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A Classification of Almost Contact Metric Manifolds (*).

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Summary. – *It is obtained a complete classification for almost contact metric manifolds through the study of the covariant derivative of the fundamental 2-form on those manifolds.*

0. – Introduction.

A $(2n+1)$-dimensional differentiable manifold M of class C^∞ is said to have an almost contact structure (J. W. GRAY [6]) if the structural group of its tangent bundle reduces to $U(n)\times 1$; equivalently (S. SASAKI and S. HATAKEYAMA [15], [17]), an almost contact structure is given by a triple (φ, ξ, η) satisfying certain conditions (see section 1). Many different types of almost contact structures are defined in the literature (cosymplectic, Sasakian, almost cosymplectic, quasi Sasakian, normal, α-Kenmotsu, α-Sasakian, trans-Sasakian, ..., [2], [3], [9], [14]). These types of structures bear sufficient resemblance to cosymplectic and Sasakian structures so that it is possible to generalize a portion of cosymplectic and Sasakian geometry to each type.

The main purpose of this paper is to fit all of these classes into a general system, which in a reasonable sense is complete. For it, we shall consider a real vector space V of dimension $2n+1$ with an almost contact metric structure and we shall study the representation of the group $U(n)\times 1$ on a certain space $\mathcal{C}(V)$. Geometrically $\mathcal{C}(V)$ can be interpreted as the space of tensors of type $(0,3)$ over V which satisfy the same symmetries as the covariant derivative of the fundamental 2-form of an almost contact metric manifold. We give a decomposition of $\mathcal{C}(V)$ into twelve irreducible invariant components $\mathcal{C}_i$, $i=1,\dots,12$, under the action of the group $U(n)\times 1$. It is possible to form 2^{12} different invariant subspaces from these twelve, corresponding to each invariant subspace a class of almost contact metric manifolds. For example, $\{0\}$ corresponds to the class of cosymplectic manifolds, $\mathcal{C}_5$ to the class of α-Kenmotsu manifolds, $\mathcal{C}_6$ to the class of α-Sasakian manifolds, $\mathcal{C}_2\oplus\mathcal{C}_9$ to the

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class of almost cosymplectic manifolds, $\mathcal{C}_6 \oplus \mathcal{C}_7$ to the class of quasi Sasakian manifolds, $\mathcal{C}_5 \oplus \mathcal{C}_6$ to the class of trans-Sasakian manifolds, and $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ to the class of normal manifolds.

In section 1, we give some results on almost contact manifolds. In order to obtain the decomposition of $\mathcal{C}(V)$, we study in section 2 the space of the invariant tensors of type $(0, p)$ under the action of $U(n) \times 1$ finding a basis for this space, and in section 3 we obtain a set of generators for the vector space of the quadratic invariants of $\mathcal{C}(V)$. In section 4, we give the decomposition of $\mathcal{C}(V)$ and the linear relations among the quadratic invariants for each of the irreducible subspaces $\mathcal{C}_i$. In section 5, we give the characterization of the twelve classes of almost contact metric structures on a manifold and we relate these classes with those studied in the literature. Finally, in section 6 we construct examples of different types of almost contact metric structures on the product of an almost Hermitian manifold with $\boldsymbol{R}$, on the hyperbolic space, on the generalized Heisenberg groups $H(p, 1)$ and $H(1, r)$, and on other Lie groups of matrices included in the Kowalski's classification for generalized symmetric Riemannian spaces of dimension $n \leqslant 5$ ([12]). These examples illustrate many types of the classification obtained in this paper.

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1. – Preliminaries.

Let M be a real $(2n+1)$-dimensional C^∞ manifold and $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. An almost contact structure on M is defined by a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η on M such that for any point $x \in M$ we have

$$\varphi_x^2 = -I + \eta_x \otimes \xi_x\,, \qquad \eta_x(\xi_x) = 1\,,$$

where I denotes the identity transformation of the tangent space $T_x M$ at x. Manifolds equipped with an almost contact structure are called almost contact manifolds. A Riemannian manifold M with metric tensor g and with an almost contact structure (φ, ξ, η) such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,,$$

where $X, Y \in \mathfrak{X}(M)$, is an almost contact metric manifold. Then g is called a compatible metric and M is said to have a (φ, ξ, η, g)-structure or an almost contact metric structure. The existence of an almost contact structure on M is equivalent to the existence of a reduction of the structural group to $U(n) \times 1$, i.e. all the matrices of $O(2n+1)$ of the form

$$\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

being A and B real (n, n)-matrices.

The fundamental 2-form Φ of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for all $X, Y \in \mathfrak{X}(M)$, and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable.

If ∇ is the Riemannian connection of g, it is easy to prove

$$(1.1) \qquad (\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi) Z),$$

$$(1.2) \qquad (\nabla_X \Phi)(Y, Z) + (\nabla_X \Phi)(\varphi Y, \varphi Z) = \eta(Z)(\nabla_X \eta)\varphi Y - \eta(Y)(\nabla_X \eta)\varphi Z,$$

$$(1.3) \qquad (\nabla_X \eta) Y = g(Y, \nabla_X \xi) = (\nabla_X \Phi)(\xi, \varphi Y).$$

The exterior derivatives of η and Φ are given by

$$(1.4) \qquad 2d\eta(X, Y) = (\nabla_X \eta) Y - (\nabla_Y \eta) X,$$

$$(1.5) \qquad 3d\Phi(X, Y, Z) = \mathfrak{S}(\nabla_X \Phi)(Y, Z),$$

where $\mathfrak{S}$ denotes the cyclic sum over $X, Y, Z \in \mathfrak{X}(M)$. If $\{X_i, \varphi X_i, \xi\}$, $i = 1 \dots n$, is a local orthonormal basis, defined on an open subset of M, the coderivatives of Φ and η are computed to be

$$(1.6) \qquad \delta\Phi(X) = -\sum_{i=1}^{n} \{(\nabla_{X_i} \Phi)(X_i, X) + (\nabla_{\varphi X_i} \Phi)(\varphi X_i, X)\} - (\nabla_\xi \Phi)(\xi, X),$$

$$(1.7) \qquad \delta\eta = -\sum_{i=1}^{n} \{(\nabla_{X_i} \eta) X_i + (\nabla_{\varphi X_i} \eta) \varphi X_i\}.$$

An almost contact structure (φ, ξ, η) is said to be normal if the almost complex structure J on $M \times \boldsymbol{R}$ given by

$$(1.8) \qquad J\left(X, a \frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta(X) \frac{d}{dt}\right),$$

where a is a C^∞ function on $M\times\boldsymbol{R}$, is integrable, which is equivalent to the condition $[\varphi,\varphi]+2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ.

An almost contact metric structure (φ,ξ,η,g) in M is said to be:

Almost cosymplectic if $d\Phi=0$ and $d\eta=0$.

Cosymplectic if it is almost cosymplectic and normal.

Quasi Sasakian if $d\Phi=0$ and (φ,ξ,η) is normal.

Almost α-Kenmotsu if $d\eta=0$ and $d\Phi(X,Y,Z)=\frac{2}{3}\alpha\mathfrak{S}\{\eta(X)\Phi(Y,Z)\}$, being α a differentiable function on M.

α-Kenmotsu if it is almost α-Kenmotsu and normal.

Almost α-Sasakian if $\alpha\Phi=d\eta$, being α differentiable function on M.

α-Sasakian if it is almost α-Sasakian and normal.

For $\alpha=$ constant our definition of almost α-Kenmotsu and almost α-Sasakian structures coincides with the structures introduced in [9].

Moreover, (φ,ξ,η,g) is said to be Kenmotsu if it is 1-Kenmotsu, contact if it is almost 1-Sasakian, and Sasakian if it is 1-Sasakian. For an extensive study of these structures we refer to [2], [3], [9], [11], [16]. On the other hand, J. Oubiña defined other classes of almost contact metric structure through the almost Hermitian structure (J,h) on $M\times\boldsymbol{R}$, where J is given by (1.8) and h is the product metric of g and the Euclidean metric on $\boldsymbol{R}$. Next, we recall some of these classes (see [13], [14]):

Nearly-K-cosymplectic if $(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=0$ and $\nabla_X\xi=0$.

Quasi-K-cosymplectic if $(\nabla_X\varphi)Y+(\nabla_{\varphi X}\varphi)\varphi Y=\eta(Y)\nabla_{\varphi X}\xi$.

Semi cosymplectic if $\delta\Phi=0$ and $\delta\eta=0$.

Trans-Sasakian if

$$(\nabla_X\Phi)(Y,Z)=-\frac{1}{2n}\{(g(X,Y)\eta(Z)-g(X,Z)\eta(Y))\,\delta\Phi(\xi)+ \\ +(g(X,\varphi Y)\eta(Z)-g(X,\varphi Z)\eta(Y))\,\delta\eta\}\,.$$

Nearly-trans-Sasakian if

$$(\nabla_X\Phi)(X,Y)=-\frac{1}{2n}\{g(X,X)\,\delta\Phi(Y)-g(X,Y)\,\delta\Phi(X)+g(\varphi X,Y)\eta(X)\,\delta\eta\}\,,$$

and

$$(\nabla_X\eta)Y=-\frac{1}{2n}\left(g(\varphi X,\varphi Y)\,\delta\eta+g(\varphi X,Y)\,\delta\Phi(\xi)\right).$$

Almost-K-contact if $\nabla_\xi\varphi=0$.

2. – Invariant tensors of $U(n)\times 1$.

Let V be a n-dimensional real vector space. We denote by $GL(V)$ the group of all nonsingular linear transformations. A tensor f of type $(0,p)$ on V is invariant by a subgroup G of $GL(V)$ if and only if

$$f(\sigma x_1, \dots, \sigma x_p) = f(x_1, \dots, x_p)\,, \tag{2.1}$$

for all $\sigma\in G$ and $x_i\in V$. We denote by $\otimes_p^0 V$ the space of the tensors of type $(0,p)$ over V and by $(\otimes_p^0 V)^{\#}(G)$ the subspace of $\otimes_p^0 V$ consisting of the tensors f on V which satisfy (2.1).

Now, let V be a real vector space of dimension $2n+1$ with an almost contact structure (φ, ξ, η) and a real positive definite inner product $\langle\,,\rangle$. We assume that $\langle\,,\rangle$ compatible with the (φ,ξ,η)-structure in the sense that

$$\langle\varphi x, \varphi y\rangle = \langle x, y\rangle - \eta(x)\eta(x)\,.$$

We consider the group $U(n)\times 1$, which can be written as

$$U(n)\times 1 = \{\sigma\in GL(V)\colon \sigma|_{\overline{V}}\in U(n) \text{ and } \sigma\xi = \xi\}\,.$$

where $\overline{V}$ denotes the orthogonal complement of the suspace spanned by ξ, i.e.,

$$\overline{V} = \{x\in V/\langle x, \xi\rangle = 0\}\,.$$

It is well-known that $(\varphi, \langle\,,\rangle)$ defines on $\overline{V}$ an almost Hermitian structure and

$$U(n)\times 1|_{\overline{V}} \simeq U(n)\,.$$

Put,

$$\lambda_n^p = \dim(\otimes_p^0 \overline{V})^{\#}(U(n)) \qquad \text{and} \qquad \mu_n^p = \dim(\otimes_p^0 V)^{\#}(U(n)\times 1)\,.$$

Then we have,

$$\lambda_n^p = 0\,, \quad \text{if } p \text{ is odd}\,,$$

and for the even case N. Iwahori has proved in [8] that

$$\lambda_n^{2p} = 2^p(2p-1)(2p-3)\dots 3.1 \qquad \text{for } p\leqslant n\,.$$

We shall say that a mapping ϱ from the set of $2p$ integers $\{1, 2, \dots, 2p\}$ onto the set of p integers $\{1, 2, \dots, p\}$ is admissible if, for every integer i, $1\leqslant i\leqslant p$, $\varrho^{-1}(i)$

consists of two integers. Let us identify two admissible mappings ϱ, τ if $\{\varrho^{-1}(1), \ldots, \varrho^{-1}(p)\}$ and $\{\tau^{-1}(1), \ldots, \tau^{-1}(p)\}$ coincide up to their orders. We denote by A_p the set of all admissible mappings.

Let us associated to $\varrho \in A_p$ the tensors of type $(0, 2p)$ $F_\varrho^{\alpha_1 \ldots \alpha_p}$, $a_i = 0, 1$, on V as follows:

$$F_\varrho^{\alpha_1 \ldots \alpha_p}(x_1, \ldots, x_{2p}) = \Omega_{\alpha_1}(x_{k_1}, x_{k_1'}) \ldots \Omega_{\alpha_p}(x_{k_p}, x_{k_p'}) , \tag{2.2}$$

where $\{k_j, k_j'\} = \varrho^{-1}(j)$, $k_j < k_j'$ $(j = 1, \ldots, p)$, $\Omega_0 = \langle\, , \rangle$, and $\Omega_1 = F$.

Then $\{F^{\alpha_1 \ldots \alpha_p}\}$ forms a base of $(\otimes_{2p}^0 V)^\#(U(n))$, [8].

THEOREM 2.1. – *If $p \leqslant 2n$, then we have*

$$\mu_n^p = \sum_{q=0}^{p} \binom{p}{q} \lambda_n^{p-q}, \qquad \textit{where } \lambda_n^0 = 1 .$$

PROOF. From $V = \bar{V} \otimes \{\xi\}$, where $\{\xi\}$ is the subspace of V spanned by ξ, we obtain the following decomposition of $\otimes_p^0 V$ into direct sum of subspaces:

$$\otimes_p^0 V = E_0 \oplus \ldots \oplus E_p ,$$

where,

$$E_0 = \otimes_p^0 \bar{V}$$

$$E_1 = (\boldsymbol{R} \otimes (\otimes_{p-1}^0 \bar{V})) \oplus (\bar{V} \otimes \boldsymbol{R} \otimes (\otimes_{p-2}^0 \bar{V})) \oplus \ldots \oplus ((\otimes_{p-1}^0 \bar{V}) \otimes \boldsymbol{R})$$

$$E_2 = (\boldsymbol{R} \otimes \boldsymbol{R} \otimes (\otimes_{p-2}^0 \bar{V})) \oplus (\boldsymbol{R} \otimes \bar{V} \otimes \boldsymbol{R} \otimes (\otimes_{p-3}^0 \bar{V})) \oplus \ldots \oplus ((\otimes_{p-2}^0 \bar{V}) \otimes \boldsymbol{R} \otimes \boldsymbol{R})$$

. . . .

$$E_p = \otimes_p^0 \boldsymbol{R}.$$

Then,

$$(\otimes_p^0 V)^\#(U(n) \times 1) = \sum_{q=0}^{p} E_q^\#(U(n) \times 1) .$$

Now, since $U(n) \times 1$ leaves invariant every element of $\boldsymbol{R}$, we have,

$$((\otimes_s^0 \bar{V}) \otimes (\otimes_t^0 \boldsymbol{R}))^\#(U(n) \times 1) = ((\otimes_s^0 \bar{V}))^\#(U(n)) \otimes \boldsymbol{R} \simeq (\otimes_s^0 \bar{V})^\#(U(n)) .$$

Thus,

$$\dim E_q^\#(U(n) \times 1) = \binom{p}{q} \lambda_n^{p-q} .$$

This proves the required result.

Consider the tensors of type $(0, 2p)\tilde{F}$, $\tilde{F}_{\varrho}^{\alpha_1 \dots \alpha_p}$, $\tilde{F}^{\alpha_1 \dots \alpha_s}_{\varrho i_{2s+1} \dots i_{2p}}$, $0 < s < p$, $\alpha_i = 0, 1$, on V defined by

$$\tilde{F}(x_1, \dots, x_{2p}) = \langle x_1, \xi \rangle \dots \langle x_{2p}, \xi \rangle ;$$

$$\tilde{F}_{\varrho}^{\alpha_1 \dots \alpha_p}(x_1, \dots, x_{2p}) = F_{\varrho}^{\alpha_1 \dots \alpha_p}(\bar{x}_1, \dots, \bar{x}_{2p}) ,$$

with $F_{\varrho}^{\alpha_1 \dots \alpha_p}$ given in (2.2), $x_i = \bar{x}_i + \beta_i \xi$, $\bar{x}_i \in \bar{V}$;

$$\tilde{F}^{\alpha_1 \dots \alpha_s}_{\varrho i_{2s+1} \dots i_{2p}}(x_1, \dots, x_{2p}) = F_{\varrho}^{\alpha_1 \dots \alpha_s}(\bar{x}_{i_1}, \dots, \bar{x}_{i_{2s}}) ,$$

with

$$1 \leqslant i_1 < \dots < i_{2s} \leqslant 2p , \quad 1 \leqslant i_{2s+1} < \dots < i_{2p} \leqslant 2p , \quad i_j \neq i_{2s+k}$$

where

$$x_{i_j} = \bar{x}_{i_j} + \beta_{i_j} \xi , \quad j = 1, \dots, 2s , \quad x_{i_{2s+k}} = \xi , \quad k = 1, \dots, 2(p - s) ,$$

and in other case:

$$\tilde{F}^{\alpha_1 \dots \alpha_s}_{\varrho i_{2s+1} \dots i_{2p}}(x_1, \dots, x_{2p}) = 0 .$$

Then,

COROLLARY 2.1. – $\{\tilde{F}, \tilde{F}_{\varrho}^{\alpha_1 \dots \alpha_p}, \tilde{F}^{\alpha_1 \dots \alpha_s}_{\varrho i_{2s+1} \dots i_{2p}}\}$ is a base of $(\otimes^0_{2p} V)^{\#}(U(n) \times 1)$ and $\{\tilde{F}^{\alpha_1 \dots \alpha_s}_{\varrho i_{2s+1} \dots i_{2p+1}}\}$ is one of $(\otimes^0_{2p+1} V)^{\#}(U(n) \times 1)$.

3. – Quadratic invariants of $\mathcal{C}(V)$.

The covariant derivative $\nabla \Phi$ of the fundamental 2-form Φ of an almost contact metric manifold M is a covariant tensor of degree 3 which has various symmetry properties. We shall define a finite dimensional vector space $\mathcal{C}(V)$ that will consist of those tensors that possess the same symmetries.

Let V be a real vector space of dimension $2n + 1$ with an almost contact structure (φ, ξ, η) and a compatible metric $\langle\, , \rangle$. Let $\mathcal{C}(V)$ be the subspace of $\otimes^0_3 V$ defined by

$$\mathcal{C}(V) = \{\alpha \in \otimes^0_3 V / \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \varphi y, \varphi z) + \\ + \eta(y)\alpha(x, \xi, z) + \eta(z)\alpha(x, y, \xi)\} .$$

The natural representation of $U(n) \times 1$ on V induces a representation of $U(n) \times 1$ on $\otimes^0_p V$ defined by

$$(a\alpha)(x_1, \dots, x_p) = \alpha(a^{-1}x_1, \dots, a^{-1}x_p) , \tag{3.1}$$

for $x_i \in V$, $a \in U(n) \times 1$, $\alpha \in \otimes^0_p V$.

The quadratic invariants of $\otimes_p^0 V$ may be written as follows

$$P(\alpha)=\sum(e_{i_1},\dots,e_{i_p})\alpha(e_{j_1},\dots,e_{j_p})F(e_{i_1},\dots,e_{i_p},e_{j_1},\dots,e_{j_p})$$

where $\{e_1,\dots,e_{2n+1}\}$ is an arbitrary orthonormal basis of V, $\alpha\in\otimes_p^0 V$ and $F\in(\otimes_{2p}^0 V)^{\#}(U(n)\times 1)$. Furthermore, all the quadratic invariants of a subspace of $\otimes_p^0 V$ are restrictions of quadratic invariants of $\otimes_p^0 V$ [1].

THEOREM 3.1. – *The space of the quadratic invariants of* $\mathcal{C}(V)$ *is generated by the following* 18 *invariants*:

$$i_1(\alpha)=\sum_{i,j,k}\alpha(e_i,e_j,e_k)^2 \qquad i_2(\alpha)=\sum_{i,j,k}\alpha(e_i,e_j,e_k)\alpha(e_j,e_i,e_k)$$

$$i_3(\alpha)=\sum_{i,j,k}\alpha(e_i,e_j,e_k)\alpha(\varphi e_i,\varphi e_j,e_k); \qquad i_4(\alpha)=\sum_{i,j,k}\alpha(e_i,e_i,e_k)\alpha(e_j,e_j,e_k)$$

$$i_5(\alpha)=\sum_{j,k}\alpha(\xi,e_j,e_k)^2; \qquad i_6(\alpha)=\sum_{i,k}\alpha(e_i,\xi,e_k)^2$$

$$i_7(\alpha)=\sum_{j,k}\alpha(\xi,e_j,e_k)\alpha(e_j,\xi,e_k); \qquad i_8(\alpha)=\sum_{j,i}\alpha(e_i,e_j,\xi)\alpha(e_j,e_i,\xi)$$

$$i_9(\alpha)=\sum_{i,j}\alpha(e_i,e_j,\xi)\alpha(\varphi e_i,\varphi e_j,\xi); \qquad i_{10}(\alpha)=\sum_{i,j}\alpha(e_i,e_i,\xi)\alpha(e_j,e_j,\xi)$$

$$i_{11}(\alpha)=\sum_{i,j}\alpha(e_i,e_j,\xi)\alpha(e_j,\varphi e_i,\xi); \qquad i_{12}(\alpha)=\sum_{i,j}\alpha(e_i,e_j,\xi)\alpha(\varphi e_j,\varphi e_i,\xi)$$

$$i_{13}(\alpha)=\sum_{j,k}\alpha(\xi,e_j,e_k)\alpha(\varphi e_j,\xi,e_k); \qquad i_{14}(\alpha)=\sum_{i,j}\alpha(e_i,\varphi e_i,\xi)\alpha(e_j,\varphi e_j,\xi)$$

$$i_{15}(\alpha)=\sum_{i,j}\alpha(e_i,\varphi e_i,\xi)\alpha(e_j,e_j,\xi); \qquad i_{16}(\alpha)=\sum_{k}\alpha(\xi,\xi,e_k)^2$$

$$i_{17}(\alpha)=\sum_{i,k}\alpha(e_i,e_i,e_k)\alpha(\xi,\xi,e_k); \qquad i_{18}(\alpha)=\sum_{i,k}\alpha(e_i,e_i,\varphi e_k)\alpha(\xi,\xi,e_k)$$

where $\{e_1,\dots,e_{2n},\xi\}$ *is an orthonormal basis of* V *and* $\alpha\in\mathcal{C}(V)$.

PROOF. – From corollary 2.1, the quadratic invariants of $\mathcal{C}(V)$ are of the following type:

$$P(\alpha)=\sum\alpha(e_{i_1},e_{i_2},e_{i_3})\alpha(e_{j_1},e_{j_2},e_{j_3})F(e_{i_1},e_{i_2},e_{i_3},e_{j_1},e_{j_2},e_{j_3}),\tag{3.2}$$

where $\alpha\in\mathcal{C}(V)$ and F is a linear combination of

$$\{\tilde F_\varrho^{\alpha_1\alpha_2\alpha_3},\ \tilde F_{\varrho i_2\dots i_6}^{\alpha_1},\ \tilde F_{\varrho i_5 i_6}^{\alpha_1\alpha_2}\}.$$

The invariants $i_1(\alpha)$, $i_2(\alpha)$, $i_3(\alpha)$ and $i_4(\alpha)$ are obtained by taking $\tilde F_\varrho^{\alpha_1\alpha_2\alpha_3}$, $i_{16}(\alpha)$ by $\tilde F_{\varrho i_2\dots i_6}^{\alpha_1}$ and using $\tilde F_{\varrho i_5 i_6}^{\alpha_1\alpha_2}$ we determine the remaining invariants $i_j(\alpha)$. Thus, taking into account the symmetries of α and (3.2), every quadratic invariant is a linear combination of $i_1(\alpha), \ldots, i_{18}(\alpha)$.

If $\dim V = 3$, we have the following linear relations between the quadratic invariants:

$$
\begin{aligned}
i_8(\alpha) &= i_{10}(\alpha) - i_9(\alpha)\,; \qquad i_{11}(\alpha) = i_{15}(\alpha) \\
i_{12}(\alpha) &= i_{10}(\alpha) - i_6(\alpha)\,; \qquad i_{14}(\alpha) = i_6(\alpha) + i_9(\alpha) - i_{10}(\alpha)\,,
\end{aligned}
\tag{3.3}
$$

being zero the invariants $i_1(\alpha), i_2(\alpha), i_3(\alpha), i_4(\alpha), i_5(\alpha), i_7(\alpha), i_{13}(\alpha), i_{17}(\alpha)$ and $i_{18}(\alpha)$.

For $\dim V = 5$, it follows that

$$
\begin{aligned}
i_1(\alpha) &= 2\{i_2(\alpha) + i_4(\alpha)\}\,, \\
i_3(\alpha) &= 2\{i_4(\alpha) - i_2(\alpha)\}\,.
\end{aligned}
$$

4. – The decomposition of $\mathcal{C}(V)$.

In this paragraph, our aim is to give a complete decomposition of $\mathcal{C}(V)$ into orthogonal irreducible factors.

The space $\mathcal{C}(V)$ has a natural inner product induced from that on V:

$$\langle \alpha, \tilde{\alpha} \rangle = \sum_{i,j,k=1}^{2n+1} \alpha(e_i, e_j, e_k)\,\tilde{\alpha}(e_i, e_j, e_k)\,,$$

where $\alpha, \tilde{\alpha} \in \mathcal{C}(V)$ and $\{e_i\}$ is an arbitrary orthonormal basis of V.

Since,

$$\langle a\alpha, a\tilde{\alpha} \rangle = \langle \alpha, \tilde{\alpha} \rangle\,, \qquad a \in U(n)\times 1\,, \qquad \alpha, \tilde{\alpha} \in \mathcal{C}(V)\,,$$

the orthogonal complement of an invariant subspace of $\mathcal{C}(V)$ is also invariant. Further, it follows from this that the standard representation of $U(n)\times 1$ on $\mathcal{C}(V)$ is completely reducible.

Now, we introduce three subspaces $\mathcal{D}_i$, $i = 1, 2, 3$, of $\mathcal{C}(V)$, as follows,

$$
\begin{aligned}
\mathcal{D}_1 &= \{\alpha \in \mathcal{C}(V) : \alpha(\xi, x, y) = \alpha(x, \xi, y) = 0\} \\
\mathcal{D}_2 &= \{\alpha \in \mathcal{C}(V) : \alpha(x, y, z) = \eta(x)\alpha(\xi, y, z) + \eta(y)\alpha(x, \xi, z) + \eta(z)\alpha(x, y, \xi)\} \\
\mathcal{D}_3 &= \{\alpha \in \mathcal{C}(V) : \alpha(x, y, z) = \eta(x)\eta(y)\alpha(\xi, \xi, z) + \eta(x)\eta(z)\alpha(\xi, y, \xi)\}\,.
\end{aligned}
$$

Then, we have,

Proposition 4.1. - *If* $\dim V = 2n+1$, $n \geqslant 2$:

$$\mathcal{C}(V) = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 .$$

If $n = 1$, *then* $\mathcal{D}_1 = \{0\}$.

Moreover, these spaces are mutually orthogonal and invariant under the action of $U(n) \times 1$.

Next we shall give a decomposition of the subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ into orthogonal irreducible factors.

Let $\bar{V}$ be the orthogonal complement of the subspace spanned by ξ. The endomorphism φ induces on $\bar{V}$ a complex structure and $\langle\, , \rangle$ is a Hermitian inner product. Gray and Hervella [5] decomposed the vector space

$$W = \{\alpha \in \otimes_3^0 \bar{V} / \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \varphi y, \varphi z),\ x, y, z \in \bar{V}\}$$

into four irreducible and invariant subspaces under the action of $U(n)$. Since $\mathcal{D}_1$ is naturally isomorphic to W, we have,

Theorem 4.1. - *If* $\dim V = 2n+1$, $n > 2$, *then*:

$$\mathcal{D}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 ,$$

where

$$\mathcal{C}_1 = \{\alpha \in \mathcal{C}(V) : \alpha(x, x, y) = \alpha(x, y, \xi) = 0\} ,$$

$$\mathcal{C}_2 = \{\alpha \in (V) : \underset{x,y,z}{\mathfrak{S}} \alpha(x, y, z) = 0,\ \alpha(x, y, \xi) = 0\} ,$$

$$\mathcal{C}_3 = \{\alpha \in \mathcal{C}(V) : \alpha(x, y, z) - \alpha(\varphi x, \varphi y, z) = 0,\ c_{12}\alpha = 0\} ,$$

$$\mathcal{C}_4 = \Big\{\alpha \in \mathcal{C}(V) / \alpha(x, y, z) = \frac{1}{2(n-1)} \big[(\langle x, y\rangle - \eta(x)\eta(y))\, c_{12}\alpha(z) -$$
$$- (\langle x, z\rangle - \eta(x)\,\eta(z))\, c_{12}\alpha(y) - \langle x, \varphi y\rangle c_{12}\alpha(\varphi z) + \langle x, \varphi z\rangle c_{12}\alpha(\varphi y)\big] , \quad \text{and } c_{12}\alpha(\xi) = 0\Big\}$$

for any $x, y, z \in V$ *and* $c_{12}\alpha(x) = \sum \alpha(e_i, e_i, x)$, *where* $\{e_i\}$, $i = 1, \ldots, 2n+1$, *is an arbitrary orthonormal basis of* V.

For $n = 1$, $\mathcal{D}_1 = \{0\}$; *and for* $n = 2$, $\mathcal{D}_1 = \mathcal{C}_2 \oplus \mathcal{C}_4$:

These subspaces are mutually orthogonal and invariant under the action of $U(n) \times 1$.

In order to decompose the subspace $\mathcal{D}_2$ we introduce the endomorphism ψ given by

$$(\psi\alpha)(x, y, z) = \frac{1}{2n} [\langle x, \varphi z\rangle \eta(y) \bar{c}_{12}\alpha(\xi) - \langle x, \varphi y\rangle \eta(z) \bar{c}_{12}\alpha(\xi)] ,$$

for any $x, y, z \in V$, where $\bar{c}_{12}\alpha(\xi) = \sum \alpha(e_i, \varphi e_i, x)$, and $\{e_i\}$ is an arbitrary orthonormal basis of V. Then ψ commutes with the action of $U(n)\times 1$, $\psi^2 = \psi$ and $\mathfrak{D}_2 = \ker\psi \oplus \operatorname{Im}\psi$. We have,

$$\operatorname{Im}\psi = \left\{\alpha \in \mathcal{C}(V)/\alpha(x, y, z) = \frac{1}{2n}\left[\langle x, \varphi z\rangle \eta(y)\bar{c}_{12}\alpha(\xi) - \langle x, \varphi y\rangle \eta(z)\bar{c}_{12}\alpha(\xi)\right]\right\}.$$

We shall denotes this subspace by $\mathcal{C}_5$.
In $\ker\psi = \{\alpha \in \mathfrak{D}_2 : \bar{c}_{12}\alpha(\xi) = 0\}$ we define the endomorphism χ by

$$(\chi\alpha)(x, y, z) = \frac{1}{2n}\left[\langle x, y\rangle \eta(z) c_{12}\alpha(\xi) - \langle x, z\rangle \eta(y) c_{12}\alpha(\xi)\right].$$

It is easy to check that χ commutes with the action of $U(n)\times 1$ and $\chi^2 = \chi$. Thus,

$$\ker\psi = \ker\chi \oplus \operatorname{Im}\chi,$$

and

$$\mathcal{C}_6 = \operatorname{Im}\chi = \left\{\alpha \in \mathcal{C}(V)/\alpha(x, y, z) = \frac{1}{2n}\left[\langle x, y\rangle \eta(z) c_{12}\alpha(\xi) - \langle x, z\rangle \eta(y) c_{12}\alpha(\xi)\right]\right\},$$

$$\ker\chi = \{\alpha \in \mathfrak{D}_2 / c_{12}\alpha(\xi) = \bar{c}_{12}\alpha(\xi) = 0\}.$$

Now, we consider the endomorphism γ on $\ker\chi$ given by

$$(\gamma\alpha)(x, y, z) = \eta(z)\alpha(\varphi x, \varphi y, \xi) + \eta(y)\alpha(\varphi x, \xi, \varphi z) - \eta(x)\alpha(\xi, y, z).$$

Since $\gamma^2 = I$, γ admits the eigenvalues $+1, -1$ and the eigenspaces

$$(\ker\chi)_+ = \{\alpha \in \mathfrak{D}_2 : \alpha(x, y, z) = \eta(z)\alpha(\varphi x, \varphi y, \xi) + \eta(y)\alpha(\varphi x, \xi, \varphi z)\}$$

$$(\ker\chi)_- = \{\alpha \in \mathfrak{D}_2 : \alpha(x, y, z) = -\eta(z)\alpha(\varphi x, \varphi y, \xi) - \eta(y)\alpha(\varphi x, \xi, \varphi z) + \eta(x)\alpha(\xi, y, z)\}$$

are invariant, mutually orthogonal and

$$\ker\chi = (\ker\chi)_+ \oplus (\ker\chi)_-.$$

In $(\ker\chi)_-$ the endomorphism τ defined by

$$(\tau\alpha)(x, y, z) = \eta(x)\alpha(\xi, y, z)$$

commutes with the action of $U(n)\times 1$ and satisfies $\tau^2 = \tau$. Thus,

$$(\ker\chi)_- = \ker\tau \oplus \operatorname{Im}\tau,$$

where

$$\ker \tau = \{\alpha \in \mathcal{D}_2 \colon \alpha(x, y, z) = -\eta(z)\alpha(\varphi x, \varphi y, \xi) - \eta(y)\alpha(\varphi x, \xi, \varphi z)\}$$

$$\operatorname{Im} \tau = \{\alpha \in \mathcal{C}(V)/\alpha(x, y, z) = -\eta(x)\alpha(\xi, \varphi y, \varphi z)\} .$$

We denote the space $\operatorname{Im} \tau$ by $\mathcal{C}_{11}$.

Finally, we introduce in the subspaces $(\ker \chi)_+$ and $\ker \tau$ the same homomorphism ϱ given by

$$(\varrho\alpha)(x, y, z) = \tfrac{1}{2}[\eta(z)(\alpha(x, y, \xi) + \alpha(y, x, \xi)) - \eta(y)(\alpha(x, z, \xi) + \alpha(z, x, \xi))];$$

ϱ is an endomorphism in each subspace which commutes with the action of $U(n)\times 1$ and satisfies $\varrho^2 = \varrho$. Hence $(\ker \chi)_+$ and $\ker \tau$ can be decomposed into mutually orthogonal and invariant subspaces as follows:

$$(\ker \chi)_+ = \mathcal{C}_7 \oplus \mathcal{C}_8 , \qquad \ker \tau = \mathcal{C}_9 \oplus \mathcal{C}_{10} ,$$

where

$$\mathcal{C}_7 = \{\alpha \in \mathcal{C}(V) \colon \alpha(x, y, z) = \eta(z)\alpha(y, x, \xi) - \eta(y)\alpha(\varphi x, \varphi z, \xi), \; c_{12}\alpha(\xi) = 0\}$$

$$\mathcal{C}_8 = \{\alpha \in \mathcal{C}(V) \colon \alpha(x, y, z) = -\eta(z)\alpha(y, x, \xi) - \eta(y)\alpha(\varphi x, \varphi z, \xi), \; \bar{c}_{12}\alpha(\xi) = 0\}$$

$$\mathcal{C}_9 = \{\alpha \in \mathcal{C}(V) \colon \alpha(x, y, z) = \eta(z)\alpha(y, x, \xi) + \eta(y)\alpha(\varphi x, \varphi z, \xi)\}$$

$$\mathcal{C}_{10} = \{\alpha \in \mathcal{C}(V) \colon \alpha(x, y, z) = -\eta(z)\alpha(y, x, \xi) + \eta(y)\alpha(\varphi x, \varphi z, \xi)\} .$$

Thus, we can conclude,

THEOREM 4.2. - *If* $\dim V = 2n + 1$, $n \geqslant 2$, *then*

$$\mathcal{D}_2 = \mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{11} ,$$

and if $n = 1$,

$$\mathcal{D}_2 = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9 .$$

These subspaces are mutually orthogonal and invariant under the action of $U(n)\times 1$.

From theorems 4.1, 4.2 and proposition 4.1, $\mathcal{C}(V)$ decomposes as a direct sum of twelve subspaces $\mathcal{C}_i$, $i = 1, \ldots, 12$, invariant under the action of $U(n)\times 1$ (where $\mathcal{C}_{12} = \mathcal{D}_3$).

In order to prove that the decomposition given before is irreducible, it is sufficient to check that the space of the quadratic invariants of each $\mathcal{C}_i$ has dimension one [1].

Using theorem 3.1 it follows that

$$\|\alpha\|^2 = i_1(\alpha) + i_5(\alpha) + 2i_6(\alpha) + i_{16}(\alpha)\,,$$

$$\|c_{12}(\alpha)\|^2 = i_4(\alpha) + i_{10}(\alpha) + i_{16}(\alpha) + 2i_{17}(\alpha)$$

and

$$\|\bar{c}_{12}(\alpha)\|^2 = i_4(\alpha) + i_{14}(\alpha)\,.$$

Moreover, we have:

(*a*) If $\alpha \in \mathfrak{D}_1$ then $i_m(\alpha) = 0$ for $m \geqslant 5$.

(*b*) If $\alpha \in \mathfrak{D}_2$, then $i_m(\alpha) = 0$ for $m = 1, 2, 3, 4, 16, 17, 18$.

From these results we give in table I the linear relations among the quadratic invariants for each of the irreducible suspaces $\mathcal{C}_i$ (Here we denote by A to the set $\{1, 2, 3, 4, 5, 7, 11, 13, 15, 16, 17, 18\}$).

TABLE I

Classes	Linear relations among the quadratic invariants ($\dim V \geqslant 7$)
$\mathcal{C}_1$	$i_1(\alpha) = -i_2(\alpha) = -i_3(\alpha) = \|\alpha\|^2$; $i_m(\alpha) = 0$ $(m \geqslant 4)$
$\mathcal{C}_2$	$i_1(\alpha) = 2i_2(\alpha) = -i_3(\alpha) = \|\alpha\|^2$; $i_m(\alpha) = 0$ $(m \geqslant 4)$
$\mathcal{C}_3$	$i_1(\alpha) = i_3(\alpha) = \|\alpha\|^2$; $i_2(\alpha) = i_m(\alpha) = 0$ $(m \geqslant 4)$
$\mathcal{C}_4$	$i_1(\alpha) = i_3(\alpha) = \dfrac{n}{(n-1)^2} i_4(\alpha) = \dfrac{n}{(n-1)^2} \sum_k^{2n} c_{12}^2(\alpha)(e_k)$; $i_2(\alpha) = i_m(\alpha) = 0$ $(m > 4)$
$\mathcal{C}_5$	$i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \dfrac{1}{2n} i_{14}(\alpha)$; $i_{10}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_6$	$i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = i_{12}(\alpha) = \dfrac{1}{2n} i_{10}(\alpha)$; $i_{14}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_7$	$i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \dfrac{\|\alpha\|^2}{2}$; $i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_8$	$i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \dfrac{\|\alpha\|^2}{2}$; $i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_9$	$i_6(\alpha) = i_8(\alpha) = -i_9(\alpha) = -i_{12}(\alpha) = \dfrac{\|\alpha\|^2}{2}$; $i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_{10}$	$i_6(\alpha) = -i_8(\alpha) = -i_9(\alpha) = i_{12}(\alpha) = \dfrac{\|\alpha\|^2}{2}$; $i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0$ $(m \in A)$
$\mathcal{C}_{11}$	$i_5(\alpha) = \|\alpha\|^2$; $i_m(\alpha) = 0$ $(m \neq 5)$
$\mathcal{C}_{12}$	$i_{16}(\alpha) = \|\alpha\|^2$; $i_m(\alpha) = 0$ $(m \neq 16)$

The corresponding results for the generators of the quadratic invariants with $\dim V = 3$ are done in table II (see (3.3)).

TABLE II

Classes	Linear relations among the quadratic invariants ($\dim V = 3$)	
$\mathcal{C}_5$	$i_6(\alpha) = i_9(\alpha) = \frac{1}{2}\bar{c}_{12}^2(\alpha)(\xi)$;	$i_{10}(\alpha) = i_{15}(\alpha) = i_{16}(\alpha) = 0$
$\mathcal{C}_6$	$i_6(\alpha) = i_9(\alpha) = \frac{1}{2} i_{10}(\alpha) = \frac{1}{2} c_{12}^2(\alpha)(\xi)$;	$i_{15}(\alpha) = i_{16}(\alpha) = 0$
$\mathcal{C}_9$	$i_6(\alpha) = - i_9(\alpha) = \frac{\|\alpha\|^2}{2}$;	$i_{10}(\alpha) = i_{15}(\alpha) = i_{16}(\alpha) = 0$
$\mathcal{C}_{12}$	$i_{16}(\alpha) = \|\alpha\|^2$;	$i_6(\alpha) = i_9(\alpha) = i_{10}(\alpha) = i_{15}(\alpha) = 0$

Finally, for $\dim V = 5$, the corresponding linear relations are the same as in table I.

So, we conclude

THEOREM 4.3. – *The decomposition of* $\mathcal{C}(V)$ *given before is irreducible under the action of* $U(n)\times 1$.

REMARK. – If $\dim V = 2n + 1$,

$$\dim \mathcal{C}_1 = \tfrac{1}{3}n(n-1)(n-2); \quad \dim \mathcal{C}_2 = \tfrac{2}{3}n(n-1)(n+1); \quad \dim \mathcal{C}_3 = n(n+1)(n-2);$$

$$\dim \mathcal{C}_4 = \dim \mathcal{C}_{12} = 2n; \quad \dim \mathcal{C}_5 = \dim \mathcal{C}_6 = 1; \quad \dim \mathcal{C}_7 = \dim \mathcal{C}_8 = n^2 - 1;$$

$$\dim \mathcal{C}_9 = n(n+1); \quad \dim \mathcal{C}_{10} = \dim \mathcal{C}_{11} = n(n-1) .$$

5. – Classification of almost contact metric structures.

Let M be a manifold of dimension $2n + 1$ with an almost contact metric structure (φ, ξ, η, g). For every $x \in M$, $(T_xM, \varphi_x, \xi_x, \eta_x)$ is an almost contact vector space with compatible metric g_x. Hence it is possible to decompose the vector space $\mathcal{C}(T_xM)$ as in the previous section. Let U be one of the invariant subspaces of $\mathcal{C}(T_xM)$. We say that M is of class U if $(\nabla\Phi)_x$ belongs to U, for all $x \in M$, where $\nabla\Phi$ is the covariant derivative of the fundamental 2-form Φ of the almost contact metric structure (φ, ξ, η, g). From equations (1.6) and (1.7) we have

$$c_{12}(\nabla\Phi)_x(X_x) = - \delta\Phi_x(X_x) ,$$

$$\bar{c}_{12}(\nabla\Phi)_x(\xi_x) = \delta\eta_x .$$

for all $x \in M$ and $X_x \in T_x M$. Then, using the defining conditions of each subspace $\mathcal{C}_i$ and formulas (1.3), (1.4) and (1.5), we deduce the following.

THEOREM 5.1. – *The defining relations for each of the twelve classes (being $n \geqslant 3$) are given in Table III. If $n = 2$, $\mathcal{C}_1 = \mathcal{C}_3 = |C|$; and if $n = 1$, $\mathcal{C}_i = |C|$ for $i = 1, 2, 3, 4, 7, 8, 10, 11$, where $|C|$ denotes the class of cosymplectic manifolds.*

TABLE III

Classes	Defining conditions
$\mathcal{C}_1$	$(\nabla_X \Phi)(X, Y) = 0, \quad \nabla\eta = 0$
$\mathcal{C}_2$	$d\Phi = \nabla\eta = 0$
$\mathcal{C}_3$	$(\nabla_X \Phi)(Y, Z) - (\nabla_{\varphi X} \Phi)(\varphi Y, Z) = 0, \quad \delta\Phi = 0$
$\mathcal{C}_4$	$(\nabla_X \Phi)(Y, Z) = -\dfrac{1}{2(n-1)} [g(\varphi X, \varphi Y)\, \delta\Phi(Z) - g(\varphi X, \varphi Z)\, \delta\Phi(Y) - \Phi(X, Y)\, \delta\Phi(\varphi Z) + \Phi(X, Z)\, \delta\Phi(\varphi Y)]; \quad \delta\Phi(\xi) = 0$
$\mathcal{C}_5$	$(\nabla_X \Phi)(Y, Z) = \dfrac{1}{2n} [\Phi(X, Z)\, \eta(Y) - \Phi(X, Y)\, \eta(Z)]\, \delta\eta$
$\mathcal{C}_6$	$(\nabla_X \Phi)(Y, Z) = \dfrac{1}{2n} [g(X, Z)\, \eta(Y) - g(X, Y)\, \eta(Z)]\, \delta\Phi(\xi)$
$\mathcal{C}_7$	$(\nabla_X \Phi)(Y, Z) = \eta(Z)(\nabla_Y \eta)\, \varphi X + \eta(Y)(\nabla_{\varphi X} \eta)\, Z, \quad \delta\Phi = 0$
$\mathcal{C}_8$	$(\nabla_X \Phi)(Y, Z) = -\eta(Z)(\nabla_Y \eta)\, \varphi X + \eta(Y)(\nabla_{\varphi X} \eta)\, Z, \quad \delta\eta = 0$
$\mathcal{C}_9$	$(\nabla_X \Phi)(Y, Z) = \eta(Z)(\nabla_Y \eta)\, \varphi X - \eta(Y)(\nabla_{\varphi X} \eta)\, Z$
$\mathcal{C}_{10}$	$(\nabla_X \Phi)(Y, Z) = -\eta(Z)(\nabla_Y \eta)\, \varphi X - \eta(Y)(\nabla_{\varphi Y} \eta)\, Z$
$\mathcal{C}_{11}$	$(\nabla_X \Phi)(Y, Z) = -\eta(X)(\nabla_\xi \Phi)(\varphi Y, \varphi Z)$
$\mathcal{C}_{12}$	$(\nabla_X \Phi)(Y, Z) = \eta(X)\, \eta(Z)(\nabla_\xi \eta)\, \varphi Y - \eta(X)\, \eta(Y)(\nabla_\xi \eta)\, \varphi Z$

Now, we explain how some classes just introduced coincide with classes studied by various authors (see section 1):

$|C|$ = the class of cosymplectic manifolds.

$\mathcal{C}_1 = |nKC|$ = the class of nearly-K-cosymplectic manifolds.

$\mathcal{C}_2 \oplus \mathcal{C}_9 = |aC|$ = the class of almost-cosymplectic manifolds.

$\mathcal{C}_5$ = the class of α-Kenmotsu manifolds, for all differentiable function α.

$\mathcal{C}_6 =$ the class of α-Sasakian manifolds, for all differentiable function α.

$\mathcal{C}_5 \oplus \mathcal{C}_6 = |tS| =$ the class of trans-Sasakian manifolds.

$\mathcal{C}_6 \oplus \mathcal{C}_7 = |qS| =$ the class of quasi-Sasakian manifolds.

$\mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = |sCN| =$ the class of semi-cosymplectic and normal manifolds.

$\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 = |ntS| =$ the class of nearly-trans-Sasakian manifolds.

$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = |qKC| =$ the class of quasi-K-cosymplectic manifolds.

$\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = |N| =$ the class of normal manifolds.

$\mathcal{D}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = |aKc| =$ the class of almost-K-contact manifolds.

$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11} \oplus \mathcal{C}_{12} = |sC| =$ the class of semi-cosymplectic manifolds.

6. – Examples.

A) Examples of almost contact metric manifolds of type $\mathcal{D}_1$.

Let (M, J, h) be an almost Hermitian manifold, $\dim M = 2n$. In $M \times \boldsymbol{R}$ we consider the almost contact metric structure (φ, ξ, η, g) given by

$$\varphi\left(X, a\frac{d}{dt}\right) = (JX, 0)\,, \qquad \xi = \left(0, \frac{d}{dt}\right), \qquad \eta\left(X, a\frac{d}{dt}\right) = a\,,$$

$$g\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = h(X, Y) + ab\,,$$

where a and b are C^∞ functions on $M \times \boldsymbol{R}$, $X, Y \in \mathfrak{X}(M)$. Then, we have

PROPOSITION 6.1.

(i) $M \times \boldsymbol{R}$ *is of class* $\mathcal{C}_1$ *iff* M *is nearly-Kaehlerian.*

(ii) $M \times \boldsymbol{R}$ *is of class* $\mathcal{C}_2$ *iff* M *is almost-Kaehlerian.*

(iii) $M \times \boldsymbol{R}$ *is of class* $\mathcal{C}_3$ *iff* M *is* W_3*-manifold.*

(iv) $M \times \boldsymbol{R}$ *is of class* $\mathcal{C}_4$ *iff* M *is* W_4*-manifold.*

In order to construct examples of class $\mathcal{D}_1$, through the previous proposition, let us consider the following manifolds:

1) $\boldsymbol{R}^{2n}$, endowed with the standard Kaehler structure.

2) $T(M)$, the total space of tangent bundle of a nonflat Riemannian manifold M, endowed with the standard almost Kaehler structure [18].

3) S^6, $S^2 \times \boldsymbol{R}^4$, $N_1 \times \boldsymbol{R}^4$ (N_1 being a nonplanar minimal surface in $\boldsymbol{R}^3$) endowed with the almost complex structure induced from the Cayley numbers [4].

4) $M_1 = S^6 \times (N_1 \times \boldsymbol{R}^4)$, $M_2 = T(M) \times (N_1 \times \boldsymbol{R}^4)$, $M_3 = S^6 \times T(M) \times (N_1 \times \boldsymbol{R}^4)$ endowed with the product almost Hermitian structures.

Then, taking into account proposition 6.1, we have

5) $R^{2n+1} \in |C|$; $S^6 \times \boldsymbol{R} \in \mathcal{C}_1 - |C|$; $T(M) \times \boldsymbol{R} \in \mathcal{C}_2 - |C|$; $N_1 \times \boldsymbol{R}^5 \in \mathcal{C}_3 - |C|$;

$S^1 \times S^{2k+1} \times \boldsymbol{R} \in \mathcal{C}_4 - |C|$, $(k \geqslant 1)$; $S^2 \times \boldsymbol{R}^5 \in \mathcal{C}_1 \oplus \mathcal{C}_2 - (\mathcal{C}_1 \cup \mathcal{C}_2)$;

$M_1 \times \boldsymbol{R} \in \mathcal{C}_1 \oplus \mathcal{C}_3 - (\mathcal{C}_1 \cup \mathcal{C}_3)$; $M_2 \times \boldsymbol{R} \in \mathcal{C}_2 \oplus \mathcal{C}_3 - (\mathcal{C}_2 \cup \mathcal{C}_3)$;

$S^{2k+1} \times S^{2q+1} \times \boldsymbol{R} \in \mathcal{C}_3 \oplus \mathcal{C}_4 - (\mathcal{C}_3 \cup \mathcal{C}_4)$; $M_3 \times \boldsymbol{R} \in \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 - (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$;

$M_1^0 \times \boldsymbol{R} \in \mathcal{C}_1 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 - (\mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$; $M_2^0 \times \boldsymbol{R} \in \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 - (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$,

where we denote M_1^0 and M_2^0 the manifolds which are obtained from the manifolds M_1 and M_2 by making a (non trivial) conformal change of the metric.

B) Almost contact metric structures on the hyperbolic space.

Let (H^{2n+1}, ds^2) be the $(2n+1)$-dimensional hyperbolic space, i.e.,

$$H^{2n+1} = \{(x_1, \ldots, x_{2n+1}) \in \boldsymbol{R}^{2n+1} / x_1 > 0\}$$

and ds^2 is the Riemannian metric given by

$$ds^2 = (cx_1)^{-2} \sum_{i=1}^{2n+1} (dx_i)^2, \qquad (c \neq 0).$$

The vector fields $E_i = cx_i(\partial/\partial x_i)$, $i = 1, \ldots, 2n+1$, form an orthonormal basis for this space. Let $(\varphi, \xi, \eta, ds^2)$ be an almost contact metric structure on H^{2n+1}, and φ_j^i the components of φ with respect to the basis $\{E_1, \ldots, E_{2n+1}\}$. If $\varphi_j^i =$ constant and $n \geqslant 2$, we have that $(\varphi, \xi, \eta, ds^2)$ is

(i) $\mathcal{C}_5$ iff $\xi = cx_1(\partial/\partial x_1)$.

In particular, if $c = -1$ $(\varphi, \xi, \eta, ds^2)$ is Kenmotsu.

(ii) $\mathcal{C}_4 \oplus \mathcal{C}_{12} - (\mathcal{C}_4 \cup \mathcal{C}_{12})$ iff $\xi = \sum_{i=2}^{2n+1} x_1 k_i(\partial/\partial x_i)$, $(k_i = \text{cte})$

[iii) $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12} - (\mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_{12})$ iff $\xi = \sum_{i=1}^{2n+1} x_1 k_i(\partial/\partial x_i)$, where $k_1 \neq 0$ and $k_i \neq 0$ for some $i > 1$.

Finally, in (H^3, ds^2) the almost contact metric structures $(\varphi, \xi, \eta, ds^2)$ given by

$$\varphi^i_j = \text{constant} \qquad \xi = x_1 k_2 \frac{\partial}{\partial x_2} + x_1 k_3 \frac{\partial}{\partial x_3}\,, \qquad k_1, k_3 = \text{cte}\,,$$

are of class $\mathcal{C}_{12}$.

C) Almost contact metric structures on the generalized Heisenberg group $H(p, 1)$, $p \geqslant 1$.
Let $H(p, 1)$ be the group of matrices of real numbers of the form

$$a = \begin{bmatrix} 1 & A & c \\ 0 & I_p & {}^tB \\ 0 & 0 & 1 \end{bmatrix}$$

where I_p denotes the identity $p \times p$ matrix, $A = (a_1, \ldots, a_p)$, $B = (b_1, \ldots, b_p) \in \boldsymbol{R}^p$ and $c \in \boldsymbol{R}$. $H(p, 1)$ is a connected simply connected nilpotent Lie group of dimension $2p + 1$ which is called a generalized Heisenberg group (see [7]). Moreover, $H(p, 1)$ is a Heisenberg group ([10]).

A global system of coordinates (x_i, x_{p+i}, z), $1 \leqslant i \leqslant p$, on $H(p, 1)$ is defined by

$$x_i(a) = a_i\,, \qquad x_{p+i}(a) = b_i\,, \qquad z(a) = c\,, \qquad (1 \leqslant i \leqslant p)\,.$$

A basis for the left invariant 1-forms on $H(p, 1)$ is given by

$$\alpha_i = dx_i\,, \qquad \alpha_{p+i} = dx_{p+i}\,, \qquad \gamma = dz - \sum_{j=1}^{p} x_j\, dx_{p+j}\,,$$

and its dual basis of left invariant vector fields on $H(p, 1)$ is given by

$$X_i = \frac{\partial}{\partial x_i}\,, \qquad X_{p+i} = \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial z}\,, \qquad Z = \frac{\partial}{\partial z}\,, \qquad i = 1, \ldots, p\,.$$

Define a left invariant metric on $H(p, 1)$ by

$$g = \sum_{k=1}^{2p} \alpha_k \otimes \alpha_k + \gamma \otimes \gamma\,.$$

With respect to this metric the basis $\{X_k, Z\}$, $k = 1, \ldots, 2p$, is orthonormal.

Now let (φ, ξ, η, g) be an almost contact metric structure on $H(p, 1)$ and φ^m_n the components of φ with respect to basis $\{X_k, Z\}$. Then, using the Riemannian connection of the metric g, we obtain:

If $Z = \xi$, $\varphi^m_n =$ constant and

$$\varphi^{p+i}_j = -\,\varphi^i_{p+j}\,, \qquad \varphi^{p+i}_{p+j} = \varphi^i_j\,, \qquad 1 \leqslant i, j \leqslant p$$

then (φ, ξ, η, g) is of class $C_6 \oplus C_7 - |C|$. Moreover, it is $\mathcal{C}_7$ iff $\sum_{i=1}^{p} \varphi_i^{p+i} \equiv 0$, and it is $\mathcal{C}_6$ iff $\varphi_i^{p+i} = \varphi_j^{p+j} = \lambda$, ($\lambda = \text{constant} \neq 0$), and the other components of φ are zero.

D) Almost contact metric structures on the generalized Heisenberg group $H(1, r)$, $r > 1$.

The generalized Heisenberg group $H(1, r)$, $r > 1$, is the Lie group of real matrices of the form

$$a = \begin{bmatrix} I_r & {}^tA & c \\ 0 & 1 & {}^tB \\ 0 & 0 & 1 \end{bmatrix}$$

where I_r denotes the identity $r \times r$ matrix, $A = (a_1, \ldots, a_r)$, $B = (b_1, \ldots, b_r) \in \boldsymbol{R}^r$ and $c \in \boldsymbol{R}$. This group is a connected and simply connected nilpotent group of dimension $2r + 1$. The dimension of its center is $r > 1$ and so $H(1, r)$ is not a Heisenberg group.

A global system of coordinates (x_i, x_{r+i}, z) $1 \leqslant i \leqslant r$, on $H(1, r)$ is defined by

$$x_i(a) = a_i\,, \qquad x_{r+i}(a) = b_i\,, \qquad z(a) = c\,.$$

A basis for the left invariant 1-forms on $H(1, r)$ is given by

$$\alpha_i = dx_i\,, \qquad \alpha_{r+i} = dx_{r+i}\,, \qquad \gamma = dz\,,$$

and its dual basis by

$$X_i = \frac{\partial}{\partial x_i}\,, \qquad X_{r+i} = \frac{\partial}{\partial x_{r+i}}\,, \qquad Z = \frac{\partial}{\partial z} + \sum_{j=1}^{r} x_j \frac{\partial}{\partial x_{r+j}}\,.$$

This basis is orthonormal with respect to the left invariant metric defined by

$$g = \sum_{k=1}^{2r} \alpha_k \otimes \alpha_k + \gamma \otimes \gamma\,.$$

Now, let (φ, ξ, η, g) be an almost contact metric structure on $H(1, r)$ and φ_n^m the components of φ with respect to basis $\{X_k, Z\}$, $k = 1, \ldots, 2r$. Then, using the Riemannian connection of the metric g, we obtain:

1) If $Z = \xi$, $\varphi_n^m = \text{constant}$ and

$$\varphi_j^{r+i} = -\varphi_{r+j}^{i} = 0 \qquad \varphi_{r+j}^{r+i} = \varphi_j^i$$

then (φ, ξ, η, g) is of class $\mathcal{C}_8$.

2) If $Z = \xi$, $\varphi_n^m =$ constant and

$$\varphi_j^{r+i} = -\varphi_{r+j}^i \qquad \varphi_{r+j}^{r+i} = \varphi_j^i = 0$$

then (φ, ξ, η, g) is of class $\mathcal{C}_9$.

REMARK. – Let $\Gamma(p, 1)$ and $\Gamma(1, r)$ be the subgroups of matrices of $H(p, 1)$ and $H(1, r)$, respectively, with integer entries and define $N(p, 1) = \Gamma(p, 1)\backslash H(p, 1)$ and $N(1, r) = \Gamma(1, r)\backslash H(1, r)$ be the spaces of right cosets. Then $N(p, 1)$ and $N(1, r)$ are compact nilmanifolds [7]. Denote by $((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the projections on $N(p, 1)$ and $N(1, r)$ of the almost contact metric structures defined in C and D, respectively. It's easy to check that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ belong to the same class that the corresponding structure (φ, ξ, η, g) on $H(p, 1)$ and $H(1, r)$.

E) Other examples.

Let G be the Lie group of real matrices of the form

$$a = \begin{bmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{bmatrix}$$

with the left invariant metric

$$g = e^{2z}\,dx^2 + e^{-2z}\,dy^2 + \lambda^2\,dz^2\,, \qquad \lambda > 0\,.$$

(G, g) is a 4-symmetric space, which is isomorphic to the semi-direct product of $\boldsymbol{R}$ and $\boldsymbol{R}^2$, both with the additive group structure, and where the action of $\boldsymbol{R}$ and $\boldsymbol{R}^3$ is given by the matrix

$$\begin{bmatrix} e^{z} & 0 \\ 0 & e^{-z} \end{bmatrix},$$

i.e., the group $E(1, 1)$ of rigid motions of the Minkowski 2-space.

With respect to the metric g, the basis of invariant vector fields $\{X_1, X_2, X_3\}$ given by

$$X_1 = e^{-z}\frac{\partial}{\partial x}\,, \qquad X_2 = e^{z}\frac{\partial}{\partial y}\,, \qquad X_3 = \frac{1}{\lambda}\frac{\partial}{\partial z}$$

if orthonormal.

It is easy to see that an almost contact metric structure (φ, ξ, η, g) on G is of class $\mathcal{C}_{12}$ if $\xi = X_1$ or $\xi = X_2$; and it is of class $\mathcal{C}_9$ if $\xi = X_3$.

Finally, we obtain an example of manifold belong to $\mathcal{C}_{11}$. For it, we consider the complex matrix group G of the form

$$a = \begin{bmatrix} e^{it} & 0 & z \\ 0 & e^{-it} & w \\ 0 & 0 & 1 \end{bmatrix}.$$

Here z, w denote complex variables and t a real variable. This Lie group is diffeomorphic to $\boldsymbol{C}^2(z, w)\times\boldsymbol{R}(t)$. A left invariant metric on G is

$$g = dz\,d\bar{z} + dw\,d\bar{w} + dt^2 . \tag{6.1}$$

The vector fields $\{Z_1, \bar{Z}_1, Z_2, \bar{Z}_2, W\}$ given by

$$Z_1 = e^{it}\frac{\partial}{\partial z}, \qquad Z_2 = e^{-it}\frac{\partial}{\partial w}, \qquad W = \frac{\partial}{\partial t},$$

are invariant under the action of G and they form an orthonormal basis of the Lie algebra of G. Put

$$X_1 = \sqrt{2}\,\mathrm{Re}\,(Z_1 + Z_2)\,, \qquad X_2 = \sqrt{2}\,\mathrm{Im}\,(Z_1 + Z_2)\,,$$
$$X_3 = \sqrt{2}\,\mathrm{Im}\,(Z_2 - Z_1)\,, \qquad X_4 = \sqrt{2}\,\mathrm{Re}\,(Z_1 - Z_2)\,.$$

Identifying $\boldsymbol{C}^2\times\boldsymbol{R}$ with the real cartesian space $\boldsymbol{R}^5$ with invariant Riemannian metric obtained from (6.1), it follows that $\{X_1, X_2, X_3, X_4, W\}$ is an orthonormal basis on this space.

Now, let (φ, ξ, η, g) be an almost contact metric structure on G and φ^i_j the components of φ with respect to basis $\{X_1, X_2, X_3, X_4, W\}$. Then we obtain:

1) If $\xi = W$, $\varphi^i_j =$ constant and

$$\varphi^3_2 = \varphi^4_1 \quad \text{and } \varphi^1_2 = \varphi^3_4$$

then (φ, ξ, η, g) is cosymplectic.

2) If $\xi = W$, $\varphi^i_j =$ constant and

$$\varphi^3_2 \neq \varphi^4_1 \quad \text{and } \varphi^1_2 \neq \varphi^3_4$$

then (φ, ξ, η, g) is of class $\mathcal{C}_{11}$.

An example of almost contact metric structure satisfying the last condition, and so of class $\mathcal{C}_{11}$, is the following:

$$\varphi Z_1 = iZ_2\,, \qquad \varphi Z_2 = iZ_1\,, \qquad \varphi \bar{Z}_1 = -\,i\bar{Z}_2\,,$$
$$\varphi \bar{Z}_2 = -\,i\bar{Z}_1; \qquad \xi = W; \qquad \eta = dt\,.$$

Note. - We learned later (by a private communication of L. VANHECKE) on that the decomposition given in this paper has also been obtained by F. BOUTEN during the preparation of hers doctoral dissertation (unpublished but announced in the abstracts of the IX Osterreichischer Mathematiker Kongress, Salsburg 1977, p. 83).

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