On Partial Asymptotic Stability by the Method of Limiting Equation (*).

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Summary. – *The extensions of the Barbashin-Krasovskij theorem to the partial asymptotic stability of the zero solution of a differential system require the boundedness of the uncontrolled coordinates along the solutions. In this paper the Barbashin-Krasovskij method is generalized without supposing «a priori» knowledges on the solutions. At the same time, the results extend one of C. Risito's theorem to nonautonomous differential equations. As an application, stability properties of the equilibrium state of nonholonomic dissipative mechanical systems are studied.*

1. – Introduction.

One of the most important results both in the theory and the practice of Ljapunov's direct method is the BARBASHIN-KRASOVSKIJ theorem [1], which establishes asymptotic stability for the zero solution of an autonomous differential system by a Ljapunov function V whose derivative $\dot{V}$ is not negative definite but only negative semi-definite. It requires that the zero set of $\dot{V}$ should contain no complete trajectories of the system except the origin. J. P. LASALLE [2] called the key argument of the proof of this theorem «invariance principle» and generalized it to nonautonomous systems even to abstract dynamical systems. Using the invariance principle L. SALVADORI [3] proved that an isolated equilibrium position of a holonomic scleronomic mechanical system is asymptotically stable provided that the potential energy has a minimum at the equilibrium position and the system is under the action of dissipative forces with total dissipation.

In 1957 V. V. RUMJANCEV [4] introduced the concept of the partial asymptotic stability. This means stability and attractivity with respect to a so called controlled part y of the coordinates of the vector $x = (y, z)$ of the phase variables. V. V. RUMJANCEV [5], A. C. OZIRANER [6] and C. RISITO [7] generalized the Barbashin-Krasovskij method to the partial asymptotic stability. Rumjancev and Oziraner supposed that the zero set of $\dot{V}$ contains no complete trajectories of the system except the origin. Risito weakened this assumption requiring that the zero set of $\dot{V}$ should contain no trajectories except those that lie in the set $\{(y, z) : y = 0\}$.

In the course of the extension of the Barbashin-Krasovskij method to the partial

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asymptotic stability a new difficulty has appeared: every generalization requires that the uncontrolled coordinates z should be bounded along the solutions.

In this paper we deal with the problem how to replace the condition of the boundedness of the uncontrolled coordinates in Risito's theorem not requiring « a priori » knowledges of the solutions. At the same time, we generalize Risito's theorem to nonautonomous differential systems. Connecting with investigations of V. V. RUMJANCEV [8] and C. RISITO [7] we apply our results to study stability properties of the equilibrium of dissipative nonholonomic mechanical systems.

2. – Notations and definitions.

Consider the system of differential equations

$$\dot{x} = X(x, t) \qquad (X(0, t) \equiv 0) \tag{2.1}$$

where $t \in R_+ := [0, \infty)$, and $x = (x^1, x^2, \ldots, x^k) \in R^k$ with a norm $|x|$. Denote by $B_k(\varrho)$ the open ball in R^k with center at the origin and radius $\varrho > 0$; $\bar{B}_k(\varrho)$ is its closure in R^k.

We distinguish two types of coordinates: controlled and uncontrolled ones. Accordingly, we consider a partition $x = (y, z)$ ($y \in R^m$, $z \in R^n$; $1 \leqq m \leqq k$, $n := k - m$); vector y consists of the *controlled coordinates*, z contains the *uncontrolled* ones. Assume that the function X is defined on the set $\Gamma_m(H)$:

$$\Gamma_m(H) := G_m(H) \times R_+ \qquad (G_m := B_m(H) \times R^n;\ 0 < H \leqq \infty)\,,$$

it is continuous in x, measurable in t, and satisfies the Carathéodory condition locally (i.e. for every compact set $K \subset R^k$ there is a locally integrable $f_K\colon R_+ \to R_+$ such that $|X(x, t)| \leqq f_K(t)$ for all $(x, t) \in K \times R_+$). We denote by $x(t) = x(t; x_0, t_0)$ any non-continuable to the right solution of (2.1) with $x(t_0) = x_0$.

We always assume that the solutions are *z-continuable* [9]. This means that if $x(t) = (y(t), z(t))$ is a solution of (2.1) and $|y(t)| \leqq H' < H$ for $t \in [t_0, T)$, then $x(t)$ can be continued to the closed interval $[t_0, T]$.

To avoid any ambiguity, we recall the definitions of the stability concepts used in this paper. The zero solution of (2.1) is said to be *y-stable* [4] if for every $\varepsilon > 0$, $t_0 \in R_+$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta(\varepsilon, t_0)$ implies $|y(t; x_0, t_0)| < \varepsilon$ for all $t \geqq t_0$. The zero solution is said to be *asymptotically y-stable* [4] if it is y-stable and, in addition, for every $t_0 \in R_+$ there exists a $\sigma(t_0) > 0$ such that $|x_0| < \sigma(t_0)$ imples $|y(t; x_0, t_0)| \to 0$ as $t \to \infty$. Finally, we say that the solutions of (2.1) are *z-bounded* if for every $t_0 \in R_+$ there is a $\gamma(t_0) > 0$ such that $|x_0| < \gamma(t_0)$ implies that the function $|z(t; x_0, t_0)|$ is bounded in $[t_0, \infty)$.

A continuous function $V\colon \Gamma_m(H') \to R$ $(0 < H' < H)$ is a *Ljapunov function* to (2.1) if $V(0, t) \equiv 0$, V is locally Lipschitzian, and its *derivative* $\dot V$ *with respect* to (2.1) is non-positive:

$$\dot V(x, t) = \dot V_{(2.1)}(x, t) := \limsup_{h\to 0+} \frac{V(x + hX(x, t), t + h) - V(x, t)}{h} \leqq 0$$

for all $(x, t) \in \Gamma_m(H')$.

We say that a function $a\colon R_+ \to R_+$ belongs to the *class* $\mathcal{K}$ if it is continuous, strictly increasing and $a(0) = 0$. *Class* $\mathcal{KL}$ consists of the functions $A\colon R_+^2 \to R_+$ which are continuous, strictly increasing in its first variable and strictly decreasing in its second variable, and for which

$$\lim_{\substack{\delta\to 0+\\ \gamma\to\infty}} A(\delta, \gamma) = 0$$

is satisfied. A function $W\colon \Gamma_m(H) \to R$ is said to be *positive y-definite* if there is a function $a \in \mathcal{K}$ such that $a(|y|) \leqq W(y, z, t)$ for all $(y, z, t) \in \Gamma_m(H)$.

As is well-known [1, 10], in the autonomous case (i.e. if $X = X(x)$, $V = V(x)$) it was the key idea of the proof of the Barbashin-Krasovskij theorem that the positive limit set of every solution lay in the so called «dangerous set» $V^{-1}(c) \cap \cap \dot V^{-1}(0)$ for some $c \in R_+$, provided that V is a Ljapunov function to (2.1). Now we define the successor in our theory of this set. Let us given a Ljapunov function $V\colon \Gamma_m(H') \to R$. For $c \in R$ denote by $V_m^{-1}[c, \infty]_0$ the set of the points $y \in R^m$ for which there exists a sequence $\{(y_i, z_i, t_i)\}$ such that $y_i \to y$, $|z_i| \to \infty$, $t_i \to \infty$, $V(y_i, z_i, t_i) \to c$ and $\dot V(y_i, z_i, t_i) \to 0$ as $i \to \infty$. It is easy to see that (1) $V_m^{-1}[c, \infty]_0$ is closed relative to $\Gamma_m(H')$; (2) if V and $\dot V$ is continuous in y uniformly with respect to $(z, t) \in R^n \times R_+$, then $y \in V_m^{-1}[c, \infty]_0$ iff there exists a sequence $\{(z_i, t_i)\}$ such that $|z_i| \to \infty$, $t_i \to \infty$, $V(y, z_i, t_i) \to c$ and $\dot V(y, z_i, t_i) \to 0$ as $i \to \infty$. Namely, if $V = V(y)$, $\dot V = \dot V(y)$, then $V_m^{-1}[c, \infty]_0 = V^{-1}(c) \cap \dot V^{-1}(0)$.

We need some concepts and results from topological dynamics given in [11]-[13]. As is known, every initial value problem for an ordinary differential equation is equivalent to an integral equation. In the method of limiting equation there occur functional equations containing more general operator than the integral with a kernel. An *ordinary integral-like operator* I is a mapping which associates with each continuous function $\varphi\colon [\alpha, \beta) \to R^k$ and $a \in [\alpha, \beta)$ a continuous function $I_a\varphi$ so that (1) if $\varphi_i\colon [\alpha, \beta) \to R^n$ are continuous and $\varphi_i(t) \to \varphi(t)$ uniformly, then $I_a\varphi_i(t) \to I_a\varphi(t)$ uniformly in $t \in [a, b]$ as $i \to \infty$ for all $[a, b] \subset [\alpha, \beta)$; (2) $I_a\varphi(t) = I_a\varphi(s) + I_s\varphi(t)$ for all $a, s, t \in [\alpha, \beta)$. We shall denote by $u = Iu$ the functional equation $u = u(a) + I_a u$ associated with the ordinary integral-like operator I.

For $t \in R_+$ the *translate* X^t *of* X is defined by

$$X^t(x, s) := X(x, t + s) \qquad (s \in R_+).$$

We denote by tran (X) the collection of all transletes X^t of X $(t \in R_+)$. An ordinary integral-like operator equation $u = Iu$ is a *limiting equation* of (2.1) if there exists a sequence $\{t_i\}$ converging to infinity so that X^{t_i} *integrally converges* to I as $i \to \infty$, i.e. whenever φ_i: $[a, b] \to R^k$ converges uniformly to φ then

$$\int_a^b X(\varphi_i(s), t_i + s)\, ds \to I_a \varphi(b) \qquad (i \to \infty)\,.$$

The set tran (X) is said to be *precompact* if every sequence in it has an integrally converging subsequence.

A point $p \in R^k$ is a *limit point* of a solution φ: $[t_0, \infty) \to R^k$ if there exists a sequence $\{t_i\}$ such that $t_i \to \infty$ and $\varphi(t_i) \to p$ as $i \to \infty$. The *limit set* $\Omega(\varphi)$ of the solution φ consists of all the limit points of φ. As is known [10], if (2.1) is autonomous then $\Omega(p)$ is invariant in the sense that for every $p \in \Omega(\varphi)$ there exists a trajectory of (2.1) through p that lies in $\Omega(\varphi)$. In the nonautonomous case the set $\Omega(\varphi)$ is said to be *semiinvariant* with respect to the family of the limiting equations of (2.1) if for every $p \in \Omega(\varphi)$ there is a limiting equation $u = Iu$ of (2.1) and a solution ξ of the equation $u = p + I_0 u$ so that $\xi(t) \in \Omega(\varphi)$ for all t in the domain D_ξ of ξ, i.e. the *trajectory*

$$\gamma = \gamma(\xi) := \bigcup_{t \in D_\xi} \xi(t)$$

of the solution ξ lies in $\Omega(\varphi)$.

3. – The assumption on the right-hand side.

In the proofs of the theorems we will need the properties that the set of the translates of (2.1) is precompact, and the limit sets of the solutions are semiinvariant with respect to the limiting equations. Moreover, these properties will be needed also for the equation

$$\dot{y} = Y(y, \chi(t), t)\,, \qquad (h \in R^m) \tag{3.1}$$

where Y derives from the partition $X = (Y, Z)$, and χ: $R_+ \to R^n$ is a given continuous function with $|\chi(t)| \to \infty$ as $t \to \infty$.

In order to assure these properties we assume the following: for every H', K $(0 < H' < H,\ K \subset R^n\text{-compact})$ there are functions $\mu_{H',K} \in \mathcal{K}$, $\mu_{H'} \in \mathcal{KL}$, $\nu_{H',K}$: $R_+^2 \to R_+$, $\nu_{H'}$: $R_+^3 \to R_+$ such that $\nu_{H',K}(\delta, t)$, $\nu_{H'}(\delta, \gamma, t)$ are locally integrable in t, increasing in δ, decreasing in γ.

$$\int_s^{s+1} \nu_{H',K}(\delta, t)\, dt \leqq N_{H',K}(\delta) \qquad (s.\ \delta \in R_+)$$

$$\int_s^{s+1} \nu_{H'}(\delta, \gamma, t)\, dt \leqq N_{H'}(\delta, \gamma) \qquad (s, \delta, \gamma \in R_+)$$

with appropriate function $N_{H',K} \in \mathcal{K}$, $N_{H'} \in \mathcal{KL}$, and such that the following assumptions are satisfied.

(A$_1$) for every continuous function $u\colon [a, b] \to \bar{B}_m(H') \times K$ we have

$$\left|\int_a^b X(u(t), t)\, dt\right| \leqq \mu_{H',K}(|b-a|);$$

(A$_2$) $|X(x', t) - X(x'', t)| \leqq \nu_{H',K}(|x' - x''|, t)$ for all x', $x'' \in \bar{B}_m(H) \times K$, $t \in R_+$;

(A$_3$) for every continuous function $v\colon [a, b] \to \bar{B}_m(H')$ and $z \notin B_n(\gamma)$ we have

$$\left|\int_a^b Y(v(t), z, t)\, dt\right| \leqq \mu_{H'}(|b-a|, \gamma);$$

(A$_4$) $|Y(y', z, t) - Y(y'', z, t)| \leqq \nu_{H'}(|y' - y''|, \gamma, t)$ for all y', $y'' \in \bar{B}_m(H')$, $z \notin B_n(\gamma)$, $t \in R_+$.

LEMMA 3.1. – *a) Assumptions* (A$_1$)-(A$_2$) *imply that the family of all translates of* (2.1) *is precompact, and the limit set of every solution of* (2.1) *is semiinvariant with respect to the family of the limiting equations.*

b) Assumptions (A$_3$)-(A$_4$) *imply the same properties with respect to equation* (3.1) *in which the continuous function* $\chi\colon R_+ \to R^n$ *is arbitrarily fixed so that* $|\chi(t)| \to \infty$ *as* $t \to \infty$.

Assertion *a*) was proved in [13] (see Theorems 7.3 and 8.1).

The proof of assertion *b*) can be obtained by a straightforward modification of that of assertion *a*), thus it is omitted.

REMARK 3.1. – Assumptions (A$_1$)-(A$_4$) are obviously satisfied if (2.1) is autonomous and *asymptotically z-independent*, i.e. $Y(y, z) \to Y^*(y)$ as $|z| \to \infty$ uniformly with respect to y on every compact set (see [14], Section 3).

Considering more general cases, in Section 5 of [14] we required the equicontinuity condition (A$_4$) with $\nu_{H'}(\delta, \gamma, t) = \varkappa(t)\delta$. Besides it is linear in δ, it is also uniform with respect to $\gamma \in R_+$. The non-uniformity of (A$_4$) can be essential, as the following simple example shows. The function

$$Y(y, z, t) := \frac{\sin(yt)}{1 + z^2}$$

obviously satisfies assumption (A_4). On the other hand,

$$\int_a^{a+1} \frac{\sin y_1 t - \sin y_2 t}{1 + z^2}\, dt \nrightarrow 0 \qquad (|y_1 - y_2| \to 0)\,,$$

thus Y does not satisfy the uniform version of assumpiton (A_4).

4. – The main theorems.

Consider a further partition of the vector of uncontrolled coordinates: $z = (z_1, z_2)$ $(z_1 \in R^{n_1}, z_2 \in R^{n_2}, 0 \leqq n_1 \leqq n, n_1 + n_2 = n)$. In the first theorem we require $V(y, z_1, z_2, t) \to 0$ uniformly in $(z_2, t) \in R^{n_2} \times R_+$ as $|(y, z_1)| \to 0$.

THEOREM 4.1. – *Let assumptions* (A_1)-(A_4) *be satisfied for the system of differential equations*

$$\dot{y} = Y(y, z, t)\,, \qquad \dot{z} = Z(y, z, t)\,. \tag{4.1}$$

Suppose, in addition, that there exists a positive y-definite Ljapunov function $V: \Gamma_m(H') \to \to R_+$ *having the following properties in* $\Gamma_m(H')$:

(i) *there is a function* $b \in \mathcal{K}$ *such that*

$$V(y, z_1, z_2, t) \leqq b(|y| + |z_1|);$$

(ii) *for* $c > 0$ *the set* $V_k^{-1}[c, \infty]_0$ *may contain the trajectory* γ *of a solution of some limiting equation of* (4.1) *only if*

$$\gamma \subset \{(y, z_1, z_2): y = 0, z_1 = 0, z_2 \in R^{n_2}\};$$

(iii) *for every* $c > 0$ *and* $\chi: R_+ \to R^n$ *such that* $|\chi(t)| \to \infty$ *as* $t \to \infty$ *the set* $V_m^{-1}[c, \infty]_0$ *contains no trajectory of any limiting equation of*

$$\dot{y} = Y(y, \chi(t), t) \tag{4.2}$$

except the origin of R^m.

Then the zero solution of (4.1) *is asymptotically y-stable.*

If (4.1) is autonomous, $V = V(x)$ and all of the coordinates are controlled, then Theorem 4.1 coincides with the Barbashin-Krasovskij theorem. Indeed, in this case $y = x$, assumptions (A_1), (A_2) and (i) are obviously satisfied; (A_3), (A_4) and (iii) are redundant, and assumption (ii) requires of the set $\dot{V}^{-1}(0) \cap V^{-1}(c)$ to contain no complete trajectory of the system except the origin.

To achieve assumption (i) to be satisfied is often very difficult (see the mechanical applications later). The next theorem shows how it can be dropped by restricting the function Z in (4.1).

THEOREM 4.2. – *Let all but assumptions* (i) *and* (ii) *in Theorem* 4.1 *are satisfied. Suppose, in addition, that*

(i′) *for every continuous function* $(v, w)\colon R_+ \to \bar{B}_m(H') \times R^n$ *the function*

$$\int_t^{t+1} Z(v(s), w(s), s)\, ds$$

is bounded in R_+;

(ii′) *for* $c > 0$ *the set* $V_k^{-1}[c, \infty]_0$ *may contain the trajectory* γ *of any solution of some limiting equation of* (4.1) *only if*

$$\gamma \subset \{(y, z)\colon y = 0,\ z \in R^n\} .$$

Then the zero solution of (4.1) *is asymptotically y-stable.*

Let us now consider the case when it is « a priori » known that the solutions are z-bounded. Then assumptions (A_3), (A_4) are redundant, and (i′) has to be required only with bounded functions $\boldsymbol{w}$. But this modification of (i′) is a consequence of assumption (A_1), thus we obtain the following

COROLLARY 4.1. – *Suppose that assumptions* (A_1), (A_2) *are satisfied, and the solutions of* (4.1) *are z-bounded. Let there exist a positive definite Ljapunov function* $V\colon \Gamma_m(H') \to R_+$ *such that for every* $c > 0$ *the set* $V_k^{-1}[c, \infty]_0$ *may contain the trajectory* γ *of any solution of some limiting equation of* (4.1) *only if* $\gamma \subset \{(y, z)\colon y = 0, z \in R^n\}$.

Then the zero solution of (4.1) *is asymptotically y-stable.*

If system (4.1) is autonomous, the set $\{(y, z)\colon y = 0,\ z \in R^n\}$ is invariant, and the set $\dot{V}^{-1}(0) \setminus \{(y, z)\colon y = 0,\ z \in R^n\}$ contains no complete trajectory of the system, then the last condition of Corollary 4.1 is satisfied, so Corollary 4.1 is an extension of RISITO's theorem ([7], Teorema 3.3) to nonautonomous systems.

REMARK 4.1. – In the course of the application of our results to the study of nonholonomic mechanical system we will have a case in which only some of the uncontrolled coordinates are bounded along the motions. In order to be able to take into account this property we introduce a further partion $z = (z_3, z_4)$ $(z_3 \in R^{n_3}$, $z_4 \in R^{n_4}$, $0 \leqq n_3 \leqq n$, $n_3 + n_4 = n)$, and assume that the solutions of (4.1) are z_3-bounded. We can adapt Theorems 4.1 and 4.2 for this case by the following modifications of the assumptions:

a) assumptions (A_3), (A_4) have to be required of (Y, Z_3) with the partition $x = ((y, z_3), z_4)$;

b) instead of assumption (iii) we assume that for every $c > 0$ and $\chi_4 \colon R_+ \to R^{n_4}$ such that $|\chi_4(t)| \to \infty$ as $t \to \infty$ the set $V^{-1}_{m+n_3}[c, \infty]_0$ may contain the trajectory ν of any solution of some limiting equation of

$$\dot{y} = Y(y, z_3, \chi_4(t), t) , \qquad \dot{z}_3 = Z_3(y, z_3, \chi_4(t), t)$$

only if $\nu \subset \{(y, z_3) \colon y = 0, z_3 \in R^{n_3}\}$;

c) assumption (i′) in Theorem 4.2 is required of $Z_4(v(s), w_3(s), w_4(s), s)$ for all continuous functions $(v, w_3, w_4) \colon R_+ \to \bar{B}_m(H') \times K_3 \times R^{n_4}$, where $K_3 \subset R^{n_3}$ is compact.

After these modifications the assertions remain true.

5. – The proofs of the main theorems.

The proofs of Theorems 4.1 and 4.2 alike need the following

LEMMA 5.1. – *Let assumptions* (A_1), (A_2) *be satisfied. Suppose in addition, that there exists a positive y-definite Ljapunov function* $V \colon \Gamma_m(H') \to R_+$ *to* (4.1) *satisfying condition* (ii) *in Theorem* 4.1. *Then*

a) *the zero solution of* (4.1) *is y-stable*;

b) *for every solution* $\varphi = (\psi, \chi_1, \chi_2) \colon [t_0, \infty) \to \Gamma_m(H'')$ $(0 < H'' < H')$ *of* (4.1) *from*

$$t_i \to \infty , \qquad (\psi(t_i), \chi_1(t_i)) \to (q, r_1) \neq (0, 0),\ V(\varphi(t_i), t_i) \to v_0 > 0$$

it follows that $|\chi_2(t_i)| \to \infty$ *as* $i \to \infty$.

PROOF. – *a*) The positive definiteness of V implies y-stability of the zero solution [9].

b) Suppose that the assertion is not true, i.e. $|\chi_2(t_i)| \not\to \infty$ as $i \to \infty$. Then there exists a subsequence of $\{t_i\}$, let it be denoted by $\{t_i\}$ again, and an $r \in R^n$ such that $\chi(t_i) \to r$. We shall prove that $p := (q, r) \in V_k^{-1}[v_0, \infty]_0$.

Suppose the contrary. Because the set $V_k^{-1}[v_0, \infty]_0$ is closed, there is an $\varepsilon > 0$ such that

$$\bar{B}_k(p, 2\varepsilon) \cap V_k^{-1}[v_0, \infty]_0 = \emptyset ,$$

where $\bar{B}_k(p, 2\varepsilon)$ denotes the closed ball in R^k with center p and radius 2ε. We show that

$$\alpha := \lim_{T\to\infty} \sup \{\dot{V}(\varphi(t), t) : t \geqq T,\ \varphi(t) \in \bar{B}_k(p, 2\varepsilon)\} > 0 . \tag{5.1}$$

Indeed, if $\alpha = 0$ then there exist $x_0 \in \bar{B}_k(p, 2\varepsilon)$ and a sequence $\{\tau_i\}$, for which $\tau_i \to \infty$, $\dot{V}(\varphi(\tau_i), \tau_i) \to 0$, $\varphi(\tau_i) \in \bar{B}_k(p, 2\varepsilon)$ and $\varphi(\tau_i) \to x_0$ as $i \to \infty$. But the function $V(\varphi(t), t)$ is decreasing, thus $x_0 \in V_k^{-1}[v_0, \infty]_0 \cap \bar{B}_k(p, 2\varepsilon)$, which is a contradiction.

Since $V \geqq 0$, (5.1) implies that $\varphi(t) \in \bar{B}_k(p, 2\varepsilon)$ cannot be satisfied on any whole interval $[T, \infty)$. Hence from $p \in \Omega(\varphi)$ it follows that there exist sequences $\{t_i'\}$, $\{t_i''\}$ with the properties

$$\begin{cases} t_i' < t_i'' < t_{i+1}', \quad t_i' \to \infty, \quad |\varphi(t_i') - p| = \varepsilon, \quad |\varphi(t_i'') - p| = 2\varepsilon, \\ \varepsilon \leqq |\varphi(t) - p| \leqq 2\varepsilon \quad (t_i' \leqq t \leqq t_i'';\ i = 1, 2, \ldots) . \end{cases} \tag{5.2}$$

Then $|\varphi(t_i'') - \varphi(t_i')| \geqq \varepsilon$; therefore, by assumption (A_1), $t_i'' - t_i' \geqq \beta > 0$ for all i with some constant β. In consequence of (5.1) we have the estimate

$$v(t_i'') - v(t_{i_0}') \leqq \sum_{j=i_0}^{i} \int_{t_j'}^{t_j''} \dot{V}(\varphi(t), t)\, dt \leqq (i - i_0) \frac{\alpha\beta}{2} \to -\infty$$

which is a contradiction.

So, we have proved that the limit set $\Omega(\varphi)$ is not empty and $\Omega(\varphi) \subset V_k^{-1}[v_0, \infty]_0$. By Theorem 7.3 in [13] the set $\Omega(\varphi)$ is semiinvariant with respect to the limiting equations of (4.1). Consequently, the set $V_k^{-1}[v_0, \infty]_0$ contains a trajectory through (q, r) of a limiting equation of (4.1). By assumption (ii) in Theorem 4.1, this implies either $q = 0$ or $v_0 = 0$, whichever is a contradiction.

The lemma is proved.

The PROOF of THEOREM 4.1. – By Lemma 5.1 the zero solution of (4.1) is y-stable. Let $\sigma(t_0) := \delta(H'', t_0)$, where $0 < H'' < H'$, and $\delta(H'', t_0) > 0$ is associated with H'', t_0 in the sense of the definition of y-stability (see Section 2). We shall prove that for every solution $\varphi = (\psi, \chi): [t_0, \infty) \to R^k$ with $|\varphi(t_0)| < \sigma(t_0)$ we have $\lim_{t\to\infty} |\psi(t)| = 0$.

Let us introduce the notation $v(t) := V(\varphi(t), t)$. Since v is nonincreasing and nonnegative, $v(t) \to v_0 \geqq 0$ as $t \to \infty$.

There are two possibilities:

$$a)\ |\chi(t)| \to \infty; \qquad b)\ |\chi(t)| \not\to \infty \quad (t \to \infty) .$$

ad a) Consider the function $U(y, t) := V(y, \chi(t), t)$. Its derivative with respet to the equation

$$\dot{y} = Y(y, \chi(t), t) \tag{5.3}$$

can be expressed by

$$\dot{U}_{(5.3)}(y, t) = \dot{V}_{(4.1)}(y, \chi(t), t);$$

consequently, U is a Ljapunov function to (5.3) and

$$U_m^{-1}[c, \infty]_0 \subset V_m^{-1}[c, \infty]_0 \quad (c \geqq 0) . \tag{5.4}$$

Let $u(t) := U(\psi(t), t)$. We shall prove that $\Omega(\varphi) \subset U_m^{-1}[u_0, \infty]_0$, where

$$u_0 := \lim_{t \to \infty} u(t) .$$

Suppose the contrary. Then, because the set $U_m^{-1}[u_0, \infty]_0$ is closed, there exist $q \in \Omega(\varphi)$ and $\varepsilon > 0$ such that

$$\bar{B}_m(q, 2\varepsilon) \cap U_m^{-1}[u_0, \infty]_0 = \emptyset .$$

Similarly to (5.1), (5.2) one can see that

$$\xi := \lim_{T \to \infty} \sup \{\dot{V}(\varphi(t), t) : t \geqq T, \psi(t) \in \bar{B}_m(q, 2\varepsilon)\} < 0 \tag{5.5}$$

and there is a sequence $\{(t_i', t_i'')\}$ having the properties

$$\begin{cases} t_i' < t_i'' < t_{i+1}', \quad t_i' \to \infty, \quad |\psi(t_i') - q| = \varepsilon, \quad |\psi(t_i'') - q| = 2\varepsilon \\ \varepsilon \leqq |\psi(t) - q| \leqq 2\varepsilon \quad (t_i' \leqq t \leqq t_i''; \ i = 1, 2, \ldots) . \end{cases} \tag{5.6}$$

Since the function $\mu_{H'}$ in assumption (A_3) belongs to class $\mathcal{KL}$, there are $\eta(\varepsilon) > 0$ and $\varrho(\varepsilon)$ such that $|b - a| \leqq \eta(\varepsilon)$, $\gamma \geqq \varrho(\varepsilon)$ imply $\mu_{H'}(|b - a|, \gamma) < \varepsilon$. Let i_0 be so large that $|\chi(t)| > \varrho(\varepsilon)$ for all $t \geqq t_i$. By virtue of (5.5) and assumption (A_3) we have the estimate

$$\varepsilon \leqq |\psi(t_i') - \psi(t_i'')| = \left| \int_{t_i'}^{t_i''} Y(\psi(s), \chi(s), s) \, ds \right| \leqq \mu_{H'}(t_i'' - t_i', \varrho(\varepsilon)) \quad (i \geqq i_0) . \tag{5.7}$$

In consequence of the definition of $\eta(\varepsilon)$, $\varrho(\varepsilon)$ we obtain the inequality $t_i'' - t_i' \geqq \eta(\varepsilon)$ for all $i \geqq i_0$. By (5.5), (5.6) it follows that

$$u(t_i'') - u(t_{i_0}') \leqq \sum_{j=i_0}^{i} \int_{t_j'}^{t_j''} \dot{U}(\psi(t), t) \, dt = (i - i_0) \frac{\xi \eta(\varepsilon)}{2} \to -\infty \quad (i \to \infty) , \tag{5.8}$$

which is a contradiction.

Thus, we have proved the inclusions

$$\Omega(\varphi) \subset U_m^{-1}[u_0, \infty]_0 \subset V_m^{-1}[u_0, \infty]_0 \tag{5.9}$$

(see (5.4)). By Theorem 7.3 in [13] the limit set $\Omega(\psi)$ is semiinvariant with respect to the limiting equations of (5.3). Hence condition (iii) and (5.9) imply either $\Omega(\psi) = \{0\}$ or $u_0 = 0$. But the Ljapunov function V, and consequently U, is positive y-definite, so we obtain $\Omega(\psi) = \{0\}$ also in the latter case, which concludes the proof in case a).

ad b) We prove that in this case $v_0 = 0$, which implies our statement since V is positive y-definite.

Suppose that $v_0 > 0$ and $\{t_i\}$ is a sequence for which $t_i \to \infty$, $\psi(t_i) \to q$, $\chi_1(t_i) \to r_1$ and $\{\chi_2(t_i)\}$ is also convergent as $i \to \infty$. By Lemma 5.1 $(q, r_1) = (0, 0)$. Then, as condition (i) shows, $v(t_i) \to 0$. Thus, being nonincreasing the function $v(t)$ tends to zero as $t \to \infty$, i.e. $v_0 = 0$, which is a contradiction.

The theorem is proved.

THE PROOF OF THEOREM 4.2. – Consider the solution $\varphi = (\psi, \chi)\colon [t_0, \infty) \to \to B_m(H') \times R^n$, the function $v(t)$ and the number $v_0 \geqq 0$ defined at the beginning of the proof of Theorem 4.1. If $v_0 = 0$, then the proof is complete. Suppose that $v_0 > 0$. We prove that this assumption implies $\Omega(\psi) = \{0\}$.

First we show the inclusion

$$\Omega(\psi) \subset N(v_0) := V_m^{-1}[v_0, \infty]_0 . \tag{5.10}$$

Since now $|\chi(t)| \to \infty$ $(t \to \infty)$ is not supposed we have to modify the argument given in case a) in the proof of Theorem 4.1.

The set $N(v_0)$ is closed, $\Omega(\psi)$ is compact and connected, so it is sufficient to prove that $(\Omega(\psi) \setminus \{0\}) \subset N(v_0)$. If it is not true then there exist $q \in \Omega(\psi)$ $(q \neq 0)$ and $\varepsilon > 0$ such that

$$\bar{B}_m(q, 2\varepsilon) \cap (N(v_0) \cup \{0\}) = \emptyset . \tag{5.11}$$

We show that inequality (5.5) holds. Indeed, otherwise there is a sequence $\{s_i\}$ for which $s_i \to \infty$, $\psi(s_i) \to q' \neq 0$, $\dot{V}(\varphi(s_i), s_i) \to 0$ as $i \to \infty$. By Lemma 5.1 these properties imply $|\chi(s_i)| \to \infty$; consequently $q' \in N(v_0)$, which contradicts (5.11).

Since $V \geqq 0$ and $q \in \Omega(\psi)$, it follows from (5.5) the existence of a sequence $\{(t_i, t_i', t_i'')\}$ and a constant c such that

$$t_i < t_i' < t_i'' < t_{i+1}, \quad t_i'' - t_i \leqq c \quad (i = 1, 2, \ldots)$$

and conditions (5.6) are satisfied.

By Lemma 5.1 we have $|\chi(t_i)| \to \infty$ as $i \to \infty$. Using assumption (i') we show that $|\chi(t_i + t)| \to \infty$ uniformly in $t \in [0, T]$ for every $T > 0$. Indeed the estimate

$$|\chi(t_i + t) - \chi(t_i)| = \left|\int_{t_i}^{t_i+t} Z(\varphi(s), s)\, ds\right| \leqq jK \qquad (i = 1, 2, \ldots)$$

holds with some positive constant K, where $j := [T] + 1$, and $[T]$ denotes the integral part of T.

Let $\eta(\varepsilon)$, $\varrho(\varepsilon)$ be defined as in case *a*) in the proof of Theorem 4.1. From the last property of χ with $T = c$ it follows that there is an i_0 such that

$$|\chi(t)| > \varrho(\varepsilon) \qquad (i \geqq i_0,\; t_i' \leqq t \leqq t_i'')\,.$$

Therefore, (5.7) holds and, similarly to (5.8) we get the contradiction $v(t_i'') \to -\infty$ as $i \to \infty$.

This completes the proof of inclusion (5.10).

Consider now the sequence of solution $\psi_i(t)$ of the initial value problems

$$\begin{cases} \dot{y} = Y^{t_i}(y, \chi(t), t) \\ y(0) = \psi(t_i) \end{cases} \qquad (i = 0, 1, 2, \ldots)\,.$$

By Lemma 3.1, the sequence of translates $Y^{t_i}(y, \chi(t), t)$ is precompact, so we can assume that it integrally converges to an ordinary integral-like operator I as $i \to \infty$. Since $\psi(t_i) \to q$, by Theorem 5.3 in [13] there exists a subsequence of $\{\psi_i\}$ converging to a solution of the equation $y = q + I_0 y$, which is a limiting equation of (4.2). The trajectory γ of this solution differs from $\{0\}$ because of $q \neq 0$. On the other hand, obviously $\gamma \subset \Omega(\psi)$ and, by (5.10), $\gamma \subset V_m^{-1}[v_0, \infty]_0$, which contradicts assumption (iii).

The proof is complete.

6. – An application to nonholonomic mechanical systems.

Consider a scleronomic mechanical system with independent Lagrangian coordinates $q = \operatorname{col}(q^1, q^2, \ldots, q^r)$ and assume that the system is subjected to a nonintegrable kinematical constraint

$$\dot{q}_2 = B(q_1, q_2)\dot{q}_1 \qquad (q_1 \in R^{r_1},\; q_2 \in R^{r_2},\; r_1 + r_2 = r)$$

where B is a given continuously differentiable $r_2 \times r_1$ matrix-function. Denote by $U = U(q)$ the force function (i.e. $-U$ is the potential energy; $U(0) = 0$); by $T = T(q, \dot{q}) = \frac{1}{2}\dot{q}^T A(q)\dot{q}$ the kinetic energy where $\dot{q}^T$ is the transposed of the vector $\dot{q}$ of the generalized velocities, $A(q)$ is an $r \times r$ matrix-function; $\boldsymbol{U}$ and $\boldsymbol{A}$ are continuously differentiable. Let us assume that the system is under the action of dissipative and non-energic forces. This means that if $Q = Q(q, \dot{q})$ denotes their resultant then $Q^T(q, \dot{q})\dot{q} \leqq 0$ for all $q, \dot{q} \in R^r$.

Consider the partition of matrix A generated by the partition $\dot{q} = \operatorname{col}(\dot{q}_1, \dot{q}_2)$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are $r_1 \times r_1, r_1 \times r_2, r_2 \times r_1, r_2 \times r_2$, respectively. Introduce the notations

$$\begin{aligned} &\Theta(q, \dot{q}_1) := T(q, \dot{q}_1, B(q)\dot{q}_1) = \tfrac{1}{2}\dot{q}_1^T \tilde{A}(q)\dot{q}_1 \,, \\ &\tilde{A}(q) := A_{11} + A_{12}B + B^T A_{21} + B^T A_{22} B \,, \\ &\hat{Q}(q, q_1) := Q_1(q, \dot{q}_1, B(q)\dot{q}_1) + B^r(q) Q_2(q, \dot{q}_1, B(q)\dot{q}_1) \,. \end{aligned}$$

Denote by $\tilde{\lambda}(q)$ the smallest eigenvalue of the symmetric positive definite matrix $\tilde{A}(q)$.

The motions are described by the Voronec equation ([15], p. 109, see also [7], [8], [16], [17]):

$$\text{(6.1)} \qquad \begin{cases} \dfrac{d}{dt}\dfrac{\partial \Theta}{\partial \dot{q}_1} = \dfrac{\partial(\Theta + U)}{\partial q_1} + \dfrac{\partial(\Theta + U)}{\partial q_2} B + \hat{Q} + \dot{q}_1^T G \\ \dot{q}_2 = B(q)\dot{q}_1 \end{cases}$$

where

$$\begin{aligned} &G = G(q, \dot{q}_1) := (A_{12} + B^T A_{22}) M \,, \\ &M^{ij} := \sum_{s=1}^{r_1} M_s^{ij} \dot{q}^s \qquad (i = 1, \dots, r_2;\ j = 1, \dots, r_1) \\ &M_s^{ij} := \sum_{\nu=1}^{r_2} \left(\frac{\partial B^{ij}}{\partial q_2^\nu} B^{\nu s} - \frac{\partial B^{is}}{\partial q_2^\nu} B^{\nu j} \right), \end{aligned}$$

and M^{ij} denotes the element staying in the i-th row and j-th column in matrix M. Obviously, $M_s^{ij} = -M_j^{is}$ for all i, j, s; consequently

$$\dot{q}_1^T G(q, \dot{q}_1)\dot{q}_1 = 0 \qquad (q \in R^r,\ \dot{q}_1 \in R^{r_1}) \,,$$

i.e. $\dot{q}_1^T G$ can be considered as a « generalized gyroscopic (non-energic) force ».

V. V. RUMJANCEV [8] and C. RISITO [7] investigated the nonholonomic mechanical system (6.1) in the case when $Q_1 = Q_1(q, \dot{q}_1)$ (in Rumjancev's paper $Q_1(q, \dot{q}_1) = D\dot{q}_1$, D is an $r_1 \times r_1$ constant matrix), $Q_2 = 0$ and the dissipation is complete, i.e. $Q_1^T \dot{q}_1 = 0$ if and only if $\dot{q}_1 = 0$.

V. V. RUMJANCEV proved the following

THEOREM A [8]. – *If for every compact set* $K \subset R^{r_1}$

(i) $B(q_1, q_2)$ *is bounded in* $K \times R^{r_2}$;

(ii) $(\partial U/\partial q_1) + (\partial U/\partial q_2)B$ *is negative* q_1*-definite in* $K \times R^{r_2}$;

(iii) U *is negative* q_1*-definite, and this fact can be checked considering only the second derivatives of* U;

then the equilibrium position $q = \dot{q} = 0$ *is asymptotically* $(q_1, \dot{q})$*-stable.*

C. RISITO succeeded in weakening conditions (ii) and (iii), but in his paper condition (i) became stronger. He proved the following

THEOREM B [7]. – *If*

(i′) *the solutions of* (6.1) *starting from some neighbourhood of* $q = 0$, $\dot{q} = 0$ *are uniformly* q_2*-bounded for* $t \geqq 0$;

(ii′) *the equilibrium positions of* (6.1) *are expressed by* $q_1 = 0$, $q_2 = c = \text{const}$;

(iii′) U *is negative* q_1*-definite in every compact set* $K \subset R^r$, *then the equilibrium* $q = \dot{q} = 0$ *is asymptotically* $(q_1, \dot{q})$*-stable.*

Condition (i′) is rather restrictive because it requires « a priori » knowledge of the solutions. Our purpose is to complete the system of conditions (i), (ii′), (iii′) to be sufficient for asymptotic $(q_1, \dot{q})$-stability. In order to do it we apply Theorem 4.2 to system (6.1).

First we have to rewrite (6.1) into a normal form consisting first order differential equations. Introducing the vector $p_1 \in R^{r_1}$ of the generalized momenta and the Hamiltonian function $\boldsymbol{H}$ by

$$p_1 := \left(\frac{\partial \Theta}{\partial \dot{q}_1}\right)^T = \tilde{A}(q)\dot{q}_1 , \qquad H(p_1, q_1) := \frac{1}{2} p_1^T \tilde{A}(q) p_1 - U(q) ,$$

equation (6.1) can be rewritten into the canonical form

$$(6.2) \qquad \begin{cases} \dot{p}_1 = -\left(\dfrac{\partial H}{\partial q_1}\right)^T - B^T \left(\dfrac{\partial H}{\partial q_2}\right)^T + \tilde{Q} + \tilde{G}^T \tilde{A}^{-1} p_1 \\ \dot{q}_1 = \left(\dfrac{\partial H}{\partial p_1}\right)^T \\ \dot{q}_2 = B\tilde{A}^{-1} p_1 , \end{cases}$$

where

$$\tilde{Q} = \tilde{Q}(p_1, q) := \hat{Q}(q, \tilde{A}^{-1}(q, p_1)); \qquad \tilde{G} = \tilde{G}(p_1, q) := G(q, \tilde{A}^{-1}(q), p_1) .$$

The systems in which functions A, U, B, Q are independent of q_2 were investigated by A. S. CHAPLYGIN. Nowadays these systems are called of Chaplygin type ([15], p. 110). We introduce the following

DEFINITION 6.1. – System (6.1) is said to be of *asymptotically Chaplygin type with total dissipation* if

a) for every compact set $K \subset R^{r_1}$ there are $\lambda_K > 0$ and $c_K \in \mathcal{K}$ such that

$$\tilde{\lambda}(q_1, q_2) \geqq \lambda_K; \quad \tilde{Q}^T \tilde{A}^{-1} p_1 \leqq -c_K(|p_1|) \tag{6.3}$$

for all $(q_1, q_2) \in K \times R^{r_2}$, $p_1 \in R^{r_1}$;

b) system (6.2) permits a limiting process as $|q_2| \to \infty$, i.e. $\tilde{A}(q_1, q_2) \to A_*(q_1)$, $B(q_1, q_2) \to B_*(q_1)$ as $|q_2| \to \infty$; and for every pair of compact sets $K, L \subset R^{r_1}$

$$U(q_1, q_2) \to U_*(q_1) , \qquad \frac{\partial U}{\partial q_1} + \frac{\partial U}{\partial q_2} B \to W_*(q_1)$$
$$\tilde{Q}(p_1, q_1, q_2) \to Q_*(p_1, q_1)$$

uniformly in $q_1 \in K$ and $p_1 \in L$ as $|q_2| \to \infty$; moreover, functions $\partial \tilde{A}/\partial q$, $\partial B/\partial q$ converge uniformly in $q_1 \in K$ as $|q_2| \to \infty$.

Now we can apply Theorem 4.2 to system (6.2) in order to get sufficient conditions for asymptotic (p_1, q_1)-stability. The role of the Ljapunov function will be taken by the total mechanical energy

$$H(p_1, q_1, q_2) := \tfrac{1}{2} p_1^T \tilde{A}^{-1}(q_1, q_2) p_1 - U(q_1, q_2) .$$

If the force function U is negative q_1-definite, and (6.1) is an asymptotically Chaplygin system then by the first property in *a*) of Definition 6.1 $\boldsymbol{H}$ is positive (p_1, q_1)-definite.

By (6.3) the derivative of $\boldsymbol{H}$ with respect to (6.2) can be estimated as follows:

$$\dot{H} = \dot{H}(p_1, q_1, q_2) = p_1^T \tilde{A}^{-1} \tilde{Q} \leqq -c_K(|p_1|) \leqq 0 \tag{6.4}$$

for all $p_1 \in R^{r_1}$, $q_1 \in K$, $q_2 \in R^{r_2}$, provided that K is compact. Consequently,

$$H^{-1}_{r_1+r}[c, \infty]_0 = \dot{H}^{-1}(0) \cap H^{-1}(c) = \{(p_1, q_1, q_2) \colon p_1 = 0,\ U(q_1, q_2) = -c\} \qquad (c \in R) ,$$

and the trajectories of (6.2) contained by $H^{-1}_{r_1+r}[c, \infty]_0$ are the equilibria $q = q^0$ such that $U(q^0) = c$. To give the set $H^{-1}_{2r_1}[c, \infty]_0$ we need the following notation. For $c \in R$ denote by $U^{-1}_{r_1}[c, \infty]$ the set of the points q_1 for which there is a sequence $\{(q_1^{(i)}, q_2^{(i)})\}$ such that $q_1^{(i)} \to q_1$, $|q_2^{(i)}| \to \infty$, and $U(q_1^{(i)}, q_2^{(i)}) \to c$ as $i \to \infty$. Then by (6.4) we obtain

$$H^{-1}_{2r_1}[c, \infty]_0 = \{(p_1, q_1): p_1 = 0,\ q_1 \in U^{-1}_{r_1}[\infty, -c]\} \qquad (c \in R) .$$

But $U(q_1, q_2) \to U_*(q_1)$ uniformly in q_1 as $|q_2| \to \infty$, consequently $U^{-1}_{r_1}[c, \infty] = U^{-1}_*(c)$, and

$$H^{-1}_{2r_1}[c, \infty]_0 = \{(p_1, q_1): p_1 = 0,\ U_*(q_1) = c\} . \tag{6.5}$$

If (6.1) is an asymptotically Chaplygin system, then the limiting system of (6.2) as $|q_2| \to \infty$ reads as follows:

$$\begin{cases} \dot p_1 = -\left[\dfrac{\partial}{\partial q_1}\left(\dfrac{1}{2}\, p_1^T A_*^{-1}(q_1)\, p_1\right)\right]^T + W_*(q_1)^T + Q_*(p_1, q_1) + G_*^T A_*^{-1} p_1 \\ \dot q_1 = A_*^{-1}(q_1)\, p_1 , \end{cases} \tag{6.6}$$

where $G_*(p_1, q_1)$ denotes the uniform limit of $\tilde G(p_1, q_1, q_2)$ as $|q_2| \to \infty$. Accordingly to (6.5) the set of the complete trajectories of (6.6) lying in the set $H^{-1}_{2r_1}[c, \infty]_0$ is

$$\{(p_1, q_1): p_1 = 0,\ U_*(q_1) = -c,\ W_*(q_1) = 0\} . \tag{6.7}$$

The limit function $U_*(q_1)$ of the negative q_1-definite function $U(q_1, q_2)$ is also negative definite, so for $c > 0$ set (6.7) consists of the single point $(0, 0)$ if and only if $W_*(q_1) = 0$ implies $q_1 = 0$.

It is easy to see that assumptions (A_1)-(A_4) are satisfied if we assume that $B(q_1, q_2)$ is bounded while q_1 belongs to a compact set. Therefore, applying Theorem 4.2 (see also Remark 4.1) we obtain the following

THEOREM 6.1. – *Suppose that*

(i) *for every compact set* $K \subset R^{r_1}$ *the function* $B(q_1, q_2)$ *is bounded in* $K \times R^{r_2}$;

(ii) *the force function* U *is negative* q_1*-definite*;

(iii) *system* (6.1) *is of Chaplying type asymptotically with total dissipation.*

Then the equilibrium state $q = \dot q = 0$ *of* (6.1) *is* $(q_1, \dot q)$*-stable and asymptotically* $\dot q$*-stable.*

If, in addition, we suppose that

(iv) *in case* (q_1, q_2) *is an equilibrium position of* (6.1) *then* $q_1 = 0$, *i.e. the equality*

$$\frac{\partial U}{\partial q_1}(q_1, q_2) + \frac{\partial U}{\partial q_2}(q_1, q_2)B(q_1, q_2) = 0$$

implies $q_1 = 0$;

(v) *in case* q_1 *is an asymptotic equilibrium position of* (6.1) *then* $q_1 = 0$, *i.e. the relation*

$$\lim_{|q_2|\to\infty}\left(\frac{\partial U}{\partial q_1}(q_1, q_2) + \frac{\partial U}{\partial q_2}(q_1, q_2)B(q_1, q_2)\right) = 0$$

implies $q_1 = 0$,

then the equilibrium state $q = \dot{q} = 0$ *of* (6.1) *is asymptotically* $(q_1, \dot{q}_1)$*-stable.*

Up to now we have studied stationary mechanical systems like V. V. RUMJANCEV and C. RISITO. But our method is applicable also to nonstationary systems. To illustrate this we give a generalization of Risito's theorem to the case of time-dependent friction.

THEOREM 6.2. – *Assume that the dissipative forces* Q *in system* (6.1) *may depend also the time, but for every pair of compact sets* $K, L \subset R^r$, $Q(q, \dot{q}, t) \to Q_*(q, \dot{q})$ *uniformly in* $(q, \dot{q}) \subset K \times L$ *as* $t \to \infty$, *and there exists a* $c_{K,L} \in \mathcal{K}$ *such that*

$$Q_*^T(q, \dot{q})\dot{q} \leqq -c_{K,L}(|\dot{q}|) \qquad (q \in K,\ \dot{q} \in L)\,.$$

If, in addition, assumptions (i′)-(iii′) *of Theorem B are satisfied, then the equilibrium* $q = \dot{q} = 0$ *of* (6.1) *is asymptotically* $(q_1, \dot{q})$*-stable.*

The proof is very similar to that of Theorem 6.1, so it is omitted.

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