Annali di Matematica pura ed applicata
(IV), Vol. CLVI (1990), pp. 159-180

p-Harmonic Obstacle Problems (*).

PART III. - Boundary Regularity.

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Summary. - *Let Ω denote a bounded domain in some Riemannian manifold X with smooth boundary $\partial\Omega$ and consider a submanifold Y of Euclidean space $\boldsymbol{R}^L$ with or without boundary. We show that if $U: \Omega \to Y$ minimizes the p-energy functional*

$$E_p(U, \Omega) := \int_\Omega \|DU\|^p \, d\,\mathrm{Vol}$$

for smooth boundary data $g: \partial\Omega \to Y$, then U is continuous in a neighborhood of $\partial\Omega$. This completes the interior partial regularity results of Part I. As an application we obtain an existence theorem concerning small solutions of the Dirichlet problem for p-harmonic maps.

0. - Introduction.

Given two Riemannian manifolds X (with boundary ∂X) and Y (embedded in some Euclidean space $\boldsymbol{R}^L$) of dimensions n and N respectively and a smooth bounded domain $M \subset Y$ such that $\mathrm{dist}(\bar{M}, \partial Y) > 0$ we study the question of boundary regularity for the associated p-harmonic obstacle problem: suppose that $U: X \to Y$ minimizes the p-energy functional

$$E_p(U, \Omega) := \int_\Omega \|DU\|^p \, d\,\mathrm{Vol}\,, \tag{0.1}$$

$\Omega := \mathrm{Int}(X)$, for smooth boundary values $g: \partial X \to \bar{M}$ and under the additional side condition $\mathrm{Im}(\cdot) \subset \bar{M}$ for the class of comparison functions. Here $p \geqslant 2$ denotes a given real number. In [F1, 2] we showed

$$\begin{cases} \boldsymbol{H}\text{-dim}(\mathrm{Sing}\,U) \leqslant n - [p] - 1\,, & n > p + 1\,; \\ \mathrm{Sing}(U) \text{ is discrete if } n - 1 \leqslant p < n \end{cases}$$

Entrata in Redazione il 24 maggio 1988.
Indirizzo dell'A.: Mathematisches Institut der Universität Düsseldorf, Universitätsstr. 1, D-4000 Düsseldorf.

for the set Sing(U) of interior singularities, here we want to prove that a minimizer U is continuous in a neighborhood of ∂X if the boundary map $g: \partial X \to \bar{M}$ is sufficiently regular by the way extending earlier results of SCHOEN-UHLENBECK [SU] and HARDT-LIN [HL] for unconstrained Riemannian problems.

As a corollary we obtain: if $U: X \to \bar{M}$ minimizes (0.1) for prescribed smooth boundary data $g: \partial X \to \bar{M}$ and if $n-1 \leqslant p < n$, then the number of singular points is finite and there are only interior singularities.

The proof of the boundary regularity theorem is carried out in two steps: we start with a partial boundary result saying that U is of class $C^{0,\alpha}$ in a one-sided neighborhood of $x \in \partial X$ iff

$$\liminf_{r \downarrow 0} r^{p-n} E_p(U, \Omega \cap B_r(x)) = 0 \tag{0.2}$$

for a sequence of «half-balls» $B_r(x) \cap \Omega$. In a second step we analyze the behaviour of boundary tangent maps by the way showing that the regularity criterion (0.2) is satisfied at every boundary point x.

1. – Notations and statement of the result.

In the *Riemannian case* let Ω denote a bounded open subset of a n-dimensional Riemannian manifold X with boundary $\partial\Omega$ of class C^2. Y is a N-dimensional submanifold of Euclidean space $\boldsymbol{R}^L$ containing a bounded open region M with the following properties: $\partial\Omega$ is of class C^3 and the closure of M is compactly contained in Int(Y). For $p \geqslant 2$ we introduce the restricted Sobolev class

$$H^{1,p}(\Omega, \bar{M}) = \{u \in H^{1,p}(\Omega, \boldsymbol{R}^L): \ u(x) \in \bar{M} \text{ almost everywhere}\}$$

and define the p-energy of $u \in H^{1,p}(\Omega, \boldsymbol{R}^L)$ as

$$E_p(u, \Omega) := \int_\Omega \|Du\|^p \, d\boldsymbol{H}^n , \tag{1.1}$$

$\boldsymbol{H}^n$ being the n-dimensional Hausdorff-measure on X.

In the *Euclidean case* Ω is a bounded subdomain of $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and M denotes a smooth bounded subregion of Euclidean space $\boldsymbol{R}^L$. The p-energy functional (1.1) is replaced by the splitting functional

$$F_p(u, \Omega) := \int_\Omega (a_{\alpha\beta} B^{ij}(\cdot, u) D_\alpha u^i D_\beta u^j)^{p/2} dx$$

with smooth elliptic and symmetric coefficients

$$a_{\alpha\beta}: \bar{\Omega} \to \boldsymbol{R} , \quad B^{ij}: \bar{\Omega} \times \boldsymbol{R}^L \to \boldsymbol{R} ,$$

and the definition of the restricted Sobolev space $H^{1,p}(\Omega, \overline{M})$ is as above.

Now we state our

MAIN THEOREM. – *Suppose that $u \in H^{1,p}(\Omega, \overline{M})$ has the property*

$$E_p(u, \Omega) \leqslant E_p(v, \Omega)$$

$$(\text{or} \quad F_p(u, \Omega) \leqslant F_p(v, \Omega))$$

for all $v \in H^{1,p}(\Omega, \overline{M})$ such that $u - v \in \mathring{H}^{1,p}(\Omega, \boldsymbol{R}^L)$. Moreover, assume that the boundary values of u are given by a function $g \in C^2(\partial\Omega, \overline{M})$. Then u is continuous in a neighborhood of $\partial\Omega$ and the singular set of u is compactly contained in the interior of Ω.

We wish to remark that our theorem is valid for *unconstrained Riemannian problems,* that is in the case of p-energy minimizing mappings $\Omega \to Z$ with range in a compact submanifold Z of $\boldsymbol{R}^L$ without boundary.

In order to avoid technical complications we concentrate on a special case of the theorem from which it is not too hard to derive the general statement, the interested reader is referred to the papers [DF] and [SU] where one finds some comments concerning the general case.

From now on we fix the following notations: for $x \in \boldsymbol{R}^n$ and $r, \Lambda > 0$ we let

$$B_r^+(x) := \{z \in \boldsymbol{R}^n\colon |x - z| < r,\ z_n > 0\} = B_r(x) \cap \boldsymbol{R}_+^n\,,$$

$$\Gamma := \{z \in \boldsymbol{R}^n\colon z_n = 0\}\,, \qquad \Gamma_r := \Gamma \cap B_r(0)\,,$$

$$\mathcal{B}_\Lambda := \{g \in C^{0,1}(B_1(0), \boldsymbol{R}^L)\colon \operatorname{Im}(g) \subset \overline{M},\ \operatorname{Lip}(g) \leqslant \Lambda\}\,.$$

The class $\mathcal{B}_\Lambda$ represents the admissible boundary functions g. We say that $u \in H^{1,p}(B_1(0), \overline{M})$ is a minimizer of p-energy with Γ-boundary values $g \in \mathcal{B}_\Lambda$ iff u has trace g on Γ and

$$E_p(u, B_r^+(0)) := \int_{B_r^+(0)} |Du|^p\, dx \leqslant E_p(v, B_r^+(0))$$

for all $v \in H^{1,p}(B_1^+(0), \overline{M})$ such that $v = g$ on Γ and $u = v$ on $B_1^+(0) \setminus B_r^+(0)$, $0 < r < 1$. Let $\mathcal{M}_\Lambda$ denote the class of all p-energy minimizers with Γ-boundary values in $\mathcal{B}_\Lambda$.

If u is in $\mathcal{M}_\Lambda$ with boundary function $g \in \mathcal{B}_\Lambda$ we define the odd extension

$$\tilde{u}(x) := \begin{cases} u(x) - g(x) & \text{if } x_n \geqslant 0 \\ -\big(u(x', -x_n) - g(x', -x_n)\big) & \text{if } x_n \leqslant 0 \end{cases}$$

for points $x = (x', x_n) \in B_1(0)$ and abbreviate

$$\operatorname{Reg}(\tilde{u}) := \{x \in B_1(0) \colon \tilde{u} \text{ is Hölder continuous in a neighborhood of } x\},$$

$$\operatorname{Sing}(\tilde{u}) := B_1(0) \setminus \operatorname{Reg}(\tilde{u}).$$

Then we have the following

THEOREM 1. – *Suppose that $u \in \mathcal{M}_\Lambda$ for some $\Lambda > 0$. Then*

$$\operatorname{Reg}(\tilde{u}) = \left\{x \in B_1(0) \colon \liminf_{r \downarrow 0} r^{p-n} \int_{B_r(x)} |D\tilde{u}|^p \, dz = 0\right\},$$

especially

$$\boldsymbol{H}^{n-p}(\operatorname{Sing}(\tilde{u})) = 0$$

and

$$\operatorname{Sing}(\tilde{u}) = \emptyset \quad \text{for } p \geq n.$$

In order to prove Theorem 1 we can concentrate on minimizers $u \in \mathcal{M}_\Lambda$ with Λ and

$$E_p(u, B_1^+(0)) = \int_{B_1^+(0)} |Du|^p \, dx$$

sufficiently small, more precisely it suffices to verify

THEOREM 2. – *There are constants $\alpha, \varepsilon \in (0, 1)$, $t > p$, $c_1, c_2 > 0$ depending only on dimensions, on p and on geometric data of M with the following property: suppose that $u \in \mathcal{M}_\Lambda$ satisfies*

$$\Lambda + E_p(u, B_1^+(0)) < \varepsilon^p.$$

Then:

a) $D\tilde{u} \in L^t(B_{1/2}(0))$ *and*

$$\left(\fint_{B_{r/2}(x)} |D\tilde{u}|^t \, dz\right)^{1/t} \leq c_1 \left(\fint_{B_r(x)} [1 + |D\tilde{u}|^p] \, dz\right)^{1/p}$$

for all $x \in B_{1/2}(0)$, $r < \frac{1}{2}$.

b) $\tilde{u} \in C^{0,\alpha}(B_{1/2}(0))$ *and*

$$[\tilde{u}]_{C^{0,\alpha}(B_{1/2}(0))} \leq c_2.$$

Clearly, Theorem 1 is a simple consequence of Theorem 2: if $x \in \Gamma$ is chosen to satisfy

$$\liminf_{r \downarrow 0} r^{p-n} \int_{B_r(x)} |D\tilde{u}|^p \, dz = 0 ,$$

then the scaled function

$$u_\lambda(z) := u(x + \lambda z) , \qquad z \in B_1^+(0) ,$$

is in $\mathcal{M}_{\lambda\Lambda}$ and satisfies the hypothesis of Theorem 2 for $\lambda \ll 1$ which gives continuity of $\tilde{u}$ near x.

The major part of our paper (section 3) is devoted to the proof of Theorem 2. In order to obtain complete boundary regularity we first of all have to replace the regularity criterion of Theorem 2 by a condition involving the smallness of the mean oscillation of the minimizer (see Lemma 4.1). This new regularity criterion is stable under blow ups and implies the existence of well-behaved tangent maps. At this stage the proof of the Main Theorem can be completed along standard lines: regularity at the boundary follows from the fact that the constants are the only tangent maps which was proved by HARDT-LIN [HL] who formulated their result for unconstrained Riemannian problems but it is easy to check that the presence of the obstacle does not change the argument.

2. – Background material.

Most of our results are based on the following

LEMMA 2.1 (extension theorem for half balls). – There are constants ε_1, δ, $\gamma \in (0, 1)$, q, $\tilde{q}$, $C_1 > 0$ depending on dimensions, on p and the geometry of M with the following property:

Let $\varphi \in H^{1,p}(\partial B_r^+(x), \bar{M})$, $x \in \Gamma$, and $a \in \mathbf{R}^L$ be given such that

$$E_p(\varphi, \partial B_r^+(x)) \cdot W_p(\varphi, \partial B_r^+(x))^\gamma \leqslant \varepsilon^q \delta^{1+\gamma} \cdot r^{(1+\gamma)(n-1)-p}$$

for some $0 < \varepsilon \leqslant \varepsilon_1$. Then we find $\Phi \in H^{1,p}(B_r^+(x), \bar{M})$ with boundary values φ and

$$E_p(\Phi, B_r^+(x)) \leqslant C_1 \{ r \cdot \varepsilon \cdot E_p(\varphi, \partial B_r^+(x)) + \varepsilon^{-\tilde{q}} \cdot r^{1-p} \cdot W_p(\varphi, \partial B_r^+(x)) \} .$$

Here and in the sequel we use the notation

$$E_p(f, A) := \int_A |Df|^p , \qquad W_p(f, A) := \int_A |f - a|^p$$

for functions $f : A \to \mathbf{R}^L$.

A detailed proof of this extension theorem can be found in the paper [F3]; in order to get the above version for half balls $B_r^+(x)$, $x \in \Gamma$, one has to apply a suitable Bi-Lipschitz transformation.

LEMMA 2.2 (monotonicity formula). – There are constants $\varepsilon_2 \in (0, 1)$, $C_2 > 0$ depending on absolute data with the following property: suppose that $u \in \mathcal{M}_\Lambda$ with $\Lambda \leqslant \varepsilon_2$. Then for all $x \in B_{1/2}(0)$ and $0 < \sigma < \varrho < \frac{1}{2}$

$$\sigma^{p-n} E_p(\tilde{u}, B_\sigma(x)) \leqslant C_2 \{\varrho^{p-n} E_p(\tilde{u}, B_\varrho(x)) + \Lambda\varrho\},$$

provided $p < n$.

PROOF. – For balls $B_\varrho(x) \subset B_1^+(0)$ Lemma 2.2 is a consequence of [F2], Theorem 2.4. To obtain the general statement one may easily modify the proof of [SU], Lemma 1.3.

Finally we need a Campanato type estimate for p-harmonic systems on half balls with prescribed boundary data on Γ.

LEMMA 2.3. – Suppose that $g \in \mathcal{B}_\Lambda$. Moreover, let $v \in H^{1,p}(B_R^+(0), \mathbf{R}^L)$ be a solution of

$$D_\alpha(|Dv|^{p-2} D_\alpha v) = 0 \tag{2.1}$$

on $B_R^+(0)$ with $v = g$ on Γ_R. Then for all $\varepsilon < 1$ and $r \leqslant R$ the following estimate is valid

$$\int_{B_r^+(0)} |Dv|^p dz \leqslant c \cdot \left\{\left[\left(\frac{r}{R}\right)^n + \varepsilon\right] \int_{B_R^+(0)} |Dv|^p dz + \varepsilon^{1-p} \Lambda^p \cdot R^n\right\},$$

c denoting a constant depending on n, L and p.

PROOF. – Let $w \in H^{1,p}(B_R^+(0), \mathbf{R}^L)$ denote the solution of

$$\int_{B_R^+(0)} |Dw|^p dz \to \min$$

in the space $(v - g) + \mathring{H}^{1,p}(B_R^+(0), \mathbf{R}^L)$. Clearly w satisfies (2.1) on $B_R^+(0)$ with $w = 0$ on Γ, and it is not hard to check that the odd extension $\tilde{w}$ of w is a weak solution of (2.1) on the whole ball $B_R(0)$ which gives the estimate (compare [U] and [F2], Section 3)

$$\int_{B_r^+(0)} |Dw|^p dx \leqslant c \cdot \left(\frac{r}{R}\right)^n \int_{B_R^+(0)} |Dw|^p dx. \tag{2.2}$$

On the other hand we have

$$\int_{B_R^+(0)} |Dv - Dw|^p\,dx \leqslant c\cdot \int_{B_R^+(0)} \big(|Dv|^{p-2}Dv - |Dw|^{p-2}Dw\big)\cdot(Dv - Dw)\,dx =$$

$$= c\cdot \int_{B_R^+(0)} \big(|Dv|^{p-2}Dv - |Dw|^{p-2}Dw\big)\cdot Dg\,dx,$$

since $v - w$ has boundary values g. Therefore:

$$\int_{B_R^+(0)} |Dv - Dw|^p\,dx \leqslant c\cdot \Lambda \int_{B_R^+(0)} \big(|Dv|^{p-1} + |Dw|^{p-1}\big)\,dx \leqslant$$

$$\leqslant c\cdot\Big\{\varepsilon\cdot \int_{B_R^+(0)} \big(|Dv|^p + |Dw|^p\big)\,dx + \Lambda^p \varepsilon^{1-p} R^n\Big\}.$$

Since w minimizes p-energy for boundary values $v - g$ on $\partial B_R^+(0)$ we obtain

$$\int_{B_R^+(0)} |Dw|^p\,dx \leqslant c\cdot\Big\{\int_{B_R^+(0)} |Dv|^p\,dx + \int_{B_R^+(0)} |Dg|^p\,dx\Big\} \leqslant c\cdot\Big\{\int_{B_R^+(0)} |Dv|^p\,dx + \Lambda^p R^n\Big\},$$

hence

$$\int_{B_R^+(0)} |Dv - Dw|^p\,dx \leqslant c\cdot\Big\{\varepsilon\cdot \int_{B_R^+(0)} |Dv|^p\,dx + \Lambda^p \varepsilon^{1-p} R^n\Big\}. \tag{2.3}$$

Finally we combine (2.2), (2.3) to get

$$\int_{B_r^+(0)} |Dv|^p\,dx \leqslant c\cdot\Big\{\Big(\frac{r}{R}\Big)^n \int_{B_R^+(0)} |Dw|^p\,dx + \varepsilon\cdot \int_{B_R^+(0)} |Dv|^p\,dx + \Lambda^p \varepsilon^{1-p} R^n\Big\} \leqslant$$

$$\leqslant c\cdot\Big\{\Big[\Big(\frac{r}{R}\Big)^n + \varepsilon\Big] \int_{B_R^+(0)} |Dv|^p\,dx + \varepsilon^{1-p}\Lambda^p R^n\Big\}. \qquad \blacksquare$$

3. – Partial higher integrability and partial regularity near the boundary.

In this section we give a proof of Theorem 2. We start with a partial higher integrability result. So assume that $u \in \mathcal{M}_\Lambda$ is given with boundary function $g \in \mathcal{B}_\Lambda$ and that

$$\Lambda + E_p(u, B_1^+) < \varepsilon^p \tag{3.1}$$

holds for some $0 < \varepsilon < \varepsilon_2$ (defined in Lemma 2). Moreover, we fix $z \in B^+_{1/2}(0) \cup \Gamma$ and $0 < R < \frac{1}{2}$. In what follows we denote all constants depending on dimensions, on p and on geometric data with the symbols $c_1, c_2, \ldots$.

Case 1: $z_n > \frac{3}{4}R$ (i.e. $B_{(3/4)R}(z) \subset\subset B^+_1(0)$).

Choosing $a := \fint_{B_{(3/4)R}(z)} u\,dx$ we find a radius $r \in [R/2, 3R/4]$ such that

$$E_p(u, \partial B_r(z)) \leqslant c_1 \cdot R^{-1}\, E_p(u, B_{(3/4)R}(z)),$$

$$W_p(u, \partial B_r(z)) \leqslant c_1 \cdot R^{-1}\, W_p(u, B_{(3/4)R}(z)).$$

This gives on account of Poincaré's inequality

$$(3.2) \qquad E_p(u, \partial B_r(z))\, W_p(u, \partial B_r(z))^\gamma \leqslant c_2 \cdot r^{(1+\gamma)(n-1)-p} \{R^{p-n} E_p(u, B_{(3/4)R}(z))\}^{1+\gamma}.$$

From Lemma 2.2 we infer

$$R^{p-n} E_p(u, B_{(3/4)R}(z)) \leqslant c_3 \{R^{p-n} E_p(\tilde{u}, B_{(3/4)R}(z)) + R^p \Lambda^p\} \leqslant$$
$$\leqslant c_4 \{E_p(\tilde{u}, B_{1/2}(z)) + \Lambda\} \leqslant c_5 \{E_p(u, B^+_1(0)) + \Lambda\} \underset{(3.1)}{\leqslant} c_5\, \varepsilon^p$$

and (3.2) turns into

$$(3.3) \qquad \delta^{-\gamma-1} r^{-(1+\gamma)(n-1)+p} E_p(u, \partial B_r(z))\, W_p(u, \partial B_r(z))^\gamma \leqslant c_6\, \delta^{-\gamma-1} \varepsilon^{p\cdot(1+\gamma)} =: \tau^q.$$

Since the extension-lemma 2.1 is also valid for functions defined on spheres (compare [F1, 2, 3], Theorem E) we may apply it to the function

$$u\colon \partial B_r(z) \to \overline{M}$$

provided we assume $\tau \leqslant \varepsilon_1$. Then (3.3) is just the smallness condition required in Lemma 2.1 and we find an extension $\bar{u}\colon B_r(z) \to \overline{M}$ of the boundary function u satisfying the energy estimate of Lemma 2.1. A trivial calculation gives:

$$\fint_{B_{R/2}(z)} |D\tilde{u}|^p\,dx \leqslant c_7 \left[\tau \cdot \fint_{B_R(z)} |D\tilde{u}|^p\,dx + \tau^{-\bar{q}} R^{-n-p} \cdot \left(\int_{B_R(z)} |D\tilde{u}|^{np/(n+p)}\,dx\right)^{1+p/n} + \tau^{-\bar{q}} \cdot \Lambda^p\right]$$

where we have made use of the Sobolev-Poincaré inequality. We choose τ to satisfy $c_7 \cdot \tau \leqslant \frac{1}{2}$, i.e.

$$(3.4) \qquad \varepsilon \leqslant [c_6^{-1} \delta^{\gamma+1} 2^{-q} c_7^{-q}]^{1/(p\cdot(\gamma+1))}.$$

Then the last estimate reads

$$\fint_{B_{R/2}(z)}|D\tilde u|^p\,dx \le \frac{1}{2}\fint_{B_R(z)}|D\tilde u|^p\,dx + c_8\left\{\left(\fint_{B_R(z)}|D\tilde u|^{np/(n+p)}\,dx\right)^{1+n/p} + \Lambda^p\right\}. \tag{3.5}$$

Case 2: $(0\le)\, z_n \le \frac{3}{4}R$.

Let $a := \fint_{B_R^+(z)} u\,dx$ and choose $r\in[\frac{7}{8}R, R]$ such that

$$\begin{cases} E_p(u, \partial B_r^+(z)\setminus\Gamma) \le c_9 R^{-1} E_p(u, B_R^+(z)), \\ W_p(u, \partial B_r^+(z)\setminus\Gamma) \le c_9 R^{-1} W_p(u, B_R^+(z)). \end{cases} \tag{3.6}$$

We let $\varphi := u$ on $\partial B_r^+(z)\setminus\Gamma$, $\varphi := g$ on $\partial B_r^+(z)\cap\Gamma$. Then

$$E_p(\varphi, \partial B_r^+(z))\, W_p(\varphi, \partial B_r^+(z))^\gamma \le (3.6)$$

$$c_{10}\cdot\{R^{-1}E_p(u, B_R^+(z)) + \Lambda^p R^{n-1}\}\cdot\{R^{-1}W_p(u, B_R^+(z)) + W_p(g, \partial B_r^+(z)\cap\Gamma)\}^\gamma,$$

$$W_p(u, B_R^+(z)) \le c_{11}\left\{\int_{B_R^+(z)}\left|(u-g) - \fint_{B_R^+(z)}(u-g)\right|^p dx + \int_{B_R^+(z)}\left|g - \fint_{B_R^+(z)} g\right|^p dx\right\} \le$$

(by Poincaré's inequality applied to the function $w(x) = u(x) - g(x)$, if $x_n \ge 0$, $w(x) = 0$, if $x_n \le 0$, $x\in B_1(0)$)

$$c_{12}\left\{R^p\int_{B_R^+(z)}|D(u-g)|^p\,dx + R^{n+p}\Lambda^p\right\} \le c_{13}\{R^p E_p(u, B_R^+(z)) + R^{n+p}\Lambda^p\}.$$

Next we estimate $W_p(g, \partial B_r^+(z)\cap\Gamma)$:

$$W_p(g, \partial B_r^+(z)\cap\Gamma) \le c_{14}\left\{\int_{\partial B_r^+(z)\cap\Gamma}|g - (g)^+|^p\,d\boldsymbol{H}^{n-1} + R^{n-1}|(g)^+ - (u)^+|^p\right\} \le$$

$$\le c_{15}\{R^{n-1+p}\Lambda^p + R^{n-1}|(g)^+ - (u)^+|^p\},$$

where we have used the notation $(f)^+ := \fint_{B_R^+(z)} f\,dx$:

$$|(g)^+ - (u)^+|^p \le c_{16}\fint_{B_R^+(z)}|g-u|^p\,dx =$$

$$= c_{16}\fint_{B_R(z)}|w|^p\,dx \le \text{(Sobolev-Poincaré + Hölder inequality)}$$

$$c_{17}R^{p-n}\int_{B_R(z)}|Dw|^p\,dx \le c_{18}\{R^{p-n}E_p(u, B_R^+(z)) + \Lambda^p R^p\}.$$

Putting together our estimates we arrive at:

$$(3.7)\qquad E_p(\varphi,\partial B_r^+(z))\,W_p(\varphi,\partial B_r^+(z))^\gamma\leqslant c_{19}R^{-1}R^{(p-1)\gamma}\big[E(u,B_R^+(z))+\Lambda^pR^n\big]^{\gamma+1}\Rightarrow$$
$$\Rightarrow \delta^{-\gamma-1}r^{(1-n)(\gamma+1)+p}E_p(\varphi,\partial B_r^+(z))\,W_p(\varphi,\partial B_r^+(z))^\gamma\leqslant$$
$$\leqslant c_{20}\delta^{-\gamma-1}\big[R^{p-n}E_p(u,B_R^+(z))+\Lambda^pR^p\big]^{1+\gamma}.$$

The monotonicity formula implies

$$R^{p-n}E_p(u,B_R^+(z))\leqslant c_{21}\{R^{p-n}E_p(\tilde u,B_R(z))+\Lambda^pR^p\}\leqslant c_{22}\{E_p(\tilde u,B_1(0))+\Lambda\}\underset{(3.1)}{\leqslant} c_{23}\cdot\varepsilon^p$$

and (3.7) turns into the inequality:

$$(3.8)\qquad \delta^{-\gamma-1}r^{(1-n)(\gamma+1)+p}E_p(\varphi,\partial B_r^+(z))\,W_p(\varphi,\partial B_r^+(z))^\gamma\leqslant c_{24}\cdot\delta^{-(1+\gamma)}\varepsilon^{p(\gamma+1)}=:\tau^a.$$

$B_r^+(z)$ is not exactly a half-ball, but according to our choice of the radius r we may apply the extension-lemma 2.1 after a suitable Bi-Lipschitz transformation. Since the hypothesis of Lemma 2.1 is satisfied by (3.8) (provided $\tau\leqslant\varepsilon_1$) we find $\bar u\colon B_r^+(z)\to\bar M$ with boundary values φ such that

$$E_p(\bar u,B_r^+(z))\leqslant c_{25}\{\tau\cdot r\,E_p(\varphi,\partial B_r^+(z))+\tau^{-\bar q}r^{1-p}\,W_p(\varphi,\partial B_r^+(z))\}.$$

Since u minimizes for Γ-boundary values g we get

$$E_p(\tilde u,B_{R/2}(z))\leqslant c_{26}\{E_p(u,B_r^+(z))+\Lambda^p\cdot R^n\}\leqslant c_{26}\{E_p(\bar u,B_r^+(z))+\Lambda^pR^n\}\leqslant$$
$$\leqslant c_{27}\{\tau\cdot R\,E_p(\varphi,\partial B_r^\pm(z))+\tau^{-\bar q}R^{1-p}\,W_p(\varphi,\partial B_r^+(z))+\Lambda^pR^n\}\leqslant$$
$$\leqslant c_{28}\{\tau E_p(u,B_R^+(z))+\Lambda^p\cdot R^n+\tau^{-\bar q}R^{1-p}[R^{-1}W_p(u,B_R^+(z))+W_p(g,\partial B_r^+(z)\cap\Gamma)]\};$$

as before we estimate

$$W_p(u,B_R^+(z))\leqslant c_{29}\Big\{\int_{B_R^+(z)}|(u-g)-(u-g)^+|^p\,dx+R^{n+p}\Lambda^p\Big\}\leqslant$$
$$\leqslant c_{30}\Big\{\int_{B_R^+(z)}|u-g|^p\,dx+R^{n+p}\Lambda^p\Big\}\leqslant$$

(by the Sobolev-Poincaré inequality)

$$c_{31}\bigg\{\bigg(\int_{B_R^+(z)}|D(u-g)|^{np/(n+p)}dx\bigg)^{1+p/n}+R^{n+p}\Lambda^p\bigg\}\leqslant c_{32}\bigg\{\bigg(\int_{B_R(z)}|D\tilde u|^{np/(n+p)}dx\bigg)^{1+p/n}+R^{n+p}\Lambda^p\bigg\},$$

$$W_p(\varphi,\partial B_R^+(z)\cap\Gamma)\leqslant c_{33}\bigg\{R^{n-1+p}\Lambda^p+R^{-1}\int_{B_R(z)}|w|^p\,dx\bigg\}\leqslant$$
$$\leqslant c_{34}R^{-1}\bigg\{R^{n+p}\Lambda^p+\bigg(\int_{B_R(z)}|D\tilde u|^{np/(n+p)}dx\bigg)^{1+p/n}\bigg\}.$$

Let us remark that we have made use of the Sobolev-Poincaré inequality

$$\|w\|_{L^p(B_R(z))} \leqslant c_{35} \|Dw\|_{L^s(B_R(z))}, \qquad s = np/(n+p),$$

which is valid since

$$L^n(\{x \in B_R(z) | w(x) = 0\}) \geqslant c_{36} R^n.$$

Putting together our results we have shown

$$E_p(\tilde{u}, B_{R/2}(z)) \leqslant c_{37} \left\{ \tau E_p(u, B_R^+(z)) + \Lambda^p R^n \cdot \tau^{-\tilde{q}} + \tau^{-\tilde{q}} R^{-p} \left(\int_{B_R(z)} |D\tilde{u}|^{(p\cdot n)/(n+p)} dx \right)^{1+p/n} \right\} \leqslant$$

$$\leqslant c_{38} \left\{ \tau E_p(\tilde{u}, B_R(z)) + \tau^{-\tilde{q}} R^{-p} \left(\int_{B_R(z)} |D\tilde{u}|^{(p\cdot n)/(n+p)} dx \right)^{1+p/n} + \tau^{-\tilde{q}} \Lambda^p R^n \right\}$$

or equivalently

$$\fint_{B_{R/2}(z)} |D\tilde{u}|^p dx \leqslant c_{39} \left\{ \tau \fint_{B_R(z)} |D\tilde{u}|^p dx + \tau^{-\tilde{q}} \left(\fint_{B_R(z)} |D\tilde{u}|^{np/(n+p)} dx \right)^{1+p/n} + \Lambda^p \tau^{-\tilde{q}} \right\}.$$

If we require $c_{39} \cdot \tau \leqslant \frac{1}{2}$ which means by definition

$$\varepsilon \leqslant [c_{24}^{-1} c_{39}^{-q} 2^{-q} \delta^{\gamma+1}]^{1/(p\cdot(\gamma+1))} \tag{3.9}$$

then we have inequality (3.5) also in Case 2.

From (3.4), (3.9), (3.5) and [G], Prop. 1.1 (page 124), we finally deduce

LEMMA 3.1. – There are constants $\varepsilon_3 \in (0, 1)$, $t > p$ and $c > 0$ depending on absolute data with the following property: if u is in $\mathcal{M}_\Lambda$ and

$$\Lambda + E_p(u, B_1^+(0)) < \varepsilon_3^p, \tag{3.10}$$

then $D\tilde{u} \in L^t(B_{1/2}(0))$ and

$$\left(\fint_{B_{R/2}(z)} |D\tilde{u}|^t dx \right)^{1/t} \leqslant c \cdot \left\{ \left(\fint_{B_R(z)} |D\tilde{u}|^p dx \right)^{1/p} + \Lambda \right\} \tag{3.11}$$

for all $z \in B_{1/2}(0)$ and $R \in (0, \frac{1}{2})$.

Next we show that a smallness condition of the form (3.10) is also sufficient to prove regularity of $\tilde{u}$ near the origin, more precisely we have

LEMMA 3.2. – There are constants ε_4, $\alpha \in (0, 1)$ and $c > 0$ with the following property: suppose that u is in $\mathcal{M}_\Lambda$ and

$$\Lambda + E_p(u, B_1^+(0)) < \varepsilon_4^p .$$

Then $\tilde{u} \in C^{0,\alpha}(B_{1/4}(0))$ and

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant c \cdot |x - y|^\alpha \tag{3.12}$$

for all $x, y \in B_{1/4}(0)$.

PROOF. – We assume

$$\Lambda + E_p(u, B_1^+(0)) < \varepsilon^p \tag{3.13}$$

for some $0 < \varepsilon \leqslant \varepsilon_3$ being specified later. Fix $x_0 \in \Gamma_{1/2}$ and $0 < r < R < \frac{1}{4}$ and consider the solution v of

$$\int_{B_R^+(x_0)} |Dv|^p \, dx \to \min$$

in the space $H^{1,p}(B_R^+(x_0), \boldsymbol{R}^L)$ for boundary values u on $\partial B_R^+(x_0)$. According to [F1, 2] we know

$$- D_\alpha(|Du|^{p-2} D_\alpha u) = F(\cdot, u, Du)$$

on $B_R^+(x_0)$ for some function F satisfying the structure condition

$$|F(\cdot, u, Du)| \leqslant c_1 |Du|^p . \tag{3.14}$$

Combining the equations satisfied by u and v we get an upper bound for the energy of $u - v$:

$$\int_{B_R^+(x_0)} |Du - Dv|^p \, dx \leqslant c_2 \cdot \int_{B_R^+(x_0)} (Du|Du|^{p-2} - Dv|Dv|^{p-2}) \cdot (Du - Dv) \, dx =$$

$$= \int_{B_R^+(x_0)} F(\cdot, u, Du) \cdot (u - v) \, dx \cdot c_2 \leqslant (3.14)$$

$$c_3 \cdot \left(\int_{B_R^+(x_0)} |Du|^t \, dx \right)^{p/t} \cdot \left(\int_{B_R^+(x_0)} |u - v|^{t/(t-p)} \, dx \right)^{1-p/t} \leqslant$$

$$\leqslant c_4 \cdot \left\{ \left(\int_{B_R(x_0)} |D\tilde{u}|^t \, dx \right)^{p/t} + \Lambda^p \cdot R^{n(p/t)} \right\} \cdot \left(\int_{B_R^+(x_0)} |u - v|^p \, dx \right)^{1-p/t} .$$

Since $\varepsilon < \varepsilon_3$ we may apply estimate (3.11) of Lemma 3.1 to handle the integral of $D\tilde{u}$:

$$\int_{B_R^+(x_0)} |Du - Dv|^p\,dx \leqslant c_5 \left\{ \Lambda^p R^{n(p/t)} + R^{n(p/t-1)} \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx \right\} \left(\int_{B_R^+(x_0)} |u - v|^p\,dx \right)^{1-p/t} =$$

$$= c_5 \left\{ \Lambda^p R^n + \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx \right\} \left(\fint_{B_R^+(x_0)} |u - v|^p\,dx \right)^{1-p/t} \leqslant$$

$$\leqslant (\text{Poincaré's inequality} + \text{minimality of } v) \leqslant$$

$$\leqslant c_6 \cdot \left[R^{p-n} \int_{B_R^+(x_0)} |Du|^p\,dx \right]^{1-p/t} \cdot \left\{ \Lambda^p R^n + \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx \right\}.$$

We combine this result with the estimate of Lemma 2.3: for arbitrary $\theta \in (0, 1)$

$$\int_{B_R^+(x_0)} |D\tilde{u}|^p\,dx \leqslant c_7 \left\{ \int_{B_R^+(x_0)} |Du|^p\,dx + \int_{B_R^+(x_0)} |Du - Dv|^p\,dx \right\} \leqslant$$

$$\leqslant c_8 \left\{ \int_{B_R^+(x_0)} |Du - Dv|^p\,dx + \left[\left(\frac{r}{R} \right)^n + \theta \right] \int_{B_R^+(x_0)} |Dv|^p\,dx + \theta^{1-p} \Lambda^p R^n \right\} \leqslant$$

$$\leqslant c_9 \left\{ \int_{B_R^+(x_0)} |Du - Dv|^p\,dx + \left[\left(\frac{r}{R} \right)^n + \theta \right] \int_{B_R^+(x_0)} |Du|^p\,dx + \theta^{1-p} \Lambda^p R^n \right\} \leqslant$$

$$\leqslant c_{10} \left\{ \psi(u, B_R^+(x_0))^{1-p/t} \cdot \left(\Lambda^p R^n + \int_{B_{2R}(0_0)} |D\tilde{u}|^p\,dx \right) + \left[\left(\frac{r}{R} \right)^n + \theta \right] \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx + \theta^{1-p} \Lambda^p R^n \right\} \leqslant$$

$$\leqslant c_{10} \left\{ \left[\left(\frac{r}{R} \right)^n + \theta + \psi(u, B_R^+(x_0))^{1-p/t} \right] \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx + \Lambda^p R^n \left[\theta^{1-p} + \psi(u, B_R^+(x_0))^{1-p/t} \right], \right.$$

where we have abbreviated

$$\psi(u, B_R^+(x_0)) := R^{p-n} \int_{B_R^+(x_0)} |Du|^p\,dx .$$

Thus we have shown

$$\text{(3.15)} \quad \int_{B_r(x_0)} |D\tilde{u}|^p\,dx \leqslant c_{11} \left\{ \left[\left(\frac{r}{R} \right)^n + \theta + \psi(u, B_R^+(x_0))^{1-p/t} \right] \int_{B_{2R}(x_0)} |D\tilde{u}|^p\,dx + \Lambda^p R^n \left[\theta^{1-p} + \psi(u, B_R^+(x_0))^{1-p/t} \right] \right\}.$$

In a next step we use (3.15) with $r = \tau \cdot R$ for some small τ being specified below:

$$\psi(\tilde{u}, B_{\tau R}(x_0)) \leqslant c_{12} \cdot \tau^p \cdot \Big\{ \big[1 + \tau^{-n}\theta + \tau^{-n}\psi(u, B_R^+(x_0))^{1-p/t}\big] \cdot \\ \cdot \psi(\tilde{u}, B_{2R}(x_0)) + \tau^{-n}\Lambda^p R^p \big(\theta^{1-p} + \psi(u, B_R^+(x_0))^{1-p/t}\big)\Big\}.$$

Choose τ to satisfy $c_{12} \cdot \tau^p = \frac{1}{8}$ and let $\theta := \tau^n$. Moreover we require

$$\tau^{-n}\psi(u, B_R^+(x_0))^{1-p/t} \leqslant 1\,. \tag{3.16}$$

In this case the last inequality implies:

$$\psi(\tilde{u}, B_{\tau R}(x_0)) \leqslant \tfrac{1}{2}\psi(\tilde{u}, B_{2R}(x_0)) + c_{13}\Lambda^p R^p\,. \tag{3.17}$$

We have

$$R^{p-n}\int_{B_R^+(x_0)} |Du|^p\,dx \leqslant c_{14}\Big\{R^{p-n}\int_{B_R(x_0)} |D\tilde{u}|^p\,dx + \Lambda^p R^p\Big\} \leqslant c_{15}\Big\{\int_{B_1^+(0)} |Du|^p\,dx + \Lambda\Big\} \underset{(3.13)}{\leqslant} c_{15}\,\varepsilon^p$$

so that (3.16) is satisfied if we impose the smallness condition

$$\tau^{-n}(c_{15}\,\varepsilon^p)^{1-p/t} \leqslant 1 \tag{3.18}$$

on the parameter ε. Finally we iterate (3.17) to get

$$\psi(\tilde{u}, B_{\tau^k R}(x_0)) \leqslant 2^{-k}\psi(\tilde{u}, B_{2R}(x_0)) + c_{16}\Lambda^p R^p \tag{3.19}$$

for all $k \in \mathbb{N}$. Abbreviating $\beta := -(\log 2)/\log \tau$ (3.19) implies the growth condition

$$\psi(\tilde{u}, B_r(x_0)) \leqslant c_{17}\left\{\left(\frac{r}{R}\right)^\beta \psi(\tilde{u}, B_R(x_0)) + \Lambda^p R^p\right\} \tag{3.20}$$

for all $x_0 \in \Gamma_{1/2}$, $0 < r \leqslant R \leqslant \frac{1}{4}$.

To complete the proof of Lemma 3.2 we recall the interior estimate of [F1, 2]: there are constants ε_4, $\gamma \in (0, 1)$ and $c_{18} > 0$ such that

$$\psi(u, B_r(x_1)) \leqslant c_{18}\left(\frac{r}{R}\right)^\gamma \psi(u, B_R(x_1)) \tag{3.21}$$

holds for $x_1 \in B_{1/2}^+(0)$ and $0 < r < R \leqslant \min\big(\operatorname{dist}(x_1, \Gamma), \frac{1}{2}\big)$ provided

$$\psi(u, B_R(x_1)) \leqslant \varepsilon_4^p\,.$$

By the monotonicity formula, (3.13) and our choice of ε we may assume that the last condition is satisfied, moreover we assume that (3.21) holds with $\gamma = \beta$.

Now we choose $x_1 \in B^+_{1/2}(0) \cup \Gamma$ and fix $x_0 \in \Gamma$ with the property

$$d := \operatorname{dist}(x_1, \Gamma) = |x_1 - x_0|\,.$$

Moreover suppose $0 < r < R \leqslant \frac{1}{8}$.

Case 1: $d > r$.

Then

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{19}\{\psi(u, B_r(x_1)) + \Lambda^p r^p\} \leqslant (3.21)\,,$$
$$c_{20}\left\{\left(\frac{r}{\varrho}\right)^{\beta} \psi(u, B_\varrho(x_1)) + \Lambda^p \varrho^p\right\},$$

for $\varrho := \min(d, R)$. In case $\varrho = R$ we get

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{21}\left\{\left(\frac{r}{R}\right)^{\beta} \psi(\tilde{u}, B_R(x_1)) + \Lambda^p R^p\right\}. \tag{3.22}$$

Assume now $\varrho = d$, that is $R > d$. Then

$$\psi(u, B_\varrho(x_1)) = \psi(u, B_d(x_1)) \leqslant c_{22}\{\psi(\tilde{u}, B_d(x_1)) + \Lambda^p R^p\} \leqslant$$
$$\leqslant c_{23}\{\psi(\tilde{u}, B_{2d}(x_0)) + \Lambda^p R^p\} \leqslant (3.20)\,,$$
$$c_{24}\left\{\left(\frac{d}{R}\right)^{\beta} \psi(\tilde{u}, B_{2R}(x_0)) + \Lambda^p R^p\right\} \leqslant (B_{2R}(x_0) \subset B_{3R}(x_1))\,,$$
$$c_{25}\left\{\left(\frac{d}{R}\right)^{\beta} \psi(\tilde{u}, B_{3R}(x_1)) + \Lambda^p R^p\right.$$

and (3.22) has to be replaced by

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{26}\left\{\left(\frac{r}{R}\right)^{\beta} \psi(\tilde{u}, B_{3R}(x_1)) + \Lambda^p R^p\right\}. \tag{3.23}$$

Case 2: $d \leqslant r$

Then

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{27} \cdot \psi(\tilde{u}, B_{2r}(x_0)) \leqslant (3.20)\,,$$
$$c_{28}\left\{\left(\frac{r}{R}\right)^{\beta} \psi(\tilde{u}, B_{2R}(x_0)) + \Lambda^p R^p\right\} \leqslant c_{29}\left\{\left(\frac{r}{R}\right)^{\beta} \psi(\tilde{u}, B_{3R}(x_1)) + \Lambda^p R^p\right\},$$

so that (3.23) remains valid in Case 2.

Clearly (3.23) implies

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{30}\left\{\left(\frac{r}{R}\right)^{\beta} \psi(\tilde{u}, B_R(x_1)) + \Lambda^p R^p\right\} \tag{3.24}$$

for all $x_1 \in B_{1/2}(0)$, $0 < r < R < \frac{1}{8}$.

Let us set $\alpha := \frac{1}{2}\beta$. If we apply [G], Lemma 2.1 (page 87), then (3.24) gives

$$\psi(\tilde{u}, B_r(x_1)) \leqslant c_{31} r^{\alpha}[R^{-\alpha}\psi(\tilde{u}, B_R(x_1)) + \Lambda^p]$$

for all x_1, r, R as in (3.24). Choosing $R = \frac{1}{8}$ and recalling (3.13) (where ε now is fixed) the proof of Lemma 3.2 is complete (with α replaced by α/p). ∎

4. – Complete boundary regularity.

Following [F1, 2] we combine Lemma 3.2 and the extension-lemma of section 1 to get a regularity criterion relying on the smallness of the mean oscillation.

LEMMA 4.1. – Given $B > 0$ there exist constants $\theta_1 = \theta_1(B)$, $c > 0$ and $\alpha \in (0,1)$ with the following property: suppose that $u \in \mathcal{M}_\Lambda$ with boundary function $g \in \mathcal{B}_\Lambda$ is given such that

$$E_p(u, B_1^+(0)) \leqslant B\,, \qquad \int_{B_1^+(0)} |u-g|^p\,dx + \Lambda \leqslant \theta_1^p\,.$$

Then $\tilde{u} \in C^{0,\alpha}(B_{1/4}(0))$ and for $x, y \in B_{1/4}(0)$

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant c\cdot|x-y|^{\alpha}\,.$$

PROOF. – Clearly all the statements of Lemma 3.2 remain valid if we require

$$\Lambda + E_p(u, B_{3/4}^+(0)) \leqslant \varepsilon_4^p\,. \tag{4.1}$$

Choose $r \in [\frac{3}{4}, 1]$ such that

$$\begin{cases} E_p(u, \partial B_r^+(0)\setminus\Gamma) \leqslant c_1\cdot E_p(u, B_1^+(0))\,,\\ W_p(u, \partial B_r^+(0)\setminus\Gamma) \leqslant c_1\cdot W_p(u, B_1^+(0))\,, \end{cases} \tag{4.2}$$

where W_p is calculated with respect to $a := g(0) \in \overline{M}$. Then the function

$$\varphi := \begin{cases} u & \text{on } \partial B_r^+(0)\setminus\Gamma\,,\\ g & \text{on } \Gamma_r\,, \end{cases}$$

satisfies

$$E_p(\varphi, \partial B_R^+(0))\, W_p(\varphi, \partial B_r^+(0))^{\gamma} \leqslant (4.2)$$

$$c_2(B+\Lambda^p)\cdot\Big(\int_{\partial B_r(0)\setminus\Gamma} |u-g(0)|^p\,d\boldsymbol{H}^{n-1} + \int_{\Gamma_r} |g-g(0)|^p\,d\boldsymbol{H}^{n-1}\Big)^{\gamma} \leqslant$$

$$\leqslant c_3(B+\Lambda^p)\cdot\Big(\int_{B_1^+(0)} |u-g(0)|^p\,dx + \Lambda^p\Big)^{\gamma} \leqslant \quad (\text{if } \theta_1 \leqslant B) \quad \leqslant c_4 B\cdot(\theta_1^p + \Lambda^p)^{\gamma} \leqslant c_5 B\theta_1^{p\cdot\gamma}\,.$$

We assume

$$c_5 \cdot B \cdot \theta_1^{p\cdot\gamma} \leqslant \varepsilon^q \delta^{\gamma+1} \tag{4.3}$$

for some small $\varepsilon > 0$ being specified later. Then by Lemma 2.1 we find an extension $\Phi\colon B_r^+(0) \to \bar{M}$ of the boundary mapping φ such that

$$E_p(\Phi, B_r^+(0)) \leqslant c_6\{\varepsilon \cdot E_p(\varphi, \partial B_r^+(0)) + \varepsilon^{-\tilde{q}} W_p(\varphi, \partial B_r^+(0))\},$$

hence by the minimality of u

$$E_p(u, B_{3/4}^+(0)) \leqslant E_p(\Phi, B_{3/4}^+(0)) \leqslant c_7\{\varepsilon \cdot B + \varepsilon^{-\tilde{q}} \theta_1^p\}.$$

We choose $\varepsilon = \varepsilon(B)$ to satisfy $c_7 \cdot \varepsilon \cdot B \leqslant \frac{1}{4} \cdot \varepsilon_4^p$. With ε fixed we calculate θ_1 with (4.3) and in addition $c_7 \varepsilon^{-\tilde{q}} \theta_1^p \leqslant \frac{1}{4} \varepsilon_4^p$. This gives $E_p(u, B_{3/4}^+(0)) \leqslant \frac{1}{2} \varepsilon_4^p$ so that (4.1) is satisfied. ■

LEMMA 4.2. – Suppose that $\{u_i\} \in \mathcal{M}_A$ is weakly convergent to some function $u_0 \in H^{1,p}(B_1^+(0), \mathbf{R}^L)$. Let $\{g_i\} \in \mathcal{B}_A \cap C^1(\bar{B}_1(0))$ denote the corresponding sequence of boundary functions and suppose $g_i \to g_0$ in $C^1(\bar{B}_1(0))$. Then $\tilde{u}_i \to \tilde{u}_0$ strongly in $H^{1,p}(B_{1/2}(0))$, $\tilde{u}_0$ denoting the odd extension of $u_0 - g_0$, and $\mathbf{H}^{n-p}(\operatorname{Sing}(\tilde{u}_0)) = 0$. Moreover $\tilde{u}_i \to \tilde{u}_0$ uniformly on compact subsets of $B_{1/2}(0) \setminus \operatorname{Sing}(\tilde{u}_0)$.

PROOF. – Exactly the same arguments as in [F1], Lemma 6.2, and [F2], Lemma 4.2, give using Lemma 4.1

$$\operatorname{Sing}(\tilde{u}_0) \subset \left\{x \in B_1(0)\colon \liminf_{r \downarrow 0} \fint_{B_R(x)} |\tilde{u}_0 - (\tilde{u}_0)_r|^p \, dz > 0\right\}$$

and $\tilde{u}_i \to \tilde{u}_0$ on compact subsets of $B_{1/2}(0) \setminus \operatorname{Sing}(\tilde{u}_0)$. In order to prove strong convergence $\tilde{u}_i \to \tilde{u}_0$ in $H^{1,p}(B_{1/2}(0))$ we cover $\operatorname{Sing}(\tilde{u}_0) \cap B_{1/2}(0)$ with balls $B_i = B_{r_i}(x_i)$ such that

$$\sum_{i=1}^{\infty} r_i^{n-p} < \varepsilon$$

for a prescribed small ε. Let $U := \bigcup_{i=1}^{\infty} B_i$. As in the interior case we get

$$\int_U |D\tilde{u}_j|^p \, dx \leqslant \text{const} \cdot \varepsilon .$$

Consider a cut-off function $\eta \in C_0^1(B_1(0), [0,1])$ such that $\eta \equiv 1$ on $\bar{B}_{1/2}(0) \setminus U$

and $\operatorname{spt}\eta \cap \operatorname{Sing}(\tilde{u}_0) = \emptyset$. We know

$$\int_{B_1^+(0)} (Lu_i - Lu_j)\cdot D\phi\, dx = \int_{B_1^+(0)} (F_i - F_j)\cdot \phi\, dx$$

for all $\phi \in \overset{\circ}{H}{}^{1,p} \cap L^\infty(B_1^+(0), \mathbf{R}^L)$ where we have abbreviated

$$Lu := |Du|^{p-2} Du\ .$$

Moreover, we have the growth estimate

$$|F_i| \leqslant K\cdot |Du_i|^p$$

for some finite K independent of i. Let $\phi := \eta^p(\tilde{u}_i - \tilde{u}_j)$ which is admissible in the above equation since ϕ vanishes on the boundary of $B_1^+(0)$. We get

$$\int_{B_1^+(0)} (Lu_i - Lu_j)\cdot D(u_i - u_j)\eta^p\, dx =$$

$$\int_{B_1^+(0)} (Lu_i - Lu_j)\cdot D(\eta^p)(u_j - u_i)\, dx + \qquad (a)$$

$$\int_{B_1^+(0)} (Lu_i - Lu_j)\cdot D(\eta^p[g_i - g_j])\, dx + \qquad (b)$$

$$\int_{B_1^+(0)} (F_i - F_j)\cdot \eta^p(\tilde{u}_i - \tilde{u}_j)\, dx\ . \qquad (c)$$

The left-hand-side is bounded below by a constant times

$$\int_{B_1^+(0)} \eta^p |Du_i - Du_j|^p\, dx\ ,$$

the terms (a), (c) clearly vanish since $u_j - u_i \rightrightarrows 0$ on $\operatorname{spt}\eta$. The remaining integral (b) vanishes on account of our assumption $g_i \to g_0$ in $C^1(\bar{B}_1(0))$. ∎

LEMMA 4.3. – Consider $u \in \mathcal{M}_A$ with boundary function of class $C^1(\bar{B}_1(0))$ and a sequence $\lambda_i \downarrow 0$. Then we have for a subsequence

$$u_i(z) := u(\lambda_i z) \to u_0$$

weakly in $H^{1,p}(B_1^+(0))$, $u_i \to u_0$ strongly in $H^{1,p}(B_r^+(0))$ for all $r < 1$ and the convergence is uniform away from the singular set of u_0. Moreover, the limit u_0 is radially independent and constant on Γ.

PROOF. – Let $g \in \mathcal{B}_A$ denote the boundary values of u. By Lemma 2.2 $E_p(u_i, B_1^+(0))$ is bounded independent of i so that we may extract a weakly convergent subsequence. Clearly $g_i \to g(0)$ in $C^1(\bar{B}_1(0))$. Thus all statements of Lemma 4.3 follow from Lemma 4.2, it only remains to show $D_r u_0 \equiv 0$ (radial independence). To this purpose we observe the following stronger version of Lemma 2.2:

There are constants ε_2, c depending on absolute data with the following property: if $v \in \mathcal{M}_A$ and $\mu \leqslant \varepsilon_2$, then

$$\int\limits_{B_t^+(0)\setminus B_s^+(0)} |D_r \tilde{v}|^p |x|^{p-n} dx \leqslant c\cdot[t^{p-n}E_p(\tilde{v}, B_t^+(0)) - s^{p-n}E_p(\tilde{v}, B_s^+(0)) + \mu(t-s)] \tag{4.4}$$

for $0 < s < t \leqslant \frac{1}{2}$.

For the proof one has to modify the arguments of [SU], Lemma 1.3, in an obvious way. We apply (4.4) to the maps $u_i \in \mathcal{M}_{\lambda_i A}$. Since $\tilde{u}_i \to \tilde{u}_0$ strongly (4.4) gives

$$\int\limits_{B_t^+(0)\setminus B_s^+(0)} |D_r \tilde{u}_0|^p \cdot |x|^{p-n} dx \leqslant c\cdot\{t^{p-n}E_p(\tilde{u}_0, B_t^+(0)) - s^{p-n}E_p(\tilde{u}_0, B_s^+(0))\}\,. \tag{4.5}$$

Thus $\phi(t) = t^{p-n} \int_{B_t^+(0)} |D\tilde{u}_0|^p dx$ is an increasing function and $L := \lim_{t\downarrow 0} \phi(t)$ exists. On the other hand we have for positive t by the strong convergence

$$\phi(t) = t^{p-n} \lim_{i\to\infty} \int\limits_{B_t^+(0)} |D\tilde{u}_i|^p dx = \lim_{i\to\infty} \phi(t\lambda_i) = L$$

so that ϕ is constant and (4.5) implies $D_r \tilde{u}_0 = 0$. ■

In order to prove the Main Theorem we have to show that the tangent map defined in Lemma 4.3 is trivial. This would give

$$0 = \lim_{i\to\infty} \lambda_i^{p-n} \int\limits_{B_{\lambda_i}^+(0)} |Du|^p dx$$

and hence regularity of u at $0 \in \Gamma$ (and by a trivial modification we obtain regularity at every boundary point). Now $u_0 \equiv$ const is a consequence of [HL], Theorem 5.7 and proof of Corollary 5.8: exactly as in [HL], Theorem 5.7, we can show $u_0 \equiv$ const if we assume that u_0 is minimizing. By reduction it is sufficient to consider tangent maps u_0 which are smooth except $0 \in \Gamma$. (Compare the comments in [HL], proof of Corollary 5.8.) But for this class of tangent maps minimality follows by direct calculation or along the lines of the proof of Lemma 10 in [DF]. ■

5. – Application: existence of small solutions of the Dirichlet-problem for p-harmonic maps.

Suppose that $\Omega \subset X$ and Y are as in Section 1 (« Riemannian case ») and that $g: \partial\Omega \to Y$ is a prescribed smooth boundary map with range contained in a regular geodesic ball $B_r(Q)$ of the target manifold Y (compare [H] for the definition of regular balls). In a famous paper Hildebrandt-Kaul-Widman [HKW] proved the existence of a continuous map $V: \bar{\Omega} \to Y$ which is harmonic (and by continuity smooth) on Ω and satisfies $V|_{\partial\Omega} = g$ as well as $V(\bar{\Omega}) \subset B_r(Q)$. In this section we want to prove the analogous result for p-harmonic mappings with exponent $p \geqslant 2$.

DEFINITION. – $U \in H^{1,p}(\Omega, Y)$ is a weakly p-harmonic mapping $\Omega \to Y$ iff U is a weak solution of the Euler-Lagrange equation associated to the p-energy functional $E_p(\cdot, \Omega)$ introduced in Section 1, formula (1.1).

THEOREM 5.1. – *Suppose that we are in the Riemannian case described in Section 1 and suppose that $p \geqslant 2$ is a fixed real number. Assume that $g: \partial\Omega \to Y$ is a smoth function with range in a regular geodesic ball $B_r(Q) \subset Y$. Then there is a map $U \in H^{1,p}(\Omega, Y)$ with the following properties:*

(i) $U \in C^{0,\alpha}(\bar{\Omega}, Y)$ *for some* $0 < \alpha < 1$,

(ii) $U \in C^{1,\alpha}(\Omega, Y)$ *and* U *is weakly p-harmonic,*

(iii) $U|_{\partial\Omega} = g$ *and* $\operatorname{Im}(U) \subset B_r(Q)$.

PROOF. – For $R > r$ sufficiently close to r the ball $B_R(Q)$ is also regular and we consider the obstacle problem

$$E_p(\cdot, \Omega) \to \min \tag{5.1}$$

in $H^{1,p}(\Omega, B_R(Q))$ for boundary values g. The direct method gives the existence of a minimizer U which according to the Main Theorem is Hölder continuous in a neighborhood of $\partial\Omega$. On the other hand Theorem F of [F1] (compare also [F4], Theorem) shows that there are no interior singularities so that (i), $U|_{\partial\Omega} = g$ and in addition

$$U \in C^{1,\alpha}(\Omega, Y)$$

are clearly satisfied. It remains to show

$$\operatorname{Im}(U) \subset B_r(Q) \tag{5.2}$$

which immediately implies p-harmonicity of U. In order to prove (5.2) we introduce normal coordinates $(u^1, \ldots, u^N)$ on $B_R(Q)$ with center Q and assume for technical simplicity that Ω is just a domain in $\boldsymbol{R}^n$ equipped with the flat metric. Then the minimum property (5.1) implies after a simple calculation ([F1], Section 7, proof of Theorem F, and [F4])

$$\int_\Omega A(u, Du)[Du\cdot D(\eta u) - \Gamma^l_{ik}(u)\,\eta\, D_\alpha u^i D_\alpha u^k u^l]\,dx \leq 0 \tag{5.3}$$

for all $\eta \in \mathring{H}^{1,p}(\Omega) \cap L^\infty$, $\eta \geq 0$. Here we have abbreviated:

$$\begin{cases} g_{ij} = \text{metric tensor on } Y,\ \Gamma^l_{ik} = \text{Christoffel symbols on } Y, \\ A(u, Du) = \big(g_{ij}(u)\,D_\alpha u^i D_\alpha u^j\big)^{p/2-1}. \end{cases}$$

We use (5.3) with

$$\eta := \max(v - r^2, 0)\,, \qquad v := |u|^2\,,$$

which is admissible since $|u| \leq r$ on $\partial\Omega$. By [H], inequality (6.11), the quantity

$$|Du|^2 - \Gamma^l_{ik}(u)\,D_\alpha u^i D_\alpha u^k u^l$$

is nonnegative and we find

$$\int_{\Omega\cap[|u|\geq r]} A(u, Du)\,|Dv|^2\,dx \leq 0$$

so that $A(u, Du)\cdot|D\eta|^2 \equiv 0$ on Ω. Since

$$|Dv| \leq 2R|Du|\,, \qquad A(u, Du) \geq c\cdot|Du|^{p-2}$$

for some $c > 0$ we deduce $D\eta = 0$ on Ω, hence $|u| \leq r$. ∎

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