions in $D_1 \setminus NB$ and in $D_2 \setminus \tau B$,

$$d(I - T, D_1, 0) = d(I - T, NB, 0)$$

and

$$d(I - T, D_2, 0) = d(I - T, \tau B, 0) .$$

This means that the products must satisfy

$$\eta(-N)\eta(-\tau) < 0 , \qquad \eta(\tau)\eta(N) < 0 ,$$

and therefore the function $\eta(t)$ must possess an additional zero in each of the intervals $(-N, -\tau)$ and (τ, N). This proves the theorem. $\square$

REMARK 1. – The first proof of at least three solutions to this type of problem was by AMBROSETTI and MANCINI, in [1]. They have severe restrictions on the nonlinearity, including the requirement that f has to satisfy $sf''(s) > 0$. We know of no proof which allows such a lack of smoothness in f, requiring only the global Lipschitz conditions and the limiting behavior at zero and infinity.

REMARK 2. – Unlike the variational methods of [3] and [4], our methods apply to the case where the linear operator is nonselfadjoint, as in the problem

$$u_t = \Delta u + f(x, t, u) \quad \text{in } \mathbf{R} \times \Omega ,$$

$$u = 0 \quad \text{on } \partial\Omega , \qquad u(x, t + T) = u(x, t) ,$$

with $f(x, t, u)$ T-periodic in t. One can apply the method of this section by making the period sufficiently small that the calculation of norms is not affected by the imaginary eigenvalues, i.e., by taking $2\pi/T > \|(-\Delta - c)^{-1}\|$. Alternatively, as in section 2, one could decompose $H = H_1 \oplus H_2$ into the space spanned by all eigenfunctions corresponding to eigenvalues inside the circle centered on the real line and encompassing $\lambda_p, \ldots, \lambda_q$.

Remark 3. – The methods of this paper can be applied to the hyperbolic problem

$$u_{tt} - u_{xx} - f(x, t, u) = 0 \quad \text{in } [0, \pi] \times [0, 2\pi] ,$$
$$u(0, t) = u(\pi, t) = 0 ,$$
$$u(x, t + 2\pi) = u(x, t)$$

if one assumes in addition that $f(x, t, u)$ is monotone in u. One then reduces the problem to one on a subspace, on which the linear operator has a compact inverse, and where the usual method of degree theory would apply. This method has been used in [11].

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