Periodic Solutions of Liénard Systems at Resonance (*).

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Summary. – *We use classical Leray-Schauder techniques in order to derive the existence of periodic solutions for Liénard differential systems.*

1. – Introduction.

In [13], [16], R. REISSIG proved the existence of 2π-periodic solutions for the Liénard differential equation

$$x'' + f(x)x' + g(t, x) = e(t) \tag{1.1}$$

where f, g and e are continuous and e has mean value zero, provided that one of the two following conditions is satisfied:

(i) there exists $R > 0$ such that

$$xg(t, x) \leqslant 0 \quad \text{for all } x \in \boldsymbol{R} \text{ with } |x| \geqslant R; \tag{1.2}$$

(ii) there exists $R > 0$ such that

$$xg(t, x) \geqslant 0 \quad \text{for all } x \in \boldsymbol{R} \text{ with } |x| \geqslant R \tag{1.2$'$}$$

and

$$\limsup_{|x|\to\infty} x^{-1} g(t, x) \leqslant q < 1 \quad \text{uniformly in } t \in [0, 2\pi] \tag{1.3}$$

Starting by A. C. LAZER [8], similar problems have been already studied by J. BEBERNES [1], J. BEBERNES and M. MARTELLI [2], S. H. CHANG [5], M. MARTELLI [9], J. MAWHIN [11], R. REISSIG [13-16].

Recently J. MAWHIN and J. R. WARD [12] considered the equation (1.1) with f continuous, g a Caratheodory's function and e Lebesgue integrable with mean value

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zero. They proved the existence of 2π-periodic solutions for (1.1) under assumptions (1.2)′ and

$$\limsup_{|x|\to\infty} x^{-1}g(t, x) \leqslant \Gamma(t) \quad \text{uniformly in } t\in[0, 2\pi]\,, \tag{1.4}$$

where $\Gamma(t)$ is measurable and such that

$$\Gamma(t) \leqslant 1 \quad \text{for a.e. } t\in[0, 2\pi] \tag{1.5}$$

with the strict inequality on a subset of positive measure.

Conditions (1.2)′, (1.3) or (1.2)′, (1.4) are usually called resonance conditions; indeed they reduce in the linear case ($g = 0$) to the necessary and sufficient conditions for the solvability of the resonant 2π-periodic problem

$$\begin{cases} x'' + cx' = e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\,. \end{cases}$$

The aim of the present paper is to extend these results concerning the existence of 2π-periodic solutions to the case of the following differential system of Liénard type

$$x'' + \frac{d}{dt}\big(\text{Grad}\, F(x)\big) + g(t, x) = e(t) \tag{1.6}$$

where $F\colon \mathbf{R}^n \to \mathbf{R}$ is of class C^2, $g\colon [0, 2\pi]\times\mathbf{R}^n \to \mathbf{R}^n$ is a generalized Caratheodory's map (see Section 3), $e\colon [0, 2\pi] \to \mathbf{R}^n$ is integrable. Some results concerning this problem or related problems are contained in the papers [2], [3], [4], [10], [11], [17], [19].

We derive the existence of 2π-periodic solutions for vector differential equation (1.6) from the classical Leray-Schauder techniques (see e.g. [6]).

In Theorem 1 we prove that system (1.6) has a 2π-periodic solution under the assumption that for each $i = 1, \ldots, n$

(B^-) there exists $R_i > 0$ such that $x_i g_i(t, x) \leqslant 0$ for all $x \in \mathbf{R}$ with $|x_i| \geqslant R_i$.

This hypothesis seems to be a natural extension of (i) to the case of system (1.6). Moreover Theorem 1 improves the previous results of the authors cited above.

In Theorem 2 we replace condition (1.2)′ by the following ones

(B_1^+) there exists $R_i > 0$ such that

$$x_i g_i(t, x) \geqslant 0 \quad \text{for all } x \in \mathbf{R}^n \text{ with } |x_i| \geqslant R_i \quad (1 \leqslant i \leqslant n)\,,$$

and, extending (1.4), we consider the assumption

(B_2^+)
$$\limsup_{|x_i|\to\infty} x_i^{-1} g_i(t, x) \leqslant \Gamma_i(t) \leqslant 1$$

uniformly a.e. $t \in [0, 2\pi]$, where $\Gamma_i \in L^1[0, 2\pi]$ is such that $\Gamma_i(t) < 1$ on a subset of $[0, 2\pi]$ of positive measure. $(1 \leqslant i \leqslant n)$.

In this case we follow a technique due to J. MAWHIN and J. R. WARD [12] in order to obtain the required a priori estimates.

Finally we give a result (Theorem 3) concerning the existence of 2π-periodic solutions for system (1.6), allowing that each component of g satisfies either (B^-) or (B_1^+) together with (B_2^+).

The hypotheses of these statements compare the asymptotic behaviour of $x_i^{-1} g_i(t, x)$ to the first eigenvalue 0 and the two first eigenvalue 0 and 1 of the problem

$$\begin{cases} x'' + \lambda x = 0 \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 . \end{cases}$$

At the end of this paper, employing a technique due to C. P. GUPTA and J. MAWHIN [7], we consider the case where for some $i = 1, \ldots, n$, together with (B_1^+), $\limsup_{|x_i|\to\infty} x_i^{-1} g_i(t, x)$ can cross 1 as far as this crossing takes place in a subset of $[0, 2\pi]$ of sufficiently small measure.

2. – Notations and definitions.

Let us set $J = [0, 2\pi]$. We will use the symbol $x = \operatorname{col}(x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and the symbol $\|\cdot\|$ for the Euclidean norm in $\boldsymbol{R}^n$.

We will use the following spaces.

1) $L^p(J, \boldsymbol{R}^n)$ are the usual Lebesgue spaces, $1 \leqslant p \leqslant +\infty$.

2) $H^1(J, \boldsymbol{R}^n) = \{x \colon J \to \boldsymbol{R}^n,\ x$ absolutely continuous, $x' \in L^2(J, \boldsymbol{R}^n)$,

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\},$$

with the norm $\|x\|_{H^1} = \left\{\sum_{i=1}^{n}\left(\left((2\pi)^{-1}\int_J x_i(t)\, dt\right)^2 + (2\pi)^{-1}\int_J (x_i'(t))^2\, dt\right\}^{\frac{1}{2}}$.

3) $\tilde{H}^1(J, \boldsymbol{R}^n) = \left\{x \in H^1(J, \boldsymbol{R}^n) \colon \int_J x(t)\, dt = 0\right\}$.

4) $W^{2,1}(J, \boldsymbol{R}^n) = \{x\colon J \to \boldsymbol{R}^n,\ x$ and x' absolutely continuous,

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}$$

with the norm $\|x\|_{W^{1,2}} = \left\{\sum_{i=1}^{n}\left((2\pi)^{-1}\sum_{k=0}^{2}\int_J |x_i^{(k)}(t)|\,dt\right)^2\right\}^{\frac{1}{2}}$.

For sake of simplicity in the notations of the spaces we will omit $\boldsymbol{R}^n$ when $n = 1$.

Clearly if $x = \operatorname{col}(x_1, \ldots, x_n) \in H^1(J, \boldsymbol{R}^n)$, then any $x_i \in H^1(J)$ for $i = 1, \ldots, n$.

We recall that every $x_i \in H^1(J)$ can be written in the form $x_i(t) = \bar{x}_i + \tilde{x}_i(t)$ with $\tilde{x}_i \in \tilde{H}^1(J)$ and $\bar{x}_i = (2\pi)^{-1}\int_J x_i(t)\,dt$. Moreover $\|x_i\|_{H^1} = \{\bar{x}_i^2 + (2\pi)^{-1}\int_J (x_i'(t))^2\,dt\}^{\frac{1}{2}}$, so that we have $\|x\|_{H^1} = \left\{\sum_{i=1}^{n}\|x_i\|_{H^1}^2\right\}^{\frac{1}{2}}$ and we use the same symbol for the norm in $H^1(J, \boldsymbol{R}^n)$ and $H^1(J)$.

In the sequel we will use the following results.

LEMMA 1 (J. MAWHIN - J. R. WARD [12]). – *Let $\Gamma \in L^1(J)$ be such that $\Gamma(t) \leqslant 1$ a.e. on J and the stricty inequality holds on a subset of positive measure. Then there exists $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{x} \in \tilde{H}^1(J)$ we have*

$$(2\pi)^{-1}\int_J \left((\tilde{x}'(t))^2 - \Gamma(t)\tilde{x}^2(t)\right)dt \geqslant \delta\|\tilde{x}\|_{H^1}^2 .$$

LEMMA 2 (J. MAWHIN - J. R. WARD [12]). – *Let Γ like in Lemma 1. Let $\delta > 0$ be associated to Γ by that Lemma 1 and let $\varepsilon > 0$. Then for all $p \in L^1(J)$ satisfying $\bar{p} > 0$ and $p(t) \leqslant \Gamma(t) + \varepsilon$ a.e. on J, all continuous function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ and all $x \in W^{2,1}(J)$ one has*

$$(2\pi)^{-1}\int_J (\bar{x} - \tilde{x}(t))\left(x''(t) + f(x(t))x'(t) + p(t)x(t)\right)dt > (\delta - \varepsilon)\|\tilde{x}\|_{H^1}^2 .$$

3. – The case of assumption B^-.

Let $g\colon J \times \boldsymbol{R}^n \to \boldsymbol{R}^n$, $(t, x) \mapsto g(t, x) = \left(g_1(t, x), \ldots, g_n(t, x)\right)$ be such that $g(\cdot, x)$ is measurable on J for each $x \in \boldsymbol{R}^n$ and $g(t, \cdot)$ is continuous on $\boldsymbol{R}^n$ for almost all $t \in J$. Assume moreover that

(A) *for each real number $r > 0$ and for any $i = 1, \ldots, n$ there exists $a_i, b_i \in L^1(J)$ (which depends also on r) such that*

$$|g_i(t, x)| \leqslant a_i(t) + b_i(t)\left(\sum_{k=1}^{n}|x_k|^\alpha\right) \qquad a.e.\ t \in J ,$$

for each $x \in \boldsymbol{R}^n$ with $|x_i| \leqslant r$ and $0 \leqslant \alpha < 1$.

Such a g will be called a *generalized Caratheodory's map.*

REMARK. – We observe that for any $\bar{r}>0$ there exists $\alpha_r \in L^1(J)$ such that $\|g(t,x)\| \leqslant \alpha_r(t)$ a.e. on J for every $x \in \boldsymbol{R}^n$ with $\|x\| \leqslant \bar{r}$.

Consider the following periodic boundary value problem for the Liénard system

$$(1) \qquad \begin{cases} x''(t) + \dfrac{d}{dt}\big(\operatorname{Grad} F(x(t))\big) + g\big(t, x(t)\big) = e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \,. \end{cases}$$

We prove the following existence result for (1).

THEOREM 1. – *Let $F \in C^2(R^n, R)$ and let g be a generalized Caratheodory's map. Assume that g verifies the following conditions for any $i = 1, \dots, n$*

(B^-) *there exists $R_i > 0$ such that $g_i(t,x)x_i \leqslant 0$ for a.e. $t \in J$, for every $x \in \boldsymbol{R}^n$ with $|x_i| \geqslant R_i$.*

Then Problem (1) *has at least one solution provided that $e \in L^1(J)$, $\bar{e} = 0$.*

PROOF. – For every $u \in H^1(J, \boldsymbol{R}^n)$ consider the following system

$$(3.1) \qquad x_i''(t) - x_i(t) = e_i(t) - u_i(t) - g_i\big(t, u(t)\big) - \frac{d}{dt}\frac{\partial F}{\partial u_i}\big(u(t)\big)\,, \qquad i = 1, \dots, n\,.$$

From classical results it follows that the homogeneous system associated to (3.1) has only the trivial 2π-periodic solution. Hence for every $u \in H^1(J, \boldsymbol{R}^n)$, system (3.1) has only one 2π-periodic solution, which depends continuously on u.

Let $T\colon H^1(J, \boldsymbol{R}^n) \to H^1(J, \boldsymbol{R}^n)$ be the operator which associates to any $u \in H^1(J, \boldsymbol{R}^n)$ the solution of system (3.1). Clearly T is continuous; moreover for the compact imbedding of $W^{1,2}(S, \boldsymbol{R}^n)$ into $H^1(S, \boldsymbol{R}^n)$ we have that T is a compact operator.

Since the fixed points of T are solutions for problem (1), we will apply the classical Leray-Schauder degree theory to the operator T. For this purpose it suffices to show that the set of 2π-periodic solutions forthe system

$$(3.2) \qquad x_i''(t) - (1-\lambda)x_i(t) = \lambda\left(-\frac{d}{dt}\frac{\partial F}{\partial x_i}\big(x(t)\big) - g_i\big(t, x(t)\big) + e_i(t)\right), \qquad i = 1, \dots, n$$

is bounded independently of $\lambda \in \,]0, 1[$.

In order to obtain this a priori estimate, we need some inequalities we are going to get into two steps.

1*st step.* – For any $i = 1, \dots, n$, let R_i be given by hypothesis (B^-). We set $R = \max\{R_i, i = 1, \dots, n\}$. We claim that, if $x \in H^1(J, \boldsymbol{R}^n)$ is a 2π-periodic solution of system (3.2) for some $\lambda \in]0, 1[$, then

$$(2\pi)^{-1}|\bar{x}_i| \leqslant A + \|\tilde{x}_i\|_{H^1} \quad \text{for } i = 1, \dots, n \tag{3.3}$$

where $A = (2\pi)^{-1}R$.

First we will show that for any $i = 1, \dots, n$ there exists $t_i \in J$ such that $|x_i(t_i)| \leqslant R_i$. Assume that for some $i = 1, \dots, n$ we have

$$|x_i(t)| > R_i \quad \text{for every } t \in J\,. \tag{3.4}$$

Integrating the i-th equation of (3.2) we obtain

$$\lambda \int_J g_i(t, x)\, dt = (1 - \lambda) \int_J x_i(t)\, dt\,.$$

Since $\lambda \in]0, 1[$, using (3.4) and hypothesis (B^-) on g, we will reach a contradiction.

On the other hand, for any $i = 1, \dots, n$, there exists $\bar{t}_i \in J$ such that $\bar{x}_i = x_i(\bar{t}_i)$. Therefore we can write

$$\bar{x}_i = x_i(\bar{t}_i) = x_i(t_i) + \int_{t_i}^{\bar{t}_i} x_i'(t)\, dt \qquad (i = 1, \dots, n)\,.$$

Hence by the Cauchy-Schwarz's inequality we get

$$|\bar{x}_i| \leqslant R_i + 2\pi\|\tilde{x}_i\|_{H^1} \qquad (i = 1, \dots, n)\,.$$

And so the inequalities (3.3) are proved.

2nd step. – For any $i = 1, \dots, n$ consider the constants R_i given by hypothesis (B^-). Then from (A), for each $i = 1, \dots, n$ there exist $a_i, b_i \in L^1(J)$ (which depend also on R_i) such that

$$|g_i(t, x)| \leqslant a_i(t) + b_i(t)\Big(\sum_{k=1}^{n} |x_k|^\alpha\Big)\,, \quad \text{for a.e. } t \in S\,, \tag{3.5}$$

$$\text{for any } x \in \mathbf{R}^n,\ |x_i| \leqslant R_i,\ 0 \leqslant \alpha < 1\,.$$

Let us set

$$L = \max\big\{\|e_i\|_{L^1} + 2\|a_i\|_{L^1},\ i = 1, \dots, n\big\} \tag{3.6}$$

and

$$M = 2\max\big\{\|b_i\|_{L^1},\ i = 1, \dots, n\big\}\,. \tag{3.7}$$

Now we shall prove that if $x \in H^1(J, \mathbf{R}^n)$ is a 2π-periodic solution of system (3.2) for some $\lambda \in]0, 1[$, then for any $i = 1, \ldots, n$

$$0 \geqslant \|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}(|\bar{x}_i| + \|\tilde{x}_i\|_C)\Big\{L + M\sum_{k=1}^{n}\|x_k\|^\alpha_C\Big\} . \tag{3.8}$$

For each $i = 1, \ldots, n$, we define $\gamma_i \colon J \times \mathbf{R}^n \to \mathbf{R}^n$ as follows

$$\gamma_i(t, x) = \begin{cases} x_i^{-1} g_i(t, x) & \text{if } |x_i| \geqslant R_i \\ R_i^{-1} g_i(t, x_1, \ldots, x_{i-1}, R_i, x_{i+1}, \ldots, x_n)\dfrac{x_i}{R_i} - (1 - x_i/R_i) & \text{if } 0 \leqslant x_i < x_i \\ R_i^{-1} g_i(t, x_1, \ldots, x_{i-1}, -R_i, x_{i+1}, \ldots, x_n)\dfrac{x_i}{R_i} - (1 + x_i/R_i) & \text{if } -R_i < x_i < 0 \end{cases}$$

By construction, from hypothesis (B^-) on g it follows that $\gamma_i(t, x) \leqslant 0$ for a.e. $t \in J$, for any $x \in \mathbf{R}^n$, $i = 1, \ldots, n$.

Moreover the map $\tilde{g}_i \colon J \times \mathbf{R}^n \to \mathbf{R}^n$, $(t, x) \mapsto \tilde{g}_i(t, x) = \gamma_i(t, x)x_i$ is a generalized Caratheodory's map such that

$$\begin{cases} \tilde{g}_i(t, x) = g_i(t, x) & \text{for any } x \in \mathbf{R}^n \text{ with } |x_i| \geqslant R_i \\ \tilde{g}_i(t, x)x_i \leqslant 0 & \text{for any } x_i \in \mathbf{R} . \end{cases} \tag{3.9}$$

On the other hand, setting

$$h_i(t, x) = g_i(t, x) - \tilde{g}_i(t, x) , \qquad t \in J,\ x \in \mathbf{R}^n ,$$

we have by construction and by hypothesis (A) that

$$|h_i(t, x)| \leqslant 2\Big(a_i(t) + b_i(t)\Big(\sum_{k=1}^{n}|x_k|^\alpha\Big)\Big) \tag{3.10}$$

for almost all $t \in J$, $x \in \mathbf{R}^n$.

Consider now the i-th equation of system (3.2) written in the equivalent form

$$\begin{aligned} x_i''(t) + (1 - \lambda)(-x_i(t)) + \\ + \lambda\Big(\tilde{g}_i(t, x(t)) + h_i(t, x(t)) + \frac{d}{dt}\Big(\frac{\partial F}{\partial x_i}(x(t))\Big) - e_i(t)\Big) = 0 . \end{aligned} \tag{3.11}$$

If $x \in H^1(J, \mathbf{R}^n)$ is a 2π-periodic solution of system (3.2) for some $\lambda \in]0, 1[$, multiplying (3.11) by $(-x_i(t))$ and integrating, we get

$$\begin{aligned} 0 = (2\pi)^{-1}\Big[\int_J (\tilde{x}_i'(t))^2\, dt + (1 - \lambda)\int_J x_i^2(t)\, dt - \lambda\int_J \tilde{g}_i(t, x(t))x_i(t)\, dt + \\ + \lambda\int_J \big(e_i(t) - h_i(t, x(t))\big)x_i(t)\, dt\Big] + (1 - \lambda)\int_J (-x_i(t))\frac{\partial F}{\partial x_i}(x(t))\, dt . \end{aligned}$$

Taking into account inequalities (3.9) and (3.10), from (3.5) it follows that for any $i = 1, \dots, n$

$$0 \geq \|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\|x_i\|_C\Big\{\|e_i\|_{L^1} + 2\|a_i\|_{L^1} + 2\|b_i\|_{L^1}\Big(\sum_{k=1}^{n}\|x_k\|^\alpha_C\Big)\Big\} - \\ - (1-\lambda)\int_J x_i(t)\frac{\partial F}{\partial x_i}(x(t))\,dt\,.$$

Now, using (3.6) and (3.7) we can write

$$0 \geq \|\tilde{x}_i\|^2_H - (2\pi)^{-1}\|x_i\|_C\Big\{L + M\Big(\sum_{k=1}^{n}\|x_k\|^\alpha_C\Big)\Big\} - (1-\lambda)\int_J x_i(t)\frac{\partial F}{\partial x_i}(x(t))\,dt\,,$$

from which inequalities (3.8) follow.

On the other hand, taking into account inequalities (3.3) proved in first step and the well known inequalities (see e.g. [18], pg. 208)

$$\|\tilde{x}_i\|_C \leq (\pi/\sqrt{3})\|\tilde{x}_i\|_{H^1}$$

we observe that for any $i = 1, \dots, n$

$$(3.12) \qquad (2\pi)^{-1}\|x_i\|_C \leq (2\pi)^{-1}(|\bar{x}_i| + \|\tilde{x}_i\|_C) \leq A + a\|\tilde{x}_i\|_{H^1}$$

where $a = 1 + (2\sqrt{3})^{-1}$.

Hence recalling that $A = (2\pi)^{-1}R$, with $R = \max\{R_i,\ i = 1, \dots, n\}$, we can write for any $k = 1, \dots, n$

$$\|x_k\|_C \leq R + 2\pi a\|\tilde{x}_k\|_{H^1}\,.$$

Since $(b+c)^\alpha \leq b^\alpha + c^\alpha$ for $0 \leq \alpha < 1$, $b, c \geq 0$, by the last inequalities we get also

$$(3.13) \qquad \sum_{k=1}^{n}\|x_k\|^\alpha_C \leq nR^\alpha + (2\pi a)^\alpha\Big(\sum_{i=1}^{n}\|\tilde{x}_k\|^\alpha_{H^1}\Big)\,.$$

Let us set $(2\pi a)^\alpha M = K$, $L + nR^\alpha M = H$; then using (3.12) and (3.13) in (3.8) it follows that

$$(3.14) \quad 0 \geq \|\tilde{x}_i\|^2_{H^1} - (A + a\|\tilde{x}_i\|_{H^1})\Big[H + K\Big(\sum_{k=1}^{n}\|\tilde{x}_k\|^\alpha_{H^1}\Big)\Big] - (1-\lambda)\int_J x_i(t)\frac{\partial F}{\partial x_i}(x(t))\,dt\,,$$

Now, adding for $i = 1, \dots, n$ the inequalities (3.14), we easily find that there exist constants $c_1, c_2, c_3, c_4 \geq 0$ such that

$$0 \geq \|\tilde{x}\|^2_{H^1} - c_1\|\tilde{x}\|^{\alpha+1}_{H^1} - c_2\|\tilde{x}\|^\alpha_{H^1} - c_3\|\tilde{x}\|_{H^1} - c_4\,.$$

Hence there exists $s > 0$ such that $\|\tilde{x}\|_{H^1} \leqslant s$. From this fact and from inequalities (3.3) we have that

$$\|x\|_{H^1} < d$$

for some constant $d > 0$ and we are done.

4. – The case of assumption B^+.

THEOREM 2. – *Let $F \in C^2(R^n, R)$ and let g be a generalized Caratheodory's map. Assume that g verifies the following conditions for any $i = 1, \ldots, n$:*

(B_1^+) *there exists $R_i > 0$ such that $g_i(t, x)x_i \geqslant 0$ for every $x \in \mathbf{R}^n$ with $|x_i| \geqslant R_i$*

and

(B_2^+) $\limsup\limits_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \leqslant \Gamma_i(t)$ *uniformly for almost all $t \in J$ and uniformly for $x_j \in \mathbf{R}$, $j \neq i$, where $\Gamma_i \in L^1(J)$ is such that $\Gamma_i(t) \leqslant 1$ a.e. on J, with the stricty inequality on a subset of positive measure.*

Then Problem (1) *has at least one solution provided $e \in L^1(J, R^n)$, $\bar{e} = 0$.*

PROOF. – For any $i = 1, \ldots, n$, hypothesis (B_1^+) implies that $\Gamma_i(t) \geqslant 0$ a.e. on J. Without loss of generality we can assume that $\Gamma_i(t) \not\equiv 0$ a.e. on J for $i = 1, \ldots, n$.

For every $u \in H^1(J, R^n)$ consider as in Theorem 1 the system

$$x_i''(t) + \Gamma_i(t)x_i(t) = e_i(t) - \Gamma_i(t)u_i(t) - g_i(t, u(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial u_i}(u(t))\right), \qquad i = 1, \ldots, n \,. \tag{4.1}$$

Taking into account the hypotheses on Γ_i, the homogeneous system associated to (4.1) has only by Lemma 1 the trivial solution. Hence for every $u \in H^1(J, R^n)$, system (4.1) has only one 2π-periodic solution which depends continuously on u.

As in Theorem 1 the operator solution $T\colon H^1(J, R^n) \to H^1(J, R^n)$ for system (4.1) is a compact operator. Therefore it suffices to show that the set of 2π-periodic solutions for the system

$$x_i''(t) + (1 - \lambda)\Gamma_i(t)x_i(t) = \lambda\left(-\frac{d}{dt}\left(\frac{\partial F}{\partial x_i}(x(t))\right) - g_i(t, x(t)) + e_i(t)\right), \qquad i = 1, \ldots, n \,, \tag{4.2}$$

is bounded for every $\lambda \in]0, 1[$ independently of λ.

Let $x \in H^1(J, R^n)$ be a 2π-periodic solution for system (4.2). Proceeding like in the proof of Theorem 1 we obtain by means of hypothesis (B_1^+):

$$(2\pi)^{-1}|\bar{x}_i| \leqslant A + \|\tilde{x}_i\|_{H^1} \quad \text{for } i = 1, \ldots, n .$$

Let $\delta_i > 0$ be associated to the function Γ_i of hypothesis (B_2^+) by Lemma 1. We set

$$\delta = \tfrac{1}{2} \min\{\delta_i, \ i = 1, \ldots, n\} .$$

By hypotheses (B_1^+) and (B_2^+), for any $i = 1, \ldots, n$ there exists $\bar{r}_i \geqslant 0$ such that

$$0 \leqslant x_i^{-1} g_i(t, x) \leqslant \Gamma_i(t) + \delta$$

a.e. $t \in J$, $x \in R^n$ with $|x_i| \geqslant \bar{r}_i$.

Define $\gamma_i \colon J \times R^n \to R$ as follows

$$\gamma_i(t, x) = \\ = \begin{cases} (x_i^{-1} g_i(t, x) & \text{if } |x_i| \geqslant \bar{r}_i \\ (\bar{r}_i)^{-1} g_i(t, x_1, \ldots, x_{i-1}, \bar{r}_i, x_{i+1}, \ldots, x_n) \dfrac{x_i}{\bar{r}_i} + (1 - x_i|\bar{r}_i) & \text{if } 0 \leqslant x_i < \bar{r}_i \\ (\bar{r}_i)^{-1} g_i(t, x_1, \ldots, x_{i-1}, -\bar{r}_i, x_{i+1}, \ldots, x_n) \dfrac{x_i}{\bar{r}_i} + (1 + x_i|\bar{r}_i) & \text{if } -\bar{r}_i < x_i < 0 . \end{cases}$$

We have $0 \leqslant \gamma_i(t, x) \leqslant \Gamma_i(t) + \delta$ for a.e. $t \in J$, $x \in R^n$. Moreover the map $\tilde{g}_i$: $J \times R^n \to R$, $(t, x) \mapsto \tilde{g}_i(t, x) = \gamma_i(t, x) x_i$ is a generalized Caratheodory's map such that $\tilde{g}_i(\cdot, x)$ is measurable on J for each $x \in R^n$, $\tilde{g}_i(t, \cdot)$ is continuous on R^n for almost all $t \in J$ and

$$\tilde{g}_i(t, x) = g_i(t, x) \qquad \text{for any } x \in R^n \text{ with } |x_i| \geqslant \bar{r}_i .$$

Let $h_i(t, x) = g_i(t, x) - \tilde{g}_i(t, x)$; by construction, from hypothesis (A) on g_i it follows that

$$|h_i(t, x)| \leqslant 2\Big(a_i(t) + b_i(t)\Big(\sum_{k=1}^{n} |x_k|^\alpha()\Big)\Big) , \qquad \text{a.e. } t \in J, \ x \in R^n \tag{4.3}$$

where $a_i, b_i \in L^1(J)$ are given by hypothesis (A) with $r_i = \bar{r}_i$.

Let $x \in H^1(J, R^n)$ be a 2π-periodic solution of system (4.2) for some $\lambda \in]0, 1[$. Multiplying by $(\bar{x}_i - \tilde{x}_i(t))$ and integrating the i-th equation of (4.2) written in the form

$$x_i''(t) + (1 - \lambda)\Gamma_i(t) x_i(t) + \lambda\Big(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t)) + \tilde{g}_i(t, x(t)) + h_i(t, x(t)) + e_i(t)\Big) = 0$$

we obtain, using Lemma 2 with

$$p_i(t) = (1-\lambda)\Gamma_i(t) + \lambda\gamma_i(t, x(t)) \leqslant \Gamma_i(t) + \delta_i\,,$$

$$0 \geqslant (2\pi)^{-1}\Big(\int_J \tilde{x}_i'^2(t)\,dt + \int_J \big((1-\lambda)\Gamma_i(t) + \lambda\gamma_i(t, x(t))\big)\Big)x_i(t)$$

$$(\bar{x}_i - \tilde{x}_i(t))\,dt + \lambda\int_J \big(h_i(t, x(t)) - e_i(t)\big)(\bar{x}_i - \tilde{x}_i(t))\,dt + \lambda\int_J (\bar{x}_i - \tilde{x}_i(t))\frac{\partial F}{\partial x_i}(x(t))\,dt \geqslant$$

$$\geqslant \delta\|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\lambda\int_J \Big(h_i(t, x(t)) - e_i(t)(\bar{x}_i - \tilde{x}_i(t))\Big)\,dt +$$

$$+ \lambda\int_J (\bar{x}_i - \tilde{x}_i(t))\frac{\partial F}{\partial x_i}\,\partial x_i\,(x(t))\,dt \geqslant \delta\|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\big(|\bar{x}_i| + \|\tilde{x}_i\|_C\big)\cdot$$

$$\cdot\int_J \big(|h_i(t, x(t))| + |e_i(t)|\big)\,dt + \lambda\int_J (\bar{x}_i - \tilde{x}_i(t))\frac{\partial F}{\partial x_i}(x(t))\,dt\,.$$

Using (4.3) and setting

$$L' = \max\{\|e_i\|_{L^1} + 2\|a_i\|_{L^1},\ i = 1, \ldots, n\}$$

$$M' = 2\max\{\|b_i\|_{L^1},\ i = 1, \ldots, n\}$$

we get for any $i = 1, \ldots, n$

$$0 \geqslant \|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\big(|\bar{x}_i| + \|\tilde{x}_i\|_C\big)\Big(L' + M'\Big(\sum_{k=1}^{n}\|x_k\|^{\alpha}_C\Big)\Big) + \lambda\int_J (\bar{x}_i - \tilde{x}_i(t))\frac{\partial F}{\partial x_i}(x(t))\,dt\,.$$

Putting $a = \big(1 + (2\sqrt{3})^{-1}\big)$, $(2\pi a)^{\alpha}M' = K'$ and $L' + nR^{\alpha}M' = H'$, in the same way as in the proof of Theorem 1 we obtain

$$0 \geqslant \delta\|\tilde{x}\|^2_{H^1} - \big(\mathrm{I}H' + K'n(\sqrt{n})^{\alpha}\|\tilde{x}\|^{\alpha}_{H^1}\big)\big(An + a\sqrt{n}\|\tilde{x}\|_{H^1}\big)\,.$$

Hence there exist constants $c_1', c_2', c_3', c_4' > 0$ such that

$$0 \leqslant \delta\|\tilde{x}\|^2_{H^1} - c_1'\|\tilde{x}\|^{\alpha+1}_{H^1} - c_2'\|\tilde{x}\|_{H^1} - c_3'\|\tilde{x}\|^{\alpha}_{H^1} - c_4'$$

and then there exists a constant $d > 0$ such that $\|x\|_{H^1} < d$ and the proof is complete.

REMARK. – Clearly hypothesis (B_2^+) is more restrictive than the norm extension of condition (1.4)

$$\limsup_{\|x\|\to\infty} \|x\|^{-1}\|g(t, x)\| \leqslant \Gamma(t) \leqslant 1\,.$$

Nevertheless the following simple example shows that hypothesis (B_2^+) is better in some situations.

EXAMPLE. – Consider the following system

$$\begin{cases} x'' + y' + (\sin^2 t)x = e_1(t) \\ y'' + x' + (\cos^2 t)y = e_2(t) \end{cases}$$

where $e_1, e_2: J \to \mathbf{R}$ are integrable functions with $\bar{e}_1 = \bar{e}_2 = 0$.

It is easy to see that the conditions of Theorem 1 are satisfied, hence there exists a 2π-periodic solution for this system.

Put $g_1(t, x) = (\sin^2 t)x$ and $g_2(t, y) = (\cos^2 t)y$. By standard calculations we obtain that

$$\limsup_{|x|+|y|\to\infty} \left\{\frac{g_1^2(t, x) + g_2^2(t, y)}{x^2 + y^2}\right\}^{\frac{1}{2}} \geqslant 1/2$$

hence we cannot apply the results in [3], [4], [10].

5. – Mixed case.

THEOREM 3. – *Let $F \in C^2(R^n, R)$, let g be a generalized Caratheodory's map and let $n_0 \in [1, n]$ be an integer.*

Assume that, for any $i = 1, \dots, n_0$, g verifies the following conditions

(B^-) *there exists $R_i > 0$ such that $g_i(t, x)x_i \leqslant 0$ for every $x \in R^n$ with $|x_i| \geqslant R_i$,*

and for every $i = n_0 + 1, \dots, n$

(B_1^+) *there exists $R_i > 0$ such that $g_i(t, x)x_i \geqslant 0$ for every $x \in R^n$ with $|x_i| \geqslant R_i$,*

and

(B_2^+) $\limsup x_i^{-1} g_i(t, x) \leqslant \Gamma_i(t)$ *uniformly a.e. $t \in J$ and uniformly for $x_j \in \mathbf{R}$, $j \neq i$, where $\Gamma_i \in L^1(J)$ is such that $\Gamma_i(t) \leqslant 1$ a.e. $t \in J$, with the strict inequality on a subset of positive measure.*

Then Problem (1) *has at least one solution provided that $e \in L^1(J, R^n)$, $\bar{e} = 0$.*

PROOF. – Like in Theorem 1 and Theorem 2 it sufficies to show that the set of solutions for the system

$$(5.1)\quad \begin{cases} x_i''(t) - (1-\lambda)x_i(t) = \lambda\left(-\dfrac{d}{dt}\big(\operatorname{Grad} F(x(t))\big)_i - g_i\big(t, x(t)\big) + e_i(t)\right), \\ \qquad\qquad\qquad\qquad i = 1, \dots, n_0 \\ x_i''(t) + (1-\lambda)\Gamma_i(t)x_i(t) = \lambda\left(-\dfrac{d}{dt}\big(\operatorname{Grad} F(x(t))\big)_i - g_i\big(t, x(t)\big) + e_i(t)\right), \\ \qquad\qquad\qquad\qquad i = n_0 + 1, \dots, n \end{cases}$$

is bounded for any $\lambda \in]0, 1[$ independently of λ.

Let $x \in H^1(J, R^n)$ be a solution of system (5.1).

In the same way as in Theorems 1 and 2 respectively, we obtain that there exist $L_1, L_2, M_1, M_2 > 0$ such that

$$(5.2) \qquad 0 \geqslant \|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\|x_i\|_C\Big(L_1 + M_1\Big(\sum_{k=1}^{n}\|x_k\|^{\alpha}_C\Big)\Big), \qquad i = 1, \ldots, n_0$$

and

$$(5.3) \qquad 0 \geqslant \|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\|x_i\|_C\Big(L_2 + M_2\Big(\sum_{k=1}^{n}\|x_k\|^{\alpha}_C\Big)\Big), \qquad i = n_0 + 1, \ldots, n.$$

Let $\bar{L} = \max\{L_1, L_2\}$ and $\bar{M} = \max\{M_1, M_2\}$.

We observe that in the inequalities (5.3) $\delta \leqslant 1$, hence by (5.2) and (5.3) we get for any $i = 1, \ldots, n_0, n_0 + 1, \ldots, n$

$$0 \geqslant \delta\|\tilde{x}_i\|^2_{H^1} - (2\pi)^{-1}\|x_i\|_C\Big(\bar{L} + \bar{M}\Big(\sum_{k=1}^{n}\|x_k\|^{\alpha}_C\Big)\Big).$$

With a technique similar to that used in Theorem 1 we can show that there exist constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4 > 0$ such that

$$0 \geqslant \delta\|\tilde{x}\|^2_{H^1} - \bar{c}_1\|\tilde{x}\|^{\alpha+1}_{H^1} - \bar{c}_2\|\tilde{x}\|_{H^1} - \bar{c}_3\|\tilde{x}\|^{\alpha}_{H^1} - \bar{c}_4.$$

This implies that there exists a constant $d > 0$ such that $\|x\|_{H^1} < d$ and the proof is complete.

COROLLARY 1. – *Theorem 1 remains valid if we suppose instead of* (B^-) *and* $\bar{e} = 0$ *that there exist constants* a_i, b_i *and* $R_i > 0$, $i = 1, \ldots, n$ *such that*

$$g_i(t, x) \leqslant b_i \leqslant a_i, \qquad \text{a.e. for } t \in J \text{ and } x_i \geqslant R_i;$$

$$g_i(t, x) \geqslant a_i, \qquad \text{a.e. for } t \in J \text{ and } x_i \leqslant -R_i;$$

$$b_i \leqslant \bar{e}_i \leqslant a_i.$$

PROOF. – System (1.6) is equivalent to the system

$$x_i'' + \frac{d}{dt}\big(\operatorname{Grad} F(x)\big)_i + g_i(t, x) - \bar{e}_i = e_i(t) - \bar{e}_i, \qquad i = 1, \ldots, n.$$

Observing that $e_i(t) - \bar{e}_i$ has mean value 0 and $g_i(t, x) - \bar{e}_i$ verifies condition (B^-) for every $i = 1, \ldots, n$, we are done.

In the same way we obtain the corresponding Corollaries for Theorem 2 and 3 which are not stated for the sake of brevity.

THEOREM 4. – *The assertion of Theorem 2 (Theorem 3) holds also if in the hypothesis* $((B_2^+)\ (\bar{B}_2^+))$ *for any* $i = 1, \ldots, n_0$ $(i = n_0 + 1, \ldots, n)$ Γ_i *verifies the following conditions*

$$\Gamma_i = \Gamma_i^0 + \Gamma_i^1 + \Gamma_i^\infty$$

with $\Gamma_i^1 \in L^1(J)$, $\Gamma_i^\infty \in L^\infty(J)$, Γ_i^0 *is measurable on* J, $\Gamma_i^0(t) \leqslant 1$ *a.e.* $t \in J$ *with the strictly inequality on a subset of positive measure and*

$$(\pi^2/3)\|\Gamma_i^1\|_{L^1} + \|\Gamma_i^\infty\|_{L^\infty} < \delta_i(\Gamma_i^0)$$

where $\delta_i(\Gamma_i^0)$ *is given by Lemma* 1.

PROOF. – The scheme of the proof is similar to that of the proof given in Theorem 2 (Theorem 3) and we omit it for sake of brevity.

The required a priori extimates are obtained setting

$$\delta = \tfrac{1}{2} \min \{\delta_i(\Gamma_i^0) - \|\Gamma_i^\infty\|_{L^\infty} - (\pi^2/3)\|\Gamma_i^1\|_{L^1}\}$$

for $i = 1, \ldots, n_0$ $(i = n_0 + 1, \ldots, n)$, and proceeding like in the proof of Theorem 2 in [7].

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REFERENCES

[1] J. BEBERNES, *A simple alternative problem for finding periodic solutions of second order ordinary differential equations*, Proc. Amer. Math. Soc., **48** (1974), pp. 121-127.

[2] J. BEBERNES - M. MARTELLI, *Periodic solutions for Liénard systems*, Boll. Un. Mat. Ital., (5) **16**-A (1979), pp. 398-405.

[3] L. CESARI - R. KANNAN, *Solutions in the large of Liénard systems with forcing terms*, Ann. Mat. Pura Appl., (4) **111** (1976), pp. 101-124.

[4] L. CESARI - R. KANNAN, *Periodic solutions in the large of Liénard systems with forcing terms*, Boll. Un. Mat. Ital., (6) **1**-A (1982), pp. 217-223.

[5] S. H. CHANG, *Periodic solutions of certain second order nonlinear differential equations*, J. Math. Anal. Appl., **49** (1975), pp. 263-266.

[6] J. CRONIN, *Fixed Points and Topological Degree in Nonlinear Analysis*, Math. Survey No. 11, Amer. Math. Soc., Providence, R.I., 1964.

[7] C. P. GUPTA - J. MAWHIN, *Asymptotic conditions at the two first eigenvalues for the periodic solutions of Liénard differential equations and an inequality of E. Schmidt*, Rapport n. 10 Sem. Math., (1982), Université Catholique de Louvain.

[8] A. C. LAZER, *On Schauder's fixed point theorem and forced second order nonlinear oscillations*, J. Math. Anal. Appl., **81** (1968), pp. 421-425.

[9] M. MARTELLI, *On forced nonlinear oscillations*, J. Math. Anal. Appl., **69** (1979), pp. 456-504.

[10] M. MARTELLI - J. D. SCHUUR, *Periodic solutions of Liénard type second order ordinary differential equations*, Tohoku Math. J., **38** (1980), pp. 201-207.

[11] J. MAWHIN, *An extension of a theorem of A.C. Lazer on forced nonlinear equations*, J. Math. Anal. Appl., **40** (1972), pp. 20-29.

[12] J. MAWHIN - J. R. WARD, *Periodic solutions of some forced Liénard differential equations at resonance*, to appear.

[13] R. REISSIG, *Schwingungssätze für die verallgemeinerte Liénardsche Differentialgleichung*, Abh. Math. Seminar Univ. Hamburg, **44** (1975), pp. 45-51.

[14] R. REISSIG, *Continua of periodic solutions of the Liénard equation*, in « Constructive Methods for Nonlinear Boundary Value Problems and Nonlinear Oscillations », I.S.N.M., vol. 48, Birkhàuser, Basel, (1979), pp. 126-133.

[15] R. REISSIG, *Über einen allgemeinen Typ erzwungener nichtlinearer Schwingungen zweiter Ordnung*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **56** (1974), pp. 297-302.

[16] R. REISSIG, *Extension of some results concerning the generalized Liénard equations*, Ann. Mat. Pura Appl., **104** (1975), pp. 269-281.

[17] R. REISSIG, *Contractive mappings and periodically perturbed nonconservative systems*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., **52** (1975), pp. 698-702.

[18] N. ROUCHE - J. MAWHIN, *Equations differentielles ordinaires* (2 Volumes), Masson, Paris, 1972.

[19] F. ZANOLIN, *Remark on Multiple Periodic Solutions for Nonlinear Ordinary Differential Systems of Liénard Type*, Boll. Un. Mat. Ital., (6) **1**-B (1982), pp. 683-698.