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Generalized Solutions to Free Boundary Problems for Hyperbolic Systems of Functional Partial Differential Equations (*).

PIERO BASSANINI - JAN TURO

Summary. – *Local a.e. solutions to a free boundary (Stefan) problem for a quasilinear hyperbolic system of functional PDE's of first order in two independent variables and diagonal form are investigated. The formulation includes retarded arguments and hereditary Volterra terms.*

1. – Introduction.

Let us denote by I_{a_0} the curvilinear rectangle

$$I_{a_0} = \{(x, y) : x \in [0, a_0],\ S_1(x) \leqslant y \leqslant S_2(x)\}\ ,$$

where $S_k(0) = \alpha_k$ and $a_0 > 0$, α_k are given constants ($k = 1, 2$). Let the unknown line $y = \varphi(x)$ divide the set I_{a_0} into two sets $I^-_{a_0}$, where $S_1(x) \leqslant y \leqslant \varphi(x)$, and $I^+_{a_0}$, where $\varphi(x) \leqslant y \leqslant S_2(x)$, with $S_1(x) < \varphi(x) < S_2(x)$ for $x \in [0, a_0]$. We will denote by « $+$ » and « $-$ » the value of the considered functions on I^+_a and I^-_a, respectively, and by $|z|_n = \max_{1\leqslant i\leqslant n} |z_i|$ the norm of z in $\boldsymbol{R}^n$.

We consider quasilinear hyperbolic systems of functional partial differential equations in diagonal form

$$D_x z_i + \lambda_i(x, y, z, Vz) D_y z_i = f_i(x, y, z, Vz)\ , \qquad i \in \mathfrak{J} = 1, \ldots, n \tag{1}$$

($z = z(x, y)$, $Vz = (Vz)(x, y)$), with the initial conditions

$$z(0, y) = \gamma(y)\ , \qquad y \in [\alpha_1, \alpha_2]\ , \tag{2}$$

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Indirizzo degli AA.: P. BASSANINI: Dip. di Matematica, Università di Roma I, Città Universitaria, 00185 Roma, Italy; J. TURO: Dept. of Mathematics, Technical University of Gdansk, 80-952 Gdansk, 11/12 Majakowski Str., Poland.

the boundary conditions on the lines $y = S_k(x)$, $k = 1, 2$,

$$z_i(x, S_k(x)) = R_k^i\big(x, \hat{z}^k(x, S_k(x))\big)\,, \quad i \in \mathfrak{J}^k \tag{3}$$

$\big(\mathfrak{J}^k = \{i\colon \operatorname{sgn}[\lambda_i(0, \alpha_k, 0, 0) - S_k'(0)] = (-1)^{k+1}\}\big)$, and the boundary conditions on the free boundary $y = \varphi(x)$

$$z_i^{\mp}(x, \varphi(x) \mp 0) = R_i^{\mp}\big(x, \varphi(x), \varphi^{(1)}(x), \dots, \varphi^{(m-1)}(x), \hat{z}^{\mp}(x, \varphi(x))\big)\,, \quad i \in \mathfrak{J}^{\mp}\,, \tag{4}$$

which satisfies the equation

$$\frac{d^m\varphi}{dx^m} = H\big(x, \varphi(x), \varphi^{(1)}(x), \dots, \varphi^{(m-1)}(x), z^-(x, \varphi(x)), z^+(x, \varphi(x))\big)\,, \tag{5}$$

and the initial conditions

$$\varphi(0) = \beta_0 \quad (\alpha_1 < \beta_0 < \alpha_2)\,, \qquad \varphi^{(k)}(0) = \beta_k\,, \quad k = 1, \dots, m-1\,. \tag{6}$$

Here

$$\hat{z}^- = \{z_i^-\colon i \in \mathfrak{J} \setminus \mathfrak{J}^-\}\,, \quad \hat{z}^+ = \{z_i^+\colon i \in \mathfrak{J} \setminus \mathfrak{J}^+\}\,, \quad \hat{z}^k = \{z_i\colon i \in \mathfrak{J} \setminus \mathfrak{J}^k\}\,, \quad k = 1, 2\,,$$

and

$$\mathfrak{J}^{\mp} = \{i\colon \operatorname{sgn}[\lambda_i(0, \beta_0, 0, 0) - \beta_1] = \mp 1\}\,.$$

Let $C_L^{m-1}[0, a]$ be the set of real functions of class C^{m-1} on $[0, a]$ whose $(m-1)$-th derivatives are Lipschitzian.

In this paper we are interested in local generalized (a.e.) solutions of mixed (initial-boundary) problems (1)-(6) with the free (unknown) boundary $y = \varphi(x)$, whose initial values (6) are known. We seek the function $z\colon I_a \to \boldsymbol{R}^n$, whose restrictions to the sets I_a^- and I_a^+ are Lipschitzian, and the function $\varphi\colon [0, a] \to \boldsymbol{R}$ of class $C_L^{m-1}[0, a]$, satisfying equations (1), (5) a.e., initial conditions (2), (6) and boundary conditions (3), (4), respectively.

Generalized solutions have been investigated in the past by various authors: for hyperbolic systems in bicharacteristic form with initial or boundary conditions by Z. KAMONT, J. TURO [7], [8] and J. TURO [12], [13], for system (1) with mixed conditions by J. TURO [14], for pure differential systems with mixed conditions (with a different definition of generalized solution) by V. E. ABOLINIA, A. D. MYSHKIS [1] and A. D. MYSHKIS, A. M. FILIMONOV [11]. Of fundamental importance for our approach are the ideas and methods for pure differential systems with initial or boundary conditions developed by L. CESARI in a series of papers (see [3]-[6] and references therein). Cesari's method has been subsequently applied by P. BASSANINI (see [2] and references therein). Classical solutions of free boundary problems (1)-(6) (without functional argument) have been considered by K. YU. KASAKOV, S. F. MOROZOV [9].

Our work is aimed at hyperbolic free boundary and Stefan problems which arise from applications [15], [16]. This motivates the formulation of the problem, adopted here. In particular $z(x, y)$ has, in general, a jump discontinuity across the free boundary $y = \varphi(x)$ (cf. [15]). The functional operator V includes retarded arguments and Volterra hereditary operators [7], [13].

2. – Basic assumptions.

ASSUMPTION H_1. – Suppose that, for given $\Omega > 0$,

1) there is a constant $s \geqslant 0$ such that for all $x, \bar{x} \in [0, a_0]$ we have

$$|S_k(x) - S_k(\bar{x})| \leqslant s|x - \bar{x}|, \qquad k = 1, 2 ;$$

2) the functions

$$\operatorname{sgn}[\lambda_i(\cdot, S_k(\cdot), \cdot, \cdot) - S_k'(\cdot)], \qquad \operatorname{sgn}[\lambda_i(\cdot, \varphi(\cdot), \cdot, \cdot) - \varphi'(\cdot)] \qquad (i \in \mathfrak{I},\ \varphi \in C_L^{m-1}[0, a_0],\ k = 1, 2)$$

are constant in $[0, a_0] \times E_{a_0}$, where

$$\bar{\Omega} = [-\Omega, \Omega]^n \subset \boldsymbol{R}^n, \qquad E_{a_0} = [0, a_0] \times \bar{\Omega} \times \bar{\Omega} ;$$

3) the functions $\lambda_i(x, y, u, v)$ $((x, y, u, v) \in \tilde{E}_{a_0} = I_{a_0} \times \bar{\Omega} \times \bar{\Omega},\ i \in \mathfrak{I})$ are measurable with respect to x and continuous with respect to (y, u, v);

4) there are a constant $\Lambda > 0$ and integrable functions

$$l_j \colon [0, a_0] \to \boldsymbol{R}_+ \qquad (\boldsymbol{R}_+ = [0, +\infty),\ j = 1, 2, 3)$$

such that for all (x, y, u, v), $(x, \bar{y}, \bar{u}, \bar{v}) \in \tilde{E}_{a_0}$ we have

$$|\lambda_i(x, y, u, v)| \leqslant \Lambda, \qquad i \in \mathfrak{I},$$

$$|\lambda_i(x, y, u, v) - \lambda_i(x, \bar{y}, \bar{u}, \bar{v})| \leqslant l_1(x)|y - \bar{y}| + l_2(x)|u - \bar{u}|_n + l_3(x)|v - \bar{v}|_n ;$$

5) there are constants $\varepsilon_0 \in (0, b_0)$ and $\Lambda_0 > 0$, such that

$$\lambda_i(x, y, u, v) - S_1'(x) \geqslant \Lambda_0 \quad \text{for } i \in \mathfrak{I}^1, \quad y \in [S_1(x), S_1(x) + \varepsilon_0], \quad (x, u, v) \in E_{a_0},$$

$$S_2'(x) - \lambda_i(x, y, u, v) \geqslant \Lambda_0 \quad \text{for } i \in \mathfrak{I}^2, \quad y \in [S_2(x) - \varepsilon_0, S_2(x)], \quad (x, u, v) \in E_{a_0},$$

$$\lambda_i(x, y, u, v) - \varphi'(x) \geqslant \Lambda_0 \quad \text{for } i \in \mathfrak{I}^+, \quad y \in [\varphi(x), \varphi(x) + \varepsilon_0], \quad (x, u, v) \in E_{a_0},$$

$$\varphi'(x) - \lambda_i(x, y, u, v) \geqslant \Lambda_0 \quad \text{for } i \in \mathfrak{I}^-, \quad y \in [\varphi(x) - \varepsilon_0, \varphi(x)], \quad (x, u, v) \in E_{a_0},$$

where

$$b_0 = \min\left\{\min_{[0,a_0]}[S_2(x) - \varphi(x)],\ \min_{[0,a_0]}[\varphi(x) - S_1(x)]\right\}.$$

ASSUMPTION H_2.

1) Assumption H_1, 3) is satisfied by the functions $f_i(x, y, u, v)$, $i \in \mathfrak{I}$;

2) There are a constant $F > 0$ and integrable functions k_j: $[0, a_0] \to \boldsymbol{R}_+$ $(j = 1, 2, 3)$ such that Assumption H_1, 4) is satisfied by $f_i(x, y, u, v)$ with Λ replaced by F and $l_j(x)$ by $k_j(x)$.

ASSUMPTION H_3.

1) There are constants $r_j \geqslant 0$ $(j = 1, 2)$ such that for all (x, u), $(\bar{x}, \bar{u})$ in $[0, a_0] \times \bar{\Omega}$, we have

$$|R_i^k(x, u) - R_i^k(\bar{x}, \bar{u})| \leqslant r_1|x - \bar{x}| + r_2|u - \bar{u}|_n\,, \qquad i \in \mathfrak{I}^k\,,\ k = 1, 2\,.$$

2) There are constants $r, \bar{r}_j, \bar{r} \geqslant 0$ $(j = 0, 1, \ldots, m-1)$ such that for all

$$(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u)\,,\ (\bar{x}, \bar{\varphi}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m-1}, \bar{u}) \in \tilde{G}_{a_0}\,,$$

we have

$$|R_i^{\mp}(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u) - R_i^{\mp}(\bar{x}, \bar{\varphi}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m-1}, \bar{u})| \leqslant$$
$$\leqslant r|x - \bar{x}| + \sum_{j=0}^{m-1} \bar{r}_j|\varphi_j - \bar{\varphi}_j| + \bar{r}|u - \bar{u}|_n\,, \qquad i \in \mathfrak{I}^{\mp}$$

where $\tilde{G}_{a_0} = [0, a_0] \times \boldsymbol{R}^m \times \bar{\Omega}$.

3) The compatibility conditions $R_i^k(0, \gamma(\alpha_k)) = \gamma_i(\alpha_k)$, $i \in \mathfrak{I}^k$, $k = 1, 2$,

$$R_i^{\mp}(0, \beta_0, \beta_1, \ldots, \beta_{m-1}, \hat{\gamma}(\beta_0 \mp 0)) = \gamma_i(\beta_0 \mp 0)\,, \qquad i \in \mathfrak{I}^{\mp}\,,$$

are satisfied.

4) There are constants $\omega, \Gamma \geqslant 0$ s.t. for all $y, \bar{y} \in [\alpha_1, \beta_0]$ or $y, \bar{y} \in [\beta_0, \alpha_2]$, we have

$$|\gamma_i(y) - \gamma_i(\bar{y})| \leqslant \Gamma|y - \bar{y}|\,, \qquad i \in \mathfrak{I}\,; \qquad \max_{[\alpha_1, \alpha_2]} |\gamma(y)|_n = \omega < \Omega\,.$$

ASSUMPTION H_4.

1) The function $H(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u, v)$ is measurable with respect to the first variable and continuous with respect to the remaining $m + 2$ variables in $G_{a_0} = \tilde{G}_{a_0} \times \bar{\Omega}$.

2) There are a constant $h > 0$ and an integrable function $\bar{h}$: $[0, a_0] \to \boldsymbol{R}^+$ s.t. for all $(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u, v)$, $(x, \bar{\varphi}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m-1}, \bar{u}, \bar{v}) \in G_{a_0}$ we have

$$|H(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u, v)| \leqslant h ,$$

$$|H(x, \varphi, \varphi_1, \ldots, \varphi_{m-1}, u, v) - H(x, \bar{\varphi}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m-1}, \bar{u}, \bar{v})| \leqslant$$
$$\leqslant \bar{h}(x)\Big[\sum_{j=0}^{m-1} |\varphi_j - \bar{\varphi}_j| + |u - \bar{u}|_n + |v - \bar{v}_n|\Big] .$$

We denote by $\mathfrak{D}(a)$ the set of all functions z: $I_a \to \boldsymbol{R}^n$, whose restrictions to the sets I_a^- and I_a^+ are continuous and Lipschitzian with respect to both variables; by $B(a)$ the subset $\{z\colon z \in \mathfrak{D}(a),\ |z(x, y)|_n \leqslant \Omega\}$; by $B(a, P, Q)$ the set of all functions in $B(a)$ s.t. $|z(x, y) - z(\bar{x}, \bar{y})|_n \leqslant P|x - \bar{x}| + Q|y - \bar{y}|$ for all (x, y), $(\bar{x}, \bar{y})$ in I_a^- or I_a^+. We assume $Q \geqslant \Gamma$ $(P \geqslant 0)$, so that the closed set

$$B(a, P, Q, \varrho) = \Big\{z\colon z \in B(a, P, Q),\ \max_{I_a} |z(x, y) - \gamma(y)|_n \leqslant \varrho,\ z(0, y) = \gamma(y)\Big\} ,$$

$(0 < \varrho \leqslant \Omega - \omega]$ is not empty.

ASSUMPTION H_5. – Suppose that, for every $a \in (0, a_0]$,

1) V: $B(a) \to B(a)$.

2) There are integrable functions c, d: $[0, a_0] \to \boldsymbol{R}_+$ s.t. for every $z \in B(a)$ we have

$$[\![(Vz(x, \cdot)]\!] \leqslant c(x)[\![z(x, \cdot)]\!] + d(x) , \qquad x \in [0, a] ,$$
$$[\![z(x, \cdot)]\!] := \sup \{|y - \bar{y}|^{-1} |z(x, y) - z(x, \bar{y})|_n\colon y, \bar{y} \in [S_1(x), S_2(x)]\} .$$

3) There is an integrable function m: $[0, a_0] \to \boldsymbol{R}_+$ s.t. for all $z, \bar{z} \in B(a)$ and $x \in [0, a]$ we have

$$\| Vz - V\bar{z}\|_x \leqslant m(x) \|z - \bar{z}\|_x ,$$

$$\|z\|_x := \sup \{|z(t, y)|_n\colon (t, y) \in I_x\} , \qquad I_x = \{(t, y)\colon S_1(t) \leqslant y \leqslant S_2(t),\ t \in [0, x]\} .$$

3. – Preliminary lemmas.

We consider, for $z \in B(a)$, the characteristic problem

$$(7) \qquad \begin{cases} D_t g(t; x, y) = \lambda_i\big(t, g(t; x, y), z(t, g(t; x, y)), (Vz)(t, g(t; x, y))\big) , \\ \qquad\qquad (i \in \mathfrak{J} ,\ \text{for a.e. } t \in [0, a] ,\ \text{every } (x, y) \in I_a^- \text{ or } I_a^+) , \\ g(x; x, y) = y . \end{cases}$$

Because of Assumptions 3), 4) of H_1, 2) of H_5, and $z \in B(a)$, we conclude that the functions $\lambda_i(\cdot, z(\cdot), (Vz)(\cdot)): I_a \to \boldsymbol{R}$, $i \in \mathfrak{I}$, satisfy the Carathéodory conditions. Thus for every $z \in B(a)$, there is a unique maximal solution $g_i = g_i^z(t; x, y)$ of problem (7) (« a unique characteristic of the i-th family through every x, y ») in I_a^+, I_a^-. We denote by $\tau_i(x, y, z)$ the smallest value of t for which the maximal solution $g_i = g_i^z(t; x, y)$ exists, and we consider the following subsets of $I_a^\mp$ (where $\tau_i = \tau_i(x, y, z)$):

$$I_{\gamma_i}^{\mp z} = \{(x, y) \colon (x, y) \in I_a^\mp,\ \tau_i = 0\}\,,$$
$$I_{S_1 i}^{z} = \{(x, y) \colon (x, y) \in I_a^-,\ \tau_i > 0,\ g_i^z(\tau_i; x, y) = S_1(\tau_i)\}\,,$$
$$I_{S_2 i}^{z} = \{(x, y) \colon (x, y) \in I_a^+,\ \tau_i > 0,\ g_i^z(\tau_i; x, y) = S_2(\tau_i)\}\,,$$
$$I_{\varphi_i}^{\mp z} = \{(x, y) \colon (x, y) \in I_a^\mp,\ \tau_i > 0,\ g_i^z(\tau_i; x, y) = \varphi_i(\tau_i)\}\,,$$

defined according to the « starting points » of the characteristics. We will need the following constants, depending on a:

$$L_1 := \exp\left\{\int_0^a [l_1(t) + l_2(t)Q + l_3(t)\{c(t)Q + d(t)\}]\,dt\right\},$$
$$L_2 := \int_0^a [l_2(t) + l_3(t)m(t)]\,dt\,.$$

LEMMA 1. – *Let Assumptions* 3), 4) *of* H_1 *and* H_5 *hold,* $u, v \in B(a, P, Q)$, *and* $(x, y), (\bar{x}, \bar{y}) \in I_a^+$ *or* I_a^-. *Then, if* g_i^u, g_i^v *are the* (*maximal*) *solutions of problem* (7) *in* I_a^+ *or* I_a^-, *respectively, the following inequality* (*for every* t *in the maximal interval of existence*)

$$|g_i^u(t; x, y) - g_i^v(t; \bar{x}, \bar{y})| \leqslant L_1(\Lambda|x - \bar{x}| + |y - \bar{y}| + L_2\|u - v\|)$$

holds, where $\|z\| := \max_{I_a} |z(x, y)|_n$.

The proof follows, as in [14], from the previous inequalities and Gronwall's Lemma.

LEMMA 2. – *Suppose Assumptions* H_1 *and* H_5 *are satisfied, and* a, $0 < a \leqslant a_0$, *is sufficiently small so that* $\Lambda a \leqslant \varepsilon_0$, *where* ε_0 *is given in* 5) *of* H_1. *Then, for all* (x, y), $(x, \bar{y})$ *in* $\bar{I}_{S_1 i}^z$ *or* $\bar{I}_{S_2 i}^z$ *or* $\bar{I}_{\varphi_i}^{\mp z}$ (*the bar denotes closure*) *and* $z \in B(a, P, Q)$, *we have*

$$|\tau_i(x, y, z) - \tau_i(x, \bar{y}, z)| \leqslant \Lambda_0^{-1} L_1 |y - \bar{y}|\,. \tag{8}$$

Moreover for (x, y) *in* $\bar{I}_{S_1 i}^u \cap \bar{I}_{S_1 i}^v$ *or* $\bar{I}_{S_2 i}^u \cap \bar{I}_{S_2 i}^v$ *or* $\overline{I_{\varphi_i}^{-u}} \cap \overline{I_{\varphi_i}^{-v}}$ *or* $\bar{I}_{\varphi_i}^{+u} \cap \bar{I}_{\varphi_i}^{+v}$, *and* $u, v \in B(a, P, Q)$, *we have*

$$|\tau_i(x, y, u) - \tau_i(x, y, v)| \leqslant \Lambda_0^{-1} L_1 L_2 \|u - v\|\,, \quad i \in \mathfrak{I}\,. \tag{9}$$

PROOF. – First we prove inequality (8). Let us suppose that (x, y), $(x, \bar{y})$ are in $\bar{I}^z_{S_1 i}$ and $y > \bar{y}$. Then, since the characteristic lines of the same family corresponding to the function z cannot intersect, we have $\tau_i(x, y, z) < \tau_i(x, \bar{y}, z)$. From Lemma 1 it follows that

$$g_i^z(\tau_i(x, \bar{y}, z); x, y) - g_i^z(\tau_i(x, \bar{y}, z); x, \bar{y}) \leqslant L_1(y - \bar{y}) . \tag{10}$$

Writing the characteristic equation in integral form yields

$$g_i^z(\bar{\tau}_i; x, y) = S_1(\tau_i) + \int_{\tau_i}^{\bar{\tau}_i} \lambda_i(t, \breve{g}_i, z(t, \breve{g}_i), (Vz)(t, \breve{g}_i))\, dt ,$$

where

$$\tau_i = \tau_i(x, y, z) , \qquad \bar{\tau}_i = \tau_i(x, \bar{y}, z) , \qquad \breve{g}_i = g_i^z(t; x, y) .$$

Hence

$$\begin{aligned} g_i^z(\bar{\tau}_i; x, y) - g_i^z(\bar{\tau}_i; x, \bar{y}) &= S_1(\tau_i) - S_1(\bar{\tau}_i) + \int_{\tau_i}^{\bar{\tau}_i} \lambda_i(t, \breve{g}_i, z(t, \breve{g}_i), (Vz)(t, \breve{g}_i))\, dt = \\ &= \int_{\tau_i}^{\bar{\tau}_i} [\lambda_i(t, \breve{g}_i, z(t, \breve{g}_i)(Vz)(t, \breve{g}_i)) - S_1'(t)]\, dt . \end{aligned} \tag{11}$$

From the estimate $|g_i^z(t; x, y) - S_1(x)| \leqslant \Lambda a$ for any characteristic in $\bar{I}^z_{S_1 i}$, it follows that $g_i(t; x, y) \in [S_1(x), S_1(x) + \varepsilon_0]$, provided $(x, y) \in \bar{I}^z_{S_1 i}$, so that $i \in \mathfrak{I}^1$. Thus, by assumptions $\Lambda a \leqslant \varepsilon_0$ and 5) of H_1, we have

$$[\lambda_i(t, \breve{g}_i, z(t, \breve{g}_i), (Vz)(t, \breve{g}_i)) - S_1'(t)] \geqslant \Lambda_0 .$$

Hence, by (10), (11) we obtain

$$\bar{\tau}_i - \tau_i \leqslant \Lambda_0^{-1} [g_i^z(\bar{\tau}_i; x, y) - g_i^z(\bar{\tau}_i; x, \bar{y})] \leqslant \Lambda_0^{-1} L_1 |y - \bar{y}| . \tag{12}$$

The remaining cases can be handled similarly. To prove (9), let us assume that $(x, y) \in \bar{I}^u_{S_1 i} \cap \bar{I}^v_{S_1 i}$ and $\tau_i(x, y, u) < \tau_i(x, y, v)$ (again, the remaining cases can be dealt with in a similar way). By Lemma 1, we have

$$|g_i^u(\tau_i(v); x, y) - g_i^v(\tau_i(v); x, y)| \leqslant L_1 L_2 \|u - v\| , \tag{13}$$

where $\tau_i(v) = \tau_i(x, y, v)$. From the characteristic equation,

$$g_i^u(\tau_i(v); x, y) - g_i^v(\tau_i(v); x, y) = \int_{\tau_i(u)}^{\tau_i(v)} [\lambda_i(t, \breve{g}_i^u, u(t, \breve{g}_i^u), (Vu)(t, \breve{g}_i^u)) - S_1'(t)]\, dt .$$

Thus the assertion follows from 5) or H_1 and inequality (13). ///

Integrating the differential system of first order, equivalent to problem (5), (6), we obtain

$$(14)\qquad \begin{cases} \varphi_k(x) \;\;= \beta_k + \int_0^x \varphi_{k+1}(t)\, dt\,, \qquad k = 0, 1, \ldots, m-2\,, \\ \varphi_{m-1}(x) = \beta_{m-1} + \int_0^x H\big(t, \varphi(t), \ldots, \varphi_{m-1}(t), z^-(t, \varphi(t)), z^+(t, \varphi(t))\big)\, dt\,, \end{cases}$$

where $\varphi_0 = \varphi$. Because of H_4 we see that this system satisfies Carathéodory conditions. Thus, $\forall z \in B(a)$, there is a unique maximal solution $\varphi_j^z(x)$, $j = 0, \ldots, m-1$, in $[0, a]$, which is Lipschitzian with respect to x, say with the constant Φ.

LEMMA 3. – *If Assumption* H_4 *is satisfied, then, for all* $u, v \in B(a, P, Q)$, *and* $x \in [0, a]$, *we have*

$$|\varphi_j^u(x) - \varphi_j^v(x)| \leq M\|u - v\|\,, \qquad j = 0, 1, \ldots, m-1\,.$$

PROOF. – Put

$$W(x) = (1 + 2Q)|\varphi^u(x) - \varphi^v(x)| + \sum_{j=1}^{m-1} |\varphi_j^u(x) - \varphi_j^v(x)|\,.$$

By 2) of H_4 we have, for $j = 0, \ldots, m-1$:

$$(15)\qquad |\varphi_j^v(x) - \varphi_j^u(x)| \leq \Big[2\int_0^x \bar{h}(t)\, dt \cdot \|u - v\| + \int_0^x \bar{h}(t)\, W(t)\, dt\Big] a^{m-j-1}\,.$$

Combining inequalities (15) we obtain

$$W(x) \leq 2\int_0^x \bar{\bar{h}}(t)\, dt \cdot \|u - v\| + \int_0^x \bar{\bar{h}}(t)\, W(t)\, dt\,, \qquad \bar{\bar{h}}(t) = \Big[(1 + 2Q)\sum_{k=0}^{m-1} a^k\Big]\, h(t)\,,$$

whence, by Gronwall's Lemma, $W(x) \leq \tilde{M}(x)\|u - v\|$, with

$$\tilde{M}(x) = 2\int_0^x \bar{\bar{h}}(t)\, dt \cdot \exp\Big[\int_0^a \bar{\bar{h}}(t)\, dt\Big]\,.$$

Thus, by (15), the assertion follows, with

$$M = \int_0^a [2 + \tilde{M}(t)]\, \bar{h}(t)\, dt \cdot \max\,[1, a^m]\,.$$

4. – An integral-functional operator and its properties.

Now we consider in $B(a)$ the operator S defined by

$$Sz = Tz + Uz\,,$$

where

$$(16)\qquad (Tz)_i(x, y) = \begin{cases} \gamma_i\big(g_i(0; x, y)\big)\,, & (x, y) \in I^{\mp z}_{\gamma_i}\,, \\ R^k_i\big(\tau_i, z^k(\tau_i, S_k(\tau_i))\big)\,, & (x, y) \in I^z_{S_k i}\,, \\ R^{\mp}_i\big(\tau_i, \varphi(\tau_i), \dots, \varphi^{(m-1)}(\tau_i), \hat{z}^{\mp}(\tau_i, \varphi(\tau_i))\big)\,, & (x, y) \in I^{\mp z}_{\varphi_i}\,, \end{cases}$$

(the « starting value » of $z(x, y)$),

$$(17)\qquad (Uz)_i(x, y) = \int_{\tau_i}^{x} f_i(t, \breve{g}_i, z(t, \breve{g}_i), (Vz)(t, \breve{g}_i))\, dt\,, \qquad i \in \mathfrak{I}\,,$$

(the « evolution along characteristics »), where $\breve{g}_i = g_i(t; x, y)$, and $\tau_i = \tau_i(x, y, z)$.

From now on we assume: $2\Lambda a < b_0$, which yields $\bar{I}^u_{S_1 i} \cap \bar{I}^{-v}_{\varphi_i} = \emptyset$, $\bar{I}^u_{S_2 i} \cap \bar{I}^{+v}_{\varphi_i} = \emptyset$, and $\Lambda a \leqslant \varepsilon_0$, which guarantees that 5) of H_1 is satisfied in the sets $\bar{I}^u_{S_k i}$, $\bar{I}^{\mp u}_{\varphi_i}$.

Using the previous estimates and the compatibility conditions we can prove the following

LEMMA 4. – *Let Assumptions* H_1-H_5 *hold. Then, for every* $z \in B(a, P, Q, \varrho)$ *the function* Sz *restricted to* I^-_a *and* I^+_a *is continuous with respect to* (x, y).

The proof is similar to that of Lemma 3 of [14]. ///

Put

$$K = \int_0^a [k_1(t) + k_2(t)Q + k_3(t)\{c(t)Q + d(t)\}]\, dt\,.$$

LEMMA 5. – *If Assumptions* H_1-H_5 *are satisfied, then, for every* z *in* $B(a, P, Q, \varrho)$ *the function* Sz *satisfies in* $\bar{I}^{\mp z}_{\gamma_i}$ *a Lipschitz condition in* y *with the constant*

$$Q^S_\gamma = (\Gamma + K)L_1\,.$$

The proof of this lemma is similar to that of Lemma 4 of [14].

LEMMA 6. – *Let Assumptions* H_1-H_5 *hold. Then, for every* $z \in B(a, P, Q, \varrho)$, *the function* Sz *is Lipschitzian in* $\bar{I}^{\mp z}_{\gamma_i}$ *with respect to* x *with the constant* $P^S_\gamma = Q^S_\gamma \Lambda + F$.

The proof can be carried out as in [14], Lemma 5.

LEMMA 7. – *If Assumptions* H_1-H_5 *are satisfied, then in the sets* $\bar{I}^z_{S_k i}$, $k = 1, 2$, *the function* Sz *satisfies a Lipschitz condition in* y *with the constant*

$$Q^S_0 = L_1\{\Lambda_0^{-1}[r_1 + r_2(P + Qs) + F] + K\}.$$

For the proof, see [14].

LEMMA 8. – *Let Assumptions* H_1-H_5 *hold. Then, for every* $z \in B(a, P, Q, \varrho)$, *the function* Sz *satisfies in* $\bar{I}^{\mp z}_{\varphi_i}$ *a Lipschitz condition in* y *with the constant*

$$Q^S_\varphi = L_1[(R + F)\Lambda_0^{-1} + K].$$

PROOF. – Suppose $(x, y), (x, \bar{y}) \in \bar{I}^{-z}_{\varphi_i}$ (similarly for $\bar{I}^{+z}_{\varphi_i}$) and $y \leqslant \bar{y}$. Then Lemma 2 implies

$$|(Tz)_i(x, y) - (Tz)_i(x, \bar{y})| \leqslant R\Lambda_0^{-1}L_1|y - \bar{y}|, \qquad R := r + \Phi\sum_{j=0}^{m-1}\bar{r}_j + \bar{r}(P + Q\Phi).$$

Furthermore, since $\tau_i(x, y, z) \geqslant \tau_i(x, \bar{y}, z)$, by Lemma 2, we find

$$|(Uz)_i(x, y) - (Uz)_i(x, \bar{y})| \leqslant L_1(K + \Lambda_0^{-1}F)|y - \bar{y}|.$$

Therefore $|(Sz)_i(x, y) - (Sz)_i(x, \bar{y})| \leqslant Q^S_\varphi|y - \bar{y}|$, and the lemma is proved.

REMARK 1. – From Lemmas 5, 7 and 8 it follows that the function Sz satisfies in I^-_a and I^+_a a Lipschitz condition in y with the constant $Q^s = \max[Q^s_\gamma, Q^s_0, Q^s_\varphi]$. As $a \to 0^+$ we have $Q^s_\gamma \sim \Gamma$, $Q^s_\varphi \sim \Lambda_0^{-1}(R + F)$, $Q^s_0 \sim \Lambda_0^{-1}[r_1 + F + r_2(P + Qs)]$, so that $\hat{W} = \max[Q^s_\gamma, Q^s_\varphi, Q^s_0]$ depends on P, Q.

If the points (x, y) and $(x, \bar{y})$ belong to different sets $\bar{I}^{\cdot}_{\cdot i}$, then, in view of Lemma 4 and by introducing an intermediate point, this case reduces to the one already considered.

LEMMA 9. – *If Assumptions* H_1-H_5 *are satisfied, the function* Sz *satisfies in* I^-_a *and* I^+_a *a Lipschitz condition in* x *with the constant* $P^s = Q^s\Lambda + F$.

The proof of this lemma is similar to that of Lemma 7 in [14].

REMARK 2. – In particular, without loss of generality, we may assume that $\Lambda \geqslant 1$. Then, by Lemmas 5, 7, 8 and 9, we conclude that the function Sz satisfies in I^-_a and I^+_v a Lipschitz condition with respect to (x, y) with the constant P^s.

LEMMA 10. – *Suppose $a \in (0, a_0]$ is sufficiently small, so that*

$$a \leqslant \varrho[r_1 + r_2(P + Qs) + R + \Gamma\Lambda + F]^{-1} . \tag{18}$$

Then, under Assumptions H_1-H_5, *the operator S maps $B(a, P, Q, \varrho)$ into $B(a, P^S, Q^S, \varrho)$.*

PROOF. – This will be proved by showing that, for $z \in B(a, P, Q, \varrho)$,

$$|(Sz)(x, y) - \gamma(y)|_n \leqslant \varrho , \tag{19}$$

and

$$(Sz)(0, y) = \gamma(y) , \qquad y \in [\alpha_1, \alpha_2] . \tag{20}$$

First, let $(x, y) \in \bar{I}^{-z}_{\gamma_i}$; then we have

$$|(Sz)_i(x, y) - \gamma_i(y)| \leqslant \Gamma |g_i(0; x, y) - y| + \int_0^x F\, dt \leqslant (\Gamma\Lambda + F)a .$$

Next, let $(x, y) \in \bar{I}^z_{S_k i}$, then taking into consideration the compatibility condition 3) of H_3 and the initial condition (2), we see that

$$|(Tz)_i(x, y) - \gamma_i(y)| \leqslant \left|R^k_i\big(\tau_i, z^k(\tau_i, S_k(\tau_i))\big) - R^k_i\big(0, z^k(0, S_k(0))\big)\right| + \\ + |R^k_i(0, \gamma(\alpha_k)) - \gamma_i(y)| \leqslant [r_1 + r_2(P + Qs) + \Gamma\Lambda]a ,$$

where $\tau_i = \tau_i(x, y, z)$. Finally, suppose that $(x, y) \in \bar{I}^{\mp z}_{\varphi_i}$, then by compatibility condition 3) of H_3 we get

$$|(Tz)_i(x, y) - \gamma_i(y)| \leqslant \left|R^{\mp}_i\big(\tau_i, \varphi(\tau_i), \dots, \varphi^{(m-1)}(\tau_i), \hat{z}^{\mp}(\tau_i, \varphi(\tau_i))\big) - \right. \\ \left. - R^{\mp}_i\big(0, \varphi(0), \dots, \varphi^{(m-1)}(0), \hat{z}^{\mp}(0, \varphi(0))\big)\right| + \\ + |R^{\mp}_i(0, \beta_0, \dots, \beta_{m-1}, \hat{\gamma}(\beta_0 \mp 0)) - \gamma_i(y)| \leqslant (R + \Gamma\Lambda)a ,$$

where again $\tau_i = \tau_i(x, y, z)$. Since $|(Uz)_i(x, y)| \leqslant Fa$, combining the previous estimates yields

$$|(Sz)_i(x, y) - \gamma_i(y)| \leqslant [r_1 + r_2(P + Qs) + R + \Gamma\Lambda + F]a .$$

Hence, by (18) we conclude that (19) holds, while (20) is obviously satisfied. Lemmas 5-9 imply that Sz satisfies in $I^{\mp}_a$ a Lipschitz condition with respect to both variables, with the appropriate constants P^S, Q^S. Thus the proof is complete. ///

From now on we shall assume that (18) is satisfied.

Notice that, generally speaking, $P^S \geqslant P$, $Q^S \geqslant Q$, so that

$$B(a, P, Q, \varrho) \subset B(a, P^S, Q^S, \varrho) .$$

The operator S is defined on all $B(a)$. We shall use the same symbol S to denote the restriction of S to the set $B(a, P, Q, \varrho)$. Put

$$\tilde{K} = \int_0^a [k_2(t) + k_3(t) m(t)]\, dt, \quad \bar{P} = \min[P, P^S], \quad \bar{Q} = \min[Q, Q^S].$$

LEMMA 11. – *Let Assumptions* H_1-H_5 *hold. Then, for all* $(x, y) \in \bar{I}_{\gamma_i}^{\mp u} \cap \bar{I}_{\gamma_i}^{\mp v}$, $u, v \in B(a, P, Q, \varrho)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant q_1 \|u - v\| ,$$

where $q_1 = L_1 L_2(\Gamma + K) + \tilde{K}$, *so that* $q_1 \to 0^+$ *as* $a \to 0^+$.

The proof of this lemma can be carried out as in Lemma 10 of [14].

Similarly as in [14] we can also prove the following

LEMMA 12. – *If Assumptions* H_1-H_5 *are satisfied, then, for all* (x, y) *in* $\bar{I}_{S_1 i}^u \cap \bar{I}_{S_1 i}^v$ *or* $\bar{I}_{S_2 i}^u \cap \bar{I}_{S_2 i}^v$, *and* $u, v \in B(a, P, Q, \varrho)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant q_2 \|u - v\| ,$$

where $q_2 = L_1 L_2[\Lambda_0^{-1}(r_1 + r_2(P + Qs) + F) + K] + \tilde{K} + r_2$, *so that* $q_2 \to r_2$ *as* $a \to 0$.

LEMMA 13. – *Suppose Assumptions* H_1-H_5 *hold. Then, for all* $(x, y) \in \bar{I}_{\varphi_i}^{\mp u} \cap \bar{I}_{\varphi_i}^{\mp v}$ *and* $u, v \in B(a, P, Q, \varrho)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant q_3 \|u - v\| ,$$

where $q_3 = L_1 L_2[\Lambda_0^{-1}(R + F) + K] + M\left(\bar{r} + \sum_{j=0}^{m-1} \bar{r}_j\right) + \tilde{K}$, *so that* $q_3 \to 0$ *as* $a \to 0$.

PROOF. – Let $(x, y) \in \bar{I}_{\varphi_i}^{-u} \cap \bar{I}_{\varphi_i}^{-v}$, then by Lemmas 2 and 3 we have

$$|(Tu)_i(x, y) - (Tv)_i(x, y)| \leqslant \left[R\Lambda_0^{-1} L_1 L_2 + M\left(\bar{r} + \sum_{j=0}^{m-1} \bar{r}_j\right)\right] \|u - v\| .$$

Assume, for definiteness, that $\tau_i(x, y, u) \leqslant \tau_i(x, y, v)$, then by Lemmas 1 and 2 we see that

$$|(Uu)_i(x, y) - (Uv)_i(x, y)| \leqslant [L_1 L_2(K + F\Lambda_0^{-1}) + \tilde{K}] \|u - v\| , \tag{21}$$

whence the assertion follows.

LEMMA 14. – *If Assumptions* H_1-H_5 *are satisfied, then for all* (x, y) *in* $\bar{I}_{\gamma_i}^{+u} \cap \bar{I}_{\varphi_i}^{+v}$ *or* $\bar{I}_{\gamma_i}^{-u} \cap \bar{I}_{\varphi_i}^{-v}$ *or* $\bar{I}_{\varphi_i}^{-u} \cap \bar{I}_{\gamma_i}^{-v}$ *or* $\bar{I}_{\varphi_i}^{+u} \cap \bar{I}_{\gamma_i}^{+v}$, *and* u, v *in* $B(a, P, Q, \varrho)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant q_4 \|u - v\|$$

where $q_4 = L_1 L_2[\Gamma + \Lambda_0^{-1}(R + F) + K] + \tilde{K}$, *so that* $q_4 \to 0$ *as* $a \to 0$.

PROOF. – We consider only the case when $(x, y) \in \bar{I}_{\gamma_i}^{+u} \cap \bar{I}_{\varphi_i}^{+v}$; the remaining cases can be handled similarly.

Since $\tau_i(x, y, u) = 0$ for $(x, y) \in \bar{I}_{\gamma_i}^{+u}$, by Lemma 2, we have

$$|\tau_i| \leqslant \Lambda_0^{-1} L_1 L_2 \|u - v\| ,$$

where, here and in the rest of the proof, $\tau_i = \tau_i(x, y, v)$. Moreover $\lambda_i(x, y, u, v) - \varphi'(x) \geqslant \Lambda_0$, since $i \in \mathfrak{I}^+$, $0 \leqslant y - \varphi(x) \leqslant \Lambda a \leqslant \varepsilon_0$ and because of 5) of H_1. Therefore the function $g_i^u(t; x, y) - \varphi(t)$ is increasing in t on the interval $[0, x]$, whence

$$g_i^u(0; x, y) - \beta_0 \leqslant g_i^u(\tau_i; x, y) - \varphi(\tau_i) .$$

In view of Lemma 1, we get

$$|g_i^u(\tau_i; x, y) - \varphi(\tau_i)| = |g_i^u(\tau_i; x, y) - g_i^v(\tau_i; x, y)| \leqslant L_1 L_2 \|u - v\| .$$

Hence we obtain

$$|g_i^u(0; x, y) - \beta_0| \leqslant L_1 L_2 \|u - v\| .$$

On account of compatibility condition 3) of H_3 and initial condition (2), we find

$$\begin{aligned}
|(Tu)_i(x, y) - Tv)_i(x, y)| &\leqslant \\
&\leqslant \left|\gamma_i(g_i^u(0; x, y)) - R_i^+\left(\tau_i, \varphi(\tau_i), \ldots, \varphi^{(m-1)}(\tau_i), \hat{z}^{\mp}(\tau_i, \varphi(\tau_i))\right)\right| \leqslant \\
&\leqslant \left|R_i^+\left(\tau_i, \varphi(\tau_i), \ldots, \varphi^{(m-1)}(\tau_i), \hat{z}^{\mp}(\tau_i, \varphi(\tau_i))\right) - \right.\\
&\left. - R_i^+\left(0, \varphi(0), \ldots, \varphi^{(m-1)}(0), \hat{z}^{\mp}(0, \varphi(0))\right)\right| + \left|\gamma_i(g_i^u(0; x, y)) - \gamma_i(\beta_0 + 0)\right| \leqslant \\
&\leqslant \Gamma |g_i^u(0; x, y) - \beta_0| + R|\tau_i| .
\end{aligned}$$

Combining the estimates above with (21) yields the assertion.

LEMMA 15. – *Let Assumptions* H_1-H_5 *hold. Then for all* (x, y) *in* $\bar{I}_{\gamma_i}^{-u} \cap \bar{I}_{S_1 i}^{v}$ *or* $\bar{I}_{\gamma_i}^{+u} \cap \bar{I}_{S_2 i}^{v}$ *or* $\bar{I}_{S_1 i}^{u} \cap \bar{I}_{\gamma_i}^{-v}$ *or* $\bar{I}_{S_2 i}^{u} \cap \bar{I}_{\gamma_i}^{+v}$, *and* $u, v \in B(a, P, Q, \varrho)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant q_5 \|u - v\| ,$$

where $q_5 = L_1 L_2[\Gamma + (r_1 + r_2(P + Qs) + F)\Lambda_0^{-1} + K] + \tilde{K}$, *so that* $q_5 \to 0$ *as* $a \to 0$.

The proof of this lemma is similar to that of Lemma 14 (cf. also Lemma 12 of [14]).

REMARK 3. – From the assumption $2\Lambda a < b_0$ it follows that

$$\bar{I}^u_{S_1 i} \cap \bar{I}^{-v}_{\varphi_i} = \bar{I}^{-u}_{\varphi_i} \cap \bar{I}^v_{S_1 i} = \bar{I}^{+u}_{\varphi_i} \cap \bar{I}^v_{S_2 i} = \bar{I}^u_{S_2 i} \cap \bar{I}^{+v}_{\varphi_i} = \emptyset$$

(« the characteristics issuing from $(0, \alpha_1)$, $(0, \beta_0)$ and $(0, \alpha_2)$ do not intersect »). Thus, the cases considered above cover all sets I^-_a and I^+_a. Lemmas 11-15 show that in I^-_a and I^+_a we have

$$\|Su - Sv\| \leqslant q_6 \|u - v\| \,, \tag{22}$$

where $q_6 = q_5 + L_1 L_2(r_1 + r_2(P + Qs))\Lambda_0^{-1} + M\left(\bar{r} + \sum_{j=0}^{m-1} \bar{r}_j\right) + r_2$, so that

$$q_6 \to r_2 \quad \text{as} \quad a \to 0^+ \,.$$

Now to impose $r_2 < 1$ is too restrictive (for instance, in [16] one has $r_2 = 1$). Thus, in general, the operator S is *not* a contraction. However, under suitable assumptions, the operator S^2 is, as will be shown in the next Section.

5. – Properties of the operator S^2.

We begin with the following

LEMMA 16. – *Let Assumptions* H_1-H_5 *hold. Then, for* $a \in (0, a_0]$ *sufficiently small and* P, Q *sufficiently large, the operator* S^2 *maps* $B(a, P, Q, \varrho)$ *into itself, and likewise* $B(a, P^S, Q^S, \varrho)$ *into itself.*

PROOF. – Applying Lemma 10 to the function $Sz \in B(a, P^S, Q^S, \varrho)$ we get

$$|(S^2 z)(x, y) - \gamma(y)|_n \leqslant \varrho \,, \qquad |(S^2 z)(x, y)|_n \leqslant \Omega \,, \qquad (S^2 z)(0, y) = \gamma(y) \,,$$

provided $a \leqslant \varrho[r_1 + r_2(P^S + Q^S s) + R^S + \Gamma\Lambda + F]^{-1}$, where R^S is defined by R with P and Q replaced by P^S and Q^S, respectively.

From Lemmas 5-9 it follows that the function $S^2 z$ satisfies in I^-_a and I^+_a a Lipschitz condition with respect to both variables with constants P^{SS} and Q^{SS}, respectively. But now the arguments of the operator S are not arbitrary functions of $B(a, P^S, Q^S, \varrho)$, but functions in the range of S. Therefore, and by exploiting the postulated form of equations (3), (4), the Lipschitz constants of the function $S^2 z$ can be made more precise.

Indeed, for any two points $(x, y)(x, \bar{y}) \in \bar{I}^{-z}_{\gamma_i}$ (or $\bar{I}^{+z}_{\gamma_i}$), by Lemma 5 we have

$$|(S^2 z)_i(x, y) - (S^2 z)_i(x, \bar{y})| \leq L^s_1(\Gamma + K^s)|y - \bar{y}| ,$$

where L^s_1 and K^s are defined by L_1 and K, with P and Q replaced by P^s and Q^s respectively.

Let now $(x, y), (x, \bar{y}) \in \bar{I}^z_{S_1 i}$ (and similarly for $\bar{I}^z_{S_2 i}$), then $i \in \mathfrak{I}^1$, therefore for $j \in \mathfrak{I}\setminus\mathfrak{I}^1$ the point $(x, S_1(x))$ belongs to the set $\bar{I}^{-z}_{\gamma_j}$ (« any point on boundary lines can be reached by a characteristic starting from the initial line »). Hence, by Lemmas 5 and 6, we obtain

$$|(Sz)_j(x, S_1(x)) - (Sz)_j(\bar{x}, S_1(\bar{x}))| \leq (P^s_\gamma + Q^s_\gamma s)|x - \bar{x}| , \qquad j \in \mathfrak{I}\setminus\mathfrak{I}^1 .$$

Hence, for $(x, y), (x, \bar{y}) \in \bar{I}^{Sz}_{S_1 i}$, $\tau_i = \tau_i(x, y, Sz)$, $\bar{\tau}_i = \tau_i(x, \bar{y}, Sz)$, by Lemmas 2, 5, 6 we have:

$$|(TSz)_i(x, y) - (TSz)_i(x, \bar{y})| \leq r_1|\tau_i - \bar{\tau}_i| + r_2 \max_j |(Sz)_j(\tau_i, S_1(\tau_i)) - (Sz)_j(\bar{\tau}_i, S_1(\bar{\tau}_i))| \leq$$
$$\leq \Lambda_0^{-1} L^s_1[r_1 + r_2(P^s_\gamma + Q^s_\gamma s)]\,|y - \bar{y}| \qquad (j \in \mathfrak{I}\setminus\mathfrak{I}^1) .$$

Let now $(x, y), (x, \bar{y}) \in \bar{I}^{-z}_{\varphi_i}$ (or $\bar{I}^{+z}_{\varphi_i}$), then $i \in \mathfrak{I}^-$, therefore for $j \in \mathfrak{I}\setminus\mathfrak{I}^-$ the point $(x, \varphi(x))$ belongs to the set $\bar{I}^{-z}_{\gamma_j}$ (« any point on the free boundary can be reached by a characteristic starting from the initial line »). Then, similarly as above, we find

$$|(TSz)_i(x, y) - (TSz)_i(x, \bar{y})| \leq \Lambda_0^{-1} L^s_1 \tilde{R}|y - \bar{y}| ,$$

where $\tilde{R} = r + \Phi \sum_{j=0}^{m-1} \bar{r}_j + [L_1(\Gamma + K)(\Lambda + \Phi) + F]\bar{r}$. Furthermore, by Lemma 8, we get

$$|USz)_i(x, y) - (USz)_i(x, \bar{y})| \leq L^s_1(K^s + F\Lambda_0^{-1})|y - \bar{y}| .$$

Combining the estimates above, we find

$$|(S^2 z)_i(x, y) - (S^2 z)_i(x, \bar{y})| \leq Q^{ss}\,|y - \bar{y}| ,$$

where

$$Q^{ss} = L^s_1\Big\{\max\,[\Gamma, \Lambda_0^{-1}\{\max\,[r_1 + r_2(P^s_\gamma + sQ^s_\gamma), \tilde{R}] + F\}] + K^s\Big\} .$$

Consequently, in virtue of Lemma 9, we conclude that the function $S^2 z$ satisfies in I^-_a and I^+_a a Lipschitz condition in x with the constant

$$P^{ss} = \Lambda Q^{ss} + F ,$$

so that (assuming, without loss of generality, $\Lambda \geqslant 1$) we can take P^{ss} as a Lipschitz constant of $S^2 z$ with respect to both variables.

Thus, in order to show that the operator S maps $B(a, P, Q, \varrho)$ into $B(a, P^s, Q^s, \varrho)$, and S^2 maps $B(a, P, Q, \varrho)$ into itself, we need the following restrictions on the constants $\varrho \in (0, \Omega - \omega]$, $P \geqslant 0$, $Q \geqslant \Gamma$, $a \in (0, a_0]$:

$$\text{(23)} \qquad \begin{cases} [r_1 + r_2(P + Qs) + R + \Gamma\Lambda + F]a \leqslant \varrho\,, & \Lambda a \leqslant \varepsilon_0\,,\ 2\Lambda a < b_0\,, \\ [r_1 + r_2(P^s + Q^s s) + R^s + \Gamma\Lambda + F]a \leqslant \varrho\,, & P^{ss} \leqslant P\,,\ Q^{ss} \leqslant Q\,. \end{cases}$$

Now if ϱ, P, Q are fixed, and $a \to 0^+$, we have

$$P^s \to \Lambda \hat{W}(P, Q) + F\,, \quad Q^s \to \hat{W}(P, Q)\,, \quad P^{ss} \to \Lambda Z + F\,, \quad Q^{ss} \to Z\,,$$

where $\hat{W}(P, Q)$ is the greatest of Q^s_γ, Q^s_φ, Q^s_0 for $a = 0$ (Remark 1), and

$$Z = \max\Big[\Gamma, \Lambda_0^{-1}\Big\{\max\Big[r_1 + r_2(\Gamma(\Lambda + s) + F) + F,\; r + \Phi\sum_{j=0}^{m-1}\bar{r}_j + \{\Gamma(\Lambda + \Phi) + F\}\bar{r}\Big] + F\Big\}\Big]$$

does not depend on P, Q. Therefore, for arbitrary $\varrho \in (0, \Omega - \omega]$, if

$$P > \Lambda Z + F\,, \quad Q > Z\,,$$

then, for sufficiently small a in $(0, a_0]$, all inequalities (23) are satisfied. Thus

$$\text{(24)} \qquad S^2\colon B(a, P, Q, \varrho) \to B(a, P, Q, \varrho)\,.$$

But, since $Z \leqslant \hat{W}(P^s, Q^s)$, and $\hat{W}(P, Q) \geqslant \hat{W}(P^{ss}, Q^{ss})$ (as can easily be verified, using (23)), we see that $P^{sss} \leqslant P^s$, $Q^{sss} \leqslant Q^s$, so that also

$$\text{(24')} \qquad S^2\colon B(a, P^s, Q^s, \varrho) \to B(a, P^s, Q^s, \varrho)$$

and the lemma is proved.

LEMMA 17. – *If Assumptions* H_1-H_5 *and* (23) *are satisfied, then, for all* u *and* v *in* $B(a, P, Q, \varrho)$ *or in* $B(a, P^s, Q^s, \varrho)$, *we have*

$$|(S^2 u)(x, y) - (S^2 v)(x, y)|_n \leqslant q^s\,\|u - v\|\,,$$

where the coefficient $q^s \to 0^+$ *as* $a \to 0^+$.

PROOF. – Let $(x, y) \in \bar{I}^{-Su}_{\gamma_i} \cap \bar{I}^{-Sv}_{\gamma_i}$ (similarly for $\bar{I}^{+Su}_{\gamma_i} \cap \bar{I}^{+Sv}_{\gamma_i}$), then by using Lemma 11 we obtain, for $u, v \in B(a, P, Q, \varrho)$:

$$|(S^2u)(x, y) - (S^2v)(x, y)|_n \leqslant \hat{q}^S \, \|Su - Sv\| , \tag{25}$$

with $\hat{q}^S = L_1^S L_2(\Gamma + K^S) + \tilde{K} =: q_1^S$.

Next, let $(x, y) \in \bar{I}^{-Su}_{\gamma_i} \cap \bar{I}^{-Sv}_{S_1 i}$ (similarly for $\bar{I}^{+Su}_{\gamma_i} \cap \bar{I}^{Sv}_{S_2 i}$, $\bar{I}^{Su}_{S_2 i} \cap \bar{I}^{+Sv}_{\gamma_i}$, $\bar{I}^{-Su}_{S_1 i} \cap \bar{I}^{-Sv}_{\gamma_i}$), then the assumptions of Lemma 15 are satisfied, and we have that inequality (25) is satisfied, with $\hat{q}^S = q_5^S$, where

$$q_5^S = L_1^S L_2[\Gamma + (r_1 + r_2(P^S + Q^S s) + F)\Lambda_0^{-1} + K^S] + \tilde{K} .$$

Let now $(x, y) \in \bar{I}^{+Su}_{\gamma_i} \cap \bar{I}^{+Sv}_{\varphi_i}$ (similarly for $\bar{I}^{-Su}_{\gamma_i} \cap \bar{I}^{-Sv}_{\varphi_i}$, $\bar{I}^{-Su}_{\varphi_i} \cap \bar{I}^{-Sv}_{\gamma_i}$, $\bar{I}^{+Su}_{\varphi_i} \cap \bar{I}^{+Sv}_{\gamma_i}$), then the assumptions of Lemma 14 are satisfied, and (25) follows, with

$$\hat{q}^S = q_4^S := L_1^S L_2[\Gamma + (R^S + F)\Lambda_0^{-1} + K^S] + \tilde{K} .$$

Further, let $(x, y) \in \bar{I}^{-Su}_{\varphi_i} \cap \bar{I}^{-Sv}_{\varphi_i}$ (similarly for $\bar{I}^{+Su}_{\varphi_i} \cap \bar{I}^{+Sv}_{\varphi_i}$), then $i \in \mathfrak{J}^-$. Thus, the point $(x, \varphi(x))$ belongs to the set $\bar{I}^{u}_{\gamma_j} \cap \bar{I}^{v}_{\gamma_j}$ for $j \in \mathfrak{J} \setminus \mathfrak{J}^-$. Consequently, by Lemma 11, we obtain

$$|(Su)_j(x, \varphi(x)) - (Sv)_j(x, \varphi(x))| \leqslant q_1 \|u - v\| .$$

Hence

$$\begin{aligned} |(TSu)_i(x, y) - (TSv)_i(x, y)| &\leqslant r|\tau_{Su} - \tau_{Sv}| + \\ &+ \sum_{k=0}^{m-1} \bar{r}_k |\varphi^{(k)}_{Su}(\tau_{Su}) - \varphi^{(k)}_{Sv}(\tau_{Sv})| + \bar{r} \max_j |(Su)_j(\tau_{Su}, \varphi_{Su}(\tau_{Su})) - (Sv)_j(\tau_{Sv}, \varphi_{Sv}(\tau_{Sv}))| \leqslant \\ &\leqslant \Big[\Lambda_0^{-1} L_1^S L_2 R^S + M\Big\{\sum_{k=0}^{m-1} \bar{r}_k + L_1(\Gamma + K)\bar{r}\Big\}\Big] \|Su - Sv\| + \bar{r} q_1 \|u - v\| , \end{aligned}$$

where $\tau_{Su} = \tau_i(x, y, Su)$, $\tau_{Sv} = \tau_i(x, y, Sv)$, and $j \in \mathfrak{J} \setminus \mathfrak{J}^-$.

In a similar way we can show that for $(x, y) \in \bar{I}^{Su}_{S_1 i} \cap \bar{I}^{Sv}_{S_1 i}$ (and analogously for $\bar{I}^{Su}_{S_2 i} \cap \bar{I}^{Sv}_{S_2 i}$), we have

$$\begin{aligned} &|(TSu)_i(x, y) - (TSv)_i(x, y)| \leqslant \\ &\qquad \leqslant \Lambda_0^{-1} L_1^S L_2\{r_1 + r_2[(\Gamma + K) L_1(\Lambda + s) + F]\} \|Su - Sv\| + r_2 q_1 \|u - v\| . \end{aligned}$$

Assuming, for definiteness, $\tau_i(x, y, Su) \leqslant \tau_i(x, y, Sv)$, we obtain (cf. (21)):

$$|(USu)_i(x, y) - (USv)_i(x, y)| \leqslant [L_1^S L_2(K^S + F\Lambda_0^{-1}) + \tilde{K}] \, \|Su - Sv\| .$$

Combining the above estimates we find in I_a^- and I_a^+ (since $\bar{I}_{S_1 i}^{Su} \cap \bar{I}_{\varphi_i}^{-Sv} = \bar{I}_{\varphi_i}^{-Su} \cap \cap \bar{I}_{S_1 i}^{Sv} = \emptyset$):

$$|(S^2u)(x, y) - (S^2v)(x, y)|_n \leqslant q_7^S \|Su - Sv\| + (\bar{r} + r_2) q_1 \|u - v\| ,$$

where

$$q_7^S = L_1^S L_2 \Big\{\Gamma + [r_1 + r_2(P^S + Q^S s + (\Gamma + K) L_1(\Lambda + s) + F) + F + R^S] \Lambda_0^{-1} + K^S\Big\} + \\ + M\Big[\sum_{k=0}^{m-1} \bar{r}_k + L_1(\Gamma + K)\bar{r}\Big] + \tilde{K} .$$

Finally, by (22), we obtain

$$|(S^2u)(x, y) - (S^2v)(x, y)|_n \leqslant q^S \|u - v\| ,$$

where $q^S = q_7^S q_6 + (\bar{r} + r_2) q_1$, so that $q^S \to 0^+$ as $a \to 0^+$.

A similar reasoning shows that for $u, v \in B(a, P^S, Q^S, \varrho)$ the same conclusion holds, with constant q^S obtained from the previous expression by replacing P^S, Q^S with P^{SS}, Q^{SS}, so that again $q^S \to 0$ as $a \to 0$. Thus S^2 is a contraction in both $B(a, \tilde{P}, \tilde{Q}, \varrho)$ and its subspace $B(a, \bar{P}, \bar{Q}, \varrho)$, with

$$\bar{P} = \min[P, P^S], \quad \bar{Q} = \min[Q, Q^S], \quad \tilde{P} = \max[P, P^S], \quad \tilde{Q} = \max[Q, Q^S] .$$

6. – The existence theorem.

THEOREM. – *Let Assumptions* H_1-H_5 *hold. Then, for given* $\Omega > 0$, *any* $\varrho \in (0, \Omega - \omega]$ *and any sufficiently large constants* P, Q, *there are a number* a $(0 < a \leqslant a_0)$ *and functions* $\bar{z}: I_a \to \boldsymbol{R}^n$, $\bar{z} \in B(a, \bar{P}, \bar{Q}, \varrho)$, *and* $\bar{\varphi}$, $\bar{\varphi} \in C_L^{m-1}[0, a]$, *which satisfy* (1), (5), *a.e. in* I_a^+ *and* I_a^-, *and* $[0, a]$, *respectively, as well as conditions* (2)-(4), (6). *Furthermore,* $\bar{z}$ *is unique in* $B(a, P, Q, \varrho)$.

PROOF. – Let us choose P, Q and a such that inequalities (23) are satisfied. Then, by Lemma 16, we see that

$$S\colon B(a, \bar{P}, \bar{Q}, \varrho) \to B(a, \tilde{P}, \tilde{Q}, \varrho) , \qquad S^2\colon B(a, \tilde{P}, \tilde{Q}, \varrho) \to B(a, \tilde{P}, \tilde{Q}, \varrho) ,$$

$$S\colon B(a, P, Q, \varrho) \to B(a, \tilde{P}, \tilde{Q}, \varrho) , \qquad S^2\colon B(a, \bar{P}, \bar{Q}, \varrho) \to B(a, \bar{P}, \bar{Q}, \varrho) ,$$

where $\tilde{P}$, $\tilde{Q}$, $\bar{P}$, $\bar{Q}$ are defined at the end of previous Section. Let us take $a \in (0, a_0]$ such that $q^S < 1$. Then, by Lemma 17, S^2 is a contraction in $B(a, \tilde{P}, \tilde{Q}, \varrho)$ and in its subspace $B(a, \bar{P}, \bar{Q}, \varrho)$. Since both are complete, there exists a function $\bar{z}$ in

$B(a, \bar{P}, \bar{Q}, \varrho)$ s.t.

$$S^2\bar{z} = \bar{z}\,, \qquad \bar{z} \in B(a, \bar{P}, \bar{Q}, \varrho) \subset B(a, \tilde{P}, \tilde{Q}, \varrho)$$

and this is the unique fixed point of S^2 in $B(a, \tilde{P}, \tilde{Q}, \varrho)$. Then, from $S^2S\bar{z} = S\bar{z}$, $S\bar{z} \in B(a, \tilde{P}, \tilde{Q}, \varrho)$, we conclude that

$$S\bar{z} = \bar{z}\,, \qquad \bar{z} \in B(a, \bar{P}, \bar{Q}, \varrho) \subseteq B(a, P, Q, \varrho)$$

and $\bar{z}$ is the unique fixed point of S in $B(a, P, Q, \varrho)$ (cf. [10], p. 83). Proceeding as in [14] we can prove, using the groupal property of characteristic lines and the Chain Rule Differentiation Lemma of [4], that $\bar{z}$ satisfies (1) a.e. in I_a^+ and I_a^-, and (2)-(4) everywhere in $[\alpha_1, \alpha_2]$ and $[0, a]$, respectively. Finally if $(\bar{\varphi}_0, \ldots, \bar{\varphi}_{m-1})$ is the solution of (14) with $z = \bar{z}$, then $\bar{\varphi} = \bar{\varphi}_0$ yields the desired free boundary. This concludes the proof.

REMARK 4. – The case when the initial condition (2) is given on an interval (as it happens when V involves retarded arguments):

$$(2') \qquad z(x, y) = \gamma(x, y)\,, \qquad (x, y) \in [-\delta, 0] \times [\alpha_1, \alpha_2]\,, \qquad \delta > 0$$

can also be studied analogously.

REMARK 5. – The solution and the free boundary depend continuously on the initial data. In fact, keeping for simplicity the β_k's fixed ($k = 0, \ldots, m-1$), we find

$$\|\bar{z}[\gamma] - \bar{z}[\tilde{\gamma}]\| \leqslant (1 - q^S)^{-1} \|\tilde{\gamma} - \gamma\|\,,$$

and continuous dependence for the free boundary follows from Lemma 3.

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